THE AFFINE GRASSMANNIAN AS A PRESHEAF QUOTIENT

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Abstract. For a reductive group G over a ring A, its affine Grassmannian Gr_G plays important roles in a wide range of subjects and is typically defined as the étale sheafification of the presheaf quotient LG/L^+G of the loop group LG by its positive loop subgroup L^+G. We show that the Zariski sheafification gives the same result. Moreover, for totally isotropic G (for instance, for quasi-split G), we show that no sheafification is needed at all: Gr_G is already the presheaf quotient LG/L^+G, which seems new already in the classical case of G over $\mathbb{C}$. For totally isotropic G, we also show that the affine Grassmannian may be formed using polynomial loops. We deduce all of these results from the study of G-torsors on $\mathbb{P}^1_A$ that is ultimately built on the geometry of Bun_G.

The affine Grassmannian Gr_G of a reductive group G originated in Lusztig's [Lus83, Section 11] (see also Beilinson–Drinfeld's [BD, Section 4.5]) and is instrumental in the geometric Langlands program and other subjects that study G-torsors and their moduli. The goal of this article is to show that Gr_G admits a simpler definition than previously thought: for most G, it is simply the presheaf quotient LG/L^+G of the loop group LG by its positive loop subgroup L^+G, so that the fpqc or étale sheafifications of this quotient that were used previously are not needed, see Theorems 2.5 and 3.4 for precise statements.

0.1. Conventions. As in [SGA $3_{\text{III new}}$, exposé XIX, définition 2.7], a *reductive group scheme* over a scheme S is a smooth, affine S-group scheme whose geometric S-fibers are connected reductive groups. An S-scheme is *locally constant* if fpqc locally on S it becomes isomorphic to some $\bigsqcup_{i\in I} S$.

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1. The affine Grassmannian and its modular description

In this section, we fix a base ring A and aim to review the definition and the modular interpretation of the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ for a smooth, quasi-affine $A[\![t]\!]$-group scheme $\mathcal{G}$, see Proposition 1.6.

1.1. The loop functor. For a functor $\mathcal{X}$ on the category of $A(\!(t)\!)$-algebras (resp., $A[\![t]\!]$-algebras), its *loop functor* $L\mathcal{X}$ (resp., and *positive loop functor* $L^+\mathcal{X}$) is defined on the category of A-algebras by

$$L\mathcal{X}\colon B \mapsto \mathcal{X}(B(\!(t)\!)) \quad (\text{resp.,} \quad L^+\mathcal{X}\colon B \mapsto \mathcal{X}(B[\![t]\!])).$$

The morphism $L^+\mathcal{X} \to L\mathcal{X}$ is often an inclusion, for instance, this is so whenever $\mathcal{X}$ is a subfunctor of a separated $A[\![t]\!]$-scheme (see [EGA I, corollaire 9.5.6]).

If $\mathcal{X}$ is an $A[\![t]\!]$-scheme, then, by [Bha16, Theorem 4.1 and Remark 4.6], $L^+\mathcal{X}$ is an A-scheme because

$$L^+\mathcal{X} \cong \varprojlim_{n>0} \mathrm{Res}_{(A[t]/(t^n))/A}(\mathcal{X}_{A[t]/(t^n)}), \quad \text{equivalently,} \quad (L^+\mathcal{X})(B) = \varprojlim_{n>0} \mathcal{X}(B[t]/(t^n))$$

for every A-algebra B, compare with [CLNS18, Chapter 3, Corollary 3.3.7 b)]. Similarly, if $\mathcal{X}$ is a quasi-compact and quasi-separated $A[\![t]\!]$-algebraic space, then, by [Bha16, Theorem 4.1] and [SP, Proposition 05YF and Lemmas 07SF and 05YD], $L^+\mathcal{X}$ is a quasi-compact and quasi-separated A-algebraic space. If $\mathcal{X}$ is even an affine $A[\![t]\!]$-scheme, then, by considering a presentation of its coordinate ring in terms of generators and relations, $L^+\mathcal{X}$ is an affine A-scheme and $L\mathcal{X}$ is an ind-affine A-ind-scheme, more precisely, there are affine A-schemes X_n and closed immersions

$$X_0 \hookrightarrow X_1 \hookrightarrow \cdots \quad \text{such that} \quad L^+\mathcal{X} = X_0 \quad \text{and} \quad L\mathcal{X} = \textstyle\bigcup_{n\geqslant 0} X_n \quad \text{as functors on } A\text{-algebras } B.$$

