SHIDLOVSKY'S MULTIPLICITY ESTIMATE AND IRRATIONALITY OF ZETA VALUES

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(November 10, 2017)

Abstract

In this paper we follow the approach of Bertrand-Beukers (and of later work of Bertrand), based on differential Galois theory, to prove a very general version of Shidlovsky's lemma that applies to Padé approximation problems at several points, both at functional and numerical levels (i.e., before and after evaluating at a specific point). This allows us to obtain a new proof of the Ball-Rivoal theorem on irrationality of infinitely many values of Riemann zeta function at odd integers, inspired by the proof of the Siegel-Shidlovsky theorem on values of E-functions: Shidlovsky's lemma is used to replace Nesterenko's linear independence criterion with Siegel's, so that no lower bound is needed on the linear forms in zeta values. The same strategy provides a new proof, and a refinement, of Nishimoto's theorem on values of L-functions of Dirichlet characters.

Keywords and phrases: Mathematics Subject Classification (2010): 11J72 (Primary); 11M06, 34M03 (Secondary).

1. Introduction

Very few results are known on the arithmetic nature of $\zeta(s)$, where ζ is the Riemann zeta function and $s \geq 2$ is an integer. If s is even, $\zeta(s)$ is a rational multiple of π^s and therefore a transcendental number. Apéry has proved [2] that $\zeta(3)$ is irrational, but there is no odd integer $s \geq 5$ for which $\zeta(s)$ is known to be irrational. The next major step is due to Ball-Rivoal [3, 22]:

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \ldots, \zeta(a)) \geq \frac{1+o(1)}{1+\log 2} \log a \tag{1.1}$$

as $a \to \infty$, where a is odd. In particular, $\zeta(s)$ is irrational for infinitely many odd integers s.

The proofs of these results rely on very ingenious explicit constructions; many authors have tried to make them more natural by establishing connections to various settings (see for instance [9] for a survey). One of these is Padé-approximation: the construction appears as the (unique) solution of a Padé-approximation problem. This has been done for Apéry's theorem in several ways, including works of Beukers [6] and Sorokin [26, 27]. On the other hand, the Ball-Rivoal theorem is proved by considering

$$S(z) = \sum_{t=1}^{\infty} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{(t)_{n+1}^a} z^{-t}$$

where r and n are positive integers (with $1 \leq r < a/2$ and $n \to \infty$), and $z = 1$; recall that Pochhammer's symbol is defined by $(\alpha)_p = \alpha(\alpha+1)\ldots(\alpha+p-1)$. This function $S(z)$ is the unique solution, up to proportionality, of the following Padé-approximation problem [10]: find polynomials $P_1, \ldots, P_{a+2}$ of degree at most n such that

$$\begin{cases} S(z) := P_{a+2}(z) + \sum_{i=1}^{a} P_i(z)\mathrm{Li}_i(1/z) = O(z^{-rn-1}), & z \to \infty, \\ \widetilde{S}(z) := P_{a+1}(z) + \sum_{i=1}^{a} P_i(z)(-1)^i\mathrm{Li}_i(z) = O(z^{(r+1)n+1}), & z \to 0, \\ T(z) := \sum_{i=1}^{a} P_i(z)(-1)^{i-1}\frac{(\log z)^{i-1}}{(i-1)!} = O((z-1)^{(a-2r)n+a-1}), & z \to 1. \end{cases} \tag{1.2}$$

In the present paper we give a new proof of the Ball-Rivoal theorem, in which this Padé-approximation problem plays a central role. Our strategy is inspired by the Siegel-Shidlovsky theorem on values of E-functions (see for instance [25, Chapter 3]): we prove a general version of Shidlovsky's lemma and use it to find sufficiently many values of k, bounded from above independently from n, such that the derivatives $S^{(k-1)}(1) - \widetilde{S}^{(k-1)}(1)$ are linearly independent linear forms in $1, \zeta(3), \zeta(5), \ldots, \zeta(a)$. This allows us to apply Siegel's linear independence criterion (instead of Nesterenko's) and deduce the lower bound (1.1).

In order to state our version of Shidlovsky's lemma, we need some notation. To begin with, given $\sigma \in \mathbb{C} \cup \{\infty\}$, recall that the Nilsson class at σ is the set of finite sums

$$f(z) = \sum_{e \in \mathbb{C}} \sum_{i \in \mathbb{N}} \lambda_{i,e}\, h_{i,e}(z)(z-\sigma)^e(\log(z-\sigma))^i$$

where $\lambda_{i,e} \in \mathbb{C}$, $h_{i,e}$ is holomorphic at σ, and $z - \sigma$ should be understood as $1/z$ if $\sigma = \infty$. If such a function $f(z)$ is not identically zero, we may assume that $h_{i,e}(\sigma) \neq 0$ for any i, e; then the (generalized) order of f at σ, denoted by $\mathrm{ord}_\sigma f$, is the minimal real part of an exponent e such that $\lambda_{i,e} \neq 0$ for some i.

Let q be a positive integer, and $A \in M_q(\mathbb{C}(z))$. We fix $P_1, \ldots, P_q \in \mathbb{C}[z]$ and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ such that $\deg P_i \leq n$ for any i. Then with any solution $Y = {}^t(y_1, \ldots, y_q)$ of the differential system $Y' = AY$ is associated a remainder $R(Y)$ defined by

$$R(Y)(z) = \sum_{i=1}^{q} P_i(z) y_i(z).$$

Let Σ be a finite subset of $\mathbb{C} \cup \{\infty\}$. For each $\sigma \in \Sigma$, let $(Y_j)_{j \in J_\sigma}$ be a family of solutions of $Y' = AY$ such that the functions $R(Y_j)$, $j \in J_\sigma$, are $\mathbb{C}$-linearly independent and belong to the Nilsson class at σ; here $\sigma \in \mathbb{C} \cup \{\infty\}$ might be a singularity of the differential system $Y' = AY$. The Padé-approximation problem (1.2) is a special case of this setting, with $q = a+2$, $\Sigma = \{0, 1, \infty\}$, $J_\sigma = \{\sigma\}$ for any $\sigma \in \Sigma$, $R(Y_\infty)(z) = S(z)$, $R(Y_0)(z) = \widetilde{S}(z)$, and $R(Y_1)(z) = T(z)$ (see §2 for details). In most cases all $R(Y_j)$ are holomorphic at σ for $j \in J_\sigma$, but a term in $\log(z - \sigma)$ is necessary to fit Beukers' problem for $\zeta(3)$ [6] into this setting.

We agree that $J_\sigma = \emptyset$ if $\sigma \notin \Sigma$, and let $M(z) = [P_{k,i}(z)]_{1 \leq i,k \leq q} \in M_q(\mathbb{C}(z))$ where the rational functions $P_{k,i} \in \mathbb{C}(z)$ are defined for $k \geq 1$ and $1 \leq i \leq q$ by

$$\begin{pmatrix} P_{k,1} \\ \vdots \\ P_{k,q} \end{pmatrix} = \left(\frac{\mathrm{d}}{\mathrm{d}z} + {}^tA\right)^{k-1} \begin{pmatrix} P_1 \\ \vdots \\ P_q \end{pmatrix}. \tag{1.3}$$

Obviously the poles of the coefficients $P_{k,i}$ of M are among those of A. These rational functions $P_{k,i}$ play an important role because they are used to differentiate the remainders [25, Chapter 3, §4]:

$$R(Y)^{(k-1)}(z) = \sum_{i=1}^{q} P_{k,i}(z) y_i(z). \tag{1.4}$$

The following multiplicity estimate appears essentially (see below) in [4, Théorème 2].

THEOREM 1.1. *There exists a positive constant c_1, which depends only on A and Σ, such that if*

$$\sum_{\sigma\in\Sigma}\sum_{j\in J_\sigma} \mathrm{ord}_\sigma(R(Y_j)) \geq (n+1)q - n\#J_\infty - \tau \tag{1.5}$$

with $0 \leq \tau \leq n - c_1$, then $\det M(z)$ is not identically zero.

The special case where $\Sigma = \{0\}$, $\#J_0 = 1$, and Y_j is analytic at 0 is essentially Shidlovsky's lemma (see [25, Chapter 3, Lemma 8]). When $\Sigma \subset \mathbb{C}$, $\#J_\sigma = 1$ for any σ, and all functions Y_j are obtained by analytic continuation from a single one, analytic at all $\sigma \in \Sigma$, this result was proved by Bertand-Beukers [5] with more details on the constant c_1. Then Bertrand has allowed [4, Théorème 2] an arbitrary number of solutions at each σ, proving Theorem 1.1 under the additional assumptions that $\infty \notin \Sigma$ and the functions Y_j, $j \in J_\sigma$, are analytic at σ.

Our proof of Theorem 1.1 (like that of [4, Théorème 2]) follows the strategy of [5], based on differential Galois theory. The point is that we allow Σ to contain ∞, and/or singularities of the differential system $Y' = AY$; moreover the functions Y_j, $j \in J_\sigma$, are not assumed to be holomorphic at σ. These features make Theorem 1.1 general enough to cover essentially all Padé approximation problems related to polylogarithms we have found in the literature, for instance the ones mentioned above. In such a setting, τ in Eq. (1.5) appears as the difference between the number of unknowns and the number of equations.

