RATIONAL POINTS AND OBSTRUCTIONS TO THEIR EXISTENCE
2015 ARIZONA WINTER SCHOOL PROBLEM SET

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The primary goal of the problems below is to build up familiarity with some useful lemmas and examples that are related to the theme of the Winter School. In case you get stuck on any particular question, consult the extended version of the problem set.

Notation. For a field k, we denote by $\overline{k}$ a choice of its algebraic closure, and by $k^s \subset \overline{k}$ the resulting separable closure. If k is a number field and v is its place, we write k_v for the corresponding completion. If $k = \mathbb{Q}$, we write $p \leqslant \infty$ to emphasize that p is allowed be the infinite place; for this particular p, we write $\mathbb{Q}_p$ to mean $\mathbb{R}$. For a base scheme S and S-schemes X and Y, we write $X(Y)$ for the set of S-morphisms $Y \to X$. When dealing with affine schemes we sometimes omit Spec for brevity: for instance, we write $\mathrm{Br}\, R$ in place of $\mathrm{Br}(\mathrm{Spec}\, R)$. A 'torsor' always means a 'right torsor.'

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1. Rational points

In this section, k is a field and X is a k-scheme. A *rational point* of X is an element $x \in X(k)$, i.e., a section $x\colon \mathrm{Spec}\, k \to X$ of the structure map $X \to \mathrm{Spec}\, k$.

1.1. Suppose that $X = \mathrm{Spec}\, \frac{k[T_1, \ldots, T_n]}{(f_1, \ldots, f_m)}$. Find a natural bijection

$$X(k) \longleftrightarrow \{(x_1, \ldots, x_n) \in k^n \quad \text{such that} \quad f_i(x_1, \ldots, x_n) = 0 \quad \text{for every } i = 1, \ldots, m\}.$$

1.2. (a) Prove that the image of a rational point $x\colon \mathrm{Spec}\, k \to X$ is necessarily a closed point of the underlying topological space of X; in fact, prove that x is a closed immersion.

(b) Deduce a strengthening of the first part of (a): for any finite extension L/k, the image of any k-morphism $\mathrm{Spec}\, L \to X$ is a closed point of the underlying topological space of X. In particular, a point $x \in X$ whose residue field $k(x)$ is a finite extension of k is a closed point.

(c) Prove a partial converse: if X is locally of finite type over k and $x \in X$ is a closed point, then the residue field $k(x)$ is a finite extension of k.

1.3. If k is finite and X is of finite type, prove that $X(k)$ is finite.

1.4. Use **1.2.** (a) to prove that every k-group scheme G is necessarily separated.

1.5. Suppose that X is of finite type over k and connected. If X has a rational point, prove that X is *geometrically connected*, i.e., that the base change $X_{\overline{k}}$ is still connected.

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1.6. Suppose that X is smooth over k and nonempty. Prove that there is a closed point $x \in X$ with $k(x)/k$ finite separable; in fact, prove that such x are Zariski dense.

1.7. Let $\mathfrak{o}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$, and let $\mathbb{F}$ be the residue field of $\mathfrak{o}$. Prove Hensel's lemma: for a smooth $\mathfrak{o}$-scheme $\mathcal{X}$, the pullback map $\mathcal{X}(\mathfrak{o}) \to \mathcal{X}(\mathbb{F})$ is surjective, i.e., every $\mathbb{F}$-point of $\mathcal{X}$ may be lifted to an $\mathfrak{o}$-point.

1.8. Suppose that k is a finite extension of $\mathbb{Q}_p$ and that X is of finite type, irreducible, and extends to a smooth $\mathfrak{o}$-scheme $\mathcal{X}$ of finite type whose special fiber is nonempty. Prove that the points of X valued in unramified extensions of k are Zariski dense.

1.9. Suppose that X is of finite type over k and regular. Prove that every rational point $x\colon \operatorname{Spec} k \to X$ factors through a k-smooth open subscheme $U \subset X$.

2. Rational points on torsors

In this section, k is a field and G is a k-group scheme of finite type.