For an A-algebra B, let $B\{t\}$ be the Henselization of $B[t]$ with respect to the ideal $tB[t]$, see [BČ22, Section 2.1.2] or [SP, Lemma 0A02]. It is useful to consider Henselian (resp., algebraic; resp., polynomial) variants $L_h\mathcal{X}$ and $L_h^+\mathcal{X}$ (resp., $L_{\mathrm{alg}}\mathcal{X}$ and $L_{\mathrm{alg}}^+\mathcal{X}$; resp., $L_{\mathrm{poly}}\mathcal{X}$ and $L_{\mathrm{poly}}^+\mathcal{X}$):

$$\begin{aligned} &L_h\mathcal{X}\colon B \mapsto \mathcal{X}(B\{t\}[\tfrac{1}{t}]) && \text{and} && L_h^+\mathcal{X}\colon B \mapsto \mathcal{X}(B\{t\}),\\ &L_{\mathrm{alg}}\mathcal{X}\colon B \mapsto \mathcal{X}((B[t]_{1+tB[t]})[\tfrac{1}{t}]) && \text{and} && L_{\mathrm{alg}}^+\mathcal{X}\colon B \mapsto \mathcal{X}(B[t]_{1+tB[t]}),\\ &L_{\mathrm{poly}}\mathcal{X}\colon B \mapsto \mathcal{X}(B[t,t^{-1}]) && \text{and} && L_{\mathrm{poly}}^+\mathcal{X}\colon B \mapsto \mathcal{X}(B[t]), \end{aligned}$$

granted that $\mathcal{X}$ is begins life over $A\{t\}[\frac{1}{t}]$ (resp., over $(A[t]_{1+tA[t]})[\frac{1}{t}]$; resp., over $A[t,t^{-1}]$) and, for $L_*^+\mathcal{X}$, even already over $A\{t\}$ (resp., over $A[t]_{1+tA[t]}$; resp., over $A[t]$). These variant functors are sometimes easier to handle, for instance, they all commute with filtered direct limits in B granted that so does $\mathcal{X}$, moreover, $L_{\mathrm{poly}}^+\mathcal{X}$ and $L_{\mathrm{poly}}\mathcal{X}$ are nothing else but restrictions of scalars.

By the following proposition, for many $\mathcal{X}$, the functors $L_*^{(+)}\mathcal{X}$ cannot be improved by sheafifying.

Proposition 1.2. *For a scheme $\mathcal{X}$ over $A(\!(t)\!)$, or over $A\{t\}[\frac{1}{t}]$, or over $(A[t]_{1+tA[t]})[\frac{1}{t}]$, or over $A[t,t^{-1}]$ (resp., over $A[\![t]\!]$, or over $A\{t\}$, or over $A[t]_{1+tA[t]}$, or over $A[t]$) as in §1.1 such that every quasi-compact open of $\mathcal{X}$ is quasi-affine, the functor $L_*\mathcal{X}$ (resp., and also its subfunctor $L_*^+\mathcal{X}$) is an fpqc sheaf on the category of A-algebras.*

Proof. Since $\mathcal{X}$ is separated, [EGA I, corollaire 9.5.6] ensures that the map $L_*^+\mathcal{X} \to L_*\mathcal{X}$ is indeed an inclusion. Thus, all we need to show is that for an fpqc cover $B \to B'$ of A-algebras, the sequence

$$(L_*\mathcal{X})(B) \to (L_*\mathcal{X})(B') \rightrightarrows (L_*\mathcal{X})(B'\otimes_B B') \ \ (\text{resp.,}\ (L_*^+\mathcal{X})(B) \to (L_*^+\mathcal{X})(B') \rightrightarrows (L_*^+\mathcal{X})(B'\otimes_B B'))$$

is exact. The case of the polynomial variant follows from fpqc descent [SP, Lemma 023Q] because

$$B[t] \to B'[t] \rightrightarrows (B'\otimes_B B')[t] \quad \text{and} \quad B[t,t^{-1}] \to B'[t,t^{-1}] \rightrightarrows (B'\otimes_B B')[t,t^{-1}]$$

are both fpqc cover sequences. For other variants, since each ring-valued point of $\mathcal{X}$ factors through a quasi-compact open, we may assume that $\mathcal{X}$ is quasi-affine. The maps

$$B\{t\} \to B'\{t\} \quad \text{and} \quad B[t]_{1+tB[t]} \to B'[t]_{1+tB'[t]}$$

are both faithfully flat, in particular, they induce surjections on spectra, so the Henselian and algebraic variants reduce further to affine $\mathcal{X}$. In the power series case, the reduction to affine $\mathcal{X}$ is more subtle if B and B' are not Noetherian (because then we do not know whether the map $\operatorname{Spec} B'((t)) \to \operatorname{Spec} B((t))$ is surjective) and will use ideas from [BČ22, Lemma 2.2.9 (i)] as follows.

By realizing $\mathcal{X}$ as the complement of the vanishing locus of a finitely generated ideal in an affine $A((t))$-scheme (resp., $A[[t]]$-scheme), to reduce to affine $\mathcal{X}$ we need to show that elements $b_1, \dots, b_n$ in $B((t))$ (resp., in $B[[t]]$) generate the unit ideal as soon as their images do so in $B'((t))$ (resp., in $B'[[t]]$). The case of $B[[t]]$ follows by checking modulo t and using the faithful flatness of $B \to B'$. To treat $B((t))$, we may assume that $b_1, \dots, b_n \in B[[t]]$ and use the faithfully flat cover $B[[t]] \to (B[[t]] \otimes_B B')^h_{(t)}$ (Henselization with respect to the ideal generated by t) to reduce to showing that $b_1, \dots, b_n$ generate the unit ideal in $(B[[t]] \otimes_B B')^h_{(t)}[\frac{1}{t}]$. The t-adic completion of $(B[[t]] \otimes_B B')^h_{(t)}$ is $B'[[t]]$ and, by assumption, there are $a_1, \dots, a_n \in B'[[t]]$ with $a_1b_1 + \dots + a_nb_n = t^N$ in $B'[[t]]$ for some $N > 0$. By approximating modulo t^{N+1}, therefore, there are $a'_1, \dots, a'_n \in (B[[t]] \otimes_B B')^h_{(t)}$ such that

$$a'_1b_1 + \dots + a'_nb_n \in t^N + t^{N+1}(B[[t]] \otimes_B B')^h_{(t)}.$$

This means that $b_1, \dots, b_n$ indeed generate the unit ideal in $(B[[t]] \otimes_B B')^h_{(t)}[\frac{1}{t}]$, as desired.

