OBSTRUCTIONS TO RATIONAL POINTS VIA ÉTALE HOMOTOPY

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Abstract. Recently Harpaz and Schlank have explained how to use the theory of étale homotopy to obtain new obstructions to existence of rational points on algebraic varieties X defined over a number field K. We explain the construction of their obstructions and discuss the relations to other known obstructions, such as the (étale) Brauer-Manin obstruction and the descent obstruction.

1. Introduction

1.1. **The basic question.** One of the basic questions in aritmetic geometry is given an algebraic variety X/K (that is, a finite type separated scheme over $\operatorname{Spec} K$) over a field K decide whether it has a *rational point*, i.e., whether there is a K-morphism $\operatorname{Spec} K \to X$. It is customary to denote this set of rational points (i.e., morphisms) $X(K)$ and ask whether $X(K) \neq \varnothing$, is finite, etc. In these notes we will fix K to be a number field, for instance, K could be $\mathbb{Q}$, the field of rational numbers, and will assume X to be smooth over K and geometrically integral. In particular, this means that X will be integral. These assumptions will be in place throughout, unless noted otherwise.

1.2. **The local-global principle.** Perhaps the most classical approach to finding *obstructions* to existence of rational points, i.e., reasons why $X(K) = \varnothing$, is the idea to look first for K_v-points of X; here and in the sequel K_v denotes the completion of K at a place v. Indeed, precomposing with $\operatorname{Spec} K_v \to \operatorname{Spec} K$ one gets an injection $X(K) \hookrightarrow X(K_v)$ (for a K-scheme Y we write $X(Y)$ for the set of K-morphisms $Y \to X$ and abuse notation to let $X(R) = X(\operatorname{Spec} R)$ for a ring R) and observes that the absence of K_v-points $X(K_v) = \varnothing$ implies the absence of K-points of X. Doing this for all places v one gets an injection

$$X(K) \hookrightarrow \prod_v X(K_v), \tag{1.2.1}$$

and observing that it factors through the adelic points of X

$$X(K) \hookrightarrow X(\mathbb{A}) \to \prod_v X(K_v) \tag{1.2.2}$$

one summarizes that X has no K-points if it has no adelic points. Here $\mathbb{A}$ denotes the ring of *adeles* of K, that is, the restricted direct product

$$\prod_v{}' (K_v, \mathcal{O}_v) = \{(x_v) \in \prod_v K_v \mid \text{ for all but finitely many } v \text{ one has } x_v \in \mathcal{O}_v\},$$

where $\mathcal{O}_v$ is the valuation ring of K_v for a nonarchimedean place of v and $\mathcal{O}_v = K_v$ (or whatever you like) for an archimedean place v.

Searching for obstructions in terms of local points $X(K_v)$ is natural in the sense that it is usually much easier to decide whether $X(K_v) = \varnothing$. This is because using Hensel's lemma the search for a

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K_v-point can often be reduced to the search for an $\mathbb{F}_q$-point for some finite field $\mathbb{F}_q$ (and different X) which is, afterall, a finite computation.

The converse implication, namely, the claim that X has a rational point whenever it has an adelic point is the celebrated *Hasse principle* or *local-global principle*. Unfortunately, this implication doesn't always hold. The most well-known class of varieties X for which the Hasse principle holds is quadric hypersurfaces in projective space.

Theorem 1.2.3 (Hasse-Minkowski)**.** *Let X be a quadric hypersurface in $\mathbb{P}^n_K$. Then $X(\mathbb{A}) \neq \varnothing \implies X(K) \neq \varnothing$.*

1.3. **Other obstructions.** The failure of the Hasse principle in general leads one to search for more refined obstructions to existence of rational points. We will discuss some of those obstructions in the next section and proceed to construct the étale homotopy obstruction (with its variations) recently introduced by Harpaz and Schlank [HS11] in the rest of these notes. The *étale homotopy obstruction* is a functorial (in X) set $X(\mathbb{A})^h$ with

$$X(K) \subset X(\mathbb{A})^h \subset X(\mathbb{A}),$$

so that we can say that the absence of rational points on X is explained by the étale homotopy obstruction if $X(\mathbb{A})^h = \varnothing$.

More precisely, we will construct functorial obstruction sets, which we collectively call *étale homotopy obstructions*, fitting in the diagram

$$\begin{tikzcd}
 & X(\mathbb{A})^{\mathbb{Z}h} \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{\mathbb{Z}h,n} \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{\mathbb{Z}h,1} \arrow[r, hook] & X(\mathbb{A}) \\
X(K) \arrow[r, hook] & X(\mathbb{A})^{h} \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{h,n} \arrow[u, hook] \arrow[r, hook] & \cdots \arrow[r, hook] & X(\mathbb{A})^{h,1} \arrow[u, hook] &
\end{tikzcd} \tag{1.3.1}$$

and will explain (in the next section, before actually constructing them) how they relate to classical obstructions, such as the (étale) Brauer-Manin obstruction and the descent obstruction. Note that it is clear that the étale homotopy obstruction $X(\mathbb{A})^h$ is the strongest of the ones considered in the diagram (1.3.1), in the sense that the emptiness of any other obstruction set implies the emptiness of $X(\mathbb{A})^h$, i.e., if the absence of rational points is explained by some obstruction in the diagram then it is also explained by the étale homotopy obstruction. The obstruction set $X(\mathbb{A})^{\mathbb{Z}h}$ is called the *étale homology obstruction*.

2. Relations to classical obstructions

All the obstructions that we discuss in this section are intermediate sets $X(K) \subset X(\mathbb{A})^? \subset X(\mathbb{A})$ so that we can say that the absence of rational points is explained by the obstruction at hand if $X(\mathbb{A})^? = \varnothing$.

2.1. **The Brauer-Manin obstruction.** The construction of the Brauer-Manin obstruction uses the familiar Brauer-Hasse-Noether exact sequence from global class field theory (it is usually given in the language of Galois cohomology but we recall that étale cohomology over a field is Galois cohomology)

$$0 \to H^2_{\text{ét}}(K, \mathbb{G}_m) \to \bigoplus_v H^2_{\text{ét}}(K_v, \mathbb{G}_m) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

The first map is obtained from the contravariant functoriality of étale cohomology, whereas the second one is the sum of invariant maps $H^2_{\text{ét}}(K_v, \mathbb{G}_m) \xrightarrow{\mathrm{inv}_v} \mathbb{Q}/\mathbb{Z}$ from local class field theory (for nonarchimedean v the map inv_v is an isomorphism). For a scheme X the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is often called the *Brauer group* of X (note that it is contravariant in X), so that the exact sequence above could be rewritten as

$$0 \to \mathrm{Br}(K) \to \bigoplus_v \mathrm{Br}(K_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0. \tag{2.1.1}$$

Using this sequence one can define the *Brauer-Manin pairing*

$$\mathrm{Br}(X) \times X(\mathbb{A}) \to \mathbb{Q}/\mathbb{Z} \tag{2.1.2}$$

by noting that an adelic point of X gives rise to a collection of local points $(x_v) \in \prod_v X(K_v)$, so that an element $A \in \mathrm{Br}(X)$ gives rise to a collection of local Brauer elements $(x_v^* A) \in \prod_v \mathrm{Br}(K_v)$, and we can sum up their invariants:

$$(A, (x_v)) \mapsto \sum_v \mathrm{inv}_v(x_v^* A).$$

Using a model $\mathcal{X}$ of X over the ring of S-integers for a big enough set S of places of K one can justify the finiteness of the sum on the right hand side (cf. [Sko01, p. 101]) so that the pairing is well defined. The exact sequence (2.1.1) shows that $X(K) \subset X(\mathbb{A})$ is in the right kernel $X(\mathbb{A})^{\mathrm{Br}}$ of the pairing (2.1.2). This right kernel $X(\mathbb{A})^{\mathrm{Br}}$ is the *Brauer-Manin obstruction set* and we say that the absence of rational points on X is explained by the Brauer-Manin obstruction if $X(\mathbb{A})^{\mathrm{Br}} = \varnothing$. Its relation to the étale homotopy obstructions is as follows.

