LOCAL FACTORS VALUED IN NORMAL DOMAINS

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Abstract. We give an exposition of Deligne's theory of local ϵ_0-factors over fields and discrete valuation rings under the assumption that the theory over the complex numbers is known. We then employ standard techniques from algebraic geometry to deduce the theory of local ϵ_0-factors over arbitrary normal integral schemes.

1. Introduction

In [Del73, §4], Deligne presented an elegant argument proving the existence of the theory of ϵ-factors of local Weil representations over the complex numbers; this theory had previously been predicted by Langlands. For Weil representations of a nonarchimedean local field K of residue characteristic p, Deligne proceeded to show in [Del73, §6] how to use the established $F = \mathbb{C}$ case to deduce a similar "mod l" theory over every field F of characteristic l different from p. Both the complex and the mod l theories have subsequently been of significant importance, for instance, in considerations concerning the local Langlands correspondence.

Deligne's argument concerning the mod l theory is brief, with many details left to the reader. Our goal here is to recall it in full and combine its ideas with standard techniques from algebraic geometry to deduce the theory of local ϵ_0-factors not only over fields but also over arbitrary normal integral schemes on which p is invertible. In precise terms, assuming the theory of [Del73, §4] over the complex numbers as known, we prove

Theorem 1.1. *There is a unique assignment ϵ_0 which to the data of*

- *A nonarchimedean local field K with the ring of integers $\mathcal{O}_K$ and a finite residue field of characteristic p,*
- *A separable closure K^s of K,*
- *A normal integral $\mathbb{Z}[\frac{1}{p}]$-scheme S,*
- *A continuous representation V of the Weil group $W(K^s/K)$ over S (cf. §2.1 and §2.2),*
- *A nontrivial additive character $\psi\colon (K,+) \to \Gamma(S, \mathcal{O}_S^\times)$ (cf. §2.3), and*
- *A $\Gamma(S, \mathcal{O}_S)$-valued Haar measure $C \mapsto \int_C dx$ on K such that $\int_{\mathcal{O}_K} dx \in \Gamma(S, \mathcal{O}_S^\times)$ (cf. §2.4)*

associates $\epsilon_0(V, \psi, dx) \in \Gamma(S, \mathcal{O}_S^\times)$ in such a way that

(i) The formation of $\epsilon_0(V, \psi, dx)$ is compatible with base change: for a morphism $f : S' \to S$ of normal integral $\mathbb{Z}[\frac{1}{p}]$-schemes and an S-representation V, abusing the f^* notation one has

$$\epsilon_0(f^*V, f^*\psi, f^*dx) = f^*(\epsilon_0(V, \psi, dx)).$$

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(ii) Exactness of a sequence $0 \to V' \to V \to V'' \to 0$ *entails*

$$\epsilon_0(V, \psi, dx) = \epsilon_0(V', \psi, dx)\epsilon_0(V'', \psi, dx).$$

In particular, $\epsilon_0(V, \psi, dx)$ *depends only on the class of* V *in the Grothendieck group* $R_S(W(K^s/K))$ *and makes sense for every* $v \in R_S(W(K^s/K))$ *(see §2.2 for the definition of* $R_S(W(K^s/K))$*).*

(iii) For $a \in \Gamma(S, \mathcal{O}_S^\times)$*, one has*

$$\epsilon_0(V, \psi, a \cdot dx) = a^{\operatorname{rk} V} \epsilon_0(V, \psi, dx).$$

(iv) For a finite subextension $K^s/L/K$ *and a virtual representation* v *of* $W(K^s/L)$ *of rank* 0*,*

$$\epsilon_0\left(\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)} v, \psi\right) = \epsilon_0(v, \psi \circ \operatorname{Tr}_{L/K}),$$

where $\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)} v := \mathcal{O}_S[W(K^s/K)] \otimes_{\mathcal{O}_S[W(K^s/L)]} v$*. (Omitting* dx *is justified by (iii).)*

(v) If V *is of dimension* 1 *(i.e., a line bundle) and* $\chi \colon W(K^s/K) \to \Gamma(S, \mathcal{O}_S^\times)$ *is the character giving the Weil group action, then*

$$\epsilon_0(\chi, \psi, dx) = \int_{\gamma^{-1}\mathcal{O}_K^\times} \chi^{-1}(x)\psi(x)dx \qquad (\bigstar)$$

where $\gamma \in K^\times$ *is an element of valuation* $\operatorname{Sw}(\chi) + n(\psi) + 1$ *(for the definition of the Swan conductor* $\operatorname{Sw}(\chi) \in \mathbb{Z}_{\geqslant 0}$*, that of* $n(\psi) \in \mathbb{Z}$*, and the meaning of the integral, see Proposition 2.11, §2.3, and §2.4).*

Remarks.

1.2. Restricting to the main case of interest, $S = \operatorname{Spec} \mathbb{C}$, the existence and uniqueness of an ϵ_0 satisfying (ii)–(v) was envisioned by Langlands: the automorphic side of his conjectural correspondence features decompositions of signs of functional equations of global L-functions as products of local factors, and (ii)–(v) are the defining properties of the corresponding local factor on the Galois side. In [Lan70], Langlands attempted to give a local proof of the existence of ϵ_0 but abandoned the project once Deligne found a short proof [Del73, §4], which, however, uses global arguments. Publishing a complete local proof remains an outstanding problem. For further and more accurate historical remarks, see the website of [Lan70].

1.3. For the purpose of proving Theorem 1.1, we will take the existence and uniqueness of an ϵ_0 satisfying (ii)–(v) with $S = \operatorname{Spec} \mathbb{C}$ as known. Although existence is intricate, uniqueness follows readily from a suitable version of Brauer's induction theorem due to the imposed ($\bigstar$) in the 1-dimensional case: see [Tat79, 2.3.1] or the proof of Proposition 5.8 (b) below. Tate's thesis [Tat50], which has been a major influence for the ideas mentioned in Remark 1.2, motivates ($\bigstar$) and also provides the main input for proving that ($\bigstar$) results in a global unit on S, as is implicit in Theorem 1.1 and will be argued in the course of the proof.

1.4. When $S = \operatorname{Spec} F$ for a field F, e.g., when $S = \operatorname{Spec} \mathbb{C}$, it is commonplace to work with

$$\epsilon(V, \psi, dx) := \epsilon_0(V, \psi, dx) \det(-\mathrm{Frob}_K \mid V^{I_K})^{-1},$$

which is the *local* ϵ*-factor* of V (for the choice of ψ and dx); here Frob_K is a geometric Frobenius defined in §1.7. However, Theorem 1.1 concerns ϵ_0 instead because the formation of ϵ is not compatible with base change, i.e., the analogue of (i) fails for ϵ.

1.5. We collect formulas concerning ϵ_0 in §3. The only one of these that exhibits phenomena not observed when $S = \operatorname{Spec} \mathbb{C}$ is that for the inverse in the higher-dimensional case, see §3.4.

1.6. The possibility of extending Deligne's theory of ϵ_0-factors for representations of $W(K^s/K)$ over fields to those over more general coefficient rings R has also been considered in [Yas09], where such an extension is proposed for Noetherian local rings R that have an algebraically closed residue field of characteristic different from p and satisfy $R^{\times p} = R^{\times}$. In the case of an algebraically closed field of characteristic different from p, the ϵ_0-factors of op. cit. agree with those of Theorem 1.1 due to [Yas09, Thm. 1.1 (3)]. For general R as above that are also normal domains, if the ϵ_0-factors there are compatible with base change along not necessarily local homomorphisms $R \to R'$, then one gets the similar agreement by taking an algebraic closure of $\operatorname{Frac} R$ for R'.

1.7. Notation. For a field F, its fixed choices of separable and algebraic closures are denoted by $F^s \subset \overline{F}$. As mentioned above, K is a nonarchimedean local field, whereas $\mathcal{O}_K$ and $\mathbb{F}_K$ are its ring of integers and residue field (so $\mathbb{F}_K$ is a finite field); $p = \operatorname{char} \mathbb{F}_K$ is the residue characteristic. A *geometric Frobenius* is any $\operatorname{Frob}_K \in W(K^s/K) \subset \operatorname{Gal}(K^s/K)$ whose inverse reduces to the Frobenius automorphism $x \mapsto x^{\#\mathbb{F}_K}$ in $\operatorname{Gal}(\overline{\mathbb{F}}_K/\mathbb{F}_K)$. For an integer $n \geqslant 1$, a primitive n^{th} root of unity is denoted by ζ_n. For a scheme S, the local ring and the residue field of a point $s \in S$ are denoted by $\mathcal{O}_{S,s}$ and $k(s)$.