Now that $\mathcal{X}$ is affine, the functor $\mathcal{X}(-)$ turns fiber products of rings into fiber products of sets. Thus, all we need to show is the exactness of the horizontal sequences in the commutative diagram

$$\begin{array}{ccccc}
B[t]_{1+tB[t]} & \longrightarrow & B'[t]_{1+tB'[t]} & \rightrightarrows & (B' \otimes_B B')[t]_{1+t(B'\otimes_B B')[t]} \\
\downarrow & & \downarrow & & \downarrow \\
B\{t\} & \longrightarrow & B'\{t\} & \rightrightarrows & (B' \otimes_B B')\{t\} \\
\downarrow & & \downarrow & & \downarrow \\
B[[t]] & \longrightarrow & B'[[t]] & \rightrightarrows & (B' \otimes_B B')[[t]],
\end{array}$$

which implies the corresponding exactness after further inverting t; here the bottom vertical maps are injective by [BČ22, Section 2.1.2] (a limit argument to reduce to finite type $\mathbb{Z}$-algebras). The exactness of the bottom row is seen coefficientwise. Thus, by fpqc descent, it is enough to show that

$$B'\{t\} \otimes_{B\{t\}} B'\{t\} \hookrightarrow (B' \otimes_B B')\{t\},$$
$$B'[t]_{1+tB'[t]} \otimes_{B[t]_{1+tB[t]}} B'[t]_{1+tB'[t]} \hookrightarrow (B' \otimes_B B')[t]_{1+t(B'\otimes_B B')[t]}.$$

Since every étale $(B' \otimes_B B')[t]$-algebra $(B' \otimes_B B')$-fiberwise has no embedded associated primes, both of these injections follow from [RG71, première partie, corollaire 3.2.6] and a limit argument. □

Remark 1.3. In Proposition 1.2, the assumption on quasi-compact opens holds if $\mathcal{X}$ is a subscheme of an affine scheme. In general, a scheme whose quasi-compact opens are all quasi-affine is *ind-quasi-affine*, see [BČ22, Definition 2.2.5] and the references to [SP] given there for details. We avoid this terminology here because it may be confusing in the context of ind-schemes. The automorphism scheme of a reductive group is such an $\mathcal{X}$ that need not be quasi-affine, see [Čes22b, Section 1.3.7]. Another useful example is locally constant schemes $\mathcal{X}$, for which even $L\mathcal{X} = L^+\mathcal{X}$ as follows.

Corollary 1.4. *Let A be a ring.*

(a) *In the following diagram, the squares are Cartesian and the maps are bijective on idempotents*:

$$\begin{array}{ccccccccc} A & \hookrightarrow & A[t] & \hookrightarrow & A[t]_{1+tA[t]} & \hookrightarrow & A\{t\} & \hookrightarrow & A[\![t]\!] \\ & \searrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & A[t,t^{-1}] & \hookrightarrow & (A[t]_{1+tA[t]})[\frac{1}{t}] & \hookrightarrow & A\{t\}[\frac{1}{t}] & \hookrightarrow & A(\!(t)\!). \end{array}$$

(b) *For every scheme $\mathcal{X}$ over $A[\![t]\!]$, or over $A\{t\}$, or over $A[t]_{1+tA[t]}$, or over $A[t]$ as in* §1.1 *such that $\mathcal{X}$ is locally constant (see* §0.1*), we have $L_*^+\mathcal{X} \xrightarrow{\sim} L_*\mathcal{X}$.*

Proof.

(a) As indicated, the maps are all injective, either by inspection, or by faithful flatness, or by a limit argument given in [BČ22, Section 2.1.2]. The squares are Cartesian by [SP, Lemma 0BNR]. As for idempotents, since the maps are injective, it suffices to recall from [BČ22, Corollary 2.1.19] that the map $A \to A(\!(t)\!)$ is bijective on idempotents (as one may also show directly).