Then we evaluate at a point α, going from functional to numerical linear forms (see [25, Chapter 3, Lemma 10] for the classical setting). The point here is that we allow α to be a singularity of the differential system $Y' = AY$, and/or an element of Σ (in our proof of the Ball-Rivoal theorem, $\alpha = 1$ is both).

THEOREM 1.2. *There exists a positive constant c_2, which depends only on A and Σ, with the following property. Assume that, for some $\alpha \in \mathbb{C}$:*

(i) If α is a singularity of the differential system $Y' = AY$, it is a regular one and all non-zero exponents at α have positive real parts.

(ii) Eq. (1.5) holds for some τ with $0 \leq \tau \leq n - c_1$.

(iii) All rational functions $P_{k,i}$, with $1 \leq i \leq q$ and $1 \leq k < \tau + c_2$, are holomorphic at $z = \alpha$.

Then the matrix $[P_{k,i}(\alpha)]_{1\leq i\leq q, 1\leq k<\tau+c_2} \in M_{q,\tau+c_2-1}(\mathbb{C})$ has rank at least $q - \#J_\alpha$.

In particular, assertion (i) holds if the differential system $Y' = AY$ has a basis of local solutions at α with coordinates in $\mathbb{C}[\log(z-\alpha)][[(z-\alpha)^e]]$ for some positive rational number e. As far as we know, Theorem 1.2 is the first general result in which α is allowed to be a singularity. The case where α is not a singularity is much easier, and assumptions (i) and (iii) are then trivially satisfied.

If $\alpha \notin \Sigma$ then $J_\alpha = \emptyset$ so that we obtain a matrix of maximal rank q. On the opposite, if $\alpha \in \Sigma$ then $\#J_\alpha$ linearly independent linear combinations of the rows of the matrix $[P_{k,i}(z)]_{i,k}$ are holomorphic at α and (probably) vanish at α: the lower bound $q - \#J_\alpha$ is best possible.

Using a zero estimate such as Theorem 1.2 is the key point in the classical proof of the Siegel-Shidlovsky theorem on values of E-functions. Following a similar but different strategy, Nikishin constructed explicitly [19] linearly independent linear forms in $1, \mathrm{Li}_1(\alpha), \ldots, \mathrm{Li}_a(\alpha)$ to prove that these numbers are linearly independent over $\mathbb{Q}$ when $\alpha = u/v$ is a rational number with v sufficiently large in terms of $|u|$. His approach was used by several authors, including Marcovecchio [15] to bound from below the dimension of the $\mathbb{Q}$-vector space spanned by these numbers, for any fixed algebraic number α with $|\alpha| < 1$ (thereby generalizing to non-real numbers α Rivoal's result [23] based on Nesterenko's linear independence criterion). The zero estimate used by Marcovecchio is similar to Theorem 1.2 but deals only with a specific situation in which (essentially) $\tau = 1$ in Eq. (1.5), $\alpha \notin \Sigma$, and α is not a singularity. Moreover he does not define $P_{k,i}$ for $k \geq 2$ using Eq. (1.3) (i.e., differentiating the remainders as in the proof of the Siegel-Shidlovsky theorem): following Nikishin he uses an additional parameter instead.

In this paper we use Theorem 1.2 to obtain a new proof, and a refinement, of the following result of Nishimoto [20] on L-functions $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ associated with Dirichlet characters χ. He proved it with d instead of N in the lower bound (1.6); see §4.1 for this easy improvement.

THEOREM 1.3. *Let χ be a Dirichlet character modulo d, of conductor N. Let $p \in \{0,1\}$ and $a \geq 2$. Denote by $\delta_{\chi,p,a}$ the dimension of the $\mathbb{Q}$-vector space spanned by 1 and the numbers $L(\chi, s)$ with $2 \leq s \leq a$ and $s \equiv p \bmod 2$. Then*

$$\delta_{\chi,p,a} \geq \frac{1+o(1)}{N + \log 2} \log a \tag{1.6}$$

where $o(1)$ is a sequence that depends on N and a, and tends to 0 as $a \to \infty$ (for any N).

If p and χ have the same parity then $L(\chi, s)\pi^{-s}$ is a non-zero algebraic number for any $s \geq 2$ such that $s \equiv p \bmod 2$ (see for instance [18, Chapter VII, §2]): this result is interesting when p and χ have opposite parities.

Nishimoto's proof is similar to Ball-Rivoal's, except that obtaining the lower bound necessary to apply Nesterenko's criterion is very technical: the saddle point method has to be used because cancellations take place (see [17]). In this paper we present an alternative proof of Theorem 1.3, based on the zero estimate stated above. It makes it unnecessary to use the saddle point method, since Siegel's criterion is applied instead of Nesterenko's. In the special case $d = N = 1$ (so that $\chi(n) = 1$ for any n, and $L(\chi, s) = \zeta(s)$) this is exactly the proof of the Ball-Rivoal theorem mentioned above. As Zudilin pointed out to us, using the same strategy it could be possible to generalize to any algebraic number q, $|q| > 1$, the results on

q-zeta values proved in [14] when $1/q$ is an integer. Another question asked by Zudilin is whether Galochkin's lemma ([11], see also [29, Lemma 1.4]) can be used in this approach.

We also obtain the following refinement of Theorem 1.3, by improving the arithmetic estimates.

THEOREM 1.4. *In the setting of Theorem 1.3, if N is a multiple of 4 then Eq.* (1.6) *can be replaced with*

$$\delta_{\chi,p,a} \geq \frac{1+o(1)}{(N/2)+\log 2} \log a.$$

When χ is the non-principal character mod $d = 4$, so that $N = 4$, this result was proved by Rivoal-Zudilin [24] as a first step towards the (conjectural) irrationality of Catalan's constant $L(\chi, 2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$.

The structure of this paper is as follows. We first sketch in §2 our proof of the Ball-Rivoal theorem. Then §3 is devoted to Shidlovsky's lemma: we prove Theorems 1.1 and 1.2. At last, in §4 we prove in details a general result which contains Theorem 1.3, Theorem 1.4, and the Ball-Rivoal theorem; in particular §4.6 is devoted to Siegel's linear independence criterion.

Acknowledgements: This paper has benefited from useful discussions with Daniel Bertrand, Raffaele Marcovecchio, Milton H. Nash, Tanguy Rivoal, and Wadim Zudilin.

2. A new proof of the Ball-Rivoal theorem

We sketch in this section the new proof of the Ball-Rivoal theorem obtained as a special case of the proof of Theorem 4.1 in §4 below (namely $N = 1$, $f(n) = 1$ for any n, $p = 1$, $z_0 = 1$, $i_0 = 2$, $\xi_1 = 0$, and $\xi_j = \zeta(j)$ for any $j \geq 2$). Of course we refer to §4 for more details.

Let a, r, r', n be such that a is odd and $r, r' < a/2$. It turns out that the best estimates come from the case where r and r' have essentially the same size, so we shall restrict in §4 to the case $r' = r$; however the proof works in the same way if $r' \neq r$. Consider the rational function

$$F(t) = n!^{a-r-r'} \frac{(t-rn)_{rn}(t+n+1)_{r'n}}{(t)_{n+1}^a}$$

where $(\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1)$ is Pochhammer's symbol, and let

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t) z^t, \qquad S_\infty(z) = \sum_{t=1}^{\infty} F(t) z^{-t}.$$

For any $k \geq 1$ we let

$$\Lambda_k = S_0^{(k-1)}(1) - S_\infty^{(k-1)}(1), \tag{2.1}$$

where $S^{(k-1)}$ is the $(k-1)$-th derivative of S; if $r' = r$ and $k = 1$ this is essentially the linear form used in [3] and [22]. We shall use a symmetry phenomenon to get rid of even zeta values, but it does not appear exactly as in the original proof of Ball-Rivoal. Indeed, even if $r' = r$, $S_0^{(k-1)}(1)$ and $S_\infty^{(k-1)}(1)$ involve both odd and even zeta values when $k \geq 2$: they are values at $z = 1$ of hypergeometric series which are no more well-poised. The cancellation of even zeta values comes at a different stage, by considering Λ_k in Eq. (2.1). Indeed there exist integers $s_{k,i}$, $2 \le i \le a$, and u_k, v_k such that for any $k \le (a-r-r')n+a-1$, we have both

$$d_n^a S_0^{(k-1)}(1) = u_k + \sum_{i=2}^{a} (-1)^i s_{k,i} \zeta(i)$$

and

$$d_n^a S_\infty^{(k-1)}(1) = v_k + \sum_{i=2}^{a} s_{k,i} \zeta(i)$$

where $d_n = \mathrm{lcm}(1, 2, \dots, n)$, so that $d_n^a \Lambda_k = d_n^a S_0^{(k-1)}(1) - d_n^a S_\infty^{(k-1)}(1)$ is a $\mathbb{Z}$-linear combination of 1 and odd zeta values:

$$d_n^a \Lambda_k = s_{k,a+1} - 2 \sum_{\substack{2 \le i \le a \\ i \text{ odd}}} s_{k,i} \zeta(i),$$

with $s_{k,a+1} = u_k - v_k$. Using Theorem 1.2 we prove that the matrix $[s_{k,i}]_{2 \le i \le a+1, 1 \le k \le c_2}$ has maximal rank, equal to a (see below). This enables one to apply Siegel's linear independence criterion (see §4.6) instead of Nesterenko's: no lower bound on $|\Lambda_k|$ is needed. The upper bounds on $|s_{k,j}|$ and $|\Lambda_k|$ are essentially the same as in the proof of Ball-Rivoal, because k is bounded from above by a constant c_2 (independent from n); therefore we obtain the same lower bound:

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \dots, \zeta(a)) \ge \frac{1+o(1)}{1+\log 2} \log a.$$

Let us focus now on the functional aspects of this proof, which play an important role (whereas the proof of Ball-Rivoal can be written with $z=1$ throughout). For simplicity we restrict ourselves to the case $r' = r$. The functions $S_0(z)$ and $S_\infty(z)$ are solutions of the following Padé approximation problem: find polynomials $P_1, \dots, P_{a+2}$ of degree at most n such that:

$$\left\{ \begin{array}{rl} S_0(z) := P_{a+1}(z) + \sum_{i=1}^a P_i(z)(-1)^i \mathrm{Li}_i(z) = O(z^{(r+1)n+1}), & z \to 0, \\ S_\infty(z) := P_{a+2}(z) + \sum_{i=1}^a P_i(z) \mathrm{Li}_i(1/z) = O(z^{-rn-1}), & z \to \infty, \\ \sum_{i=1}^a P_i(z)(-1)^{i-1} \frac{(\log z)^{i-1}}{(i-1)!} = O((z-1)^{(a-2r)n+a-1}), & z \to 1. \end{array} \right. \tag{2.2}$$

This is exactly the Padé approximation problem of [10, Théorème 1], stated in the introduction: it has a unique solution up to proportionality, $(n+1)(a+2)$ unknowns and $(n+1)(a+2)-1$ equations. Let $A \in M_{a+2}(\mathbb{C}(z))$ denote the following matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{z-1} & \frac{1}{z(1-z)} \\ \frac{-1}{z} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{z} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{z} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{-1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

and consider the following solutions of the differential system $Y' = AY$:

$$Y_0(z) = {}^t(-\mathrm{Li}_1(z), \mathrm{Li}_2(z), \ldots, (-1)^a \mathrm{Li}_a(z), 1, 0),$$
$$Y_\infty(z) = {}^t(\mathrm{Li}_1(1/z), \mathrm{Li}_2(1/z), \ldots, \mathrm{Li}_a(1/z), 0, 1),$$
$$Y_1(z) = {}^t(1, -\log z, \frac{(\log z)^2}{2}, \ldots, (-1)^{a-1}\frac{(\log z)^{a-1}}{(a-1)!}, 0, 0).$$

Let $\Sigma = \{0, 1, \infty\}$ and $J_0 = \{0\}$, $J_1 = \{1\}$, $J_\infty = \{\infty\}$. Then with the notation of the introduction, we have $R(Y_0) = S_0(z)$, $R(Y_\infty) = S_\infty(z)$, and $R(Y_1)$ is the left hand side of the third equation of (2.2); Eq. (1.5) stated in the introduction holds with $\tau = 1$ as a consequence of the Padé approximation problem (2.2). In general, τ corresponds in Eq. (1.5) to the difference between the number of unknowns and the number of equations. To apply Theorem 1.2 it is not useful to prove that the problem has a unique solution up to proportionality: the upper bound $\tau \le n/2$, for instance, would be sufficient since n is taken arbitrarily large.

Defining $P_{k,i}$ as in the introduction by Eq. (1.3), Eq. (1.4) yields for any $k \ge 1$:

$$S_0^{(k-1)}(z) = P_{k,a+1}(z) + \sum_{i=1}^{a} P_{k,i}(z)(-1)^i \mathrm{Li}_i(z) \text{ and } S_\infty^{(k-1)}(z) = P_{k,a+2}(z) + \sum_{i=1}^{a} P_{k,i}(z)\mathrm{Li}_i(1/z). \tag{2.3}$$

Moreover $P_{k,i}$ is a rational function of which 0 is the only possible pole if $i \le a$. If $i = a+1$ or $i = a+2$, both 0 and 1 may be poles of $P_{k,i}$; but if $k \le (a-2r)n + a - 1$, the functions $S_0^{(k-1)}(z)$ and $S_\infty^{(k-1)}(z)$ have finite limits as $z \to 1$ so that 1 is not a pole.

Finally Theorem 1.2 applies at $\alpha = 1$: the matrix $[P_{k,i}(1)]_{1\le i\le a+2, 1\le k\le c_2}$ has rank at least $a+1$. Actually $P_{k,1}(1) = 0$ for any $k \le (a-2r)n + a - 1$ (which can be seen by letting z tend to 1 in Eq. (2.3)) so that the first row of this matrix is zero (provided n is large enough) and its rank is exactly $a+1$. Since the coefficients $s_{k,i}$ defined above are given by $s_{k,i} = d_n^a P_{k,i}(1)$ for $2 \le i \le a$ and $s_{k,a+1} = d_n^a(P_{k,a+1}(1) - P_{k,a+2}(1))$, the matrix $[s_{k,i}]_{2\le i\le a+1, 1\le k\le c_2}$ has rank a: Siegel's criterion (stated and proved in §4.6) applies.

3. Zero estimates

In this section we prove Theorems 1.1 and 1.2. We start with the functional part of the proof (§3.1), in which we follow the approach of Bertrand-Beukers [5] to generalize Shidlovsky's lemma (see Theorem 3.1). Then in §3.2 we deduce Theorems 1.1 and 1.2 stated in the introduction: the important point is to evaluate at α which may be a singularity and/or an element of Σ.

3.1. Functional zero estimate Throughout this section we consider a positive integer q and a matrix $A \in M_q(\mathbb{C}(X))$. We let $P_1, \ldots, P_q \in \mathbb{C}[X]$ with $\deg P_i \le n$ for any i. We also denote by Ω a simply connected open subset of $\mathbb{C}$ in which A has no pole. We assume that Ω is obtained from $\mathbb{C}$ by removing finitely many half-lines, so that Ω is dense in $\mathbb{C}$, and denote by $\mathcal{H}$ the space of functions holomorphic on Ω. A solution Y of the differential system $Y' = AY$ will always be a column matrix in $M_{q,1}(\mathcal{H})$, identified with the corresponding element $(y_1, \ldots, y_q)$ of $\mathcal{H}^q$. Since $P_1, \ldots, P_q$ are fixed, to such a solution is associated a remainder $R(Y)$ defined on Ω by

$$R(Y)(z) = \sum_{i=1}^{q} P_i(z) y_i(z).$$

Let Σ be a finite subset of $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$. For each $\sigma \in \Sigma$, let $(Y_j)_{j\in J_\sigma}$ be a family of solutions of $Y' = AY$ such that:

- For any $j \in J_\sigma$, the function $R(Y_j)$ belongs to the Nilsson class at σ.
- The functions $R(Y_j)$, for $j \in J_\sigma$, are linearly independent over $\mathbb{C}$.

The point is that we do not assume any relation (or lack of relation) between the families $(Y_j)_{j\in J_\sigma}$ at distinct points σ, except of course that all are solutions of the same differential system. To prove Theorems 1.1 and 1.2 we may omit all pairs (j, σ) with $j \in J_\sigma$ such that $\mathrm{ord}_\sigma(R(Y_j)) \leq 0$, so that we shall assume from now on that $\mathrm{ord}_\sigma(R(Y_j)) \geq 0$ for any $j \in J_\sigma$. At last, we let $J_\sigma = \emptyset$ when $\sigma \notin \Sigma$.

Defining $P_{k,i}(z)$ and $M(z)$ as in the introduction, our functional multiplicity estimate is the following generalization of Bertrand-Beukers' version of Shidlovsky's lemma; if $\infty \notin \Sigma$ and the functions Y_j, $j \in J_\sigma$, are analytic at σ it is due to Bertrand [4, Théorème 2]. The constant c_1 is the same as in Theorem 1.1 (that we shall deduce from Theorem 3.1 at the beginning of §3.2).

Theorem 3.1. *Let μ denote the order of a non-zero differential operator $L \in \mathbb{C}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$ such that $L(R(Y_j)) = 0$ for any σ and any $j \in J_\sigma$. Then*

$$\sum_{\sigma\in\Sigma}\sum_{j\in J_\sigma} \mathrm{ord}_\sigma(R(Y_j)) \leq (n+1)(\mu - \#J_\infty) + c_1 \tag{3.1}$$

where c_1 is a constant that depends only on A and Σ.

In the special case where $\Sigma \subset \mathbb{C}$, J_σ consists of a single element j_σ, and the function Y_{j_σ} is the same for all σ, this is exactly [5, Théorème 2] except that we did not try to make the constant c_1 explicit (we refer to [5], and to [1, Appendix of Chapter III] in the Fuchsian case, for discussions on effectivity which are not relevant to our purposes). Indeed we have fixed a simply connected open subset Ω only for convenience: analytic continuation from a point of Σ to another could be performed along any fixed path.

