Princeton, 15th of July, 2015

Dear Česnavičius,

You are right. Thank you for alerting me that 4.13 of my paper with Rapoport is wrong when n is not square free.

In addition to being wrong, it gives an incorrect picture of the ramification of $\mathcal{M}_{\Gamma_0(n)}$ over $\mathcal{M}_1$ ⊗, and of the Neron model of the elliptic curve we have on the open $\mathring{\mathcal{M}}_{\Gamma_0(n)}$.

⊗ as well as of the ramification at ∞ of the classical $M_{\Gamma_0(n)}/M_1$, which is the same

Here is what I guess one could do; it is not elegant. I do not claim to have proofs, but I expect they are feasible. I only tried to understand.

I feel on solid ground in describing the completion of $\mathcal{M}_{\Gamma_0(n)}$ along infinity (= the cusps) and the elliptic curve on this completion, minus infinity.

First, as a guide, the analytic story over $\mathbb{C}$. Near ∞, an elliptic curve has a natural description $E = \mathbb{C}^*/q^{\mathbb{Z}}$ with $|q| << 1$. In families, the coverings $\mathbb{C}^*$ of E form a family, but it can have monodromy. When describing cusps, we insist on a global description as $\mathbb{C}^*/q^{\mathbb{Z}}$. For $\Gamma_0(n)$, we consider E with a cyclic subgroup H of order n. Let n_1 be the

order of the group of roots of unity in the inverse image of H in $\mathbb{C}^*$. The group H will have generators q_1 of the form $q_1 = (q^{n_1/n} \cdot (\text{a } n\text{th root of } 1))$.
Required: $q_1^{n/n_1}/q$ is a primitive n_1 th root of 1.
The cyclic subgroup H is determined by $q_1^{n_1}$: it is generated by any n_1 th root of q^{n_1}. Write $n_1^2/n = a/b$ in reduced terms. We have then
discrete invariants, labelling the cusps:

n_1, and $(q_1^{n_1})^b / q^a$, a primitive root of 1 of order $n/n_1 b$

continuous invariants: q, and $q_1^{n_1}$, related.

The completion $\mathcal{M}^\wedge_{\Gamma_0(n)}$ along the divisor at ∞ will have a similar description. I work over $\mathbb{Z}[1/n]$ and for simplicity I take n to be a prime power: I consider $\Gamma_0(p^n)$ and work over $\mathbb{Z}[1/p]$. Define $k' := \inf(k, n-k)$. Let $\mathcal{O}_{k'}$ be generated over $\mathbb{Z}[1/p]$ by a primitive root of unity ~~of~~ ζ of order $p^{k'}$. Then, $\mathcal{M}^\wedge_{\Gamma_0(p^n)}$ decomposes into pieces $\mathcal{M}^\wedge_k$ indexed by k (with n_1 above being p^k), $0 \le k \le n$. Then, $\mathcal{M}^\wedge_k$ is the spectrum of a finite extension, described below, of $\mathcal{O}_{k'}[[q]]$. Outside $q = 0$, the elliptic curve is the pull back of the Tate curve "$\mathbb{G}_m/q^{\mathbb{Z}}$", and the cyclic subgroup H

contains μ_{p^k} and is given below.

a) when $k \le \frac{n}{2}$: $\mathcal{O}_k[[q]][(q\eta)^{1/p^{n-2k}}]$;

H : generated by any p^k-th root of $(q\eta)^{1/p^{n-2k}}$

b) when $k \ge \frac{n}{2}$: $\mathcal{O}_{k'}[[q]]$, where $k' = n - k$.

H : generated by any p^k-th root of $q^{p^{2k-n}}\eta$.

On $\mathcal{M}_{\Gamma_0(p^n)}$, the elliptic curve on $\mathcal{M}^\circ_{\Gamma_0(p^n)}$ will extend as a generalized elliptic curve. At infinity, we have disjoint divisors D_k $(0 \le k \le n)$, and above D_k this generalized elliptic curve is a

$$\begin{cases} p^{n-2k}\text{-gone} & \text{if } k \le \frac{n}{2} \\ 1\text{-gone} & \text{if } k \ge \frac{n}{2}. \end{cases}$$

This gives a map

$$\mathcal{M}_{\Gamma_0(p^n)} \longrightarrow \mathcal{M}_*$$

where $\mathcal{M}_*$ is the horribly non separate stack of generalized elliptic curve (N-gone allowed for any N). This map is etale. The stack $\mathcal{M}_*$ is glued along the common open $\mathcal{M}^\circ_1$ of the compactifications $\mathcal{M}_{[N]}$ where N-gones are allowed.