(b) By [SP, Lemma 0AP8], we may check fpqc locally on the base that our locally constant $\mathcal{X}$ satisfies the assumptions of Proposition 1.2, more precisely, that every quasi-compact open of $\mathcal{X}$ is quasi-affine. In particular, in the case when $\mathcal{X}$ begins life over $A\{t\}$, the square

$$\begin{array}{ccc} \mathcal{X}(A\{t\}) & \hookrightarrow & \mathcal{X}(A\{t\}[\frac{1}{t}]) \\ \downarrow & & \downarrow \\ \mathcal{X}(A[\![t]\!]) & \hookrightarrow & \mathcal{X}(A(\!(t)\!)) \end{array}$$

is Cartesian by [BČ22, Proposition 2.2.12], and likewise with $A[t]_{1+tA[t]}$ or $A[t]$ in place of $A\{t\}$. Since we may vary A, this reduces us to showing that the inclusion $\mathcal{X}(A[\![t]\!]) \subset \mathcal{X}(A(\!(t)\!))$ is an equality. For this, since *loc. cit.* also implies that for every prime $\mathfrak{p} \subset A$ the square

$$\begin{array}{ccc} \mathcal{X}(A[\![t]\!] \otimes_A A_\mathfrak{p}) & \hookrightarrow & \mathcal{X}(A(\!(t)\!) \otimes_A A_\mathfrak{p}) \\ \downarrow & & \downarrow \\ \mathcal{X}(A_\mathfrak{p}[\![t]\!]) & \hookrightarrow & \mathcal{X}(A_\mathfrak{p}(\!(t)\!)) \end{array}$$

is Cartesian, by combining this square with spreading out, we may assume further that A is local. At this point, Proposition 1.2 allows us replace A by its strict Henselization.

Once A is strictly Henselian local, we fix any $A(\!(t)\!)$-point of $\mathcal{X}$ and form its schematic image to get a closed subscheme $Z \subset \mathcal{X}$, see [SP, Definition 01R7]. It is enough to show that

$$Z \overset{?}{\cong} \mathrm{Spec}(A[\![t]\!]).$$

We first claim that the map $Z \to \mathrm{Spec}(A[\![t]\!])$ is surjective. For this, since $\mathcal{X}$ is locally constant, we may choose a faithfully flat $A[\![t]\!]$-algebra B such that $\mathcal{X}_B \cong \bigsqcup_{i \in I} \mathrm{Spec}\, B$. By [SP, Lemma 081I], the formation of Z commutes with base change to B, so we need to show that the schematic image of any $B[\frac{1}{t}]$-point of $\mathcal{X}_B$ surjects onto $\mathrm{Spec}\, B$. It does because otherwise, by the explicit nature of $\mathcal{X}_B$, the open immersion $\mathrm{Spec}(B[\frac{1}{t}]) \subset \mathrm{Spec}\, B$ would factor through some proper closed subscheme, contradicting its schematic density.

Since $A[\![t]\!]$ is strictly Henselian local and $\mathcal{X}$ is étale, [EGA IV$_4$, théorème 18.5.11] ensures that for every $z \in Z$ above the maximal ideal of $A[\![t]\!]$, we have

$$\mathcal{X} \cong \mathrm{Spec}(\mathscr{O}_{\mathcal{X},z}) \sqcup \mathcal{X}' \quad \text{with} \quad \mathscr{O}_{\mathcal{X},z} \cong A[\![t]\!].$$

We have already argued that such a z exists, and (a) shows that $\operatorname{Spec}(A((t)))$ inherits connectedness from $\operatorname{Spec} A$, and so ensures that $Z \cap \mathcal{X}' = \varnothing$. The desired $Z \cong \operatorname{Spec}(A[[t]])$ now follows from the schematic density of the open immersion $\operatorname{Spec}(A((t))) \subset \operatorname{Spec}(A[[t]])$. $\square$

1.5. The affine Grassmannian. In the setting of §1.1, if the functor $\mathcal{X}$ is group valued, then so are the functors $L_*^+\mathcal{X}$ and $L_*\mathcal{X}$. With this in mind, for a smooth, quasi-affine $A[[t]]$-group scheme $\mathcal{G}$, its *affine Grassmannian* is the pointed set valued functor defined by

$$\mathrm{Gr}_{\mathcal{G}} := (L\mathcal{G}/L^+\mathcal{G})_{\text{ét}}, \tag{1.5.1}$$

where, as indicated, the sheafification of the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$ is formed in the étale topology. If instead $\mathcal{G}$ begins life as a smooth, quasi-affine $A\{t\}$-group scheme, then [BČ22, Example 2.2.19] (which is based on approximation and algebraization techniques) ensures that we may use Henselian loops instead, more precisely, that we have an identification of presheaf quotients

$$L\mathcal{G}/L^+\mathcal{G} \cong L_h\mathcal{G}/L_h^+\mathcal{G}$$

In many situations, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is represented by an A-ind-scheme, which is often even ind-projective over A, see, for instance, [PR08]. This proceed by embedding $\mathcal{G}$ into some GL_n, which is not always possible with our general assumptions, so we will not use representability results.

In the literature one sometimes finds the affine Grassmannian defined as the fpqc sheafification $(L\mathcal{G}/L^+\mathcal{G})_{\text{fpqc}}$. A priori this makes no mathematical sense: even for a field, isomorphism classes of its fpqc covers form a proper class, so the fpqc sheafification need not exist; however, by the following proposition, this "fpqc sheafification approach" gives the same result for our $\mathcal{G}$ as above.