Theorem 2.1.3 ([HS11, Theorem 10.1])**.** *The Brauer-Manin obstruction is equivalent to the étale homology obstruction. More precisely,*

$$X(\mathbb{A})^{\mathrm{Br}} = X(\mathbb{A})^{\mathbb{Z}h}.$$

2.2. **The descent obstruction.** Let X be a (smooth, geometrically connected) K-variety and let G be an affine algebraic K-group (so in particular, a K-variety). An (fppf) *G-torsor* over X is a scheme $Y \xrightarrow{f} X$ together with a right action of $G \times_K X$ over X such that fppf locally Y is (equivariantly) isomorphic to $G \times_K X$ with the right translation action on itself. More precisely, there should exist a family $\{U_i \to X\}$ of jointly surjective, flat, and locally of finite presentation morphisms such that each $Y \times_X U_i$ with the induced $G \times_K U_i$-action is (equivariantly) isomorphic over U_i to $G \times_K U_i$ with the right translation action over itself.

Given a G-torsor $Y \xrightarrow{f} X$ one can twist it by the elements σ of the nonabelian Galois cohomology set $H^1(K, G)$ to get torsors $Y^\sigma \xrightarrow{f^\sigma} X$. We don't want to get into explaining this here so we simply assert that by doing this one gets a partition

$$X(K) = \bigsqcup_{\sigma \in H^1(X,G)} f^\sigma(Y^\sigma(K)),$$

and refer the reader to [Sko01, §2, esp. formula (2.12)] for details. Since each $Y^\sigma(K) \subset Y^\sigma(\mathbb{A})$ we get that

$$X(K) \subset X(\mathbb{A})^f := \bigcup_{\sigma \in H^1(X,G)} f^\sigma(Y^\sigma(\mathbb{A})) \subset X(\mathbb{A}). \tag{2.2.1}$$

Doing this for all torsors over X under affine algebraic K-groups G one gets

$$X(K) \subset \bigcap_f X(\mathbb{A})^f =: X(\mathbb{A})^{\mathrm{desc}} \subset X(\mathbb{A}). \tag{2.2.2}$$

The obstruction set $X(\mathbb{A})^{\mathrm{desc}}$ is called the *descent obstruction*. One can also consider its variants where in the intersection in (2.2.2) one only takes torsors under finite or finite abelian affine algebraic

K-groups. The obtained obstruction sets are denoted $X(\mathbb{A})^{\text{fin}}$ and $X(\mathbb{A})^{\text{fin-ab}}$, respectively. Their relation to étale homotopy obstructions from diagram (1.3.1) is as follows.

Theorem 2.2.3 ([HS11, Theorem 9.3])**.**

$$X(\mathbb{A})^{\text{fin}} = X(\mathbb{A})^{h,1},$$
$$X(\mathbb{A})^{\text{fin-ab}} = X(\mathbb{A})^{\mathbb{Z}h,1}.$$

It was proved by Skorobogatov in [Sko09] that for projective X the descent obstruction is equivalent to the étale Brauer-Manin obstruction that we will introduce in the next section. In more precise terms, $X(\mathbb{A})^{\text{desc}} = X(\mathbb{A})^{\text{ét,Br}}$ so that Theorem 2.3.1 furnishes a relation of the descent obstruction with the étale homotopy obstruction for projective X.

2.3. **The étale Brauer-Manin obstruction.** The étale Brauer-Manin obstruction is a synthesis of the ideas of sections 2.1 and 2.2. Namely, one observes that in the setting of (2.2.1) one has $Y^\sigma(K) \subset Y^\sigma(\mathbb{A})^{\text{Br}}$ so that in fact

$$X(K) \subset X(\mathbb{A})^{f,\text{Br}} := \bigcup_{\sigma \in H^1(X,G)} f^\sigma(Y^\sigma(\mathbb{A})^{\text{Br}}).$$

If one does this for all torsors under *finite* affine algebraic K-groups G one gets (the intersection runs over all such f)

$$X(K) \subset \bigcap_f X(\mathbb{A})^{f,\text{Br}} =: X(\mathbb{A})^{\text{ét,Br}}.$$

The obstruction set $X(\mathbb{A})^{\text{ét,Br}}$ is called the *étale Brauer-Manin obstruction.*

Theorem 2.3.1 ([HS11, Theorem 11.1])**.** *The étale Brauer-Manin obstruction is equivalent to the étale homotopy obstruction. More precisely,*

$$X(\mathbb{A})^{\text{ét,Br}} = X(\mathbb{A})^h.$$

3. Simplicial constructions

We warn the reader that the constructions in this section are often ad hoc. This is because we do not want to detour into generalities of simplicial homotopy theory but prefer to define just what we need later and proceed. For a more appropriate treatment of the contents of this section one can consult any text on simplicial homotopy theory, for instance [GJ09].

3.1. **Simplicial objects.** Let Δ denote the category of finite ordinal numbers and order preserving maps. In other words, the objects of Δ are finite sets $[n] = \{0, 1, \ldots, n\}$, $n \geqslant 0$, while $\mathrm{Hom}_\Delta([n],[m])$ is the set of all nondecreasing functions $f\colon [n] \to [m]$.

Definition 3.1.1. *Suppose $\mathcal{C}$ is a category. A* simplicial object *in $\mathcal{C}$ is a functor $\mathbf{X}\colon \Delta^{\text{op}} \to \mathcal{C}$, and we will denote $\mathbf{X}([n]) \in \mathcal{C}$ by $\mathbf{X}_n$. A morphism between two simplicial objects is a natural transformation of corresponding functors. The resulting category of simplicial objects in $\mathcal{C}$ is denoted $\mathcal{C}^{\Delta^{\text{op}}}$.*

The objects of $\mathrm{Set}^{\Delta^{\text{op}}}$ are *simplicial sets.* For each $[n]$ the functor $\Delta^n\colon \Delta^{\text{op}} \to \mathrm{Set}$ represented by $[n]$ is called *the standard n-simplex.* For an $\mathbf{X} \in \mathrm{Set}^{\Delta^{\text{op}}}$ the set $\mathbf{X}_n$ is the set of *n-simplices* of $\mathbf{X}$; by Yoneda's lemma it identifies with the set of simplicial set maps (natural transformations) $\Delta^n \to \mathbf{X}$.