1.8. Conventions. The following assumptions are implicit throughout: all rings are commutative and unital; all representations are of finite rank; all representations of I_K or $W(K^s/K)$ are trivial on an open subgroup of I_K.

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2. Recollections

We gather several relevant concepts and constructions that are used freely in other sections.

2.1. The Weil group $W(K^s/K)$ (see also [BH06, §28] if needed). It is the subgroup of those elements of $\operatorname{Gal}(K^s/K)$ that reduce to an integral power of Frobenius in $\operatorname{Gal}(\overline{\mathbb{F}}_K/\mathbb{F}_K)$. Thus,

$$1 \to I_K \to W(K^s/K) \to \mathbb{Z} \to 1$$

is short exact, where $I_K \lhd \operatorname{Gal}(K^s/K)$ is the inertia. The Weil group is topologized by insisting that I_K with its profinite topology be an open subgroup.

The open subgroups of $W(K^s/K)$ of finite index are precisely the $W(K^s/L)$ for finite subextensions $K^s/L/K$; normality of the subgroup corresponds to L/K being Galois. The natural map

$$W(K^s/K)/W(K^s/L) \to \operatorname{Gal}(K^s/K)/\operatorname{Gal}(K^s/L)$$

is bijective. In particular, $W(K^s/K) \subset \operatorname{Gal}(K^s/K)$ is dense and the representations of $W(K^s/K)$ whose kernel is open and of finite index are identified with those of $\operatorname{Gal}(K^s/K)$. Such representations of $W(K^s/K)$ are called *Galois*.

Let $W(K^s/K)^{\text{ab}}$ be the maximal abelian Hausdorff quotient of $W(K^s/K)$. Due to the ambiguity in choosing K^s, the Weil group is determined by K only up to an inner automorphism of $\operatorname{Gal}(K^s/K)$;

this ambiguity disappears for $W(K^s/K)^{\mathrm{ab}}$, which is determined by K up to a unique isomorphism. Local class field theory furnishes the local Artin homomorphism

$$\mathrm{Art}_K\colon K^\times \to W(K^s/K)^{\mathrm{ab}}, \tag{2.1.1}$$

which is an isomorphism of topological groups. We choose to normalize it so that uniformizers are brought to geometric Frobenii. As usual, we identify continuous 1-dimensional characters of $W(K^s/K)$ valued in Hausdorff abelian groups with those of $K^\times$ by means of Art_K. We let $|\cdot|_K : K^\times \to \mathbb{Z}[\frac{1}{p}]^\times$ be the unramified character that takes the value $(\#\mathbb{F}_K)^{-1}$ on uniformizers.

2.2. Grothendieck groups. A *representation* of a discrete group G over a scheme S is an $\mathcal{O}_S$-module V that is locally free of finite rank and is endowed with an $\mathcal{O}_S$-linear action of G. For the topological groups I_K and $W(K^s/K)$, one only considers *continuous* representations, i.e., those V on which an open subgroup of I_K acts trivially (some authors call such representations *smooth*).

Let $R_S(G)$ (resp., $R_S(I_K)$ or $R_S(W(K^s/K))$) be the *Grothendieck group* of representations of G (resp., continuous representations of I_K or $W(K^s/K)$) over S, i.e., $R_S(G)$ is the quotient of the free abelian group on the set $\{[V]\}$ of isomorphism classes of representations of G over S by the subgroup generated by the relations $[V] = [V'] + [V'']$ for all exact sequences

$$0 \to V' \to V \to V'' \to 0,$$

and likewise for I_K or $W(K^s/K)$ in place of G. A *virtual representation* is an element of a Grothendieck group. The rank of V is an integer valued locally constant function on S; its additivity in short exact sequences defines the notion of the *rank* of a virtual representation.

Letting J run over the open subgroups of I_K that are normal in[1] $W(K^s/K)$, one has

$$R_S(I_K) = \varinjlim_J R_S(I_K/J) \qquad \text{and} \qquad R_S(W(K^s/K)) = \varinjlim_J R_S(W(K^s/K)/J). \tag{2.2.1}$$

Due to the $\mathcal{O}_S$-flatness of V, the tensor product endows $R_S(-)$ with the structure of a commutative ring with $[\mathcal{O}_S]$ as the multiplicative unit. The subset $R_S^0(-) \subset R_S(-)$ of virtual representations that are of rank 0 at every $s \in S$ is an ideal. Pullback along $S' \to S$ induces a ring homomorphism $R_S(-) \to R_{S'}(-)$, which maps $R_S^0(-)$ to $R_{S'}^0(-)$. If $S = \operatorname{Spec} A$, one often writes $R_A(-)$ for $R_S(-)$.

2.3. Additive characters. For an integral $\mathbb{Z}[\frac{1}{p}]$-scheme S, an *additive character*

$$\psi\colon (K, +) \to \Gamma(S, \mathcal{O}_S^\times)$$

is a locally constant abelian group homomorphism as indicated. Local constancy is equivalent to the existence of an integer n such that $\psi|_{\pi^{-n}\mathcal{O}_K} = 1$, where $\pi \in \mathcal{O}_K$ is a uniformizer, and forces the values of ψ to be p-power roots of unity (if $\operatorname{char} K = p$, these values are even p^{th} roots of unity). If ψ is nontrivial, as we assume from now on, we let $n(\psi)$ be the maximal n as above. Since p is a unit on S, a p-power root of unity in $\Gamma(S, \mathcal{O}_S^\times)$ has trivial image in $k(s)^\times$ for an $s \in S$ if and only if it is trivial to begin with. Consequently, $n(\psi) = n(\psi_{k(s)})$ for every $s \in S$, and hence $n(\psi)$ is stable under pullback along every $f\colon S' \to S$ with an integral S'.

The group $K^\times$ acts freely on the set of nontrivial ψ by setting $(a\psi)(x) := \psi(ax)$ for $a \in K^\times$. If the set is nonempty, then the action is also transitive: endowing S with a structure of an R-scheme, where $R = \mathbb{Z}[\zeta_{p^\infty}]$ if $\operatorname{char} K = 0$ and $R = \mathbb{Z}[\zeta_p]$ if $\operatorname{char} K = p$, realizes every ψ as the pullback of an additive character valued in $R^\times$, and hence reduces the transitivity claim to the classical $S = \operatorname{Spec} \mathbb{C}$ case treated, e.g., in [BH06, §1.7, Prop.]. In conclusion, if the set of nontrivial ψ is nonempty, then it has a natural structure of a $K^\times$-torsor.

[1] Every open $J' \lhd I_K$ contains such a J of the form $\bigcap_{i=0}^{n-1} (\mathrm{Frob}_K^i\, J'\, \mathrm{Frob}_K^{-i})$ for some $n \geqslant 1$: indeed, J' corresponds to a finite Galois L/K^{nr}, which descends to a Galois extension of the degree n unramified extension of K for some $n \geqslant 1$.

2.4. Haar measures valued in abelian groups ([Del73, 6.1])**.** Fix a $\mathbb{Z}[\frac{1}{p}]$-module A and let C range over the compact open subsets of K. An A-valued *Haar measure* on K is a function $C \mapsto \int_C dx \in A$ that is translation invariant and additive in disjoint unions. Since every C is a disjoint union of translates of balls centered at $0 \in K$ and A is uniquely p-divisible, a choice of an A-valued Haar measure amounts to that of the element $\int_{\mathcal{O}_K} dx \in A$. Once the choice is made, if A is in addition a ring, one can integrate locally constant compactly supported $f\colon K \to A$ and write $\int_K f(x)dx$ for the resulting finite sums; if f is implicitly multiplied by the characteristic function of C, one writes $\int_C f(x)dx$ instead. Coupled with (2.1.1), this clarifies ($\bigstar$), which takes $A = \Gamma(S, \mathcal{O}_S)$.

2.5. Sufficiently large fields. Fix a finite group G and let m be the least common multiple of the orders of elements of G. A field F is *sufficiently large* if it contains the m^{th} roots of unity. A separably closed F is sufficiently large for every G.

2.6. Virtual representations over fields. For a field F, thanks to the Jordan–Hölder theorem for abelian categories [Ses67, Thm. 2.1], $R_F(G)$ of §2.2 is the free abelian group on the set of isomorphism classes of irreducible representations of G over F, and similarly for I_K or $W(K^s/K)$ and representations that are trivial on an open subgroup of I_K (cf. [Ser77, §14.1, Prop. 40] if needed).