Interlude: the orbifold construction.

Suppose we have in a stack S a family of Cartier divisors D_i, disjoints, and integers ~~$\geq$~~ $N_i \geq 1$. I want to define a stack $(S, (\frac{1}{N_i} D_i))$, assuming the N_i are prime to the characteristics (if not, we would obtain an Artin stack rather than an algebraic stack). ~~The~~ This stack agrees with S outside of the D_i, and the construction is local along each D_i. Let me assume there is just one D_i, call it D, and put $N := N_i$. Local charts are obtained as follow: one start with local charts U for S, small enough so that $\mathcal{O}(-D)$ is trivial on U, and one picks an equation t for the inverse image D_U of D in U. Local chart: $U[t^{1/N}]$. To glue those charts, one should tell what the fiber product of two such charts is. Take the charts defined by (U, t) and (V, t'). To get the ~~[illegible]~~ fiber product over $(S, \frac{1}{N} D)$: take $U \times_S V$. On it, we have the inverse images of t and t', still noted t and t'. As t and t' are equations of the same divisor, t'/t makes sense as an invertible function on $U \times_S V$. The fiber products maps to $U \times_S V[t^{1/N}, t'^{1/N}]$. It is

$U \times_S V[t^{1/N}, (t'/t)^{1/N}]$.

Remark : on $(\mathcal{S}, \frac{1}{N}D)$ we have a divisor $\frac{1}{N}D$ (with $N(\frac{1}{N}D)$ the inverse image of D by the natural $(\mathcal{S}, \frac{1}{N}D) \to \mathcal{S}$) : local equations the $t^{1/N} = 0$.

Example : if D is the (reduced) divisor at ∞ of $\mathcal{M}_{[M]}$, then $\mathcal{M}_{[MN]}$ is naturally isomorphic to $(\mathcal{M}_{[M]}, \frac{1}{N}D)$

Remark For a reduced divisor on a smooth curve, this is how an "orbifold curve" can be viewed as a stack.

The stack $'\mathcal{M}_{\Gamma_0(p^k)}$ of the paper with Rapoport is deduced from $\mathcal{M}_{\Gamma_0(p^k)}$ we want to understand by this method, applied to the divisors D_k at infinity, using as ramification N the $p^{k'}$. Along D_k, that is when we remove from $\mathcal{M}_{\Gamma_0(p^k)}$ the D_ℓ, $\ell \neq k$, this gives a cartesian square

$$\begin{array}{ccc} '\mathcal{M}_{\Gamma_0(p^k)} \text{ minus the } D_\ell & \longrightarrow & \mathcal{M}_{[a p^{k'}]} \\ \downarrow & & \downarrow \\ \mathcal{M}_{\Gamma_0(p^k)} \text{ minus the } D_\ell & \longrightarrow & \mathcal{M}_{[a]} \end{array}$$

with $a = \begin{cases} p^{n-2k} & \text{if } k \leq \frac{n}{2} \quad (k' = k) \\ 1 & \text{if } k \geq \frac{n}{2} \quad (k' = n - k) \end{cases}$

We want to describe a moduli problem for $\mathcal{M}_{\Gamma_0(p^n)}$. To have

$$S \longrightarrow \mathcal{M}_{\Gamma_0(p^n)},$$

we should first give

a) disjoint closed subscheme D_k $(0 \leq k \leq n)$

b) a generalized elliptic curve E, elliptic over the complement of the D_k, and [illegible] over D_k

$$\begin{cases} \text{if } k \leq \frac{n}{2} : \text{ a } p^{n-2k}\text{-gone} \\ \text{if } k \geqslant \frac{n}{2} : \text{ a } 1\text{-gone} \end{cases}$$

In addition, $\coprod D_k$ should be the schematic direct image of the non smoothless locus of E/S (which can be defined as a scheme). In other words, b) defines $S \longrightarrow \mathcal{M}_*$, and $\coprod D_k$ is the inverse image the divison at ∞ of $\mathcal{M}_*$.

c1) over the complement of the D_k, a cyclic subgroup $H \subset E$ of order p^n

c2) we need more along each D_k, and this more is a local data. Let me explain it on the complement of the D_ℓ, $\ell \neq k$: suppose $D_\ell = \phi$ for $\ell \neq k$, and put $D := D_k$. Put as above $a = p^{n-2k}$, or 1 if $k \geqslant \frac{n}{2}$.