3.2. **Coskeleta.** For each $n \geqslant 0$ let $\Delta_{\leqslant n}$ denote the full subcategory of Δ spanned by $[m]$, $m \leqslant n$. A functor $\mathbf{Y}\colon \Delta_{\leqslant n} \to \mathcal{C}$ is an *n-truncated simplicial object* in $\mathcal{C}$ and the collection of such is organized into a category $\mathcal{C}^{\Delta^{\mathrm{op}}_{\leqslant n}}$. Precomposing with the inclusion $\iota\colon \Delta_{\leqslant n} \to \Delta$ induces a functor

$$\mathrm{tr}_n\colon \mathcal{C}^{\Delta^{\mathrm{op}}} \to \mathcal{C}^{\Delta^{\mathrm{op}}_{\leqslant n}},$$

called the *n-truncation*. Suppose $\mathcal{C}$ is complete (has all finite limits). The general theory of right Kan extensions then tells us (cf. [ML98, §X.3]) that tr_n admits a right adjoint Ran_ι which is computed by declaring $\mathrm{Ran}_\iota \mathbf{Y}$ to be the right Kan extension of $\mathbf{Y}$ along ι. It is customary to denote this right adjoint by cosk_n, call it the *n-coskeleton* functor, and give it in a more pedestrian (and useful) way: for $m \geqslant 0$ one has (the second equality is Yoneda's lemma)

$$(\mathrm{cosk}_n \mathbf{Y})_m = \varprojlim_{\substack{[k]\to[m]\\ k\leqslant n}} \mathbf{Y}_k = \varprojlim_{\substack{\Delta^k\to\Delta^m\\ k\leqslant n}} \mathbf{Y}_k \tag{3.2.1}$$

with the natural $(\mathrm{cosk}_n \mathbf{Y})(f)\colon (\mathrm{cosk}_n \mathbf{Y})_{m'} \to (\mathrm{cosk}_n \mathbf{Y})_m$ for nondecreasing $f\colon [m] \to [m']$ which we leave for the reader to explicate. One can verify that these formulas indeed define a right adjoint to tr_n, but in fact they are a special case of more general formulas used to compute right Kan extensions.

Remark 3.2.2. Note that for $m \leqslant n$ one has $(\mathrm{cosk}_n \mathbf{Y})_m \cong \mathbf{Y}_m$ because in this case the indexing categories in (3.2.1) have terminal (remember, $\mathbf{Y}$ is contravariant) objects $\mathrm{id}\colon [m] \to [m]$ and $\mathrm{id}\colon \Delta^m \to \Delta^m$, respectively. We say that cosk_n doesn't change the n-skeleton.

Proposition 3.2.3. *For $m \geqslant n$ there is a natural isomorphism of functors*

$$\mathrm{cosk}_m \circ \mathrm{tr}_m \circ \mathrm{cosk}_n \cong \mathrm{cosk}_n .$$

Proof. One can work this out using a combination of adjointness tricks, formula (3.2.1), and Remark 3.2.2.

Alternatively, if $\mathcal{C}$ is cocomplete (has finite colimits), like is the case for the category of sets, then in a manner analogous to the above one can define the left adjoint to tr_n called the *n-skeleton* functor sk_n. The conclusion then follows from the uniqueness of adjoints and the following computation:

$$\begin{aligned}
\mathrm{Hom}(\mathbf{X}, \mathrm{cosk}_m(\mathrm{tr}_m(\mathrm{cosk}_n(\mathbf{Y})))) &\cong \mathrm{Hom}(\mathrm{tr}_m(\mathbf{X}), \mathrm{tr}_m(\mathrm{cosk}_n(\mathbf{Y}))) \\
&\cong \mathrm{Hom}(\mathrm{sk}_m(\mathrm{tr}_m(\mathbf{X})), \mathrm{cosk}_n(\mathbf{Y})) \\
&\cong \mathrm{Hom}(\mathrm{tr}_n(\mathrm{sk}_m(\mathrm{tr}_m(\mathbf{X}))), \mathbf{Y}) \\
&\cong \mathrm{Hom}(\mathrm{tr}_n(\mathbf{X}), \mathbf{Y}) \\
&\cong \mathrm{Hom}(\mathbf{X}, \mathrm{cosk}_n(\mathbf{Y})).
\end{aligned}$$

To justify the penultimate step one needs to observe that sk_m doesn't change the m-skeleton which is justified similarly to Remark 3.2.2.

In fact, even though we were assuming that $\mathcal{C}$ is cocomplete this is not necessary. One could first do the Yoneda embedding $\iota\colon \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of $\mathcal{C}$ into the presheaf category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ which is (co)complete and then do the argument there. For this to work one would need to observe that by doing this coskeleta are brought to coskeleta. This is because ι preserves limits and coskeleta are defined in terms of limits. The one sentence "proof" that we gave in the beginning results from translating this argument into the language that avoids mentioning Yoneda. □

3.3. **Geometric realization.** Given a simplicial set $\mathbf{X}$ one can form a topological space $|\mathbf{X}|$, called the *geometric realization* of $\mathbf{X}$, in the following way: for Δ^n set $|\Delta^n|$ to be the *standard topological n-simplex* $\{(t_0, \dots, t_n) \,|\, t_i \geqslant 0, t_0 + \cdots + t_n = 1\} \subset \mathbb{R}^{n+1}$; this is functorial in that $f\colon [n] \to [m]$ gives a map $|\Delta^n| \to |\Delta^m|$ by sending $(t_i)_{0 \leqslant i \leqslant n}$ to $(\sum_{j \in f^{-1}(i)} t_j)_{0 \leqslant i \leqslant m}$ and extending linearly. For a general $\mathbf{X}$ one sets (the colimit is taken over the comma category $\Delta^n \downarrow \mathbf{X}$ of standard simplices over $\mathbf{X}$)

$$|\mathbf{X}| = \varinjlim_{\Delta^n \to \mathbf{X}} |\Delta^n| \tag{3.3.1}$$

and notes that this defines a functor $|\cdot| : \mathrm{Set}^{\Delta^{\mathrm{op}}} \to \mathrm{Top}$ which agrees with our previous definition of $|\Delta^n|$ because when $\mathbf{X} = \Delta^n$ the indexing category for the colimit above has the final object $\mathrm{id}\colon \Delta^n \to \Delta^n$.

3.4. **Kan complexes.** A morphism $\mathbf{X} \to \mathbf{Y}$ of simplicial sets is called a *cofibration* if it is a monomorphism (levelwise injective); it is called a *weak equivalence* if the induced $|\mathbf{X}| \to |\mathbf{Y}|$ is a homotopy equivalence. An *acyclic cofibration* is a map which is both a cofibration and a weak equivalence. A map of simplicial sets $\mathbf{X} \to \mathbf{Y}$ is a *Kan fibration* (or simply a *fibration*) if it has the *right lifting property* with respect to all acyclic cofibrations. What this condition means is that given a commutative diagram

$$\begin{tikzcd} \mathbf{A} \arrow[r] \arrow[d] & \mathbf{X} \arrow[d] \\ \mathbf{B} \arrow[r] \arrow[ur, dashed] & \mathbf{Y} \end{tikzcd}$$

the indicated lift exists making both triangles commute whenever $\mathbf{A} \to \mathbf{B}$ is an acyclic cofibration. A simplicial set $\mathbf{X}$ is *Kan* or *fibrant* if the unique map $\mathbf{X} \to * = \Delta^0$ to the terminal object is a Kan fibration.