If V and $\widetilde{V}$ are nonisomorphic irreducible representations of a finite group G, then the extensions of scalars $V_{F'}$ and $\widetilde{V}_{F'}$ to every overfield F'/F have no common composition factors [CR81, Ex. 7.9]. Consequently, $R_F(G) \to R_{F'}(G)$ is injective. It is also surjective if F is sufficiently large, as is clear from Brauer's induction theorem 5.3 (a), whose proof does not use this surjectivity. Therefore, for sufficiently large F, extension of scalars induces a bijection between the sets of isomorphism classes of irreducible representations of G over F and those over F'. If F contains the m^{th} roots of unity for every m, e.g., if F is separably closed, then this bijection remains in place for I_K, because $R_F(I_K) \to R_{F'}(I_K)$ is an isomorphism thanks to the previous discussion and (2.2.1).

2.7. The decomposition homomorphism. Fix a finite group G and a discrete valuation ring A with the fraction field η of characteristic 0 and the residue field $F = A/\mathfrak{m}$ of characteristic l. For a representation V of G over η, a choice of a G-stable A-lattice $\Lambda \subset V$ gives rise to the representation $\Lambda/\mathfrak{m}\Lambda$ of G over F whose class in $R_F(G)$ does not depend on Λ [Ser77, §15.2, Thm. 32 and Rem. (1)] (the $l > 0$ assumption of loc. cit. is not used for this). The resulting *decomposition homomorphism*

$$d_G\colon R_\eta(G) \to R_F(G)$$

preserves ranks and commutes with restriction and induction. If A is complete, then d_G is surjective: Cohen's structure theorem [Mat89, 28.3 (ii)] settles the equicharacteristic case, whereas [Ser77, §16.1, Thm. 33] treats the case $l > 0$. As noted in [Ser77, Ex. 16.1], surjectivity also holds if η is sufficiently large[2] because d_G commutes with $R_\eta(G) \to R_{\widehat{\eta}}(G)$, which is an isomorphism [Ser77, §12.3].

In the equicharacteristic 0 case, d_G is injective without additional assumptions on A: for every virtual character χ, one has $\langle \chi, \chi \rangle \in \mathbb{Z}_{\geqslant 0}$; moreover, $\langle \chi, \chi \rangle = 0$ if and only if $\chi = 0$. Consequently, if in this case A is complete or η is sufficiently large, then d_G is an isomorphism.

2.8. The Swan representation. Let J be a continuous finite quotient of I_K. Continuity means the openness of $\mathrm{Ker}(I_K \twoheadrightarrow J)$ and is a nonvacuous condition, as we now explain in a digression. By Krasner's lemma, there are only countably many finite degree subextensions of K^s/K because the same holds for global K. Applying this observation to finite unramified extensions of K, we conclude that the same holds for K^s/K^{nr}, in other words, that I_K has only countably many open subgroups of finite index. On the other hand, local class field theory applied to finite unramified

[2]However, d_G is not surjective in general, see [Ser77, Ex. 16.2] or [CR81, p. 512, Ex. 21.4].

extensions of K produces continuous surjections $I_K \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^n$ for arbitrarily large $n \in \mathbb{Z}_{>0}$. Thus, $\bigoplus_{i=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a quotient of I_K, so that I_K has uncountably many distinct surjections onto $\mathbb{Z}/p\mathbb{Z}$. For cardinality reasons, one of these surjections must have a kernel that is not open.

Let $J = J_0 \rhd J_1 \rhd \cdots$ be the ramification filtration in the lower numbering, so the J_i are normal in J and J_1 is the image of the wild inertia. The *Artin character* of J is the class function

$$a_J := \sum_{i=0}^{\infty} \frac{1}{[J_0 : J_i]} \operatorname{Ind}_{J_i}^{J_0} u_i, \tag{2.8.1}$$

where u_i is the augmentation representation of J_i, i.e.,

$$u_i(j) = \begin{cases} -1, & \text{if } j \in J_i \backslash \{1\}, \\ \#J_i - 1, & \text{if } j = 1, \end{cases} \qquad \text{so } r_{J_i} = \mathbf{1}_{J_i} \oplus u_i, \text{ where } r_{J_i} \text{ is the regular representation.}$$

The formula for the induced character shows that the sum in (2.8.1) is finite and a_J is $\mathbb{Z}$-valued; moreover, as the name suggests, a_J is the character of a complex representation [Ser79, VI.§2, Thm. 1 and Prop. 2], namely, the *Artin representation* of J. The *Swan character*

$$\mathrm{Sw}_J := \sum_{i=1}^{\infty} \frac{1}{[J_0 : J_i]} \operatorname{Ind}_{J_i}^{J_0} u_i$$

inherits these properties: a priori Sw_J is a virtual character with $\langle \mathrm{Sw}_J, \chi \rangle \geqslant 0$ for every character χ of J, and hence a posteriori the character of a complex representation, namely, the *Swan representation* of J. Moreover, Sw_J vanishes on $J \backslash J_1$, so whenever the Swan character can be realized as a representation over the fraction field η of characteristic 0 of a Dedekind domain A with $p \in A^\times$, due to [Swa63, Thm. 5], it can also be realized as a finite projective $A[J]$-module. Realizability over a sufficiently large η is automatic (cf. §2.6); therefore, [Ser77, §16.3, Prop. 44] realizes Sw_J uniquely as a finite projective $\mathbb{Z}_l[J]$-module for every $l \neq p$. We continue to write Sw_J for its base change to a $\mathbb{Z}_l$-algebra A, for instance, to a field of characteristic l; the resulting Sw_J is a finite projective $A[J]$-module.

If J' is a quotient of J, then $a_{J'} \cong a_J \otimes_{\mathbb{C}[J]} \mathbb{C}[J']$ [Ser79, VI.§2 Prop. 3]; the same relation holds for the augmentation representations, so also $\mathrm{Sw}_{J'} \cong \mathrm{Sw}_J \otimes_{\mathbb{C}[J]} \mathbb{C}[J']$. Moreover, uniqueness of the realization of $\mathrm{Sw}_{J'}$ as a projective $\mathbb{Z}_l[J']$-module for $l \neq p$ entails the $A = \mathbb{Z}_l$ case of the isomorphism

$$\mathrm{Sw}_{J'} \cong \mathrm{Sw}_J \otimes_{A[J]} A[J'] \qquad \text{for every } \mathbb{Z}_l\text{-algebra } A, \tag{2.8.2}$$

and the general case follows by base change to A.

2.9. The Swan conductor. Let V be a continuous representation of I_K over a field F of characteristic l with $l \neq p$, and let J be a continuous finite quotient through which the I_K-action on V factors. The *Swan conductor*[3] of V is

$$\mathrm{Sw}\, V := \dim_F \operatorname{Hom}_J(\mathrm{Sw}_J, V), \tag{2.9.1}$$

where in the $l > 0$ case one uses the projective $F[J]$-module Sw_J defined in §2.8, and in the $l = 0$ case one writes χ for the character of V and interprets the right hand side of (2.9.1) as

$$\langle \mathrm{Sw}_J, \chi \rangle = \frac{1}{\#J} \sum_{j \in J} \mathrm{Sw}_J(j)\chi(j). \tag{2.9.2}$$

Of course, if $l = 0$ and Sw_J is realizable over F, then (2.9.1) and (2.9.2) agree thanks to character theory. Moreover, one can assume realizability for the purpose of the definition, because $\mathrm{Sw}\, V$ is

[3] Some authors call $\mathrm{Sw}\, V$ *the exponent of the Swan conductor*, reserving the term *Swan conductor* for the corresponding power of the maximal ideal of $\mathcal{O}_K$.

invariant under base change to an overfield F'/F regardless of l. Likewise, $\mathrm{Sw}\, V$ is invariant under change of J due to (2.8.2) and the adjunction $-\otimes_{F[J]} F[J'] \dashv \mathrm{Hom}_{J'}(F[J'], -)$. Since it is also additive in exact sequences due to projectivity of Sw_J, it extends to a homomorphism $\mathrm{Sw}\colon R_F(I_K) \to \mathbb{Z}$.

To define $\mathrm{Sw}\, V$, it is not necessary to restrict to representations over fields, as Proposition 2.11 below shows. For its proof, we recall a well-known Lemma 2.10, which will also be used later.

Lemma 2.10. *For a locally Noetherian scheme S, a point $s \in S$, and its specialization $s' \neq s$, there is a complete discrete valuation ring A and a morphism $\mathrm{Spec}\, A \to S$ mapping the generic and the closed points of $\mathrm{Spec}\, A$ to s and s', respectively.*

Proof. Replace a discrete valuation ring provided by [EGA II, 7.1.9] by its completion. □

Proposition 2.11. *Let V be a continuous representation of I_K over an integral $\mathbb{Z}[\frac{1}{p}]$-scheme S (cf. §2.2). For varying $s \in S$, the Swan conductor of the residual representation $V_{k(s)}$ is constant.*

The common value of the $\mathrm{Sw}\, V_{k(s)}$ is the *Swan conductor* of V. It has already been used in (v).