Proposition 1.6. *For a ring A and a smooth, quasi-affine $A[[t]]$-group scheme $\mathcal{G}$, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is an fpqc sheaf and has the following modular description on the category of A-algebras B:*

$$\mathrm{Gr}_{\mathcal{G}}(B) \cong \left\{ (\mathcal{E}, \iota) \;\middle|\; \begin{array}{l} \mathcal{E} \textit{ is a } \mathcal{G}\textit{-torsor over } B[[t]], \\ \iota \in \mathcal{E}(B((t))) \textit{ is its trivialization over } B((t)) \end{array} \right\} / \sim,$$

while the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$ is a subfunctor of $\mathrm{Gr}_{\mathcal{G}}$ parametrizing those $(\mathcal{E}, \iota)$ with $\mathcal{E}$ trivial.

Proof. For the trivial $\mathcal{G}$-torsor, $L^+\mathcal{G}$ parametrizes its $\mathcal{G}$-torsor automorphisms over $(-)[[t]]$ while $L\mathcal{G}$ parametrizes its trivializations ι over $(-)((t))$, so the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$ is the claimed subfunctor of $\mathrm{Gr}_{\mathcal{G}}$, granted that the latter has the displayed modular description. Moreover, by the smoothness and quasi-affineness of $\mathcal{E}$ inherited from $\mathcal{G}$ (see [SP, Lemma 0247]) and by the infinitesimal lifting of sections (compare also with [BČ22, Proposition 2.1.4]), each $(\mathcal{E}, \iota)$ lands in this subfunctor étale locally on B, so the sought modular description will follow once we argue that it defines an fpqc sheaf. For this, since the pairs $(\mathcal{E}, \iota)$ have no nontrivial automorphisms, all that remains is to show that the groupoid-valued functor

$$B \mapsto \left\{ (\mathcal{E}, \iota) \;\middle|\; \begin{array}{l} \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor over } B[[t]], \\ \iota \in \mathcal{E}(B((t))) \text{ is its trivialization over } B((t)) \end{array} \right\}$$

is a stack for the fpqc topology on A-algebras B. For this, let $B \to B'$ be an fpqc cover of A-algebras and let $(\mathcal{E}', \iota')$ be a pair for B' equipped with a descent datum with respect to this cover. It suffices to uniquely descend $\mathcal{E}'$ to a $\mathcal{G}$-torsor $\mathcal{E}$ over $B[[t]]$: the trivialization ι' will then also descend because $L\mathcal{E}$ is an fpqc sheaf by Proposition 1.2. The usual fpqc descent for $\mathcal{G}$-torsors (see [SP, Lemma 0247]) gives us a unique compatible system of $\mathcal{G}$-torsor descents $\mathcal{E}_n$ over $B[[t]]/(t^{n+1})$ for $n \geqslant 0$. It then remains to uniquely algebraize this sequence to a $\mathcal{G}$-torsor $\mathcal{E}$ over $B[[t]]$, and there are several ways to do this. Perhaps the most direct is to apply the algebraization result [BHL17, Theorem 8.1] to the classifying stack $\mathbf{B}\mathcal{G}$. A somewhat more elementary approach is to first use [BČ22, Theorem 2.1.6] to lift $\mathcal{E}_0$ to a $\mathcal{G}$-torsor $\mathcal{E}$ over $B[\![t]\!]$ and to then inductively build a compatible sequence of isomorphisms

$$\mathcal{E}|_{B[\![t]\!]/(t^{n+1})} \simeq \mathcal{E}_n,$$

by using *loc. cit.* and the smoothness of the quasi-affine $B[\![t]\!]$-group $\underline{\mathrm{Aut}}_{\mathcal{G}}(\mathcal{E})$. □

2. The affine Grassmannian as a Zariski quotient

Even though the affine Grassmannian of $\mathcal{G}$ is defined as the étale sheafification of the presheaf quotient $L\mathcal{G}/L\mathcal{G}^+$ (see (1.5.1)), in Theorem 2.5 below we show that the Zariski sheafification suffices when $\mathcal{G}$ is reductive and descends to A. For intuition for why the Zariski topology may be enough, we recall that the inclusion $L^+\mathcal{G} \subset L\mathcal{G}$ is morally similar to the inclusion $P \subset G$ of a parabolic subgroup of a reductive group scheme, and that $(G/P)_{\mathrm{fpqc}} \cong (G/P)_{\mathrm{Zar}}$ because over any semilocal ring parabolic subgroups of the same type are conjugate, see [SGA $3_{\mathrm{III\ new}}$, exposé XXVI, corollaire 5.2].

We will deduce that the Zariski sheafification is enough from the following result about torsors over $\mathbb{P}^1_A$ that is proved by studying the geometry of the algebraic stack Bun_G that parametrizes such torsors, or in [PS23a] and [PS23b] by a different approach. Various weaker and more technical earlier variants of this result would suffice as well, for instance, [Fed22, Theorem 6] or [Čes22b, Proposition 5.3.6]. In effect, in some sense, we obtain the main results of this article by ascending geometric information along the uniformization map $\mathrm{Gr}_G \to \mathrm{Bun}_G$.

Theorem 2.1 ([ČF23, Theorem 3.6])**.** *Let G be a reductive group scheme over a semilocal ring A. Every G-torsor E over $\mathbb{P}^1_A$ is A-sectionwise constant, equivalently, $E|_{\{t=0\}} \simeq E|_{\{t=\infty\}}$.* □

We will use Theorem 2.1 through its following consequence for reductive group torsors over $A[\![t]\!]$.