- A *right action* of G on a k-scheme X is a morphism $X \times_k G \to X$ that induces a right $G(S)$-action on $X(S)$ for every k-scheme S.
- A *trivial torsor* under G is a k-scheme X equipped with the right action of G such that X is isomorphic to G equipped with its right translation action (the isomorphism is required to respect the actions of G). A choice of such an isomorphism is a *trivialization* of X.
- A *torsor* under G (or a *G-torsor*) is a k-scheme X equipped with a right action of G such that for some finite extension k'/k the base change $X_{k'}$ is a trivial torsor under $G_{k'}$.

If G is commutative and smooth (smoothness is automatic if $\operatorname{char} k = 0$), then there is a bijection

$$\{\text{isomorphism classes of } G\text{-torsors } X\} \longleftrightarrow H^1(k, G(k^s)). \qquad (\bigstar)$$

2.1. For a G-torsor X, find a natural bijection

$$\{\text{trivializations of } X\} \longleftrightarrow \{\text{rational points } x \in X(k)\}.$$

2.2. If G fits into a short exact sequence $1 \to G \to H \to Q \to 1$ of k-group schemes of finite type and $x \in Q(k)$, prove that the fiber $H_x := H \times_{Q,x} \operatorname{Spec} k$ is a G-torsor. When is it trivial?

2.3. For $a, b \in k^\times$, prove that $G := \operatorname{Spec}\left(\frac{k[x,y]}{(x^2-ay^2-1)}\right)$ has a structure of a k-group scheme and $X := \operatorname{Spec}\left(\frac{k[x,y]}{(x^2-ay^2-b)}\right)$ has a structure of its torsor.

2.4. Prove that every $\mathbb{G}_m$-torsor over k is trivial. Prove the same for $\mathbb{G}_a$-torsors. Deduce that every G-torsor is trivial if G admits a filtration whose subquotients are either $\mathbb{G}_m$ or $\mathbb{G}_a$.

2.5. If G is smooth, prove that every G-torsor trivializes over a finite *separable* extension k'/k.

2.6. Suppose that the field k is finite.

(a) If G is an abelian variety, prove that every G-torsor is trivial.

(b) If X is a proper smooth geometrically connected k-curve of genus 1, prove that $X(k) \neq \varnothing$.

2.7. Suppose that k is a finite extension of $\mathbb{Q}_p$ and that A is a nonzero abelian variety over k.

(a) Prove that up to isomorphism there are only finitely many A-torsors X for which the associated class in $H^1(k, A)$ is killed by an integer that is prime to p.

(b) Prove that up to isomorphism there are infinitely many A-torsors.

2.8. Suppose that k is a finite extension of $\mathbb{Q}_p$. Let E be an elliptic curve over k, and let X be a torsor under E. The *period* of X is the order n of the corresponding class in $H^1(k, E)$. The *index* of X is the greatest common divisor of the degrees of closed points on X. Lichtenbaum has proved in [Lic68, Thm. 3] that period equals index under our assumptions. Assuming Lichtenbaum's result, prove that X even has a closed point of degree n.

3. Brauer groups

In this section, X is a scheme.

- An *Azumaya algebra* over X is a coherent $\mathscr{O}_X$-algebra $\mathscr{A}$ such that for some étale cover $\{f_i \colon X_i \to X\}$ there are $\mathscr{O}_{X_i}$-algebra isomorphisms $f_i^* \mathscr{A} \cong \mathrm{Mat}_{n_i \times n_i}(\mathscr{O}_{X_i})$ for some $n_i \in \mathbb{Z}_{>0}$.
- Azumaya algebras $\mathscr{A}$ and $\mathscr{A}'$ over X are *similar* if there is an $\mathscr{O}_X$-algebra isomorphism
$$\mathscr{A} \otimes_{\mathscr{O}_X} \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}) \cong \mathscr{A}' \otimes_{\mathscr{O}_X} \underline{\mathrm{End}}_{\mathscr{O}_X}(\mathscr{E}')$$
for some locally free coherent $\mathscr{O}_X$-modules $\mathscr{E}$ and $\mathscr{E}'$ that are stalkwise nonzero.
- The set of similarity classes of Azumaya algebras over X forms an abelian group with $- \otimes_{\mathscr{O}_X} -$ as the group operation. This *Azumaya Brauer group* of X is denoted by $\mathrm{Br}_{\mathrm{Az}} X$.
- The *Brauer group* of X is $\mathrm{Br}\, X := H^2_{\text{ét}}(X, \mathbb{G}_m)$.
- If X is regular, Noetherian, and has an ample invertible sheaf (in the sense of [EGA II, 4.5.3]), then $\mathrm{Br}_{\mathrm{Az}} X = \mathrm{Br}\, X$ and both of these groups are torsion, see [Gro68, Prop. 1.4] and [dJ, Thm. 1.1]. For example, this holds if X is a smooth quasi-projective scheme over a field.