3.5. **Fibrant replacement.** Let $\mathbf{X}$ be a simplicial set. A *fibrant replacement* of $\mathbf{X}$ is an acyclic cofibration $\mathbf{X} \to \mathbf{Y}$ with $\mathbf{Y}$ fibrant. It turns out that a fibrant replacement exists for any simplicial set $\mathbf{X}$ and, moreover, can be done functorially: there is a functor $\mathrm{Ex}^\infty\colon \mathrm{Set}^{\Delta^{\mathrm{op}}} \to \mathrm{Set}^{\Delta^{\mathrm{op}}}$ together with a natural transformation $\mathrm{Id} \to \mathrm{Ex}^\infty$ such that for each $\mathbf{X} \in \mathrm{Set}^{\Delta^{\mathrm{op}}}$ the obtained $\mathbf{X} \to \mathrm{Ex}^\infty(\mathbf{X})$ is a fibrant replacement of $\mathbf{X}$. We do not attempt to give proofs of the assertions that we just made, the construction of Ex^∞ is carried out in [GJ09, III.§4] and we direct the reader there for details.

3.6. **Postnikov towers.** Another technicality that will make an appearance in the sequel is the notion of a *Postnikov tower* of a fibrant simplicial set $\mathbf{X}$. This is a sequence of maps

$$\cdots \to P_n(\mathbf{X}) \to \cdots \to P_1(\mathbf{X}) \to P_0(\mathbf{X}) \tag{3.6.1}$$

where $P_i(\mathbf{X}) = \mathrm{cosk}_{i+1}(\mathrm{tr}_{i+1}(\mathbf{X}))$ and the morphisms are induced from natural transformations

$$P_{i+1}(-) \to P_i(-) \tag{3.6.2}$$

which are adjoint to the natural transformations $\mathrm{tr}_{i+1}(P_{i+1}(-)) \xrightarrow{\sim} \mathrm{tr}_{i+1}(-)$ (cf. Remark 3.2.2). In a similar vein, one has natural transformations

$$\mathrm{Id} \to P_i(-) \tag{3.6.3}$$

which are compatible with (3.6.2).

The simplicial set $P_n(\mathbf{X})$ is called the n^{th} *Postnikov piece* of $\mathbf{X}$. Note that the Postnikov tower is functorial in the sense that a map $\mathbf{X} \to \mathbf{Y}$ induces maps $P_i(\mathbf{X}) \to P_i(\mathbf{Y})$ fitting into an infinite commutative ladder between the Postnikov towers of $\mathbf{X}$ and $\mathbf{Y}$. Also, one may consider truncated

Postnikov towers by which we mean that one only takes Postnikov pieces up to some fixed level n in (3.6.1). Clearly, this is a functorial operation as well.

The main utility of the Postnikov tower is to decompose the homotopical information carried by a fibrant simplicial set $\mathbf{X}$ into pieces $P_n(\mathbf{X})$ which are often easier to study. This is because the first $n+1$ homotopy groups (the homotopy groups of a simplicial set are defined to be the homotopy groups of the geometric realization) of $P_n(\mathbf{X})$ are those of $\mathbf{X}$ while the other homotopy groups $\pi_k(P_n(\mathbf{X}))$, $k > n$ vanish, so that the n^{th} Postnikov piece captures the information contained in the first $n+1$ homotopy groups of $\mathbf{X}$. For instance, from a homotopic point of view $P_0(\mathbf{X})$ only remembers the path components of $\mathbf{X}$, $P_1(\mathbf{X})$ remembers the fundamental group in addition, and so on.

4. Pro-categories

A category $\mathcal{I}$ is called *cofiltered* if it is nonempty, for any $x, y \in \mathcal{I}$ there is a $z \in \mathcal{I}$ together with morphisms $z \to x$, $z \to y$, and for any two morphisms $f, g\colon x \to y$ with the same source and target there is a morphism $h\colon z \to x$ such that $fh = gh$.

Definition 4.1. *Let $\mathcal{C}$ be a category. The pro-category of $\mathcal{C}$, denoted* $\mathrm{Pro}\,\mathcal{C}$*, is the category whose objects are functors $F\colon \mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ a small cofiltered category, and whose morphisms between $F\colon \mathcal{I} \to \mathcal{C}$ and $G\colon \mathcal{J} \to \mathcal{C}$ are*

$$\mathrm{Hom}_{\mathrm{Pro}\,\mathcal{C}}(F, G) = \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(F(i), G(j)). \tag{4.2}$$

A couple of observations are in order:

- Just by staring at the formula (4.2) it may not be immediately clear how the composition in $\mathrm{Pro}\,\mathcal{C}$ works. Key to understanding it is noting that *every* object in the target must be hit by a morphism from the source. Alternatively, one can use an equivalent approach to pro-categories outlined below (Remark 4.3), or a mixture of both approaches (objects from Definition 4.1, morphisms from Remark 4.3).
- If the indexing categories $\mathcal{I}$ and $\mathcal{J}$ are the same $\mathcal{I} = \mathcal{J}$ then any natural transformation between F and G gives rise to a morphism between F and G in $\mathrm{Pro}\,\mathcal{C}$.
- Restricting to only those functors F for which $\mathcal{I}$ contains one object and one morphism realizes $\mathcal{C}$ as a full subcategory of $\mathrm{Pro}\,\mathcal{C}$.
- One should think of $\mathrm{Pro}\,\mathcal{C}$ as obtained from $\mathcal{C}$ by "formally adding filtered limits". There is a way to make this statement precise but we will not do this here.

Remark 4.3. There is another way to deal with pro-categories which elucidates more clearly the morphisms between F and G and in particular how the composition of morphisms in $\mathrm{Pro}\,\mathcal{C}$ works. Namely, one could use the contravariant Yoneda embedding $\mathcal{C} \mapsto (\mathrm{Set}^{\mathcal{C}})^{\mathrm{op}}$ sending an object $C \in \mathcal{C}$ to the representable functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$. Under this point of view F as above is a filtered *direct* system of representable covariant functors $F(i)\colon \mathcal{C} \to \mathrm{Set}$, and we can identify it with the colimit $\varinjlim_{i \in \mathcal{I}} F(i)$ taken in the functor category. The set of natural transformations between $\varinjlim_{i \in \mathcal{I}} F(i)$ and $\varinjlim_{j \in \mathcal{J}} G(j)$ is (we use Yoneda's lemma repeatedly)

$$\begin{aligned}\mathrm{Hom}_{(\mathrm{Set}^{\mathcal{C}})^{\mathrm{op}}}(\varinjlim_{i \in \mathcal{I}} F(i), \varinjlim_{j \in \mathcal{J}} G(j)) &= \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(\varinjlim_{j \in \mathcal{J}} G(j), \varinjlim_{i \in \mathcal{I}} F(i)) = \varprojlim_{j \in \mathcal{J}} \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(G(j), \varinjlim_{i \in \mathcal{I}} F(i)) \\ &= \varprojlim_{j \in \mathcal{J}} (\varinjlim_{i \in \mathcal{I}} F(i))(G(j)) = \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} F(i)(G(j)) = \varprojlim_{j \in \mathcal{J}} \varinjlim_{i \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(F(i), G(j)).\end{aligned}$$

This is precisely the formula in (4.2). This calculation therefore shows that $\mathrm{Pro}\,\mathcal{C}$ in the sense of Definition 4.1 is (anti)equivalent to the category of *pro-representable covariant* functors (functors isomorphic to ones of the form above) and natural transformations between them. From our point of view, however, this approach is inferior to Definition 4.1 even though the resulting categories are equivalent. This is because many different diagrams (functors $F\colon \mathcal{I} \to \mathcal{C}$) can give rise to the same pro-representable covariant functor, and if one only remembers the latter, one has forgotten the properties of the diagrams which gave rise to it.