Let us prove Theorem 3.1 now, following the strategy of [5].
Given $\sigma \in \mathbb{P}^1(\mathbb{C})$, we let $\mathcal{A}_\sigma$ denote the set of all finite sums

$$\sum_{\alpha\in E}\sum_{Q\in\mathcal{P}}\sum_{j=0}^{J} u_{\alpha,Q,j}(z-\sigma)(z-\sigma)^\alpha(\log(z-\sigma))^j \exp(Q((z-\sigma)^{-1/q!})) \tag{3.2}$$

where $E \subset \mathbb{C}$ and $\mathcal{P} \subset \mathbb{C}[X]$ are finite subsets, $J \geq 0$, and $u_{\alpha,Q,j}(z-\sigma) \in \mathbb{C}[[(z-\sigma)^{1/q!}]]$ for any α, Q, j. Here and below, we agree that $z - \sigma$ stands for $1/z$ si $\sigma = \infty$. Then the differential system $Y' = AY$ has a complete system of formal solutions in $\mathcal{A}_\sigma^q$. Moreover we let $\mathcal{K}_\sigma$ denote the fraction field of $\mathcal{A}_\sigma$, and F_σ denote the differential subfield of $\mathcal{K}_\sigma$ generated over $\mathbb{C}(z)$ by all components of all solutions of $Y' = AY$ in $\mathcal{K}_\sigma^q$. Then the differential extension $F_\sigma/\mathbb{C}(z)$ is Picard-Vessiot, and we denote by G_σ its group of differential automorphisms.

To prove Theorem 3.1 we may assume that $0 \in \Sigma$, that μ is the minimal order of a non-zero differential operator that annihilates $R(Y_j)$ for any $j \in J_\sigma$ and any $\sigma \in \Sigma$, and that the coefficient of $(\frac{\mathrm{d}}{\mathrm{d}z})^\mu$ in L is 1.

Given $\sigma \in \Sigma$ and $j \in J_\sigma$, all components of Y_j are holomorphic on the cut plane Ω, and can be seen as elements of $\mathcal{A}_0$. Indeed, if 0 is a regular singularity (or an ordinary point) of the system $Y' = AY$ then all components of Y_j have a generalized Taylor expansion at the origin in $\mathcal{A}_0$ (of the form (3.2) with $\mathcal{P} = \{0\}$). In the general case, we identify each component of Y_j with its asymptotic expansion at 0 in a fixed large sector (see [21]). By definition of F_0, all components of Y_j (seen in $\mathcal{A}_0$) belong to F_0 so that $R(Y_j) \in F_0$. We consider the $\mathbb{C}$-vector space $V \subset F_0$ spanned by the images $\gamma(R(Y_j))$ of all $R(Y_j)$, $j \in J_\sigma$, $\sigma \in \Sigma$, under all $\gamma \in G_0$. Since the kernel of $L : F_0 \to F_0$ is stable under G_0, we have $V \subset \ker L$ so that $m \leq \mu$, where $m = \dim_{\mathbb{C}} V$. Let $(R_1, \ldots, R_m)$ be a basis of V, such that $R_i = \gamma_i(R(Y_{j_i}))$ with $\gamma_i \in G_0$ and $j_i \in J_{\sigma_i}$ for any $i \in \{1, \ldots, m\}$.

Arguing as in the proof of [5, Proposition 3], we have

$$Ly = \frac{1}{W(R_1, \ldots, R_m)} \det \begin{bmatrix} y & y' & \ldots & y^{(m)} \\ R_1 & R_1' & \ldots & R_1^{(m)} \\ \vdots & \vdots & & \vdots \\ R_m & R_m' & \ldots & R_m^{(m)} \end{bmatrix}$$

where $W(R_1, \ldots, R_m) = \det[R_i^{(j-1)}]_{1 \leq i,j \leq m}$ is the wronskian determinant. In particular, we have $m = \mu$ and $V = \ker L$.

Now we claim that for any $\sigma \in \mathbb{P}^1(\mathbb{C})$ there exist μ solutions $Y^{[\sigma,j]} = (y_1^{[\sigma,j]}, \ldots, y_q^{[\sigma,j]})$ of $Y' = AY$ in F_σ^q, with $1 \leq j \leq \mu$, such that $R(Y^{[\sigma,1]})$, $\ldots$, $R(Y^{[\sigma,\mu]})$ span the $\mathbb{C}$-vector space of solutions of $Ly = 0$ in F_σ. Indeed, as in [5, Corollaire], using a differential isomorphism $F_0 \to F_\sigma$ we may assume $\sigma = 0$. Then for any $i \in \{1, \ldots, m\}$, $\gamma_i(Y_{j_i}) \in F_0^q$ is a solution of $Y' = AY$ and $\pi_0(\gamma_i(Y_{j_i})) = R(\gamma_i(Y_{j_i})) = \gamma_i(R(Y_{j_i})) = R_i$ so that the claim is proved since $(R_1, \ldots, R_m)$ is a basis of $V = \ker L$.

Let us recall the following terminology from [5]: an element of $\mathcal{A}_\sigma$ has rank $\leq \kappa \in \frac{1}{q!}\mathbb{N}$ and generalized order $\geq r$ if it is of the form (3.2) with $\deg Q \leq q!\kappa$ for any $Q \in \mathcal{P}$ and $\operatorname{Re} \alpha \geq r$ for any $\alpha \in E$. The differential operator L has rank $\leq \kappa$ at σ and $(r_1, \ldots, r_\mu) \in \mathbb{R}^\mu$ is an admissible system of exponents of L at σ if the differential equation $Ly = 0$ has a complete system of solutions $(y_1, \ldots, y_\mu)$ in $\mathcal{A}_\sigma^\mu$ such that each y_i has rank $\leq \kappa$ and generalized order $\geq r_i$.

Given $\sigma \in \mathbb{P}^1(\mathbb{C})$ all functions $y_i^{[\sigma,j]}$ with $1 \leq i \leq q$ and $1 \leq j \leq \mu$ have rank $\leq \kappa_\sigma$ and generalized order $\geq r_\sigma$ for some $\kappa_\sigma \in \frac{1}{q!}\mathbb{N}$ and $r_\sigma \in \mathbb{R}$ which depend only on A and σ (see [5, Proposition 1]). If $\sigma \neq \infty$, $R(Y^{[\sigma,j]}) = \sum_{i=1}^q P_i(z) y_i^{[\sigma,j]}(z)$ has rank $\leq \kappa_\sigma$ and generalized order $\geq r_\sigma$; these functions make up a complete system of solutions of L (using the claim above). Moreover, if $\sigma \in \Sigma \setminus \{\infty\}$ then for any $j \in J_\sigma$ the function $R(Y_j)$ can be seen as an element of $\mathcal{A}_\sigma$ with rank ≤ 0 and generalized order $\geq \operatorname{ord}_\sigma(R(Y_j))$. Combining these $\mathbb{C}$-linearly independent solutions of $Ly = 0$ with suitable functions $R(Y^{[\sigma,j]})$, we obtain that L has rank $\leq \kappa_\sigma$ at σ and an admissible system of exponents of L at σ consists in r_σ repeated $\mu - \#J_\sigma$ times, and $\operatorname{ord}_\sigma(R(Y_j))$ for each $j \in J_\sigma$. In the same way, at infinity, for any $j \in J_\infty$ the function $R(Y_j) \in \mathcal{A}_\infty$ has rank ≤ 0 and generalized order $\geq \operatorname{ord}_\infty(R(Y_j))$. To obtain a complete system of solutions of $Ly = 0$ in $\mathcal{A}_\infty$ we use also $\mu - \#J_\infty$ functions $R(Y^{[\infty,j]})$, which have rank $\leq \kappa_\sigma$ and generalized order $\geq r_\infty - n$ since $\deg P_i(z) \leq n$ for any $i \in \{1, \ldots, q\}$. Therefore L has rank $\leq \kappa_\infty$ at ∞ and an admissible system of exponents of L at ∞ consists in $r_\infty - n$ repeated $\mu - \#J_\infty$ times, and $\operatorname{ord}_\infty(R(Y_j))$ for each $j \in J_\infty$.