Proof. Since every two nonempty opens of S intersect, it suffices to treat the affine case $S = \mathrm{Spec}\, A$ and assume that V is free. Writing A as a filtered direct limit of finite type $\mathbb{Z}[\frac{1}{p}]$-subalgebras, one uses limit arguments and invariance of $\mathrm{Sw}\, V_{k(s)}$ under base change to overfields to assume further that A is Noetherian. Taking s in Lemma 2.10 to be the generic point, one finally reduces to the case when A is a complete discrete valuation ring with the fraction field η and the residue field F.

In this case, if $\mathrm{char}\, F = 0$, then the claim follows from (2.9.2), which takes values in A. If, on the other hand, $\mathrm{char}\, F = l$ with $l > 0$, then A is a $\mathbb{Z}_l$-algebra and the projectivity of the finite $A[J]$-module Sw_J realizes it as a direct summand of a finite free $A[J]$-module; consequently, the A-module $\mathrm{Hom}_J(\mathrm{Sw}_J, V)$ is also finite free, and it remains to note that the rank of its pullback to η (resp., F) equals $\mathrm{Sw}\, V_\eta$ (resp., $\mathrm{Sw}\, V_F$). □

3. Formulas involving ϵ_0

For $S = \mathrm{Spec}\, \mathbb{C}$, having proved the existence and uniqueness of ϵ_0 in [Del73, §4], Deligne proceeds to establish a formulary [Del73, §5] that details its properties and facilitates its computation. We gather some of these formulas here with a twofold aim: their special cases will be used in the proof of Theorem 1.1 to argue passage to more general bases S, and with little additional effort we will establish (3.1.1), (3.2.1), and (3.2.2) for all normal integral S.

3.1. Change of additive character. As we have already observed in §2.3, $K^\times$ acts on the set of possible choices of ψ. The effect that this action bears on ϵ_0 is explicated by

$$\epsilon_0(V, a\psi, dx) = (\det V)(a) \cdot |a|_K^{-\mathrm{rk}\, V} \cdot \epsilon_0(V, \psi, dx), \tag{3.1.1}$$

where $(\det V)(a)$ is the element of $\Gamma(S, \mathcal{O}_S^\times)$ by which a acts on the line bundle $\bigwedge^{\mathrm{rk}\, V} V$.

3.2. Unramified twists. Twisting V by an unramified 1-dimensional character θ or, more generally, tensoring by an unramified W of arbitrary dimension changes ϵ_0 as follows:

$$\epsilon_0(V\theta, \psi, dx) = \theta(\mathrm{Frob}_K)^{\mathrm{Sw}\, V + \mathrm{rk}\, V \cdot (n(\psi)+1)} \epsilon_0(V, \psi, dx), \tag{3.2.1}$$

$$\epsilon_0(V \otimes W, \psi, dx) = (\det W)(\mathrm{Frob}_K)^{\mathrm{Sw}\, V + \mathrm{rk}\, V \cdot (n(\psi)+1)} \epsilon_0(V, \psi, dx)^{\mathrm{rk}\, W}. \tag{3.2.2}$$

3.3. Explicit inverse in the 1-dimensional case. Unlike the other formulas, we will deduce (3.3.1) only for spectra of fields F of characteristic different from p from the known $F = \mathbb{C}$ case[4], and this will be one of the key inputs in proving that the ϵ_0 are global units as claimed in Theorem 1.1.

Given ψ and dx, the *dual Haar measure* $\widehat{dx}$ of dx with respect to ψ is defined by insisting that

$$\int_{\mathcal{O}_K} dx \cdot \int_{\mathcal{O}_K} \widehat{dx} = (\#\mathbb{F}_K)^{-n(\psi)}.$$

The resulting $\widehat{dx}$ is well-defined thanks to the discussion of §2.4.

Suppose that $S = \operatorname{Spec}\mathbb{C}$, and let $C_c^\infty(K)$ be the space of locally constant compactly supported $\mathbb{C}$-valued functions on K. Then the composition of the Fourier transform on $C_c^\infty(K)$ with respect to ψ and dx with the Fourier transform with respect to[5] $-\psi$ and $\widehat{dx}$ is the identity: compare, e.g., [BH06, §23.1, proof of Prop.][6]. Therefore, Tate's local functional equation [Del73, 5.8.1] for a continuous $\chi \in \operatorname{Hom}(K^\times, \mathbb{C}^\times)$ gives

$$\epsilon(\chi, \psi, dx)\epsilon(\chi^{-1}\left|\cdot\right|_K, -\psi, \widehat{dx}) = 1,$$

where $\epsilon(\chi, \psi, dx) = -\chi(\mathrm{Frob}_K)^{-\operatorname{rk}\chi^I}\epsilon_0(\chi, \psi, dx)$ as in Remark 1.4. This proves the $F = \mathbb{C}$ case of

$$\epsilon_0(\chi, \psi, dx)\epsilon_0(\chi^{-1}\left|\cdot\right|_K, -\psi, \widehat{dx}) = (\#\mathbb{F}_K)^{-\operatorname{rk}\chi^I}, \tag{3.3.1}$$

which is often helpful when reasoning about the inverse of $\epsilon_0(\chi, \psi, dx)$.

3.4. Explicit inverse in the higher-dimensional case. Assume that $S = \operatorname{Spec} F$ for a field F of characteristic l with $l \neq p$. If $l = 0$, then set $I' := I$. If $l > 0$, then set I' to be the preimage in I of the compositum of the prime-to-l Sylow subgroups of the quotient of I by the wild inertia. In both cases one may interpret I' as the minimal subgroup of I for which I/I' is pro-l. The closed subgroup I' is normal in $W(K^s/K)$. Moreover, the finite quotients of I' are of order prime to l, so the functor of taking I'-invariants is exact. Therefore, this functor induces a homomorphism

$$(-)^{I'}\colon R_F(W(K^s/K)) \to R_F(W(K^s/K)/I').$$

Let V be a continuous representation of $W(K^s/K)$ over F. If $\dim V = 1$, then the action of I on V factors through a finite quotient of prime to l order; thus, $V^{I'} = V^I$ for such V. Consequently,

$$\epsilon_0(V, \psi, dx)\epsilon_0(V^*\left|\cdot\right|_K, -\psi, \widehat{dx}) = (\#\mathbb{F}_K)^{-\operatorname{rk} V^{I'}}, \tag{3.4.1}$$

where V^* denotes the dual representation $\operatorname{Hom}_F(V, F)$, is an extension of (3.3.1) beyond the 1-dimensional case. We will prove (3.4.1) for all F at once in Corollary 5.11. The formula (3.4.1) will not play a role in the proof of Theorem 1.1.

4. The case when S is restricted to spectra of fields of characteristic 0

As remarked in 1.2 and 1.3, for $S = \operatorname{Spec}\mathbb{C}$ there exists a unique ϵ_0 satisfying (ii)–(v). Consequently,

$$\text{(i) holds whenever } f \text{ arises from an element of } \operatorname{Aut}(\mathbb{C}). \tag{†}$$

Proposition 4.1. *Theorem 1.1 holds if one restricts to S of the form* $\operatorname{Spec} F$ *for a field F of characteristic* 0. *Moreover, the resulting ϵ_0 satisfies* (3.1.1), (3.2.1), (3.2.2), *and* (3.3.1).

[4] One difficulty encountered over more general bases is the incompatibility of $\operatorname{rk}\chi^I$ with reduction modulo l.

[5] Here $-\psi$ should be interpreted as $(-1)\cdot\psi$ with $-1 \in K$, see §2.3.

[6] Beware that what loc. cit. calls the level of ψ is $-n(\psi)$ in the notation used here.

Proof. Since I_K acts through a finite quotient, $V \cong V' \otimes_{F'} F$ for a subfield $F' \subset F$ of finite transcendence degree over $\mathbb{Q}$ and a representation V' of $W(K^s/K)$ over F'. Enlarging F' if needed, we assume further that ψ and the Haar measure are F'-valued. Due to (i), a choice of an embedding $\iota\colon F' \hookrightarrow \mathbb{C}$ forces us to set

$$\epsilon_0(V, \psi, dx) := \iota^{-1}(\epsilon_0(V' \otimes_{F',\iota} \mathbb{C}, \iota \circ \psi, \iota \circ dx)).$$

Once we check, as we do below, that the resulting $\epsilon_0(V, \psi, dx)$ is independent of choices, (i)–(v) as well as the claimed formulas will follow from the construction and the assumed $F = \mathbb{C}$ case.