5. Hypercoverings

5.1. **Hypercoverings.** Let $X_{\text{ét}}$ denote the étale site of X, i.e., the category of étale schemes over X whose coverings are jointly surjective families.

A *hypercovering* of X is a simplicial object $\mathcal{U}$ in $X_{\text{ét}}$ such that

- $\mathcal{U}_0 \to X$ is surjective;
- For each $n \geqslant 0$ the natural morphism $\mathcal{U}_{n+1} \to \operatorname{cosk}_n(\operatorname{tr}_n(\mathcal{U}))$ is surjective.

5.2. **Čech hypercoverings.** The principal example of a hypercovering is that of a Čech hypercovering. Namely, let $U \to X$ be an étale cover. For instance, U could be the disjoint union $\bigsqcup U_i$ where the collection $\{U_i \to X\}$ of étale morphisms is jointly surjective. Then one sets $\mathcal{U} = \operatorname{cosk}_0 U$ and notes that $\mathcal{U}$ is a hypercovering because the first condition is verified due to Remark 3.2.2, while the second one follows from Proposition 3.2.3. Using formula (3.2.1) one sees that $\mathcal{U}_n = U \times_X \cdots \times_X U$ is the $(n+1)$-fold fiber product. At this point it becomes clear that $\mathcal{U}$ is just the usual Čech simplicial object associated to the covering $U \to X$.

Note a peculiar property of $\mathcal{U}$ which follows from Proposition 3.2.3: all the morphisms $\mathcal{U}_{n+1} \to \operatorname{cosk}_n(\operatorname{tr}_n(\mathcal{U}))$ are isomorphisms. Getting rid of this and allowing arbitrary refinements at every stage is the principal feature of hypercoverings that makes them a useful generalization of Čech hypercoverings.

Even though we will later be talking about étale homotopy, the advantages of using hypercoverings surface already at the cohomology level. Namely, recall that in general one cannot use Čech cohomology to compute étale cohomology of a sheaf even if one passes to finer and finer coverings. This is mended by Verdier's hypercovering theorem which says that étale cohomology can be computed by computing Čech-like cohomology for finer and finer hypercoverings. We will not make this more precise here but see [SGA 4.2, Exposé V §7].

5.3. **Homotopies.** It is clear what a morphism of hypercoverings of X is: a natural transformation of corresponding functors. However, we will want to consider morphisms only "up to homotopy". To make sense of this first define a functor $\otimes\colon (X_{\text{ét}})^{\Delta^{\mathrm{op}}} \times \mathrm{Set}^{\Delta^{\mathrm{op}}} \to (X_{\text{ét}})^{\Delta^{\mathrm{op}}}$ which takes $(\mathcal{U}, \mathbf{K})$ to $\mathcal{U} \otimes \mathbf{K}$ given by $(\mathcal{U} \otimes \mathbf{K})_n = \bigsqcup_{k \in \mathbf{K}_n} \mathcal{U}_n$ with the structure morphisms $(\mathcal{U} \otimes \mathbf{K})(f)$ for $f\colon [m] \to [n]$ defined by $\mathcal{U}(f)\colon \mathcal{U}_n \to \mathcal{U}_m$ going from the copy of $\mathcal{U}_n$ corresponding to $k \in \mathbf{K}_n$ to the copy of $\mathcal{U}_m$ corresponding to $\mathbf{K}(f)(k) \in \mathbf{K}_m$. Now call two morphisms $f, g\colon \mathcal{U} \to \mathcal{V}$ of hypercoverings $\mathcal{U}, \mathcal{V}$ of X *strictly homotopic* if there is a commutative diagram

$$\begin{tikzcd} \mathcal{U} \sqcup \mathcal{U} \arrow[d, "i"] \arrow[dr, "f \sqcup g"] & \\ \mathcal{U} \otimes \Delta^1 \arrow[r, "h"'] & \mathcal{V} \end{tikzcd}$$

where i is the morphism which is given on each factor $\mathcal{U} \cong \mathcal{U} \otimes \Delta^0$ of the coproduct by $\mathrm{id}_{\mathcal{U}} \otimes j$ where $j\colon [0] \to [1]$ is the map with image $\{0\}$ for the first factor and the map with image $\{1\}$ for the second (by Yoneda's lemma such j give rise to corresponding morphisms $j\colon \Delta^0 \to \Delta^1$). Two morphisms $f, g\colon \mathcal{U} \to \mathcal{V}$ are *homotopic* if they are related by a chain of strict homotopies. The relation of homotopy thus defined is an equivalence relation; one also sees easily that if f is homotopic to g then uf is homotopic to ug and fv is homotopic to gv whenever the compositions make sense. Now $\mathrm{HR}(X)$, the *homotopy category of hypercoverings* of X, is defined to be the category whose objects are hypercoverings of X and whose morphisms are homotopy classes of morphisms of hypercoverings.

Remark 5.3.1. In our construction of the notion of homotopy we have not used any properties of $X_{\text{ét}}$ except that it has (finite) coproducts. Nor have we used that $\mathcal{U}$ and $\mathcal{V}$ are hypercoverings rather than arbitrary simplicial objects in $X_{\text{ét}}$. Therefore, the construction goes through for *any* category $\mathcal{C}$ with coproducts and we arrive at a notion of homotopic morphisms between simplicial objects in $\mathcal{C}$. For instance, if $\mathcal{C} = \mathrm{Set}$ one has a notion of homotopic maps between simplicial sets. Equipped with this notion one defines the *homotopy category of simplicial sets*, denoted $\mathrm{Ho}(\mathrm{Set}^{\Delta^{\mathrm{op}}})$, to be the category whose objects are *fibrant* simplicial sets (cf. section 3.4) and whose morphisms are homotopy classes of morphisms of simplicial sets in the sense just discussed.

Proposition 5.3.2. $\mathrm{HR}(X)$ *is cofiltered.*

Proof. See [AM69, Corollary 8.13]. □

This is in fact one of the main advantages of passing to the homotopy category: the second condition in the definition of cofiltered (cf. section 4) would hold anyway because products of hypercoverings are hypercoverings, but to get the third condition one passes to homotopy classes of morphisms.

5.4. **Pullbacks of hypercoverings.** Let $f\colon Y \to X$ be a morphism of schemes and $\mathcal{U}$ a hypercovering of X. The *pullback* of $\mathcal{U}$ is the simplicial object $\mathcal{U} \times_X Y$ in $Y_{\text{ét}}$ which is obtained by taking fiber products $\mathcal{U}_n \times_X Y$ levelwise.

Proposition 5.4.1. $\mathcal{U} \times_X Y$ *is a hypercovering of* Y.