So far we have found an upper bound on the rank of L, and an admissible system of exponents of L, at any $\sigma \in \Sigma$. Enlarging Σ if necessary, we may assume that it contains ∞ and all poles of A. Then for any $\sigma \in \mathbb{P}^1(\mathbb{C}) \setminus \Sigma$ the differential system $Y' = AY$ has a complete system of solutions holomorphic at σ, and therefore the same property holds for the differential equation $Ly = 0$ using the claim above. Accordingly Σ contains ∞ and all non-apparent singularities of L, so that the Corollary of [5, Théorème 3] provides an inequality involving upper bounds on the ranks of L and admissible systems of exponents of L at all points of Σ, namely:

$$(\mu - \#J_\infty)(r_\infty - n) + \Big(\sum_{j \in J_\infty} \mathrm{ord}_\infty(R(Y_j)) \Big) - (\kappa_\infty + 1)\mu(\mu-1)/2$$
$$+ \sum_{\sigma \in \Sigma \setminus \{\infty\}} \Big[(\mu - \#J_\sigma) r_\sigma + \Big(\sum_{j \in J_\sigma} \mathrm{ord}_\sigma(R(Y_j)) \Big) - (\kappa_\sigma + 1)\mu(\mu-1)/2 \Big] \leq -\mu(\mu-1)$$

so that

$$\Big(\sum_{\sigma \in \Sigma} \sum_{j \in J_\sigma} \mathrm{ord}_\sigma(R(Y_j)) \Big) - (n+1)(\mu - \#J_\infty) \leq c_1$$

where c_1 is a constant that can be written down explicitly in terms of Σ, μ, κ_σ, r_σ and $\#J_\sigma$ for $\sigma \in \Sigma$. This concludes the proof of Theorem 3.1.

3.2. Numerical zero estimate In this section we prove Theorems 1.1 and 1.2 stated in the introduction. The proof falls into 3 steps; the first one is Theorem 1.1.

Step 1: $M(z) \in M_q(\mathbb{C}(z))$ is an invertible matrix.

As in [25], if M is singular in $M_q(\mathbb{C}(z))$ then there is a non-trivial linear relation with coefficients in $\mathbb{C}(z)$ between the $\mathrm{rk}(M) + 1$ first columns of M; this provides a differential operator L of order $\mu = \mathrm{rk}(M)$ to which Theorem 3.1 applies, in contradiction with Eq. (1.5) since $\tau \leq n - c_1$. Indeed, for any solution Y of the differential system $Y' = AY$ we have

$${}^tYM = \left[\begin{array}{cccc} R(Y) & R(Y)' & \dots & R(Y)^{(q-1)} \end{array} \right].$$

Step 2: Determination of $\det M(z)$ up to factors of bounded degree.

Let S denote the set of finite singularities of the differential system $Y' = AY$, i.e. poles of coefficients of A. For any $s \in S$, let N_s denote the maximal order of s as a pole of a coefficient of A; let $N_s = 0$ for $s \in \mathbb{C} \setminus S$. Then Eq. (1.3) shows that $(z-s)^{(k-1)N_s} P_{k,i}(z)$ is holomorphic at $z = s$ for any $k \geq 1$ and any $i \in \{1, \dots, q\}$. Therefore $\det M(z) \cdot \prod_{s \in S} (z-s)^{q(q-1)N_s}$ has no pole: is it a polynomial.

Now let $\sigma \in \Sigma$, and denote by $T_\sigma \in M_{\#J_\sigma, q}(\mathcal{H})$ the matrix with rows ${}^tY_j$, $j \in J_\sigma$. The vector-valued functions Y_j, $j \in J_\sigma$, are linearly independent over $\mathbb{C}$ because the functions $R(Y_j)$ are; therefore they are the $\#J_\sigma$ first elements of a basis of solutions $\mathcal{B}$ of the differential system $Y' = AY$. The wronskian determinant of $\mathcal{B}$ may vanish at σ if σ is a singularity, but even in this case it has generalized order $\leq c_0(\sigma)$ at σ (with the terminology of §3.1) where $c_0(\sigma)$ is a constant depending only on A and σ (not on $\mathcal{B}$). On the other hand, all components of all elements of $\mathcal{B}$ have generalized order $\geq r_\sigma$ at σ (as in §3.1). Therefore there exists a subset I_σ of $\{1, \dots, q\}$, with $\#I_\sigma = q - \#J_\sigma$, such that the determinant of the submatrix of T_σ corresponding to the columns indexed by $\{1, \dots, q\} \setminus I_\sigma$ has generalized order $\leq c(\sigma)$ at σ, where $c(\sigma) = c_0(\sigma) - r_\sigma \# I_\sigma$ depends only on A and σ. Increasing $c_0(\sigma)$ and $c(\sigma)$ if necessary, we may assume that $c(\sigma) \in \mathbb{N}$.

Let $P_\sigma \in M_q(\mathcal{H})$ denote the matrix of which the $\# J_\sigma$ first rows are that of T_σ, and the other rows are the ${}^t e_i$, $i \in I_\sigma$, where $(e_1, \ldots, e_q)$ is the canonical basis of $M_{q,1}(\mathbb{C})$. Then $P_\sigma M$ has its first rows equal to $\left[\begin{array}{cccc} R(Y_j) & R(Y_j)' & \ldots & R(Y_j)^{(q-1)} \end{array}\right]$ with $j \in J_\sigma$, and its last rows equal to $\left[\begin{array}{c} P_{1,i} \ldots P_{q,i} \end{array}\right]$ with $i \in I_\sigma$. Therefore all coefficients in the row corresponding to $j \in J_\sigma$ have order at σ at least $\mathrm{ord}_\sigma R(Y_j) - q + 1$, and (if $\sigma \neq \infty$) all coefficients in the row corresponding to $i \in I_\sigma$ are either holomorphic at σ, or have a pole of order at most $(q-1)N_\sigma$ is $\sigma \in S$. Since $N_\sigma = 0$ if $\sigma \notin S$, we have for any $\sigma \in \Sigma \setminus \{\infty\}$:

$$\mathrm{ord}_\sigma \det(P_\sigma M) \geq \Big(\sum_{j \in J_\sigma} \mathrm{ord}_\sigma R(Y_j) \Big) - (q-1)\# J_\sigma - (q-1) N_\sigma (q - \# J_\sigma).$$

Since $\det P_\sigma$ has generalized order $\leq c(\sigma)$ at σ, we obtain

$$\mathrm{ord}_\sigma \det(M) \geq \Big(\sum_{j \in J_\sigma} \mathrm{ord}_\sigma R(Y_j) \Big) - c(\sigma).$$

Now let

$$Q_2(z) = \Big(\prod_{s \in S} (z-s)^{q(q-1)N_s} \Big) \cdot \Big(\prod_{\sigma \in \Sigma \setminus \{\infty\}} (z-\sigma)^{c(\sigma)} \Big)$$

so that $Q_2(z) \det M(z)$ is a polynomial and vanishes at any $\sigma \in \Sigma \setminus \{\infty\}$ with order at least $\sum_{j \in J_\sigma} \mathrm{ord}_\sigma R(Y_j)$. To bound from above the degree of this polynomial, we define P_∞ as above if $\infty \in \Sigma$, and let P_∞ denote the identity matrix (and $J_\infty = \emptyset$) otherwise. Then for some non-negative integer t we have $R(Y_j)^{(k-1)} = O(z^{-\mathrm{ord}_\infty R(Y_j)} (\log z)^t)$ as $|z| \to \infty$ for any $j \in J_\infty$ and any $k \geq 1$, and $P_{k,i}(z) = O(z^{n+(q-1)d})$ for any $i \in I_\infty$ and any $k \in \{1, \ldots, q\}$ (where d is greater than or equal to the degree of all coefficients of A). Since $\det M(z)$ is a rational function we deduce $\det M(z) = O(z^u)$ as $|z| \to \infty$, with

$$u = (q - \# J_\infty)(n + (q-1)d) - \sum_{j \in J_\infty} \mathrm{ord}_\infty R(Y_j),$$

so that

$$\deg(Q_2(z) \det M(z)) \leq u + \deg Q_2 \leq \sum_{\sigma \in \Sigma \setminus \{\infty\}} \sum_{j \in J_\sigma} \mathrm{ord}_\sigma R(Y_j) + \tau + c_1$$

using Eq. (1.5), where c_1 depends only on A and Σ (since $0 \leq \# J_\sigma \leq q$ for any σ). To sum up, we have found a polynomial Q_1 of degree at most $\tau + c_1$ such that

$$\det M(z) = \frac{Q_1(z)}{Q_2(z)} \prod_{\sigma \in \Sigma \setminus \{\infty\}} (z-\sigma)^{\lceil \sum_{j \in J_\sigma} \mathrm{ord}_\sigma R(Y_j) \rceil},$$

where $\lceil \omega \rceil$ is the least integer greater than or equal to ω.

Step 3: Evaluation at α.

To begin with, we denote by $\mathcal{L}_\alpha$ the $\mathbb{C}$-vector space spanned by the functions $h(z)(z-\alpha)^e (\log(z-\alpha))^i$ with h holomorphic at α, $i \in \mathbb{N}$, and $e \in \mathbb{C}$ such that either $e = 0$ or $\mathrm{Re}\,(e) > 0$.

Let $q_\alpha = \#J_\alpha$ and $q'_\alpha = q - q_\alpha$, where $J_\alpha = \emptyset$ if $\alpha \notin \Sigma$; for simplicity we assume that $J_\alpha = \{1, \ldots, q_\alpha\}$. Since the solutions $Y_1, \ldots, Y_{q_\alpha}$ of the differential system $Y' = AY$ are linearly independent over $\mathbb{C}$, there exist solutions $Y_{q_\alpha+1}, \ldots, Y_q$ such that $(Y_1, \ldots, Y_q)$ is a local basis of solutions at α. Let $\mathcal{Y}$ be the matrix with columns $Y_1, \ldots, Y_q$; then ${}^t\mathcal{Y}M$ is the matrix $[R(Y_i)^{(k-1)}]_{1 \leq i,k \leq q}$, and assumption (i) of Theorem 1.2 yields $\mathcal{Y} \in M_q(\mathcal{L}_\alpha)$.