Firstly, $\epsilon_0(V' \otimes_{F',\iota} \mathbb{C}, \iota \circ \psi, \iota \circ dx) \in \iota(F')^\times$ due to (†), because $(\mathbb{C}^\times)^{\mathrm{Aut}(\mathbb{C}/\iota(F'))} = \iota(F')^\times$ [BouA, V.107, Prop. 10]. Moreover, $\mathrm{Aut}(\mathbb{C})$ acts transitively on the set of embeddings of F' into $\mathbb{C}$ [BouA, V.107, Cor. 2], so (†) also shows the independence of $\epsilon_0(V, \psi, dx)$ of the choice of ι, and hence the independence of enlarging F', as well. The independence of the choice of V' follows, too, because any two choices are isomorphic over a larger F'. ☐

5. The case when S is restricted to spectra of fields

To settle this case in stages in Propositions 5.8 and 5.9, we discuss the necessary representation-theoretic preliminaries in 5.2–5.6. To prepare for those, we recall the following well known lemma.

Lemma 5.1. *Fix a finite group G, and let F be a sufficiently large field. There exists a complete discrete valuation ring A with the residue field $F = A/\mathfrak{m}$ and the field of fractions η that is sufficiently large and of characteristic* 0.

Proof. For F of characteristic 0, one takes $A = F[\![t]\!]$. For F of characteristic $l > 0$, one lets m be the least common multiple of the orders of elements of G, applies [Mat89, 29.1] to $\mathbb{Q}_l(\zeta_m)$, and replaces the resulting discrete valuation ring A by its completion. ☐

5.2. Elementary groups. For a prime p, a finite group H is *p-elementary* if it is a product of a p-group and a cyclic group of order prime to p. A finite group is *elementary* if it is p-elementary for some prime p. An elementary H is the direct product of its Sylow subgroups; consequently, every $h \in H$ is a product of commuting elements of prime power order that are powers of h. To conclude that every $H' \leqslant H$ is again the direct product of its Sylow subgroups, and hence also elementary, it remains to note that the latter are the intersections of H' with the Sylow subgroups of H.

Many subsequent arguments will be based on the following version of Brauer's induction theorem.

Proposition 5.3 ([Del73, 1.5])**.** *For a finite group G and a sufficiently large field F,*

(a) $R_F(G)$ is spanned by the elements of the form $\mathrm{Ind}_H^G[\chi]$ for elementary $H \leqslant G$ and characters $\chi \in \mathrm{Hom}(H, F^\times)$, and

(b) $R_F^0(G)$ is spanned by the elements of the form $\mathrm{Ind}_H^G([\chi] - [\mathbf{1}_H])$ for elementary $H \leqslant G$ and characters $\chi \in \mathrm{Hom}(H, F^\times)$.

Proof. Suppose initially that $\mathrm{char}\, F = 0$. By Brauer's induction theorem [Ser77, §12.6, Thm. 27],

$$[\mathbf{1}_G] = \textstyle\sum_i \mathrm{Ind}_{H_i}^G a_i \quad \text{in } R_F(G) \text{ for some elementary } H_i \leqslant G \text{ and } a_i \in R_F(H_i).$$

After multiplying both sides of this equality by a $v \in R_F(G)$ (resp., a $v \in R_F^0(G)$ for (b)), the projection formula reduces to the case when G is elementary itself. On the other hand, by [Ser77, §12.3, proof of Thm. 24], $R_F(G)$ is spanned by the elements of the form $\mathrm{Ind}_H^G[\chi]$ for subgroups

$H \leqslant G$ and characters $\chi \in \operatorname{Hom}(H, F^\times)$, so (a) follows because, as noted in §5.2, H is elementary if so is G. As for (b), this expresses every $v \in R_F^0(G)$ as

$$v = \textstyle\sum_i n_i \operatorname{Ind}_{H_i}^G[\chi_i] = \sum_i n_i \operatorname{Ind}_{H_i}^G([\chi_i] - [\mathbf{1}_{H_i}]) + \sum_i n_i \operatorname{Ind}_{H_i}^G[\mathbf{1}_{H_i}], \qquad n_i \in \mathbb{Z}$$

for elementary $H_i \leqslant G$ and characters $\chi_i \in \operatorname{Hom}(H_i, F^\times)$, so the conclusion follows by induction on $\#G$ because $\sum_i n_i \operatorname{Ind}_{H_i}^G[\mathbf{1}_{H_i}] \in R_F^0(G)$ is the inflation of an element of $R_F^0(G/Z)$ where $Z \lhd G$ is the center, which is nontrivial if so is the elementary G.

Suppose now that F is of characteristic $l > 0$ and choose A as in Lemma 5.1. As observed in §2.7, the decomposition homomorphism $d_G\colon R_\eta(G) \to R_F(G)$ is surjective, preserves ranks, and commutes with induction, so we deduce (a) and (b) for F from the case of η established above. $\square$

Proposition 5.4 ([Del73, 1.8]). *For a finite group G and a discrete valuation ring A with the residue field $F = A/\mathfrak{m}$ of characteristic l and the field of fractions η that is sufficiently large and of characteristic 0, the elements of the form $\operatorname{Ind}_H^G([\chi] - [\chi'])$ for elementary $H \leqslant G$ and characters $\chi, \chi' \in \operatorname{Hom}(H, A^\times)$ with $\chi(h) \equiv \chi'(h) \bmod \mathfrak{m}$ for all $h \in H$ generate $\operatorname{Ker}(d_G\colon R_\eta(G) \to R_F(G))$.*

Proof. Passing to $\widehat{\eta}$ as in §2.7, one may for comfort reduce to the complete case. Also, d_G is an isomorphism if $l = 0$ (see §2.7), so we assume for the remainder of the proof that $l > 0$.

Using Proposition 5.3 (a) to write

$$[\mathbf{1}_G] = \textstyle\sum_i n_i \operatorname{Ind}_{H_i}^G[\chi_i] \quad \text{ in } R_\eta(G) \text{ for some } n_i \in \mathbb{Z}, \text{ elementary } H_i \leqslant G, \text{ and } \chi_i \in \operatorname{Hom}(H_i, \eta^\times),$$

for $v \in \operatorname{Ker} d_G$ we have

$$v = \textstyle\sum_i n_i \operatorname{Ind}_{H_i}^G([\chi_i] \cdot \operatorname{Res}_{H_i}^G v).$$

Since $\operatorname{Res}_{H_i}^G v \in \operatorname{Ker} d_{H_i}$, we are reduced to the case of an elementary G.

An elementary G is a product $G = N \times P$ with $l \nmid \#N$ and $\#P = l^n$ for some $n \geqslant 0$. By [CR81, 10.33 and 17.1], the irreducible representations of G over η or F are precisely the tensor products of an irreducible representation of N with a one of P. Consequently, d_G takes the form

$$R_\eta(G) \cong R_\eta(N) \otimes_{\mathbb{Z}} R_\eta(P) \xrightarrow{d_N \otimes d_P} R_F(N) \otimes_{\mathbb{Z}} R_F(P) \cong R_F(G).$$

Since d_N is an isomorphism [Ser77, §15.5] and $R_\eta(N)$ is $\mathbb{Z}$-free,

$$\operatorname{Ker} d_G \cong R_\eta(N) \otimes \operatorname{Ker} d_P,$$

so Proposition 5.3 (a) applied to N reduces further to the case $G = P$.

However, if G is of l-power order, then every character $\chi \in \operatorname{Hom}(H, \eta^\times)$ of a subgroup $H \leqslant G$ takes values in $1 + \mathfrak{m}A \subset A^\times$, and the claim results from Proposition 5.3 (b). $\square$

5.5. Weil representation types ([Del73, 4.10]). Fix a separably closed field F, and let l be its characteristic. A continuous representation V of $W(K^s/K)$ over F is said to have a type if Frob_K^m acts on V as a scalar $a \in F^\times$ for some $m \geqslant 1$. When this is the case, $\operatorname{Frob}_K^{Nm}$ acts on V as the scalar a^N for $N \geqslant 1$, so the resulting element of the direct limit (which is indexed by the positive integers ordered by the divisibility relation)

$$\varinjlim_{m \mid n} F^\times \quad \text{ with transition maps } \quad a \mapsto a^{\frac{n}{m}} \quad \text{ between the copies of } F^\times \text{ in positions } m \text{ and } n$$

is independent of the choice of m; it is the *type* of V. To argue that the type is also independent of the choice of Frob_K, let W be the quotient through which $W(K^s/K)$ acts on V, let $J \lhd W$ be the finite image of I_K, and write $\overline{\operatorname{Frob}}_K \in W$ for the image of Frob_K. Since J is centralized by $\overline{\operatorname{Frob}}_K^m$

for all sufficiently divisible m and changing Frob_K changes $\overline{\mathrm{Frob}}_K^m$ by an element of J, we see that $\overline{\mathrm{Frob}}_K^{m\cdot\#J}$ is independent of Frob_K, and hence so is the type.

