Final Exam 30/03/2022

Duration : 3 hours

The subject consists of two independent problems. Do not hesitate to present partial solutions or ideas even when you are not sure how to formulate them rigorously. Documents are allowed.

Problem 1: Duality and Partial optimal transport

(*Duality*) Let X, Y be compact metric spaces and $c : X \times Y \to \mathbb{R}_+ \cup \{+\infty\}$ such that $c \in \mathcal{C}^0(X \times Y)$. In this first part we want to prove that strong duality holds, that is

$$(\mathrm{KP}) = (\mathrm{KD}),$$

where

$$\begin{aligned} &(\mathrm{KP}) = \inf_{\gamma \in \Pi(\mu,\nu)} \int c \mathrm{d}\gamma \\ &\text{where } \Pi(\mu,\nu) = \{\gamma \in \mathcal{P}(X \times Y) \mid \pi_{X\#}\gamma = \mu \text{ and } \pi_{Y\#}\gamma = \nu\} \end{aligned} \tag{1}$$

and

$$(\mathrm{KD}) = \sup_{\substack{(\varphi,\psi) \in \mathcal{C}^0(X) \times \mathcal{C}^0(Y) \\ \varphi \oplus \psi \leqslant c}} \int \varphi \mathrm{d}\mu + \int \psi \mathrm{d}\nu \tag{2}$$

(Q1.1) Prove that (KD) can be re-written as

$$(\mathrm{KD}) = \sup_{u \in \mathcal{C}^0(X \times Y)} \{-K_1(u) - K_2(u)\}, \tag{3}$$

where

$$K_1 : u \in \mathcal{C}^0(X \times Y) \mapsto \begin{cases} \int \varphi \mathrm{d}\mu + \int \psi \mathrm{d}\nu & \text{if } u(x,y) = \varphi(x) + \psi(y) \\ +\infty & \text{otherwise} \end{cases},$$

and

$$K_2 : u \in \mathcal{C}^0(X \times Y) \mapsto \begin{cases} 0 & \text{if } u(x,y) \leqslant c(x,y) \\ +\infty & \text{otherwise} \end{cases}.$$

(Q1.2) Show that K_1, K_2 are convex and there exists $u_0 \in \mathcal{C}^0(X \times Y)$ such that both K_1 and K_2 are finite at u_0 and K_2 is continuous at u_0.

(Q1.3) Compute the Legendre-Fenchel transforms $K_1^*, K_2^* : \mathcal{C}^0(X \times Y)' \to \mathbb{R} \cup \{+\infty\}$ where $\mathcal{C}^0(X \times Y)'$ denotes the topological dual space of $\mathcal{C}^0(X \times Y)$.

(Q1.4) Prove that

$$\sup_{u \in \mathcal{C}^0(X \times Y)} \{-K_1(u) - K_2(u)\} = \inf_{\gamma \in \mathcal{C}^0(X \times Y)'} \{K_1^*(-\gamma) + K_2^*(\gamma)\},$$

and deduce that strong duality holds.

(*Partial OT*) We consider the partial optimal transport problem between a non-negative measure $\mu \in \mathcal{M}_+(X)$ with $1 < \mu(X) < +\infty$ and a probability measure $\nu \in \mathcal{P}(Y)$:

$$\begin{aligned} &(\mathrm{KPP}) = \inf_{\gamma \in \Pi_-(\mu,\nu)} \int c \mathrm{d}\gamma \\ &\text{where } \Pi_-(\mu,\nu) = \{\gamma \in \mathcal{M}_+(X \times Y) \mid \pi_{X\#}\gamma \leqslant \mu \text{ and } \pi_{Y\#}\gamma = \nu\} \end{aligned} \tag{4}$$

(Note that for $\alpha, \beta \in \mathcal{M}_+(X)$, $\alpha \leqslant \beta$ means $\forall \varphi \in \mathcal{C}^0(X), (\varphi \geqslant 0 \implies \int \varphi \mathrm{d}\alpha \leqslant \int \varphi \mathrm{d}\beta)$.)

(Q1.5) Prove that (KPP) $\geqslant$ (KPD) where

$$(\mathrm{KPD}) = \sup_{\substack{(\varphi,\psi) \in \mathcal{C}^0(X) \times \mathcal{C}^0(Y) \\ \varphi \oplus \psi \leqslant c,\ \varphi \leqslant 0}} \int \varphi \mathrm{d}\mu + \int \psi \mathrm{d}\nu \tag{5}$$

(Q1.6) Recalling that for $\psi \in \mathcal{C}^0(Y)$, $\psi^c(x) = \inf_{y \in Y} c(x,y) - \psi(y)$, prove that

$$(\mathrm{KPD}) = \sup_{\psi \in \mathcal{C}^0(Y)} \int \min(\psi^c, 0) \mathrm{d}\mu + \int \psi \mathrm{d}\nu. \tag{6}$$

Let $\overline{Y} = Y \sqcup \{y_\infty\}$ where $y_\infty \notin Y$ is an arbitrary point, let $\overline{\nu} = \nu + \alpha \delta_{y_\infty}$ where $\alpha = \mu(X) - 1$, and let $\overline{c} : X \times \overline{Y} \to \mathbb{R}$ defined by $\overline{c}|_{X \times Y} = c$ and $\overline{c}(x, y_\infty) = 0$ for all $x \in X$.