Caution. Some authors use different definitions! For example, instead of meaning $\mathrm{Br}\, X$ the term *the Brauer group of* X may mean either $\mathrm{Br}_{\mathrm{Az}} X$ or $(\mathrm{Br}\, X)_{\mathrm{tors}}$.

3.1. Prove that similarity of Azumaya algebras over X is an equivalence relation.

3.2. Prove that an Azumaya algebra $\mathscr{A}$ over X is in particular a locally free $\mathscr{O}_X$-module whose rank at every point $x \in X$ is a square. Deduce that for every fixed $n \in \mathbb{Z}_{>0}$, the locus where the rank of $\mathscr{A}$ is n^2 is an open and closed subscheme of X.

3.3. Suppose that $X = \mathrm{Spec}\, k$ with k a field.

(a) Find a natural bijection
$$\{\mathrm{PGL}_n\text{-torsors over } X\}/\simeq \quad \longleftrightarrow \quad \{\text{Azumaya algebras over } X \text{ of rank } n^2\}/\simeq .$$
(On both sides, "$/\simeq$" means "up to isomorphism.")

(b) Find the following maps:
$$H^1(k, \mathrm{PGL}_n) \hookrightarrow H^2(k, \mu_n) \xrightarrow{\sim} (\mathrm{Br}\, k)[n].$$
Combine them with (a) to prove that every Azumaya k-algebra gives an element of $\mathrm{Br}\, k$.

3.4. Prove that if a field k is a filtered union of its subfields k_i, then
$$\mathrm{Br}\, k = \varinjlim \mathrm{Br}\, k_i.$$

3.5. Prove that the Brauer group of a finite field is trivial.

3.6. If k is a finite extension of $\mathbb{Q}_p$ and $\mathfrak{o}$ is its ring of integers, prove that $\operatorname{Br}\mathfrak{o} = 0$.

3.7. If K is a number field and K^{ab} is its maximal abelian extension, prove that $\operatorname{Br} K^{\mathrm{ab}} = 0$.

3.8. Let R be a discrete valuation ring and K its field of fractions.

(a) Prove that $H^1(R, \mathrm{PGL}_n) \to H^1(K, \mathrm{PGL}_n)$ has trivial kernel, i.e., that there is no nontrivial $(\mathrm{PGL}_n)_R$-torsor $\mathcal{T}$ whose base change to K is trivial.

(b) Prove that $\operatorname{Br} R \to \operatorname{Br} K$ is injective.

3.9. (a) Prove that $\operatorname{Br}\mathbb{Z} = 0$.

(b) For a proper smooth curve X over a finite field, prove that $\operatorname{Br} X = 0$.

3.10. Recall Tsen's theorem: if $\overline{k}(X)$ is the function field of an integral curve X over an algebraically closed field $\overline{k}$, then $\operatorname{Br}\overline{k}(X) = 0$.

Prove that if k is a perfect field, then $\operatorname{Br} k \to \operatorname{Br}\mathbb{P}^1_k$ is an isomorphism.

3.11. Suppose that X is equipped with a structure map $f\colon X \to \operatorname{Spec} k$ for some field k.

(a) If $X(k) \neq \varnothing$, prove that $\operatorname{Br} k \xrightarrow{\operatorname{Br}(f)} \operatorname{Br} X$ is injective.

(b) If k is a number field and $\prod_v X(k_v) \neq \varnothing$, prove that $\operatorname{Br} k \xrightarrow{\operatorname{Br}(f)} \operatorname{Br} X$ is injective even when $X(k) = \varnothing$.

4. The Hasse principle

In this section, K is a number field and X is a K-scheme of finite type. A rational point $x \in X(K)$ gives rise to local points $x_v \in X(K_v)$, one for each place v of K. In particular,

$$X(K) \neq \varnothing \qquad \Longrightarrow \qquad \textstyle\prod_v X(K_v) \neq \varnothing.$$

One may wonder whether the existence of local points forces the existence of a global point:

$$\textstyle\prod_v X(K_v) \neq \varnothing \qquad \overset{?}{\Longrightarrow} \qquad X(K) \neq \varnothing.$$

If it does, then X *satisfies the Hasse principle*. If it does not, then X *violates the Hasse principle*.