Lemma 5.5.1. *Every irreducible V has a type.*

Proof. Indeed, $\mathrm{End}_W V$ is a finite dimensional division algebra over the separably closed F, so $\mathrm{End}_W V \cong F'$ for a finite field extension F'/F. Thus, every $\overline{\mathrm{Frob}}_K^N$ that is central in W acts as a scalar in $F'^{\times}$, and hence $\overline{\mathrm{Frob}}_K^{Nl^a}$ acts as a scalar in $F^{\times}$ for large a. □

For a fixed type τ, let $R_{F,\tau}(W(K^s/K))$ be the subgroup of $R_F(W(K^s/K))$ spanned by the classes of representations of type τ. Due to Lemma 5.5.1 and the discussion of §2.6,

$$R_F(W(K^s/K)) = \textstyle\bigoplus_\tau R_{F,\tau}(W(K^s/K)). \tag{5.5.2}$$

The representations of type 1 are precisely the Galois representations, see §2.1.

If F is in addition algebraically closed, then for every type τ there is an unramified character $\chi_\tau \colon W(K^s/K) \to F^{\times}$ of this type. Twisting by χ_τ induces an isomorphism

$$R_{F,1}(W(K^s/K)) \xrightarrow[\sim]{-\otimes\chi_\tau} R_{F,\tau}(W(K^s/K)). \tag{5.5.3}$$

Proposition 5.6. *For an algebraically closed F, the elements of the form $\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)}([\chi] - [\chi'])$ for finite subextensions $K^s/L/K$ and continuous $\chi, \chi' \in \mathrm{Hom}(W(K^s/L), F^{\times})$ span $R_F^0(W(K^s/K))$.*

Proof. For a $v \in R_F^0(W(K^s/K))$, let $v = \sum_\tau v_\tau$ be the decomposition provided by (5.5.2). For each appearing τ, fix a χ_τ as in (5.5.3). Then $\sum_\tau \mathrm{rk}\, v_\tau \cdot [\chi_\tau]$ is in the desired span ($L = K$ suffices), and it remains to note that so is each $v_\tau - \mathrm{rk}\, v_\tau \cdot [\chi_\tau]$ thanks to Proposition 5.3 (b) and (5.5.3). □

Having settled the representation-theoretic preliminaries, we get back to studying ϵ_0.

Lemma 5.7. *Let A be a discrete valuation ring with the residue field $F = A/\mathfrak{m}$ of characteristic l with $l \neq p$ and the fraction field η of characteristic 0. Adopt the setup of Theorem 1.1 for $S = \mathrm{Spec}\, A$.*

(a) For the ϵ_0 of Proposition 4.1, one has $\epsilon_0(V_\eta, \psi_\eta, (dx)_\eta) \in A^{\times}$.

(b) For a finite Galois subextension $K^s/L/K$ with the Galois group $G = \mathrm{Gal}(L/K)$, the restriction of the ϵ_0 of Proposition 4.1 to the kernel of $R_\eta(G) \to R_F(G)$ takes values in $1+\mathfrak{m}A \subset \eta^{\times}$.

Proof.

(a) Due to (i) and the existence of a discrete valuation prolonging the given one on η to any finite extension η'/η [Mat89, Cor. to 11.7], we may replace η by any η' if needed. Moreover, for a $W(K^s/K)$-stable A-lattice in V_η, its intersection with a subrepresentation (resp. image in a quotient) is a stable A-lattice, so any subquotient of V_η is realizable over A. Therefore, we may assume that V_η is absolutely irreducible, there is an unramified character $\theta \colon W(K^s/K) \to \eta^{\times}$ with the same type as $V_{\overline{\eta}}$, and the Galois representation $V_\eta \otimes \theta^{-1}$ factors through a continuous finite quotient G of $W(K^s/K)$ for which η is sufficiently large. Since scaling by $\theta(\mathrm{Frob}_K^N)$ is an automorphism of the A-lattice $V \subset V_\eta$ if N is sufficiently divisible, θ takes values in $A^{\times}$. Thus, (3.2.1), Proposition 5.3 (b) applied to $V_\eta \otimes \theta^{-1} - \mathrm{rk}\, V \cdot [\mathbf{1}_G]$, and (iv) reduce to the 1-dimensional case, in which the A-valued ϵ_0 given by (v) has the A-valued inverse provided explicitly by (3.3.1) and (v).

(b) As in the proof of (a), we assume that η is sufficiently large. Proposition 5.4 then reduces to proving that $\epsilon_0(\mathrm{Ind}_H^G([\chi]-[\chi']),\psi,dx) \in 1+\mathfrak{m}A$ for every subgroup $H \leqslant G$ and characters $\chi,\chi' \in \mathrm{Hom}(H,A^\times)$ with $\chi(h) \equiv \chi'(h) \bmod \mathfrak{m}$ for all $h \in H$. The additive character ψ necessarily takes values in A. The property (iv) reduces further to proving that

$$\epsilon_0(\chi,\psi\circ\mathrm{Tr}_{L^H/K},dx) \equiv \epsilon_0(\chi',\psi\circ\mathrm{Tr}_{L^H/K},dx) \bmod \mathfrak{m} \qquad \text{with both sides in } A \tag{5.7.1}$$

for a dx that also takes values in A. To obtain (5.7.1), apply (v) and note that $\mathrm{Sw}(\chi) = \mathrm{Sw}(\chi')$, as can be checked over the closed point thanks to Proposition 2.11. $\square$

Proposition 5.8. *In Theorem 1.1, restrict to S of the form $\mathrm{Spec}\, F$ for an algebraically closed field F.*

(a) An ϵ_0 satisfying (ii)–(v), (3.1.1), (3.2.1), *and* (3.2.2) *exists if it does when V, θ, and W are restricted to Galois representations.*

(b) If one restricts V to Galois representations, then an ϵ_0 satisfying (ii)–(v) is unique if it exists, in which case it also satisfies (i). The same conclusion holds if one restricts further to representations that factor through a fixed finite quotient $\mathrm{Gal}(L/K)$.

(c) There exists a unique ϵ_0 satisfying (i)–(v). It also satisfies (3.1.1), (3.2.1), *and* (3.2.2).

Proof. For every A provided by Lemma 5.1, Hensel's lemma (or, if one prefers, [EGA IV$_4$, 18.5.15]) lifts every ψ valued in $F^\times$ to a unique additive character valued in $A^\times$; lifting dx amounts to lifting $\int_{\mathcal{O}_K} dx \in F^\times$ to $A^\times$, see §2.4. We continue to denote these lifts by ψ and dx (although the latter is not unique) and recall from §2.3 that $n(\psi)$ is invariant under reduction modulo $\mathfrak{m}$. Lifting is also possible for unramified characters of Weil groups and 1-dimensional characters of finite groups.

(a) For each type τ, take an unramified character $\chi_\tau \colon W(K^s/K) \to F^\times$ of this type. To define ϵ_0 when its restriction to Galois representations $v \in R_{F,1}(W(K^s/K))$ is given, set

$$\epsilon_0(v\otimes\chi_\tau,\psi,dx) := \chi_\tau(\mathrm{Frob}_K)^{\mathrm{Sw}\, v+\mathrm{rk}\, v\cdot(n(\psi)+1)}\cdot\epsilon_0(v,\psi,dx) \tag{5.8.1}$$

and extend ϵ_0 to $R_F(W(K^s/K))$ using (5.5.2) and (5.5.3). Due to (3.2.1) for Galois representations, the definition (5.8.1) does not depend on the choice of χ_τ.

The desired (ii), (iii), and (v) are immediate. So is (3.1.1), because $n(a\psi) = n(\psi) + v_K(a)$ where $v_K(a)$ is the valuation of a. In checking (3.2.2), which includes (3.2.1), one restricts to irreducible V and W, which share their types with unramified characters χ_V and χ_W. Writing $W \cong W'\otimes\chi_W$ and $V \cong V'\otimes\chi_V$, (3.2.2) results from its version for $V'\otimes W'$, (5.8.1) with $\chi_\tau = \chi_V\chi_W$, and the relation $\det W = \chi_W^{\mathrm{rk}\, W}\det W'$.