For any subset E of $\{1, \ldots, q\}$ of cardinality $q'_\alpha = q - q_\alpha$, we denote by Δ_E the determinant of the submatrix of $[R(Y_i)^{(k-1)}]$ obtained by considering only the rows with index $i \geq q_\alpha + 1$ and the columns with index $k \in E$, and by $\widetilde{\Delta}_E$ the one obtained by removing these rows and columns. Then Laplace expansion by complementary minors yields

$$\det \mathcal{Y}(z) \cdot \det M(z) = \sum_{\substack{E \subset \{1, \ldots, q\} \\ \#E = q'_\alpha}} \varepsilon_E \Delta_E(z) \widetilde{\Delta}_E(z) \tag{3.3}$$

with $\varepsilon_E \in \{-1, 1\}$. Now $\det \mathcal{Y}$ is the wronskian of $Y_1, \ldots, Y_q$: it is a solution of the first order differential equation

$$w'(z) = w(z)\mathrm{trace}(A(z)). \tag{3.4}$$

Moreover it is non-zero, and belongs to $\mathcal{L}_\alpha$. Therefore we have $\varpi_1 := \mathrm{ord}_\alpha \det \mathcal{Y}(z) \geq 0$. Moreover $\varpi_2 := \mathrm{ord}_\alpha \det M(z) \in \mathbb{N}$ using Step 1 and the assumption that all entries of $M(z)$ are rational functions holomorphic at α. Then Eq. (3.3) provides a subset E such that

$$\varpi_3 := \mathrm{ord}_\alpha \Delta_E(z) \leq \varpi_1 + \varpi_2 - \mathrm{ord}_\alpha \widetilde{\Delta}_E(z). \tag{3.5}$$

Now letting $\omega_\alpha = \lceil \sum_{j \in J_\alpha} \mathrm{ord}_\alpha R(Y_j) \rceil$ if $\alpha \in \Sigma$ and $\omega_\alpha = 0$ otherwise, Step 2 shows that $\varpi_2 \leq \omega_\alpha + \tau + c_1$. Moreover the (generalized) order at α of any non-zero solution of Eq. (3.4), and in particular ϖ_1, can be bounded from above in terms of A only. At last, for any $i \in J_\alpha = \{1, \ldots, q_\alpha\}$ and any $k \in \{1, \ldots, q\}$ we have $\mathrm{ord}_\alpha R(Y_i)^{(k-1)} \geq \mathrm{ord}_\alpha R(Y_i) - (q-1)$ so that $\mathrm{ord}_\alpha \widetilde{\Delta}_E(z) \geq \omega_\alpha - q_\alpha(q-1)$. Therefore Eq. (3.5) yields $\varpi_3 \leq \tau + c_3$ for some constant c_3 depending only on A and Σ. Using this upper bound we shall prove now that ϖ_3 is a non-negative integer, and $\Delta_E^{(\varpi_3)}(z)$ has a finite non-zero limit as z tends to α.

Since $\mathcal{Y} \in M_q(\mathcal{L}_\alpha)$ and $P_{i,k}$ has no pole at α for $k \leq q$, we have $\Delta_E(z) \in \mathcal{L}_\alpha$ so that

$$\Delta_E(z) = \sum_{e \in \mathcal{E}} \sum_{i=0}^{I} \lambda_{i,e}\, h_{i,e}(z)(z-\alpha)^e (\log(z-\alpha))^i \tag{3.6}$$

where $h_{i,e}(z)$ is holomorphic at α and $\mathcal{E}$ is a finite subset of $\mathbb{C}$ such that for any $e \in \mathcal{E}$, either $e = 0$ or $\mathrm{Re}\,(e) > 0$. Moreover we may assume that $e - e' \notin \mathbb{Z}$ for any distinct $e, e' \in \mathcal{E}$, and that for any $e \in \mathcal{E}$ there exists i such that $\lambda_{i,e} h_{i,e}(\alpha) \neq 0$. At last, the integer I can be chosen in terms of A only, since the exponents of $\log(z-\alpha)$ in local solutions at α of $Y' = AY$ are bounded.

We choose the constant c_2 of Theorem 1.2 to be $c_2 = c_3 + I + q + 1$. For any non-negative integer $\varpi \leq c_2 + \tau - q - 1$, the ϖ-th derivative $\Delta_E^{(\varpi)}(z)$ is a $\mathbb{Z}$-linear combination of determinants of matrices of the form

$$N_{k_1, \ldots, k_{q'_\alpha}} = [R(Y_{q_\alpha+i})^{(k_j-1)}]_{1 \leq i,j \leq q'_\alpha}$$

with $1 \leq k_1 < \ldots < k_{q'_\alpha} \leq q + \varpi < c_2 + \tau$. Since $Y_i \in M_{q,1}(\mathcal{L}_\alpha)$ and $P_{k,i}$ is assumed to be holomorphic at α for any i and any $k < \tau + c_2$, we have $R(Y_i)^{(k-1)} \in \mathcal{L}_\alpha$. Accordingly $\det N_{k_1,\ldots,k_{q'_\alpha}} \in \mathcal{L}_\alpha$, and finally $\Delta_E^{(\varpi)}(z) \in \mathcal{L}_\alpha$ for any non-negative integer $\varpi \leq c_2 + \tau - q - 1$. Therefore in the expression (3.6), all pairs (e, i) such that $\lambda_{i,e} h_{i,e}(\alpha) \neq 0$ and $\mathrm{Re}\,(e) + i \leq c_2 + \tau - q - 1$ satisfy $e \in \mathbb{N}$ and $i = 0$. Now recall that $\varpi_3 = \mathrm{ord}_\alpha \Delta_E(z) \leq \tau + c_3 = c_2 + \tau - q - 1 - I$. Then there is a term (e, i) in Eq. (3.6) such that $\lambda_{i,e} h_{i,e}(\alpha) \neq 0$ and $\mathrm{Re}\,(e) = \varpi_3 \leq c_2 + \tau - q - 1 - I$, and accordingly $\mathrm{Re}\,(e) + i \leq c_2 + \tau - q - 1$: we have $e \in \mathbb{N}$, $i = 0$, and no other term (e', i') such that $\lambda_{i',e'} h_{i',e'}(\alpha) \neq 0$ satisfies $\mathrm{Re}\,(e') = \varpi_3$. In particular ϖ_3 is a non-negative integer, and $\Delta_E^{(\varpi_3)}(z)$ has a finite non-zero limit as z tends to α.

Let $\mathrm{ev}_\alpha : \mathcal{L}_\alpha \to \mathbb{C}$ denote regularized evaluation at α, defined by $\mathrm{ev}_\alpha(f) = \lambda_{0,0}\, h_{0,0}(\alpha)$ if $f(z)$ is the right hand side of Eq. (3.6), and of course $\mathrm{ev}_\alpha(f) = 0$ if $0 \notin \mathcal{E}$. The important point here is that any $e \in \mathcal{E}$ satisfies either $e = 0$ or $\mathrm{Re}\,(e) > 0$, so that ev_α is a $\mathbb{C}$-algebra homomorphism; moreover $\mathrm{ev}_\alpha(f)$ is equal to the limit of $f(z)$ as $z \to \alpha$ whenever this limit exists. In particular we have $\mathrm{ev}_\alpha(\Delta_E^{(\varpi_3)}) \neq 0$. Now, as above $\mathrm{ev}_\alpha(\Delta_E^{(\varpi_3)})$ is a $\mathbb{Z}$-linear combination of $\mathrm{ev}_\alpha(\det N_{k_1,\ldots,k_{q'_\alpha}})$ with $1 \leq k_1 < \ldots < k_{q'_\alpha} \leq q + \varpi_3 < c_2 + \tau$, so that $\mathrm{ev}_\alpha(\det N_{k_1,\ldots,k_{q'_\alpha}}) \neq 0$ for some tuple $(k_1, \ldots, k_{q'_\alpha})$. For this tuple we consider the equality ${}^t\widetilde{\mathcal{Y}}\widetilde{M} = N_{k_1,\ldots,k_{q'_\alpha}}$, where $\widetilde{\mathcal{Y}} \in M_{q,q'_\alpha}(\mathcal{L}_\alpha)$ is the matrix with columns $Y_{q_\alpha+1}, \ldots, Y_q$, and $\widetilde{M} = [P_{k_j,i}]_{1\leq i\leq q, 1\leq j\leq q'_\alpha}$. The Cauchy-Binet formula yields

$$\det N_{k_1,\ldots,k_{q'_\alpha}} = \sum_{\substack{B\subset\{1,\ldots,q\}\\ \#B=q'_\alpha}} \det\ {}^t\widetilde{\mathcal{Y}}_B \cdot \det \widetilde{M}_B \tag{3.7}$$

where $\widetilde{\mathcal{Y}}_B$ (resp. $\widetilde{M}_B$) is the square matrix consisting in the rows of $\widetilde{\mathcal{Y}}$ (resp. of $\widetilde{M}$) corresponding to indices in B. Extending ev_α coefficientwise to matrices, Eq. (3.7) yields

$$\mathrm{ev}_\alpha\Big(\det N_{k_1,\ldots,k_{q'_\alpha}}\Big) = \sum_{\substack{B\subset\{1,\ldots,q\}\\ \#B=q'_\alpha}} \mathrm{ev}_\alpha\Big(\det\ {}^t\widetilde{\mathcal{Y}}_B\Big) \cdot \mathrm{ev}_\alpha\Big(\det \widetilde{M}_B\Big).$$