4.1. (a) Prove that $\operatorname{Proj}\left(\frac{\mathbb{Q}[x,y]}{(x^2 - ay^2)}\right)$ satisfies the Hasse principle for every $a \in \mathbb{Q}^\times$.

(b) Prove that $\operatorname{Proj}\left(\frac{K[x,y]}{(x^2 - ay^2)}\right)$ satisfies the Hasse principle for every $a \in K^\times$.

(c) Prove that $\operatorname{Proj}\left(\frac{\mathbb{Q}(\sqrt{7})[x,y]}{(x^8 - 16y^8)}\right)$ violates the Hasse principle (over $\mathbb{Q}(\sqrt{7})$).

4.2. Prove that $\mathbb{Z}/n\mathbb{Z}$-torsors over K satisfy the Hasse principle.

4.3. A *Severi–Brauer variety* over K is a K-scheme S for which there is a K^s-isomorphism $S_{K^s} \cong \mathbb{P}^n_{K^s}$ for some $n \geqslant 0$.

(a) Find a natural bijection

$$\{(\mathrm{PGL}_{n+1})_K\text{-torsors}\}/\simeq \quad \longleftrightarrow \quad \{n\text{-dimensional Severi–Brauer varieties over } K\}/\simeq .$$

(On both sides, "$/\simeq$" means "up to isomorphism.")

(b) Prove that an n-dimensional Severi–Brauer variety S is isomorphic to $\mathbb{P}^n_K$ if and only if $S(K) \neq \varnothing$.

(c) Prove that Severi–Brauer varieties satisfy the Hasse principle.

4.4. The goal of this question is to work out an example of Lind [Lin40] and Reichardt [Rei42]:

$$X := \operatorname{Proj}\left(\frac{\mathbb{Q}[x,y,z]}{(x^4-17y^4-2z^2)}\right), \quad \text{where the grading has } x \text{ and } y \text{ in degree 1 and } z \text{ in degree 2,}$$

violates the Hasse principle; in other words, $x^4 - 17y^4 = 2z^2$ has a nonzero solution in $\mathbb{Q}_p$ for every $p \leqslant \infty$, but does not have any nonzero solution in $\mathbb{Q}$.

(a) Prove that X is a smooth geometrically connected curve of genus 1.

(b) Prove that $x^4 - 17y^4 = 2z^2$ has a nonzero solution in $\mathbb{F}_p$ for every prime $p \notin \{2, 17\}$.

(c) Prove that $X(\mathbb{Q}_p) \neq \varnothing$ for every prime $p \notin \{2, 17\}$.

(d) Use the 2-adic logarithm to prove that $X(\mathbb{Q}_2) \neq \varnothing$. Prove that $X(\mathbb{Q}_{17}) \neq \varnothing$ by exploiting the fact that 17 splits in $\mathbb{Q}(\sqrt{2})$. Observe that $X(\mathbb{R}) \neq \varnothing$.

(e) Prove that $X(\mathbb{Q}) = \varnothing$.

5. The Brauer–Manin obstruction

In this section, K is a number field, $\mathbb{A}_K$ is its ring of adeles, and X is a separated K-scheme of finite type, so that $X(\mathbb{A}_K) \subset \prod_v X(K_v)$ (see **5.1.** (d)).

- The *Brauer–Manin set* of X is

  $$X(\mathbb{A}_K)^{\mathrm{Br}} := \{(x_v)_v \in X(\mathbb{A}_K) \quad \text{for which} \quad \textstyle\sum_v \operatorname{inv}_v(x_v^*(B)) = 0 \quad \text{for every} \quad B \in \operatorname{Br} X\}.$$

  The Brauer–Manin set fits into inclusions

  $$X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K) \subset \textstyle\prod_v X(K_v).$$

- If $\prod_v X(K_v) \neq \varnothing$ but $X(\mathbb{A}_K)^{\mathrm{Br}} = \varnothing$, so that necessarily $X(K) = \varnothing$, then X has a *Brauer–Manin obstruction* to the local-global principle. In this case, the absence of rational points of X is explained by the emptiness of the Brauer–Manin set.