Proof. Firstly, $\mathcal{U} \times_X Y$ is well-defined because étale morphisms are stable under base change. So are surjections, so that $\mathcal{U}_0 \times_X Y \to Y$ is surjective. In fact, this also verifies the second condition in the definition of a hypercovering because both truncations and coskeleta commute with fiber products: this is trivial for tr_n and true for cosk_n because of formula (3.2.1) and the fact that limits commute. □

The canonical isomorphism $(\bigsqcup U_i) \times_X Y \cong \bigsqcup (U_i \times_X Y)$ shows that strict homotopies are preserved under the functor $- \times_X Y$, so that one gets a functor $\mathrm{HR}_{Y/X}\colon \mathrm{HR}(X) \to \mathrm{HR}(Y)$. The construction of $\mathrm{HR}_{Y/X}$ is evidently functorial in the sense that there are natural isomorphisms

$$\mathrm{HR}_{X/X} \cong \mathrm{Id}_{\mathrm{HR}(X)} \quad \text{and} \quad \mathrm{HR}_{Z/Y} \circ \mathrm{HR}_{Y/X} \cong \mathrm{HR}_{Z/X} \tag{5.4.2}$$

for $Z \to Y \to X$.

6. Étale homotopy type

We are now ready to give the definition of the étale homotopy type of X following Artin and Mazur [AM69]. Observe that any scheme U in $X_{\text{ét}}$ is locally noetherian because $U \to X$ is étale hence locally of finite presentation. Therefore, the connected components of U are open ([EGA I, 6.1.9]). We denote this set of connected components by $\pi_0(U)$ and observe that this defines a functor $\pi_0\colon X_{\text{ét}} \to \text{Set}$. If one applies π_0 levelwise to a simplicial object in $X_{\text{ét}}$ one therefore gets a simplicial set. Postcomposing with the fibrant replacement functor Ex^∞ gives a fibrant simplicial set and hence a functor (cf. Remark 5.3.1)

$$\text{Ex}^\infty(\pi_0(-))\colon \text{HR}(X) \to \text{Ho}(\text{Set}^{\Delta^{\text{op}}}). \tag{6.1}$$

To see that Ex^∞ preserves homotopies one uses natural maps $\text{Ex}^\infty(\mathbf{X}) \times \Delta^n \to \text{Ex}^\infty(\mathbf{X}) \times \text{Ex}^\infty(\Delta^n) \cong \text{Ex}^\infty(\mathbf{X} \times \Delta^n)$ the existence of which can be argued using the construction of Ex^∞ (which we haven't carried out). Since $\text{HR}(X)$ is cofiltered (cf. Proposition 5.3.2) this functor can be viewed as an element of $\text{Pro}\,\text{Ho}(\text{Set}^{\Delta^{\text{op}}})$ which we denote by $\text{Ét}(X)$ and call the *étale homotopy type* of X.

Taking into account functors $\text{HR}_{Y/X}$ together with natural isomorphisms (5.4.2) we see that the construction of $\text{Ét}(X)$ is functorial in the sense that we get a functor

$$\text{Ét}\colon \text{Var}_K \to \text{Pro}\,\text{Ho}(\text{Set}^{\Delta^{\text{op}}}).$$

Here Var_K, the category of K-varieties (i.e., finite type separated schemes over K), can actually be replaced by the category of locally noetherian schemes with no significant modification to the construction.

A couple of remarks about étale homotopy type are in order, even though we will not need them later. They are quite vague, however: this is because we haven't developed enough theory to give more details, which, together with proofs, can be found in [AM69].

- Verdier's hypercovering theorem to which we have alluded to in section 5.2 implies that one can recover étale cohomology of X (say, with constant coefficients) from its étale homotopy type $\text{Ét}(X)$.
- One can recover the étale fundamental group $\pi_1^{\text{ét}}(X)$ from $\text{Ét}(X)$ (really, from a variation of $\text{Ét}(X)$ because one has to take care of basepoints).
- If K is embedded as a subfield of $\mathbb{C}$ and X is geometrically connected one can consider $X(\mathbb{C})$ with its complex topology which will be a connected topological space. Then one can recover the profinite completion of the topological homotopy groups of $X(\mathbb{C})$ from $\text{Ét}(X)$ (with the same caveat about basepoints). Coupled with the previous observation this recovers the classical isomorphism ([SGA 1, Exposé XII Corollaire 5.2]) between the étale fundamental group of X and the profinite completion of the topological fundamental group of $X(\mathbb{C})$.

7. The étale homotopy obstruction

7.1. **Relative étale homotopy type.** The definition of the étale homotopy obstruction will use a relative version (due to Harpaz and Schlank [HS11]) of the étale homotopy type functor introduced in the last section. To define it first consider the full subcategory $\widetilde{\text{HR}}(X)$ of $\text{HR}(X)$ consisting of hypercoverings that are levelwise separated and of finite type over X. Note that each $\text{HR}_{Y/X}$ restricts to a functor $\widetilde{\text{HR}}(X) \to \widetilde{\text{HR}}(Y)$.

For $\mathcal{U} \in \widetilde{\text{HR}}(X)$ one sets $\overline{X} = X \times_K \overline{K}$ (here $\overline{K}$ is a fixed algebraic closure of K) and considers $\text{HR}_{\overline{X}/X}(\mathcal{U}) = \mathcal{U} \times_X \overline{X} \cong \mathcal{U} \times_K \overline{K}$. One observes that Γ_K, the Galois group of $\overline{K}$ over K, acts on $\overline{K}$ and hence on $\text{HR}_{\overline{X}/X}(\mathcal{U})$. If one takes connected components levelwise, i.e., computes $\pi_0(\text{HR}_{\overline{X}/X}(\mathcal{U}))$, one gets a Γ_K-simplicial set, i.e., a simplicial object in the category Set_{Γ_K} of sets equipped with a continuous Γ_K-action. This is because each connected component of every $\mathcal{U}_n \times_K \overline{K}$ is defined over some finite extension of K and therefore is fixed by an open subgroup of Γ_K (this is the reason why we had to restrict to $\widetilde{\mathrm{HR}}(X)$). The construction being functorial one gets a functor (with the same remark on Ex^∞ concerning homotopies as in (6.1))

$$\mathrm{Ex}^\infty(\pi_0(\mathrm{HR}_{\overline{X}/X}(-)))\colon \widetilde{\mathrm{HR}}(X) \to \mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}). \tag{7.1.1}$$

We haven't told the reader yet what $\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$ is: its objects are $\mathbf{X} \in \mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}$ such that $\mathbf{X}^\Lambda$ (the fixed point simplicial set of Λ) is fibrant for every open subgroup $\Lambda \leqslant \Gamma_K$, its morphisms are Γ_K-equivariant simplicial set maps taken up to Γ_K-equivariant homotopy in the way analogous to Remark 5.3.1 (more checking concerning Ex^∞ that we omit).

At this point one checks that the constructions in the proof of Proposition 5.3.2 can be done within $\widetilde{\mathrm{HR}}(X)$ so that $\widetilde{\mathrm{HR}}(X)$ is cofiltered and the functor in (7.1.1) can be considered as an object $\acute{\mathrm{E}}\mathrm{t}_{/K}(X)$ in $\mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$ called the *relative étale homotopy type* of X/K. Using functors $\mathrm{HR}_{Y/X}$ one sees as in the case of Artin-Mazur étale homotopy type that the construction of this object is functorial so that one gets a functor

$$\acute{\mathrm{E}}\mathrm{t}_{/K}\colon\ \mathrm{Var}_K \to \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$$

called the relative étale homotopy type functor.