Proposition 2.2. *Let A be a semilocal ring and let $\mathcal{G}$ be an $A[\![t]\!]$-group scheme that is an extension of an $A[\![t]\!]$-group $\mathcal{G}^{\text{ét}}$ that is locally constant (see §0.1) by a reductive $A[\![t]\!]$-group scheme $\mathcal{G}^0$ such that $\mathcal{G}^0_{A(\!(t)\!)}$ descends to a reductive A-group scheme. No nontrivial $\mathcal{G}$-torsor over $A[\![t]\!]$ trivializes over $A(\!(t)\!)$, that is, we have*

$$\mathrm{Ker}\left(H^1(A[\![t]\!], \mathcal{G}) \to H^1(A(\!(t)\!), \mathcal{G})\right) = \{*\};$$

in particular, $\mathcal{G}^0$ itself descends to a reductive A-group scheme.

Proof. Let $\mathcal{E}$ be a $\mathcal{G}$-torsor over $A[\![t]\!]$ that trivializes over $A(\!(t)\!)$. By Corollary 1.4 (b), every $e \in \mathcal{E}(A(\!(t)\!))$ gives rise to an $A(\!(t)\!)$-point of the $\mathcal{G}^{\text{ét}}$-torsor $\mathcal{E}/\mathcal{G}^0$ that extends uniquely to an $A[\![t]\!]$-point $\overline{e} \in (\mathcal{E}/\mathcal{G}^0)(A[\![t]\!])$. The preimage of $\overline{e}$ in $\mathcal{E}$ is a $\mathcal{G}^0$-torsor that trivializes over $A(\!(t)\!)$. We may replace $\mathcal{E}$ by this preimage to reduce to the case when $\mathcal{G} = \mathcal{G}^0$.

Now that $\mathcal{G} = \mathcal{G}^0$, suppose first that $\mathcal{G}$ descends to a reductive A-group G. Due to the triviality over $A(\!(t)\!)$, we may patch $\mathcal{E}$ with the trivial G-torsor over $\mathbb{P}^1_A \backslash \{t = 0\}$ (see, for instance, [BČ22, Lemma 2.2.11 (b)]) to build a G-torsor E over $\mathbb{P}^1_A$ such that $E|_{\{t=0\}} \simeq \mathcal{E}|_{\{t=0\}}$ and $E|_{\{t=\infty\}}$ is trivial. By Theorem 2.1, then $\mathcal{E}|_{\{t=0\}}$ is also trivial, to the effect that, by the infinitesimal lifting of sections due to smoothness, $\mathcal{E}$ is trivial, too, as desired (compare also with [BČ22, Proposition 2.1.4]).

To complete the proof, it remains to show that the reductive A-group scheme G for which we have $G_{A(\!(t)\!)} \simeq \mathcal{G}_{A(\!(t)\!)}$ also descends $\mathcal{G}$, that is, that already $G_{A[\![t]\!]} \simeq \mathcal{G}_{A[\![t]\!]}$. By Corollary 1.4 (a) and the classification of reductive group schemes (recalled in [Čes22b, Sections 1.3.1 and 1.3.7]), our $\mathcal{G}$ corresponds to an $\underline{\mathrm{Aut}}_{\mathrm{gp}}(G)$-torsor over $A[\![t]\!]$ that trivializes over $A(\!(t)\!)$. However, as recalled in *loc. cit.*, $\underline{\mathrm{Aut}}_{\mathrm{gp}}(G)$ is an extension of a locally constant A-group by the reductive A-group G^{ad}.

In particular, the case settled in the first two paragraphs of the proof applies and shows that the $\underline{\mathrm{Aut}}_{\mathrm{gp}}(G)$-torsor in question is trivial already over $A[\![t]\!]$, so that $G_{A[\![t]\!]} \simeq \mathcal{G}$, as desired. ☐

Corollary 2.3. *For a reductive group scheme G over a semilocal ring A, no nontrivial G-torsor trivializes over $A(\!(t)\!)$, in other words, we have*

$$\mathrm{Ker}(H^1(A, G) \to H^1(A(\!(t)\!), G)) = \{*\}.$$

Proof. In the view of Proposition 2.2, it suffices to recall from [BČ22, Theorem 2.1.6] that

$$H^1(A, G) \xrightarrow{\sim} H^1(A[\![t]\!], G).$$ ☐

Remark 2.4. The results of [FG21] suggest that the reductivity assumption may be nonessential for Corollary 2.3. It would therefore be interesting to find a more general result of this type.

We are ready for the promised sufficiency of the Zariski sheafification for $\mathrm{Gr}_{\mathcal{G}}$ in the case when $\mathcal{G}$ is (constant) reductive. One may compare this to its earlier variant [Bac19, Proposition 14] that restricted to smooth A over a field and deduced the conclusion from the Grothendieck–Serre conjecture. For us, it is the study of the latter that has indirectly led to the results of this article.