Now the left hand side is non-zero, so that $\mathrm{ev}_\alpha(\det \widetilde{M}_B) \neq 0$ for some B. Since all coefficients $P_{k,i}$ are holomorphic at α, so is $\det \widetilde{M}_B$ and therefore $\det(\widetilde{M}_B(\alpha)) = \mathrm{ev}_\alpha(\det \widetilde{M}_B) \neq 0$. We have found an invertible submatrix of $M(\alpha)$ of size q'_α, so that $\mathrm{rk}M(\alpha) \geq q'_\alpha$: this concludes the proof of Theorem 1.2.

4. Diophantine part of the proof

In this section we prove Theorem 1.4 stated in the introduction, and give in details new proofs of the Ball-Rivoal theorem and Nishimoto's Theorem 1.3. To provide a unified treatment, we state a general result (namely Theorem 4.1) and deduce these results from it in §4.1. In order to help the reader, we first sketch the proof of Theorem 4.1 in §4.2, then construct the linear forms (§4.3), apply the zero estimate (namely Theorem 1.2) to obtain an invertible matrix (§4.4), and study the arithmetic and asymptotic properties (§4.5). At last we state and prove Siegel's linear independence criterion in §4.6.

4.1. Statement of the main theorem and consequences

THEOREM 4.1. *Let $N \geq 1$, and $f : \mathbb{N} \to \mathbb{C}$ be such that $f(n+N) = f(n)$ for any n. Let $p \in \{0,1\}$, $a \geq 2$, and $z_0 \in \{1, e^{i\pi/N}\}$; put*

$$\xi_j = \sum_{n=1}^{\infty} \frac{f(n) z_0^n}{n^j} \text{ for any } j \in \{1, \ldots, a\},$$

except that $\xi_1 = 0$ if $z_0 = 1$. Then as $a \to \infty$,

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\{\xi_j,\ 1 \leq j \leq a,\ j \equiv p \bmod 2\}) \geq \frac{1+o(1)}{N + \log 2} \log a.$$

We refer to §2 for the special case of the Ball-Rivoal theorem.

Let us deduce Theorems 1.3 and 1.4 stated in the introduction from this result. Let χ be a Dirichlet character mod d. Its conductor is the smallest divisor e of d for which there exists a character χ' mod e such that $\chi(n) = \chi'(n)$ for any n coprime to d. Comparing the L-functions of χ and χ' (see for instance [13, §§3.2 and 3.3]) yields

$$L(\chi, s) = L(\chi', s) \prod_{\substack{p \mid d \\ p \nmid e}} (1 - \chi'(p) p^{-s})$$

so that $\delta_{\chi,p,a} = \delta_{\chi',p,a}$ for any p, a (with the notation of Theorem 1.3). Therefore we may assume that $e = d$, i.e. χ is primitive. Then Theorem 1.3 follows from Theorem 4.1 by letting $z_0 = 1$ and $f = \chi$.

To prove Theorem 1.4, we first prove that for any primitive Dirichlet character χ modulo a multiple e of 4,

$$\chi(n + \frac{e}{2}) = -\chi(n) \text{ for any } n \in \mathbb{Z}. \tag{4.1}$$

Indeed we have $n(\frac{e}{2}+1) \equiv n + \frac{e}{2} \bmod e$ if n is odd, so that $\chi(n+\frac{e}{2}) = \chi(n)\chi(\frac{e}{2}+1)$ for any $n \in \mathbb{Z}$ (since both sides vanish if n is even). Moreover $(\chi(\frac{e}{2}+1))^2 = 1$ since $(\frac{e}{2}+1)^2 \equiv 1 \bmod e$, and $\chi(\frac{e}{2}+1) \neq 1$ because χ is primitive (so that $\chi(n+\frac{e}{2}) \neq \chi(n)$ for some n). Therefore $\chi(\frac{e}{2}+1) = -1$: this concludes the proof of (4.1).

Now let $N = e/2$ and define $f : \mathbb{N} \to \mathbb{C}$ by $f(r) = \chi(r) z_0^{-r}$ for any $r \in \{1, \ldots, N\}$, where $z_0 = e^{i\pi/N}$. Then Eq. (4.1) yields

$$\sum_{n=1}^{\infty} \frac{f(n) z_0^n}{n^j} = \sum_{r=1}^{N} f(r) z_0^r \sum_{\substack{n \geq 1 \\ n \equiv r \bmod 2N}} \Big(\frac{1}{n^j} - \frac{1}{(n+N)^j} \Big) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^j} = L(\chi, j)$$

so that Theorem 4.1 implies Theorem 1.4.

4.2. Sketch of the proof To prove Theorem 4.1, we let $r, n \geq 1$ be such that $r < \frac{a}{2N}$ and N divides n. We define $\xi'_1, \ldots, \xi'_{a+N}$ as follows:

$$\left\{ \begin{array}{rcll} \xi'_j & = & 2(-1)^p \xi_j & \text{for } j \in \{1, \ldots, a\} \text{ such that } j \equiv p \bmod 2 \\ \xi'_j & = & 0 & \text{for } j \in \{1, \ldots, a\} \text{ such that } j \not\equiv p \bmod 2 \\ \xi'_{a+1+\lambda} & = & z_0^{\lambda} f(\lambda) & \text{for any } \lambda \in \{0, \ldots, N-1\}. \end{array} \right. \tag{4.2}$$

We also let

$$\delta_n = (Nd_n)^a N^{an/N},$$

and define i_0 to be equal to 1 if $z_0 = e^{i\pi/N}$, and equal to 2 otherwise (i.e., if $z_0 = 1$).

In §4.3 (see (4.16)) we shall construct integers $s_{k,i}$, $i_0 \le i \le a+N$, such that as $n \to +\infty$:

$$\max_{i_0 \le i \le a+N} |s_{k,i}| \le \beta^{n(1+o(1))} \text{ and } \Big| \sum_{i=i_0}^{a+N} s_{k,i}\xi'_i \Big| \le \alpha^{n(1+o(1))} \tag{4.3}$$

where

$$\alpha = e^a 4^{a/N-r}(N+1)^{2r+2} r^{-a/N+4r+2} \text{ and } \beta = (2e^N)^{a/N}(rN+1)^{2r+2}.$$

Then Lemma 4.2 (that will be stated and proved in §4.4 using Theorem 1.2) provides a positive constant c_2 (which depends only on a and N) and integers $1 \le k_{i_0} < k_{i_0+1} < \ldots < k_{a+N} \le c_2$ (which depend on a, N, r, and n) such that the matrix $[s_{k_j,i}]_{i_0 \le i,j \le a+N}$ is invertible. Since $k_j \le c_2$ for any j, the symbols $o(1)$ in (4.3) with $k = k_j$ can be made uniform with respect to k. Therefore Siegel's linear independence criterion applies (see §4.6). Taking a very large, N fixed, and r equal to the integer part of $\frac{a}{(\log(a))^2}$ concludes the proof of Theorem 4.1 since $\xi'_i = 0$ if $i \le a$ and $i \not\equiv p \bmod 2$, and

$$1 - \frac{\log\alpha}{\log\beta} = \frac{1+\varepsilon_a}{N+\log 2}\log a \text{ where } \lim_{a\to+\infty} \varepsilon_a = 0.$$

4.3. Construction of the linear forms Let a, r, N be positive integers such that $1 \le r < \frac{a}{2N}$. For any integer multiple n of N we let

$$F(t) = (n/N)!^{a-2rN}\frac{(t-rn)_{rn}(t+n+1)_{rn}}{\prod_{h=0}^{n/N}(t+Nh)^a}.$$

Then F is a rational function, and its degree $-d_0$ satisfies

$$d_0 := a(\frac{n}{N}+1) - 2rn = -\deg F \ge n+a \ge 2. \tag{4.4}$$

Its partial fraction expansion reads

$$F(t) = \sum_{h=0}^{n/N}\sum_{j=1}^{a}\frac{p_{j,h}}{(t+Nh)^j}$$

with rational coefficients $p_{j,h}$. Let

$$P_j(z) = \sum_{h=0}^{n/N} p_{j,h} z^{Nh} \in \mathbb{Q}[z]_{\le n} \text{ for any } j \in \{1,\ldots,a\},$$

and also

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t)z^t, \qquad S_\infty(z) = \sum_{t=1}^{\infty} F(t)z^{-t}. \tag{4.5}$$