7.2. **Variants using Postnikov towers.** To define the étale homotopy obstruction we will need a slight modification of the relative étale homotopy type functor $\acute{\mathrm{E}}\mathrm{t}_{/K}$. Namely, given an object $\{\mathbf{X}_i\}_{i\in\mathcal{I}} \in \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$ we can apply the Postnikov tower functor to each $\mathbf{X}_i$ to get an object $\{P_n(\mathbf{X}_i)\}_{i\in\mathcal{I},n\geqslant 0}$ (note that the new indexing category is still cofiltered). There are three things to check to make sure that this is a legitimate thing to do:

1. $P_n(-)$ brings Γ_K-simplicial sets to Γ_K-simplicial sets and equivariant maps to equivariant maps (this is easily verified because we have formula (3.2.1) at our disposal and $P_n(-)$ is a functor).
2. The way we have defined $\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$ we should make sure that each $P_n(\mathbf{X})^\Lambda$ is Kan for each open $\Lambda \leqslant \Gamma_K$ given $\mathbf{X}$ satisfies this condition.
3. Each $P_n(-)$ brings homotopic maps to homotopic maps (strict homotopies are required to be Γ_K-equivariant).

We are not going to check the last two claims but will remark that both follow because $P_n(-)$ is right adjoint and hence commutes with taking fixed points or products (such as $- \times \Delta^1$).

Since taking (truncated) Postnikov towers is functorial (cf. section 3.6), from the claims above that we haven't checked one sees that for $0 \leqslant n \leqslant \infty$ we get functors

$$\begin{aligned}(-)^n\colon\ \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}) &\to \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}),\\ \{\mathbf{X}_i\}_{i\in\mathcal{I}} &\mapsto \{P_m(\mathbf{X}_i)\}_{i\in\mathcal{I},0\leqslant m\leqslant n}.\end{aligned}$$

By this we mean that for $n = \infty$ we take the full Postnikov tower objectwise, whereas for finite n we take all Postnikov pieces objectwise up to level n. Note that the indexing categories are cofiltered if $\mathcal{I}$ is cofiltered.

One has evident natural transformations

$$\mathrm{Id} \to (-)^{n'} \to (-)^n \tag{7.2.1}$$

for $n' \geqslant n$ which are compatible among one another in the obvious sense. Here the last natural transformation results from simply forgetting part of the indexing diagram, whereas the first one is an avatar of the natural transformations (3.6.3).

The functors that are used to define étale homotopy obstructions are the composed functors $(\acute{\mathrm{E}}\mathrm{t}_{/K}(-))^n$, $0 \leqslant n \leqslant \infty$ which we denote $\acute{\mathrm{E}}\mathrm{t}^n_{/K}(-)$. The natural transformations (7.2.1) translate to natural transformations

$$\acute{\mathrm{E}}\mathrm{t}_{/K} \to \acute{\mathrm{E}}\mathrm{t}^{n'}_{/K} \to \acute{\mathrm{E}}\mathrm{t}^n_{/K} \tag{7.2.2}$$

for $n' \geqslant n$ which are again compatible.

7.3. **Étale homotopy obstructions.** For each $0 \leqslant n \leqslant \infty$ a rational point $\operatorname{Spec} K \to X$ gives rise to a morphism $\acute{\mathrm{E}}\mathrm{t}_{/K}(\operatorname{Spec} K) \to \acute{\mathrm{E}}\mathrm{t}^n_{/K}(X)$ (use (7.2.2)). If we denote by $X^n(hK)$ the set of morphisms $\acute{\mathrm{E}}\mathrm{t}_{/K}(\operatorname{Spec} K) \to \acute{\mathrm{E}}\mathrm{t}^n_{/K}(X)$ in $\operatorname{Pro}\operatorname{Ho}(\mathrm{Set}^{\Delta^{\mathrm{op}}}_{\Gamma_K})$ then we get maps $X(K) \to X^n(hK)$, $0 \leqslant n \leqslant \infty$.

If v is a place of K with completion K_v we denote by $\overline{K}_v$ a fixed algebraic closure of K_v containing the fixed algebraic closure $\overline{K}$ from section 7.1. The Galois group of $\overline{K}_v$ over K_v will be denoted by Γ_v and we identify it with a closed subgroup of Γ_K. Suppose one has a K_v-point of X, i.e., a K-morphism $\operatorname{Spec} K_v \to X$. Since we have an inclusion $\overline{K} \hookrightarrow \overline{K}_v$ it gives rise to a $\overline{K}_v$-point of $\overline{X}$ and we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \overline{K}_v \arrow[r] \arrow[d] & \overline{X} \arrow[d] \\ \operatorname{Spec} K_v \arrow[r] & X \end{tikzcd}$$

which for each $0 \leqslant n \leqslant \infty$ gives rise to a natural (in X) morphism[1]

$$\acute{\mathrm{E}}\mathrm{t}_{/K_v}(\operatorname{Spec} K_v) \to \acute{\mathrm{E}}\mathrm{t}_{/K}(X). \tag{7.3.1}$$

It results from forgetting Γ_K-action and remembering only Γ_v-action for hypercoverings which are pulled back from X. Now apply Postnikov tower functors $(-)^n$ for each $0 \leqslant n \leqslant \infty$ and use (7.2.2) to get natural (in X) morphisms

$$\acute{\mathrm{E}}\mathrm{t}_{/K_v}(\operatorname{Spec} K_v) \to \acute{\mathrm{E}}\mathrm{t}^n_{/K}(X). \tag{7.3.2}$$

If one denotes $X^n(hK_v)$ the set of morphisms $\acute{\mathrm{E}}\mathrm{t}_{/K_v}(\operatorname{Spec} K_v) \to \acute{\mathrm{E}}\mathrm{t}^n_{/K}(X)$ one gets a map of sets $X^n(hK) \to X^n(hK_v)$ which fits into a commutative diagram

$$\begin{tikzcd} X(K) \arrow[r] \arrow[d] & X^n(hK) \arrow[d] \\ X(K_v) \arrow[r] & X^n(hK_v). \end{tikzcd}$$

[1]Here (and in the sequel when we consider similar maps) we are viewing $\acute{\mathrm{E}}\mathrm{t}_{/K}(X)$ as an object in $\operatorname{Pro}\operatorname{Ho}(\mathrm{Set}^{\Delta^{\mathrm{op}}}_{\Gamma_v})$. To see that this is a legitimate thing to do one needs to check that if a Γ_K-simplicial set $\mathbf{X}$ satisfies that $\mathbf{X}^{\Lambda}$ is fibrant for each open $\Lambda \leqslant \Gamma_K$ then the same holds for open $\Lambda' \leqslant \Gamma_v$. We will not check this here but the reader who knows the criterion of being a fibration in terms of lifting properties for inclusions of horns should have no trouble.