The aim of the first few questions is to solidify the understanding of these ideas.

5.1. (a) Prove that there is a nonempty open $U \subset \operatorname{Spec} \mathcal{O}_K$ and a separated U-scheme $\mathcal{X}$ of finite type for which one may fix an isomorphism $\mathcal{X}_K \cong X$. Prove uniqueness of $\mathcal{X}$ up to shrinking U: if $\mathcal{X} \to U$ and $\mathcal{X}' \to U'$ both extend X, then the composite isomorphism $\mathcal{X}_K \cong X \cong \mathcal{X}'_K$ extends to an isomorphism $\mathcal{X}_{U''} \cong \mathcal{X}'_{U''}$ for some nonempty open $U'' \subset U \cap U'$.

(b) With $\mathcal{X}$ as in (a), prove that the restricted product $\prod'_v(X(K_v), \mathcal{X}(\mathcal{O}_v))$ is an independent of $\mathcal{X}$ subset of $\prod_v X(K_v)$.

(c) For an $\mathcal{X}$ as in (a) and each finite set of places Σ containing the places that do not correspond to a closed point of U, prove that pullback maps induce an isomorphism

$$\mathcal{X}(\textstyle\prod_{v\in\Sigma} K_v \times \prod_{v\notin\Sigma} \mathcal{O}_v) \xrightarrow{\sim} \prod_{v\in\Sigma} \mathcal{X}(K_v) \times \prod_{v\notin\Sigma} \mathcal{X}(\mathcal{O}_v).$$

(d) Using (b) to interpret the restricted product, prove that $X(\mathbb{A}_K) = \prod'_v X(K_v)$.

(e) If X is proper, prove that $X(\mathbb{A}_K) = \prod_v X(K_v)$.

5.2. For an $(x_v) \in X(\mathbb{A}_K) \subset \prod_v X(K_v)$ and a $B \in \operatorname{Br} X$, prove that $\operatorname{inv}_v(x_v^*(B)) = 0$ for all but finitely many v.

5.3. Prove that $X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}}$.

5.4. Suppose that $f\colon X \to X'$ is a morphism of separated K-schemes of finite type.

(a) Prove that $f(\mathbb{A}_K)\colon X(\mathbb{A}_K) \to X'(\mathbb{A}_K)$ maps $X(\mathbb{A}_K)^{\mathrm{Br}}$ into $X'(\mathbb{A}_K)^{\mathrm{Br}}$.

(b) Assume that $\prod_v X(K_v) \neq \varnothing$, so that necessarily $\prod_v X'(K_v) \neq \varnothing$. If X' has a Brauer–Manin obstruction to the local–global principle, prove that so does X.

5.5. Recall that each $X(K_v)$ has a "v-adic topology" inherited from K_v: if X has a closed immersion into some $\mathbb{A}^n$, then the v-adic topology on $X(K_v)$ is just the subspace topology of the v-adic topology on $\mathbb{A}^n(K_v) = K_v^n$; in general, the v-adic topology on $X(K_v)$ is described by also requiring that $U(K_v) \subset X(K_v)$ be open for each affine open $U \subset X$. The identification $X(\mathbb{A}_K) = \prod' X(K_v)$ of **5.1.** (d) then endows $X(\mathbb{A}_K)$ with the restricted product topology.

For (a), (b), and (c) below, suppose that the separated finite type K-scheme X is *regular*.

(a) For a Brauer class $B \in \operatorname{Br} X$ and a place v, prove that the map

$$X(K_v) \to \mathbb{Q}/\mathbb{Z}, \qquad x_v \mapsto \operatorname{inv}_v(x_v^* B)$$

is locally constant for the v-adic topology on $X(K_v)$.

(b) Prove that the evaluation of a Brauer class $B \in \operatorname{Br} X$ defines a continuous map

$$X(\mathbb{A}_K) \to \mathbb{Q}/\mathbb{Z}, \qquad (x_v) \mapsto \textstyle\sum_v \operatorname{inv}_v(x_v^* B),$$

where $\mathbb{Q}/\mathbb{Z}$ is endowed with the discrete topology.

(c) Prove that $X(\mathbb{A}_K)^{\mathrm{Br}}$ is closed in $X(\mathbb{A}_K)$.