(Q1.7) Using an explicit bijection between $\Pi(\mu, \overline{\nu})$ and $\Pi_-(\mu,\nu)$, prove that (KPP) $=$ ($\overline{\mathrm{KP}}$) where ($\overline{\mathrm{KP}}$) is the "standard" optimal transport problem

$$(\overline{\mathrm{KP}}) = \min_{\overline{\gamma} \in \Pi(\mu, \overline{\nu})} \int_{X \times \overline{Y}} \overline{c} \mathrm{d}\overline{\gamma}.$$

(Q1.8) Show that if $\overline{\psi} : \overline{Y} \to \mathbb{R}$ is defined by $\overline{\psi}\big|_Y = \psi$ and $\overline{\psi}(y_\infty) = 0$, then $\overline{\psi}^{\overline{c}} = \min(\psi^c, 0)$. Prove that (KPP) $=$ (KPD) and that the maximum in (KPD) is attained.

Problem 2: Eulerian Optimal Transport with a source term.

Let B be the closed unit ball in $\mathbb{R}^d$, $Q = [0,1] \times B$, and let be given two smooth functions $\gamma_0 > 0$, $\gamma_1 > 0$ on B.
We set, for each continuous function $(t,x) \to f(t,x)$ on Q,

$$BT(f) = \int_B (f(1,x)\gamma_1(x) - f(0,x)\gamma_0(x))dx.$$

For all continuous functions A, B, C on Q, respectively valued in $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}$, we set $F(A,B,C) = 0$ if $A + |B|^2/2 + C^2/2 \leq 0$ pointwise on Q, and $F(A,B,C) = +\infty$ otherwise. We define J to be the supremum of $BT(\varphi)$ over all smooth functions φ on Q such that

$$\partial_t \varphi(t,x) + \frac{1}{2}(|\nabla\varphi(t,x)|^2 + \varphi(t,x)^2) \leq 0, \quad \forall (t,x) \in Q$$

and consider a maximizing family (φ_ϵ) such that $BT(\varphi_\epsilon) \geq J - \epsilon$, for $\epsilon \in]0,1]$.

(Q2.1) Using the Fenchel-Rockafellar duality theorem (involving F and a suitable function G to be found), show the existence of some bounded Borel measures (ρ, m, ν) defined on Q and respectively valued in $\mathbb{R}_+$, $\mathbb{R}^d$ and $\mathbb{R}$ such that : m and ν are absolutely continuous with respect to ρ with respective Radon-Nikodym densities $v \in L^2(Q, \rho\,; \mathbb{R}^d)$ and $w \in L^2(Q, \rho\,; \mathbb{R})$.

$$J = \int_{(t,x)\in Q} \frac{1}{2}(|v(t,x)|^2 + w(t,x)^2)\rho(t,x),$$

$$\int_{(t,x)\in Q} (\partial_t f(t,x) + v(t,x)\cdot\nabla f(t,x) + w(t,x)f(t,x))\rho(t,x) = BT(f),$$

for each C^1 function f on Q. Show that

$$\int_Q |\partial_t\varphi_\epsilon + \frac{1}{2}|\nabla\varphi_\epsilon|^2 + \frac{1}{2}\varphi_\epsilon^2|\rho \leq \epsilon, \quad \int_Q \frac{1}{2}|v - \nabla\varphi_\epsilon|^2\rho \leq \epsilon, \quad \int_Q \frac{1}{2}(w - \varphi_\epsilon)^2\rho \leq \epsilon.$$

(Q2.2) Using the results of the previous question, discuss the convergence of φ_ϵ as $\epsilon \downarrow 0$ to a limit φ, if we assume that we can write the measure ρ as $\gamma(t,x)dtdx$, for some function γ with $r \leq \gamma \leq r^{-1}$ for some constant $r \in]0,1]$. Get the following system of equations, in a suitable sense (to be precised),

$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 = 0, \quad \partial_t\gamma + \nabla\cdot(\gamma\nabla\varphi) = \gamma\varphi.$$

(Q2.3) Find the optimal φ in the special case γ_0 and γ_1 are two distinct constant functions on B. Find also v and try to explain the result.