We concentrate on the remaining (iv); it can be proved by a small computation, which, however, can also be relegated to characteristic 0, as we now explain. Proposition 5.6 reduces to considering $v \in R_F^0(W(K^s/L))$ of the form $[\chi]-[\chi']$ for continuous $\chi,\chi' \in \mathrm{Hom}(W(K^s/L),F^\times)$. For such a v, choose unramified $\theta,\theta' \in \mathrm{Hom}(W(K^s/K),F^\times)$ for which $\theta|_L$ and $\theta'|_L$ have the same types as χ and χ'. Let G be a continuous finite quotient of $W(K^s/L)$ through which $\chi^{-1}\cdot\theta|_L$ and $\chi'^{-1}\cdot\theta'|_L$ factor, and take A as in Lemma 5.1. Lifting θ, θ', $\chi^{-1}\cdot\theta|_L$, $\chi'^{-1}\cdot\theta'|_L$, ψ, and the involved Haar measures to A yields the desired conclusion thanks to Proposition 4.1, as long as we exhibit $\epsilon_0(\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)}\chi,\psi,dx_K)$, $\epsilon_0(\chi,\psi\circ\mathrm{Tr}_{L/K},dx_L)$, and their χ' analogues as reductions of the corresponding elements of $A^\times$ (cf. Lemma 5.7 (a)).

To argue this, thanks to Proposition 2.11 and the unramified twist aspect of Proposition 4.1, we only need to prove that the formation of ϵ_0 is compatible with reduction modulo $\mathfrak{m}$ for

Galois representations. Having the liberty of replacing A by its normalization in a finite extension of η, we deduce this compatibility from (iv), (v), and Proposition 5.3 (b).

(b) To show that the value of ϵ_0 on V is uniquely determined, take a finite quotient $\mathrm{Gal}(L/K)$ through which the $W(K^s/K)$-action factors and use Proposition 5.3 (b) to express

$$[V] - \mathrm{rk}\, V \cdot [\mathbf{1}_K] = \sum \mathrm{Ind}_{W(K^s/\widetilde{L})}^{W(K^s/K)} \left([\chi] - [\mathbf{1}_{\widetilde{L}}]\right) \quad \text{in } R_F(W(K^s/K))$$

for suitable subextensions $L/\widetilde{L}/K$ and characters $\chi \in \mathrm{Hom}(\mathrm{Gal}(L/\widetilde{L}), F^\times)$. The uniqueness, as well as (i), follows immediately from (ii)–(v).

(c) Thanks to (b) and (a), we seek an ϵ_0 satisfying (ii)–(v), (3.1.1), (3.2.1), and (3.2.2) for Galois representations. Furthermore, we restrict to representations that factor through a fixed finite quotient $G = \mathrm{Gal}(L/K)$, since the resulting ϵ_0 for varying G will be compatible by (b).

For an A given by Lemma 5.1 and the ϵ_0 constructed in Proposition 4.1, Lemma 5.7 (a) gives

$$\epsilon_0(-, \psi, dx) \colon R_\eta(G) \to A^\times \subset \eta^\times.$$

Thus, due to Lemma 5.7 (b) and the surjectivity of d_G (see §2.7), $\epsilon_0(-, \psi, dx) \bmod \mathfrak{m}$ induces

$$\epsilon_0(-, \psi, dx) \colon R_F(G) \to F^\times,$$

which satisfies (ii)–(v), (3.1.1), (3.2.1), and (3.2.2) by construction and Proposition 2.11. $\square$

Proposition 5.9. *Theorem 1.1 holds if one restricts to S of the form* $\mathrm{Spec}\, F$ *for a field F. Moreover, the resulting ϵ_0 satisfies* (3.1.1), (3.2.1), *and* (3.2.2).

We will show in Corollary 5.11 that ϵ_0 also satisfies (3.3.1) and (3.4.1).

Proof. Since (i) forces $\epsilon_0(V, \psi, dx) := \epsilon_0(V_{\overline{F}}, \psi, dx) \in \overline{F}^\times$, thanks to Proposition 5.8 (c), we only need to check that the resulting ϵ_0 takes values in F. Also, (i) applied to f arising from the elements of $\mathrm{Aut}(\overline{F}/F)$ reduces to the case of a separably closed F. We therefore assume that F is separably closed and imperfect of characteristic $l > 0$. We may and do further assume that V is irreducible and, scaling dx if needed, that ψ and dx take values in $\overline{\mathbb{F}}_l \subset F$. If V is a Galois representation, then the claim results from Proposition 5.3 (b), (iv), and (v).

Since I_K is a normal subgroup, every element of $W(K^s/K)$ maps every I_K-stable subspace of V to another such subspace. Consequently, the sum of the irreducible I_K-subrepresentations of V is $W(K^s/K)$-stable, so the irreducibility of V entails the semisimplicity of $V|_{I_K}$. Moreover, $W(K^s/K)$ acts transitively on the I_K-isotypic components $V' \subset V$, so $V \cong \mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)} V'$ for such a V' and its stabilizer $W(K^s/L) \leqslant W(K^s/K)$ (compare [Ser77, §8.1, proof of Prop. 24] if needed). Therefore, since $\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)}[\mathbf{1}_L]$ is a Galois representation, induction on $\dim V$ by means of (iv) allows us to assume that $V|_{I_K}$ is isotypic.

Using §2.6, we let (ρ, X) be the irreducible representation of I_K over $\overline{\mathbb{F}}_l$ for which $V|_{I_K}$ is a multiple of X_F. Since the isomorphism class of X_F is preserved under Frob_K-conjugation, so is that of X, see §2.6. A choice of an $\overline{\mathbb{F}}_l$-isomorphism $\iota \colon (\rho, X) \xrightarrow{\sim} (\rho', X)$ where $\rho'(i) = \rho(\mathrm{Frob}_K\, i\, \mathrm{Frob}_K^{-1})$ for $i \in I_K$ extends X to a representation of $W(K^s/K)$ over $\overline{\mathbb{F}}_l$ by letting Frob_K act as ι. This extension, still denoted by X, is a Galois representation because $\iota \in \mathrm{GL}(X)$ has finite order.

Since $\dim_F \mathrm{End}_{I_K}(X_F, X_F) = \dim_{\overline{\mathbb{F}}_l} \mathrm{End}_{I_K}(X, X) = 1$ (for the last equality, see the proof of Lemma 5.5.1), $\mathrm{End}_{I_K}(X_F, X_F) \cong F$, so the canonical $X_F \otimes_F \mathrm{Hom}_{I_K}(X_F, V) \to V$ is an isomorphism. Evidently, it is also $W(K^s/K)$-equivariant, so V decomposes over F as a tensor product of the Galois representation X_F and the unramified $\mathrm{Hom}_{I_K}(X_F, V)$. It remains to apply (3.2.2). $\square$

We record the following strengthening of Lemma 5.7 (a) that will be used in §6.

Proposition 5.10. *Let A be a discrete valuation ring with the residue field F of characteristic l with $l \neq p$ and the fraction field η. Adopt the setup of Theorem 1.1 for $S = \operatorname{Spec} A$. For the ϵ_0 of Proposition 5.9, one has $\epsilon_0(V_\eta, \psi_\eta, (dx)_\eta) \in A^\times$ with the image $\epsilon_0(V_F, \psi_F, (dx)_F)$ in $F^\times$.*

Proof. The proof is similar to that of Lemma 5.7 (a). Namely, if needed we again replace η by a finite extension and V by its subquotient to assume that $V \cong V' \otimes \chi_V$ for an $A^\times$-valued unramified character χ_V and a Galois representation V' that factors through a continuous finite quotient G of $W(K^s/K)$ for which η is sufficiently large. The formula (3.2.1) (together with §2.3 and Proposition 2.11) then reduces to the case $V = V'$, and we can further assume that $\dim V = 1$ thanks to Proposition 5.3 (b), (iv), and the compatibility of the decomposition homomorphism with induction (see §2.7). However, in the 1-dimensional Galois case, $\epsilon_0(V_\eta, \psi_\eta, (dx)_\eta) \in A$ thanks to ($\bigstar$), which also shows that this ϵ_0 reduces to $\epsilon_0(V_F, \psi_F, (dx)_F)$ in F. Since the latter is nonzero by Proposition 5.9, the claim follows. $\square$

Corollary 5.11. *The ϵ_0 of Proposition 5.9 also satisfies* (3.3.1) *and* (3.4.1)*.*

Proof. We begin with (3.3.1). Let $\chi_0 \in \operatorname{Hom}(W(K^s/K), F^\times)$ be the unramified character for which $\chi_0(\operatorname{Frob}_K) = \chi(\operatorname{Frob}_K)$. Replacing χ by $\chi\chi_0^{-1}$, invoking (3.2.1), and passing to $\overline{F}$, we may and do assume that χ factors through a continuous finite quotient G for which F is sufficiently large. We then use the remarks in the beginning of the proof of Proposition 5.8 to lift ψ, dx, and χ to corresponding objects over an A provided by Lemma 5.1. Since the lift of χ is ramified if and only if so is χ, Proposition 5.10 reduces to the char $F = 0$ case, which was established in Proposition 4.1.