5.6. The goal of this question is to work out an example of Birch and Swinnerton-Dyer [BSD75]:

$$X := \operatorname{Proj}\left(\frac{\mathbb{Q}[u,v,x,y,z]}{(uv - x^2 + 5y^2, (u+v)(u+2v) - x^2 + 5z^2)}\right)$$

has a Brauer–Manin obstruction to the local-global principle. In other words, $X(\mathbb{Q}_p) \neq \varnothing$ for every $p \leqslant \infty$ but $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \varnothing$, so that $X(\mathbb{Q}) = \varnothing$, too.

(a) Prove that X is a smooth, projective, geometrically connected surface over $\mathbb{Q}$.

(b) A smooth, projective, geometrically connected surface Y over a field k is a *del Pezzo surface* if the line bundle $\omega_{Y/k}^{-1}$ is ample, where $\omega_{Y/k} = \bigwedge^2 \Omega^1_{Y/k}$. Prove that X is a del Pezzo surface over $\mathbb{Q}$.

(c) The *degree* of a del Pezzo surface $Y \to \operatorname{Spec} k$ is the self-intersection number of the canonical line bundle $\omega_{Y/k}$. Prove that the degree of X is 4.

(d) Verify that the points

$$(0:0:\sqrt{5}:1:1), \qquad (1:1:1:0:\sqrt{-1}),$$
$$(1:0:0:0:\tfrac{1}{\sqrt{-5}}), \qquad (-5:1:0:1:\sqrt{\tfrac{-12}{5}})$$

lie on X. Use these points to prove that $X(\mathbb{Q}_p) \neq \varnothing$ for every $p \leqslant \infty$.

(e) Let F be the function field of X. Use the cup product

$$(\, , \,)\colon H^1(F,\mathbb{Z}/2\mathbb{Z}) \times H^1(F,\mu_2) \to H^2(F,\mu_2) = (\operatorname{Br} F)[2]$$

to make sense of the following 2-torsion classes in $\operatorname{Br} F$:

$$(5, \tfrac{u+v}{u}), \qquad (5, \tfrac{u+v}{v}), \qquad (5, \tfrac{u+2v}{u}), \qquad (5, \tfrac{u+2v}{v}).$$

(f) For a suitable finite extension F'/F, use the projection formula

$$(\,\cdot\,, \mathrm{Norm}_{F'/F}(-)) = \mathrm{Norm}_{F'/F}((\mathrm{Res}_{F'/F}(\cdot), -))$$

to prove that $(5, \frac{u+v}{u}) = (5, \frac{u+v}{v}) = (5, \frac{u+2v}{u}) = (5, \frac{u+2v}{v})$ in $\operatorname{Br} F$.

(g) Admit the existence and exactness of the residue sequences

$$0 \to \operatorname{Br} U \to \operatorname{Br} F \to \textstyle\bigoplus_{u \in U^{(1)}} H^1(k(u), \mathbb{Q}/\mathbb{Z}),$$

in which $U \subset X$ is a nonempty open, the direct sum is indexed by height 1 points $u \in U$, the residue field of u is denoted by $k(u)$, and the maps $\operatorname{Br} F \to H^1(k(u), \mathbb{Q}/\mathbb{Z})$ do not depend on the choice of an open U containing u. Use these sequences to prove that the element $b \in \operatorname{Br} F$ exhibited in (f) extends to a $B \in \operatorname{Br} X$.

(h) If $p \leqslant \infty$ is a prime different from 5 and $x_p \in X(\mathbb{Q}_p)$, prove that $\mathrm{inv}_p(x_p^* B) = 0$.

(i) If $x_5 \in X(\mathbb{Q}_5)$, prove that $\mathrm{inv}_5(x_5^* B) = \frac{1}{2}$.

(j) Prove that $X(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} = \varnothing$.

6. The étale Brauer–Manin obstruction

As in §5, we assume that K is a number field and X is a separated K-scheme of finite type.

- For an X-group scheme $\mathcal{G}$ and $\mathcal{G}$-torsors $Y \to X$ and $Y' \to X$, the *isomorphism functor* $\mathrm{Isom}_\mathcal{G}(Y, Y')$ is the fppf sheaf

  $$S \mapsto \{\mathcal{G}_S\text{-torsor isomorphisms } Y_S \xrightarrow{\sim} Y'_S\}, \qquad \text{where } S \text{ is a variable } X\text{-scheme}.$$

  If $\mathcal{G} \to X$ is affine, then $\mathrm{Isom}_\mathcal{G}(Y, Y')$ is representable[1] by an X-scheme that is X-affine.