In a glib but hopefully correct way of speaking, we want as additional data a morphism of stacks

$$S \times_{\mathcal{M}_a} \mathcal{M}_{ap^{k'}} \longrightarrow \mathcal{M}_{p^n} ,$$

mapping the inverse image of D_k where it should.

More concretely: for any $S' \to S$, and generalized elliptic curve E'/S', which is a p^{n-k}-gone over D, and isomorphisme

E pulled back to $S' \xrightarrow[\alpha]{\sim}$ contraction of E', turning a p^{n-k}-gone into a a-gone,

the data of a ~~sub~~ cyclic subgroup H of E', of order p^n, meeting each irreducible component of each fiber. This H has to be functorial in $(E'/S', \alpha)$ and on the complement of D, we want of course the structure c_1.

I now want to get rid of "for any S'/S". This is easier to explain when D is a divisor (I continue to assume ~~$D_\ell \neq$~~ $D_\ell = \phi$ for $\ell \neq k$). Locally, pick an equation t for D. Go to the covering obtained by adjoining $t^{1/p^{k'}}$. This is a local chart of the $S \times_{\mathcal{M}_a} \mathcal{M}_{ap^{k'}}$ above, and on it we should have a unique generalized elliptic curve ~~of which E is~~, a p^{n-k}-gone over D, of which E is contraction. We want H on it, as before. It will be the closure of what we have outside of D.

If D is not a divisor (for instance if $S_{red} = D$), the orbifold construction has to be generalized in a "logarithmic" fashion. The minimum we need is

α) a line bundle $\mathcal{L}$ on D

β) an epimorphism $\mathcal{L} \to I/I^2$, where I is the divisor defining D

In the orbifold construction, one then replace "equation t of D" by "equation t of D, and section ℓ of ℓ agreeing with t mod I^2. Instead of t'/t invertible, one uses any extension $\ell'/\ell^{\sim}$ of ℓ'/ℓ, and $((\ell'/\ell)^{\sim})^{1/N}$ instead $(t'/t)^{1/N}$.

In our case, from $S \longrightarrow \mathcal{M}_a$, we get more: the pull back $\mathcal{L}$ of ~~the divisor at~~ $\mathcal{O}(-D_\infty)$, for D_∞ the divisor at ∞ of $\mathcal{M}_a$, and $\mathcal{L} \longrightarrow I$ an epimorphism. ~~By~~ One can take then ℓ in $\mathcal{L}$ and its image t, an equation for $\mathcal{L}$. Given ℓ and t, one should get on $S[t^{1/p^{k'}}]$ ~~and E'~~ a canonical E', ~~a canonical~~ $\mathbb{P}^{n-k}$-gone above D; ~~it will have an action of $\mu_{p^{k'}}$. On it, we want a H. It will be $\mu_{p^{k}}$-stable.~~

This $(E'/S[t^{1/p^{k'}}])$ will have a natural action of $\mu_{p^{k'}}$, above the action $t^{1/p^{k'}} \mapsto \zeta\, t^{1/p^{k'}}$. We want an H of the correct type, invariant by the action (hence giving a H on E on $S-D$). If we change ℓ (and hence t), the new $(E'/S[t^{1/p^{k'}}])$ is deduced from the old by torsion by the $\mu_{p^{k'}}$-torsor of $p^{k'}$-roots of ℓ'/ℓ, and in this way the data for one ℓ gives it for another.

The "pull-back by $S \to \mathcal{M}_a$ of $\mathcal{O}(-D_\infty)$" is a little abstract to me. Its restriction to D is more concrete, in that it can be described purely in term of the singularity of E/S: "the tensor product of the tangent space along the two branches of E at a singular point, dualized".

Best, and my reiterated thanks,

P. Deligne

Pierre Deligne

P.S. If this letter is difficult to read, I could have it typed, but it will take a week or more.