Theorem 2.5. *For a ring A and an $A[\![t]\!]$-group scheme $\mathcal{G}$ that is an extension of a finite étale $A[\![t]\!]$-group $\mathcal{G}^{\text{ét}}$ by a reductive $A[\![t]\!]$-group scheme $\mathcal{G}^0$ that descends to a reductive A-group scheme, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is the Zariski sheafification of the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$, that is,*

$$\mathrm{Gr}_{\mathcal{G}} \cong (L\mathcal{G}/L^+\mathcal{G})_{\mathrm{Zar}}.$$

Proof. By the modular interpretation supplied by Proposition 1.6, the possibility of varying A, and the infinitesimal lifting of sections due to smoothness, all we need to show is that for any $\mathcal{G}$-torsor $\mathcal{E}$ over $A[\![t]\!]$ that trivializes over $A(\!(t)\!)$, the $\mathcal{G}$-torsor $\mathcal{E}|_{\{t=0\}}$ trivializes Zariski locally on A. However, Proposition 2.2 ensures that $\mathcal{E}|_{\{t=0\}}$ trivializes even Zariski semilocally on A. ☐

Remarks.

2.6. We do not know whether the reductivity assumption is critical for Theorem 2.5, for instance, whether the Zariski sheafification also suffices for parahoric groups. Certainly, it does in the case when $\mathcal{G}$ is a reductive $A[\![t]\!]$-group scheme and $\mathcal{P}$ is a smooth, quasi-affine $A[\![t]\!]$-group scheme equipped with an $A[\![t]\!]$-morphism $\mathcal{P} \to \mathcal{G}$ that modulo t reduces to an inclusion of a parabolic subgroup: indeed, by [BČ22, Theorem 2.1.6] and [Čes22b, equation (1.3.5.2)], these assumptions ensure that $H^1(B[\![t]\!], \mathcal{P}) \subset H^1(B[\![t]\!], \mathcal{G})$ for any semilocal A-algebra B, so

$$\mathrm{Ker}\left(H^1(B[\![t]\!], \mathcal{G}) \to H^1(B(\!(t)\!), \mathcal{G})\right) = \{*\} \implies \mathrm{Ker}\left(H^1(B[\![t]\!], \mathcal{P}) \to H^1(B(\!(t)\!), \mathcal{P})\right) = \{*\},$$

to the effect that Proposition 1.6 and Theorem 2.5 imply the claimed $\mathrm{Gr}_{\mathcal{P}} \cong (L\mathcal{P}/L^+\mathcal{P})_{\mathrm{Zar}}$.

2.7. We do not know whether Theorem 2.5 admits a version for the Witt vector affine Grassmannian. For a version of Theorem 2.5 for the B^+_{dR}-affine Grassmannian, see [ČY23, Theorem 3.1].

3. The affine Grassmannian as a presheaf quotient

For most reductive group schemes $\mathcal{G}$, even the Zariski sheafification is not needed when forming the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$: in Theorem 3.4 below, we show that the latter often agrees already with the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$. This appears to be new already for reductive groups G over $\mathbb{C}$, although for GL_n it essentially follows from [BČ22, Theorem 2.1.24], and is based on the finer variant of Theorem 2.1 recorded in Theorem 3.3 below, which, in addition to the geometry of Bun_G, uses Quillen patching for torsors over $\mathbb{A}^1_A$ to progress beyond semilocal A. For this variant, the relevant condition on $\mathcal{G}$ is the following.

Definition 3.1 ([Česk22a, Definition 8.1])**.** A reductive group scheme G over a scheme S is *totally isotropic* if in the canonical decomposition of [SGA $3_{\text{III new}}$, exposé XXIV, proposition 5.10 (i)]:

$$G^{\mathrm{ad}} \cong \prod\nolimits_{i\in\{A_n,B_n,\dots,G_2\}} \mathrm{Res}_{S_i/S}(G_i)$$

of its adjoint quotient G^{ad}, where i ranges over the types of connected Dynkin diagrams, S_i is a finite étale S-scheme, and G_i is an adjoint semisimple S_i-group with simple geometric fibers of type i, Zariski locally on S each G_i has a parabolic S_i-subgroup that contains no S_i-fiber of G_i.

Example 3.2. Slightly informally, G is totally isotropic if Zariski locally on S it has a parabolic subgroup containing no factor of the adjoint group G^{ad}. To see this, recall that parabolic subgroups of G correspond to those of G^{ad}, which correspond to collections of parabolic subgroups of G_i, one for each i, see [Čes22b, end of Section 1.3.4]. Certainly, every quasi-split (so also every split) reductive group, in particular, every torus, is totally isotropic.

Theorem 3.3 ([ČF23, Theorem 4.2])**.** *Let G be a totally isotropic reductive group scheme over a ring A. For a G-torsor E over $\mathbb{P}^1_A$, if $E|_{\{t=\infty\}}$ is trivial, then $E|_{\mathbb{A}^1_A}$ is also trivial.* □

Theorem 3.4. *For a ring A and an $A[\![t]\!]$-group scheme $\mathcal{G}$ that is an extension of a finite étale $A[\![t]\!]$-group $\mathcal{G}^{\text{ét}}$ by a reductive $A[\![t]\!]$-group scheme $\mathcal{G}^0$ that descends to a totally isotropic reductive A-group scheme, the affine Grassmannian $\mathrm{Gr}_{\mathcal{G}}$ is the presheaf quotient $L\mathcal{G}/L^+\mathcal{G}$, that is,*

$$\mathrm{Gr}_{\mathcal{G}} \cong L\mathcal{G}/L^+\mathcal{G}.$$

Proof. As in the proof of Theorem 2.5, by the modular description of Proposition 1.6 and the possibility of varying A, we need to show that no nontrivial $\mathcal{G}$-torsor $\mathcal{E}$ over $A[\![t]\!]$ trivializes over $A(\!(t)\!)$. For this, as in the proof of Proposition 2.2, Corollary 1.4 (b) immediately reduces us to the case when $\mathcal{G} = \mathcal{G}_0$ and $\mathcal{G}$ is the base change of a totally isotropic reductive A-group scheme G.