- The *étale Brauer–Manin set* of X is

  $$X(\mathbb{A}_K)^{\text{ét,Br}} := \bigcap_{G \text{ and } Y \to X} \; \bigcup_{[\mathcal{T}] \in H^1(K,G)} \mathrm{Im}\left((\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X)(\mathbb{A}_K))^{\mathrm{Br}} \to X(\mathbb{A}_K) \right),$$

  where the intersection is taken over the isomorphism classes of finite étale K-group schemes G and over the isomorphism classes of G_X-torsors $Y \to X$, and the union is taken over the isomorphism classes of G-torsors $\mathcal{T}$. The étale Brauer–Manin set fits into inclusions

  $$X(K) \subset X(\mathbb{A}_K)^{\text{ét,Br}} \subset X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K) \subset \textstyle\prod_v X(K_v).$$

- If $\prod_v X(K_v) \neq \varnothing$ but $X(\mathbb{A}_K)^{\text{ét,Br}} = \varnothing$, so that necessarily $X(K) = \varnothing$, then X has an *étale Brauer–Manin obstruction* to the local-global principle.

The questions below are intended to help internalize the above notions.

6.1. Let $\mathcal{G}$ be an X-group scheme, and let $Y \to X$ and $Y' \to X$ be $\mathcal{G}$-torsors.

[1] If $\mathcal{G} \to X$ is not assumed to be affine, then $\mathrm{Isom}_\mathcal{G}(Y, Y')$ is only representable by an algebraic space. This is "good enough" for most practical purposes.

(a) Consider the *automorphism functor* $\mathrm{Aut}_{\mathcal{G}}(Y)$:

$$S \mapsto \{\mathcal{G}_S\text{-torsor isomorphisms } Y_S \xrightarrow{\sim} Y_S\}, \qquad \text{where } S \text{ is a variable } X\text{-scheme.}$$

Prove that $\mathrm{Aut}_{\mathcal{G}}(Y)$ is a sheaf on the fppf site of X.

(b) If $\mathcal{G} \to X$ is affine, prove that $\mathrm{Aut}_{\mathcal{G}}(Y)$ is representable by an X-scheme that is X-affine.

(c) Prove that $\mathrm{Isom}_{\mathcal{G}}(Y, Y')$ is an $\mathrm{Aut}_{\mathcal{G}}(Y)$-torsor fppf sheaf.

(d) If $\mathcal{G} \to X$ is affine, prove that $\mathrm{Isom}_{\mathcal{G}}(Y, Y')$ is representable by an X-scheme that is X-affine.

6.2. (a) If G is a finite étale K-group scheme, $\mathcal{T} \to \operatorname{Spec} K$ is a G-torsor, and $Y \to X$ is a G_X-torsor, prove that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X) \to X$ is finite étale. Conclude that $\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X)$ is a separated K-scheme of finite type, so that $(\mathrm{Isom}_{G_X}(Y, \mathcal{T}_X)(\mathbb{A}_K))^{\mathrm{Br}}$ makes sense.

(b) Prove that $X(\mathbb{A}_K)^{\text{ét},\mathrm{Br}} \subset X(\mathbb{A}_K)^{\mathrm{Br}}$.

6.3. Prove that $X(K) \subset X(\mathbb{A}_K)^{\text{ét},\mathrm{Br}}$.

6.4. Suppose that $f\colon X \to X'$ is a morphism of separated K-schemes of finite type.

(a) Prove that $f(\mathbb{A}_K)\colon X(\mathbb{A}_K) \to X'(\mathbb{A}_K)$ maps $X(\mathbb{A}_K)^{\text{ét},\mathrm{Br}}$ into $X'(\mathbb{A}_K)^{\text{ét},\mathrm{Br}}$.

(b) Assume that $\prod_v X(K_v) \neq \varnothing$, so that necessarily $\prod_v X'(K_v) \neq \varnothing$. If X' has an étale Brauer–Manin obstruction to the local–global principle, prove that so does X.

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