We now turn to (3.4.1), for which we set $l := \operatorname{char} F$ and also adopt other notation of §3.4. Both sides of (3.4.1) define homomorphisms $R_F(W(K^s/K)) \to F^\times$: the left one due to (ii) and the exactness of $V \mapsto V^*$, the right one due to the exactness of $(-)^{I'}$ noted in §3.4. Moreover, by (3.3.1), the two homomorphisms agree on the classes of 1-dimensional representations. Passing to $\overline{F}$ and invoking Proposition 5.6, we therefore reduce to showing that they also agree on the elements of the form $\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)}([\chi]-[\chi'])$ for finite subextensions $K^s/L/K$ and continuous $\chi, \chi' \in \operatorname{Hom}(W(K^s/L), F^\times)$. Taking duals commutes with induction,[7] so the combination of (iii), (iv), and (3.3.1) shows that the left hand side of (3.4.1) evaluates on $\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)}([\chi] - [\chi'])$ to

$$\epsilon_0\left([\chi] - [\chi'], \psi \circ \operatorname{Tr}_{L/K}\right) \epsilon_0\left([\chi^{-1}\,|\cdot|_L] - [\chi'^{-1}\,|\cdot|_L], -\psi \circ \operatorname{Tr}_{L/K}\right) = (\#\mathbb{F}_L)^{-\operatorname{rk}\chi^{I'_L} + \operatorname{rk}\chi'^{I'_L}},$$

where $I'_L := I' \cap W(K^s/L)$, which equals the I' of $W(K^s/L)$. This agrees with what the right hand side evaluates to thanks to the following Claim 5.11.1 (b).

Claim 5.11.1. For a representation V of $W(K^s/L)$ over F, one has

(a) $\left(\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)} V\right)^{I'} \cong \operatorname{Ind}_{W(K^s/L)/I'_L}^{W(K^s/K)/I'} V^{I'_L}$;

(b) $(\#\mathbb{F}_K)^{\operatorname{rk}\left(\operatorname{Ind}_{W(K^s/L)}^{W(K^s/K)} V\right)^{I'}} = (\#\mathbb{F}_L)^{\operatorname{rk} V^{I'_L}}$ in F.

Proof.

[7] This commutation follows from the self-duality of $F[W(K^s/K)/W(K^s/L)]$, which in turn follows from the $W(K^s/K)$-invariance of the nondegenerate bilinear pairing $F[W(K^s/K)/W(K^s/L)] \times F[W(K^s/K)/W(K^s/L)] \to F$ for which the standard basis (indexed by the cosets) is its own dual basis.

(a) The argument below imitates an argument in [Del73, proof of 3.8].

After consulting [BH06, §2.5, proof of Lemma] to reconcile with the definition of induction given in (iv), we may identify

$$\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)} V = \{f\colon W(K^s/K) \to V \mid f(yx) = yf(x) \ \text{ for } \ y \in W(K^s/L)\},$$

with $w \in W(K^s/K)$ acting by $wf\colon x \mapsto f(xw)$. Likewise,

$$\mathrm{Ind}_{W(K^s/L)/I'_L}^{W(K^s/K)/I'} V^{I'_L} = \{\overline{f}\colon W(K^s/K)/I' \to V^{I'_L} \mid \overline{f}(yx) = y\overline{f}(x) \ \text{ for } \ y \in W(K^s/L)/I'_L\},$$

with $w \in W(K^s/K)/I'$ acting by $w\overline{f}\colon x \mapsto \overline{f}(xw)$.

Due to these descriptions, $\mathrm{Ind}_{W(K^s/L)/I'_L}^{W(K^s/K)/I'} V^{I'_L} \subset \left(\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)} V\right)^{I'}$. For the converse inclusion, an $f \in \left(\mathrm{Ind}_{W(K^s/L)}^{W(K^s/K)} V\right)^{I'}$ is an inflation of an $\overline{f}\colon W(K^s/K)/I' \to V$ that must take values in $V^{I'_L}$, because $\overline{f}(x) = \overline{f}(yx) = y\overline{f}(x)$ for $y \in I'_L$.

(b) We have $[W(K^s/K)/I' : W(K^s/L)/I'_L] = [W(K^s/K) : W(K^s/L)I']$, which, up to a power of l if $l > 0$, equals $[\mathbb{F}_L : \mathbb{F}_K]$. The combination of this and (a) gives the claim because the power of l does not matter: $(\#\mathbb{F}_K)^l = \#\mathbb{F}_K$ in F if $l > 0$. □

The agreement of the two homomorphisms $R_F(W(K^s/K)) \to F^\times$ establishes (3.4.1). □

6. The general case

6.1. Normal integral schemes. Recall from [EGA I, §0, 4.1.4] or [EGA IV$_2$, 5.13.5] that a scheme S is *normal* if its local rings are integrally closed domains. For an integral S, normality is equivalent to $\mathcal{O}_S(U)$ being an integrally closed domain for every open $U \subset S$ [EGA II, 8.8.6.1].

6.2. Universally Japanese rings. For an episodic appearance below, recall from [EGA IV$_2$, 7.7.1] that a ring R is *universally Japanese* if for every domain R' that is a finite type R-algebra, the integral closure of R' in $\operatorname{Frac} R'$ is a finite R'-module. For our purposes it suffices to know that every Dedekind domain whose fraction field has characteristic 0 is universally Japanese [EGA IV$_2$, 7.7.4].

Proof of Theorem 1.1 and the formulas (3.1.1), (3.2.1), *and* (3.2.2). Proposition 5.9 and (i) force us to define

$$\epsilon_0(V, \psi, dx) := \epsilon_0(V_\eta, \psi_\eta, (dx)_\eta),$$

where η is the generic point of S. Once we check, as we do below, that the resulting ϵ_0 is $\Gamma(S, \mathcal{O}_S^\times)$-valued and its image in $k(s)^\times$ is $\epsilon_0(V_s, \psi_s, (dx)_s)$ for every $s \in S$, (i)–(v), as well as (3.1.1), (3.2.1), and (3.2.2), will follow from Proposition 5.9.

The promised checking can be done locally on S, so we assume that $S = \operatorname{Spec} A$ is affine and V is free. Once we express A as a filtered direct limit of Noetherian normal $\mathbb{Z}[\frac{1}{p}]$-subalgebras containing the values of ψ and dx, since $W(K^s/K)$ acts on V through a finitely generated quotient, limit arguments and the base change aspect of Proposition 5.9 will descend V to such a subalgebra and permit to assume further that A is Noetherian. For such an expression, to capture the values of ψ, endow A with a structure of an R-algebra, where $R = \mathbb{Z}[\frac{1}{p}][\zeta_{p^\infty}]$ if $\operatorname{char} K = 0$ and $R = \mathbb{Z}[\frac{1}{p}][\zeta_p]$ if $\operatorname{char} K = p$; then ψ is the pullback of an $R^\times$-valued additive character. The normal 1-dimensional $R \subset \mathbb{Z}[\frac{1}{p}][\zeta_{p^\infty}]$ is a Dedekind domain because it is Noetherian due to Cohen's theorem: every finite prime l different from p is finitely decomposed and unramified in $\mathbb{Q}(\zeta_{p^\infty})$, so every prime $\mathfrak{p} \subset R$

is generated by the finitely generated $\mathfrak{p} \cap \mathbb{Z}[\frac{1}{p}][\zeta_{p^n}]$ for some $n \geqslant 0$. Consequently, as recalled in §6.2, R is universally Japanese, so finite type R-subalgebras of A have Noetherian normalizations in their fraction fields and contain the values of dx whenever they contain $\int_{\mathcal{O}_K} dx$ (see §2.4), which completes the reduction to the Noetherian case.

In the Noetherian case, A is an intersection of discrete valuation rings [Mat89, 12.3 and 12.4 (i)]:

$$A = \bigcap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}} \quad \text{inside Frac}\, A,$$

for which Proposition 5.10 supplies the required checking, so the desired $\epsilon_0(V_\eta, \psi_\eta, (dx)_\eta) \in A^\times$ follows. Lastly, for a nongeneric $s' \in S$ the image of $\epsilon_0(V_\eta, \psi_\eta, (dx)_\eta)$ in $k(s')^\times$ is $\epsilon_0(V_{s'}, \psi_{s'}, (dx)_{s'})$, as one sees by taking $s = \eta$ in Lemma 2.10 and combining its conclusion with the base change aspect of Proposition 5.9. $\square$

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