To treat this case, we again patch $\mathcal{E}$ with the trivial G-torsor over $\mathbb{P}^1_A\backslash\{t=0\}$ to build a G-torsor E over $\mathbb{P}^1_A$ such that $E_{A[\![t]\!]} \simeq \mathcal{E}$ and $E|_{\{t=\infty\}}$ is trivial. By Theorem 3.3, this last condition forces $E|_{\mathbb{A}^1_A}$ to be trivial. However, then $\mathcal{E}$ is trivial, too, as desired. □

We recall from [BČ22, Theorems 2.1.24 and 3.1.7] that the following global variant of Corollary 2.3 was known when G is either a pure inner form of GL_n or a torus.

Corollary 3.5. *For a totally isotropic reductive group scheme G over a ring A, no nontrivial G-torsor over A trivializes over $A(\!(t)\!)$, in other words, we have*

$$\mathrm{Ker}(H^1(A,G) \to H^1(A(\!(t)\!),G)) = \{*\}.$$

Proof. Proposition 1.6 and Theorem 3.4 show that no nontrivial G-torsor over $A[\![t]\!]$ trivializes over $A(\!(t)\!)$. Thus, it suffices to recall from [BČ22, Theorem 2.1.6] that

$$H^1(A, G) \xrightarrow{\sim} H^1(A[\![t]\!], G). \qquad \square$$

Finally, we note that the preceding proof shows that the affine Grassmannian of totally isotropic reductive groups may even be formed using polynomial loops as follows.

Theorem 3.6. *For a totally isotropic reductive group scheme G over a ring A, its affine Grassmannian Gr_G may be formed as the presheaf quotient using the polynomial loops, more precisely, we have*

$$\mathrm{Gr}_G \cong L_{\mathrm{poly}}G/L^+_{\mathrm{poly}}G,$$

explicitly, we have

$$\begin{aligned} G(A(\!(t)\!))/G(A[\![t]\!]) &\cong G(A\{t\}[\tfrac{1}{t}])/G(A\{t\}) \\ &\cong G((A[t]_{1+tA[t]})[\tfrac{1}{t}])/G(A[t]_{1+tA[t]}) \cong G(A[t,t^{-1}])/G(A[t]), \end{aligned}$$

equivalently,

$$\begin{aligned} G(A(\!(t)\!)) &= G(A[t,t^{-1}])G(A[\![t]\!]), \\ G(A\{t\}[\tfrac{1}{t}]) &= G(A[t,t^{-1}])G(A\{t\}), \\ G((A[t]_{1+tA[t]})[\tfrac{1}{t}]) &= G(A[t,t^{-1}])G(A[t]_{1+tA[t]}). \end{aligned}$$

These equalities seem elementary, but we do not know how to argue them directly even for $G = \mathrm{GL}_n$.

Proof. Since G is affine, its functor of points preserves the Cartesianness of the squares from Corollary 1.4 (a). In particular, the map

$$G(A[t,t^{-1}])/G(A[t]) \hookrightarrow G(A(\!(t)\!))/G(A[\![t]\!])$$

is injective, and so are its counterparts for algebraic or Henselian loops in place of polynomial loops. Thus, all we need to show is that this map is also surjective or, equivalently, that

$$G(A[t,t^{-1}])\backslash G(A(\!(t)\!))/G(A[\![t]\!]) = \{*\}.$$

However, if this double quotient was nontrivial, then we could use patching (for instance, [BČ22, Lemma 2.2.11 (b)]) to build a nontrivial G-torsor over $\mathbb{A}^1_A$ that would trivialize both over $\mathbb{G}_{m,A}$ and also over the formal completion along $\{t = 0\}$. We could then extend this G-torsor to all of $\mathbb{P}^1_A$ by patching with the trivial torsor at infinity, and thus obtain a contradiction to Theorem 3.3. $\square$

Remark 3.7. We do not know the extent to which the assumptions of Theorems 3.4 and 3.6 are optimal because they are imposed by our proofs, for instance, we do not know whether the affine Grassmannian Gr_G agrees with the presheaf quotient LG/L^+G for every (possibly not totally isotropic) reductive A-group G, although we expect that it does not. On the other hand, since our proofs are reductions to general results about torsors over $\mathbb{P}^1_A$, the reader will have no trouble adapting them to various close variants of the affine Grassmannian that are sometimes considered in the literature, for instance, to affine Grassmannians constructed using general relative Cartier divisors in $\mathbb{A}^1_A$ in place of $\{t = 0\}$.

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