Putting such diagrams together for all places v and remembering that $X(K) \to \prod_v X(K_v)$ factors through $X(\mathbb{A})$ (cf. (1.2.2)) one gets a diagram

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d, hook] & X^n(hK) \arrow[dd, "l_n"] \\
X(\mathbb{A}) \arrow[d] \arrow[dr, "h_n"] & \\
\prod_v X(K_v) \arrow[r] & \prod_v X^n(hK_v).
\end{tikzcd}$$

Now one sets $X(\mathbb{A})^{h,n} := h_n^{-1}(l_n(X^n(hK)))$ and lets $X(\mathbb{A})^h := X(\mathbb{A})^{h,\infty}$. These are the étale homotopy obstruction sets that appear in the bottom part of the diagram (1.3.1). Their functoriality in X is clear from the functoriality of (7.3.2) and $\acute{\mathrm{E}}\mathrm{t}^n_{/K}$. The fact that we have the natural inclusions $X(\mathbb{A})^{h,n'} \hookrightarrow X(\mathbb{A})^{h,n}$ for $n' \geqslant n$ results from the natural transformations (7.2.2) which give rise to a commutative diagram

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X^{n'}(hK) \arrow[r] \arrow[d, "l_{n'}"] & X^n(hK) \arrow[d, "l_n"] \\
X(\mathbb{A}) \arrow[r, "h_{n'}"] \arrow[rr, bend right, "h_n"'] & \prod_v X^{n'}(hK_v) \arrow[r] & \prod_v X^n(hK_v).
\end{tikzcd}$$

Since obviously $X(K) \subset X(\mathbb{A})^h$, we recover the bottom part of (1.3.1).

7.4. **Étale homology obstructions.** The étale homology obstructions are the sets $X(\mathbb{A})^{\mathbb{Z}h}$, $X(\mathbb{A})^{\mathbb{Z}h,n}$ appearing in the top part of the diagram (1.3.1). They are defined in an analogous manner to étale homotopy obstructions that we have seen in the previous section using the free abelian group functor $\mathbb{Z}(-)$ applied levelwise to the simplicial sets appearing in the diagrams for $\acute{\mathrm{E}}\mathrm{t}_{/K}(X)$ and then taking the corresponding Postnikov towers. Let us make this more precise.

Suppose you take a simplicial set $\mathbf{X}$ and apply the free abelian group functor $\mathbb{Z}(-)$ levelwise. You get a simplicial set $\mathbb{Z}\mathbf{X}$ which in fact is a simplicial (abelian) group (a simplicial object in the category of (abelian) groups). As a set naturally includes to the free abelian group on that set one has a natural transformation

$$\mathrm{Id} \to \mathbb{Z}(-). \tag{7.4.1}$$

If $\mathbf{X}$ is a Γ_K-simplicial set then so is $\mathbb{Z}\mathbf{X}$ (in fact it is even a simplicial Γ_K-module). This defines a functor

$$\mathbb{Z}(-)\colon \mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}} \to \mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}$$

such that $(\mathbb{Z}\mathbf{X})^\Lambda$ is fibrant for any open subgroup $\Lambda \leqslant \Gamma_K$ (actually, it is a general fact that the underlying simplicial set of any simplicial group is fibrant, see [GJ09, Lemma I.3.4]). It is not hard to see that Γ_K-equivariantly homotopic maps between simplicial sets satisfying the condition that we just spelled out (i.e., between the objects of $\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$) are still Γ_K-equivariantly homotopic after applying $\mathbb{Z}(-)$. Therefore, this defines a functor

$$\mathbb{Z}(-)\colon \mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}) \to \mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}).$$

By applying it to each object in the diagram we get a functor

$$\mathbb{Z}(-)\colon \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}) \to \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}}).$$

In particular, we could consider the composed functor $\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}\colon \mathrm{Var}_K \to \mathrm{Pro}\,\mathrm{Ho}(\mathrm{Set}_{\Gamma_K}^{\Delta^{\mathrm{op}}})$, as well as its decorations $(\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}(-))^n$, $0 \leqslant n \leqslant \infty$ (cf. section 7.2) together with natural transformations

$$\acute{\mathrm{E}}\mathrm{t}_{/K} \to \acute{\mathrm{E}}\mathrm{t}^n_{/K} \to (\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}(-))^n \tag{7.4.2}$$

for $0 \leqslant n \leqslant \infty$ that result from (7.2.2) and (7.4.1).

At this point one takes (7.3.1) together with (7.4.2) to see that each K_v-point gives rise to a natural (in X) morphism $\acute{\mathrm{E}}\mathrm{t}_{/K_v}(\mathrm{Spec}\,K_v) \to (\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}(X))^n$ and the story is much the same as in the previous section. Namely, for $0 \leqslant n \leqslant \infty$ one denotes by $X^{\mathbb{Z},n}(hK)$ (resp., $X^{\mathbb{Z},n}(hK_v)$) the set of morphisms $\acute{\mathrm{E}}\mathrm{t}_{/K}(\mathrm{Spec}\,K) \to (\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}(X))^n$ (resp., $\acute{\mathrm{E}}\mathrm{t}_{/K_v}(\mathrm{Spec}\,K_v) \to (\mathbb{Z}\,\acute{\mathrm{E}}\mathrm{t}_{/K}(X))^n$) one gets commutative diagrams

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X^{\mathbb{Z},n}(hK) \arrow[d] \\
X(K_v) \arrow[r] & X^{\mathbb{Z},n}(hK_v).
\end{tikzcd}$$

which fit into

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X^{\mathbb{Z},n}(hK) \arrow[dd, "l_n^{\mathbb{Z}}"] \\
X(\mathbb{A}) \arrow[d] \arrow[dr, "h_n^{\mathbb{Z}}"] & \\
\prod_v X(K_v) \arrow[r] & \prod_v X^{\mathbb{Z},n}(hK_v).
\end{tikzcd}$$

The étale homology obstruction sets are now defined $X(\mathbb{A})^{\mathbb{Z}h,n} := (h_n^{\mathbb{Z}})^{-1}(l_n^{\mathbb{Z}}(X^{\mathbb{Z},n}(hK)))$ and one lets $X(\mathbb{A})^{\mathbb{Z}h} := X(\mathbb{A})^{\mathbb{Z}h,\infty}$. The inclusions $X(\mathbb{A})^{\mathbb{Z}h,n'} \hookrightarrow X(\mathbb{A})^{\mathbb{Z}h,n}$ for $n' \geqslant n$ are obtained in the same way as for the étale homotopy obstructions that we discussed in the previous section (there are obvious analogues of the natural transformations (7.2.2)). This recovers the top part of (1.3.1). To get the vertical arrows in (1.3.1) one uses the natural transformations (7.4.2) to argue commutative diagrams

$$\begin{tikzcd}
X(K) \arrow[r] \arrow[d] & X^n(hK) \arrow[r] \arrow[d, "l_n"] & X^{\mathbb{Z},n}(hK) \arrow[d, "l_n^{\mathbb{Z}}"] \\
X(\mathbb{A}) \arrow[r, "h_n"] \arrow[rr, bend right, "h_n^{\mathbb{Z}}"'] & \prod_v X^n(hK_v) \arrow[r] & \prod_v X^{\mathbb{Z},n}(hK_v)
\end{tikzcd}$$

which explicate the inclusions $X(\mathbb{A})^{h,n} \hookrightarrow X(\mathbb{A})^{\mathbb{Z}h,n}$.

We have now justified the diagram (1.3.1). However, justification of the claims that we have made in section 2 relating étale homotopy (and homology) obstructions to classical obstructions requires much more work and we refer the reader to [HS11] for details.

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