As in [3] we have

$$S_\infty(z) = V(z) + \sum_{j=1}^{a} P_j(z)\mathrm{Li}_j(1/z)$$

where

$$V(z) = -\sum_{t=0}^{n-1} z^t \sum_{j=1}^{a} \sum_{h=\lceil (t+1)/N \rceil}^{n/N} \frac{p_{j,h}}{(Nh-t)^j} \in \mathbb{Q}[z]_{\leq n}.$$

In the same way (see [10]) we have

$$S_0(z) = U(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(z)$$

with the same polynomials $P_1, \ldots, P_a$, and

$$U(z) = -\sum_{t=1}^{n} z^t \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(t-Nh)^j} \in \mathbb{Q}[z]_{\leq n}.$$

Now let $P_{1,j} = P_j$ for any $j \in \{1, \ldots, a\}$, and define inductively $P_{k,j} \in \mathbb{Q}(z)$ by

$$P_{k,j}(z) = P'_{k-1,j}(z) - \frac{1}{z} P_{k-1,j+1}(z) \text{ for any } k \geq 2 \text{ and any } j \in \{1, \ldots, a\}, \tag{4.6}$$

where $P_{k-1,a+1} = 0$ for any k; we shall check in §4.4 below that this notation $P_{k,j}$ is consistent with the one used in the introduction. We let also $U_1 = U$, $V_1 = V$, and define U_k, V_k for any $k \geq 2$ by the recurrence relations

$$U_k(z) = U'_{k-1}(z) - \frac{1}{1-z} P_{k-1,1}(z), \tag{4.7}$$

$$V_k(z) = V'_{k-1}(z) + \frac{1}{z(1-z)} P_{k-1,1}(z). \tag{4.8}$$

Then for any $k \geq 1$ we have

$$S_0^{(k-1)}(z) = U_k(z) + \sum_{j=1}^{a} P_{k,j}(z)(-1)^j \mathrm{Li}_j(z) \tag{4.9}$$

$$\text{and } S_\infty^{(k-1)}(z) = V_k(z) + \sum_{j=1}^{a} P_{k,j}(z) \mathrm{Li}_j(1/z). \tag{4.10}$$

Moreover Eqns. (4.6), (4.7) and (4.8) show that the rational functions $P_{k,j}$ with $1 \leq j \leq a$ (resp. U_k and V_k) have only 0 (resp. only 0 and 1) as possible finite poles. Now we have

$$S_0^{(k-1)}(z) = \sum_{t=n+1}^{\infty} F(-t)(t-k+2)_{k-1} z^{t-k+1} \text{ for } |z| < 1$$

$$\text{and } S_\infty^{(k-1)}(z) = \sum_{t=1}^{\infty} F(t)(-1)^{k-1}(t)_{k-1} z^{-t-k+1} \text{ for } |z| > 1.$$

Let us assume that $k - 1 \leq d_0 - 2$, where $d_0 = -\deg F$ is defined by Eq. (4.4); then these formulas hold also when $|z| = 1$ and we may let z tend to 1 in Eqns. (4.9) and (4.10). Since

$P_{k,j}$ is holomorphic at $z=1$ for any $k \geq 1$ and any j, a possible divergence may come only from poles of U_k or V_k at $z=1$, or from the logarithmic term involving $\mathrm{Li}_1(z)$ or $\mathrm{Li}_1(1/z)$. Since a pole and a logarithmic term cannot cancel each other out, and $S_0^{(k-1)}(z)$ and $S_\infty^{(k-1)}(z)$ have finite limits as $z \to 1$, we obtain:

$$\text{For any } k \leq d_0 - 1, \qquad P_{k,1}(1) = 0 \quad \text{and} \quad U_k, V_k \text{ do not have a pole at } z=1. \tag{4.11}$$

Now let $k \leq d_0 - 1$, and $z \in \mathbb{C}$ be such that $|z| = 1$. Then Eqns. (4.9) and (4.10) hold, upon agreeing that the sums start at $j=2$ if $z=1$; the same remark applies in what follows. Since $P_j(z) \in \mathbb{Q}[z^N]$ for any $j \in \{1, \ldots, a\}$, Eq. (4.6) yields $P_{k,j} \in z^{1-k}\mathbb{Q}[z^N]$ (see the proof of Proposition 4.4 in §4.5 for details). On the other hand, since $U_k, V_k \in \mathbb{Q}[z, z^{-1}]$ for any $k \leq d_0 - 1$, we can write

$$z^{k-1}U_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) \text{ and } z^{k-1}V_k(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) \tag{4.12}$$

with $U_{k,\lambda}, V_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$. Then Eqns. (4.9) and (4.10) yield

$$z^{k-1}S_0^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda U_{k,\lambda}(z) + \sum_{j=1}^{a} z^{k-1}P_{k,j}(z)(-1)^j \mathrm{Li}_j(z) \tag{4.13}$$

$$\text{and } z^{k-1}S_\infty^{(k-1)}(z) = \sum_{\lambda=0}^{N-1} z^\lambda V_{k,\lambda}(z) + \sum_{j=1}^{a} z^{k-1}P_{k,j}(z)\mathrm{Li}_j(1/z). \tag{4.14}$$

The point now is that $U_{k,\lambda}(z)$, $V_{k,\lambda}(z)$, and $z^{k-1}P_{k,j}(z)$ depend only on z^N. For any $\ell \in \{1, \ldots, N\}$ we consider

$$\mu_\ell = \frac{1}{N}\sum_{\lambda=1}^{N} f(\lambda)\omega^{-\ell\lambda}. \tag{4.15}$$

Let $z_0 \in \{1, e^{i\pi/N}\}$ and $p \in \{0,1\}$ be as in Theorem 4.1, and recall that $\omega = e^{2i\pi/N}$. For any $k \leq d_0 - 1$ we let

$$\Lambda_k = \sum_{\ell=1}^{N} \mu_\ell \Big[(\omega^\ell z_0)^{k-1} S_0^{(k-1)}(\omega^\ell z_0) + (-1)^p (\omega^\ell z_0)^{1-k} S_\infty^{(k-1)}(\frac{1}{\omega^\ell z_0})\Big].$$

Then Eqns. (4.13) and (4.14) yield, since $U_{k,\lambda}(z)$, $V_{k,\lambda}(z)$, and $z^{k-1}P_{k,j}(z)$ depend only on z^N and $(\omega^\ell z_0)^N = (\omega^\ell z_0)^{-N} = z_0^N$:

$$\Lambda_k = \sum_{\lambda=0}^{N-1} \Big[\Big(\sum_{\ell=1}^{N} \mu_\ell (\omega^\ell z_0)^\lambda\Big) U_{k,\lambda}(z_0) + (-1)^p \Big(\sum_{\ell=1}^{N} \mu_\ell (\omega^\ell z_0)^{-\lambda}\Big) V_{k,\lambda}(z_0)\Big]$$
$$+ \sum_{j=1}^{a} z_0^{k-1} P_{k,j}(z_0) \sum_{\ell=1}^{N} \mu_\ell \mathrm{Li}_j(\omega^\ell z_0)((-1)^j + (-1)^p).$$

Now Eq. (4.15) yields

$$\sum_{\ell=1}^{N} \mu_\ell \omega^{n\ell} = f(n) \text{ for any } n \in \mathbb{Z}, \text{ so that } \sum_{\ell=1}^{N} \mu_\ell \mathrm{Li}_j(\omega^\ell z_0) = \sum_{n=1}^{\infty} \frac{f(n) z_0^n}{n^j} = \xi_j \text{ for any } j \leq a.$$

Letting $V_{k,N} = V_{k,0}$ we obtain:

$$\Lambda_k = 2(-1)^p \sum_{\substack{1\le j\le a \\ j\equiv p \bmod 2}} z_0^{k-1} P_{k,j}(z_0)\xi_j + \sum_{\lambda=0}^{N-1} (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) z_0^{\lambda} f(\lambda).$$

As announced in §4.2 we now define the coefficients $s_{k,i}$:

$$\begin{cases} s_{k,i} = \delta_n z_0^{k-1} P_{k,i}(z_0) \text{ for } 1 \le i \le a, \\ s_{k,a+1+\lambda} = \delta_n (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) \text{ for } 0 \le \lambda \le N-1, \end{cases} \tag{4.16}$$

where $\delta_n = (N d_n)^a N^{an/N}$, so that

$$\delta_n \Lambda_k = \sum_{i=i_0}^{a+N} s_{k,i} \xi'_i$$

since $\xi'_1 = 0$ if $z_0 = 1$ (recall from §4.2 that $i_0 = 2$ in this case, and $i_0 = 1$ otherwise, i.e. if $z_0 = e^{i\pi/N}$; ξ'_i is defined in Eq. (4.2)).

Since $z_0^N \in \{-1, 1\}$ and $z^{k-1}P_{k,j}(z)$, $U_{k,\lambda}(z)$ and $V_{k,N-\lambda}(z)$ are polynomials in z^N with rational coefficients, the numbers $s_{k,1}, \ldots, s_{k,a+N}$ are rational. We shall prove in Proposition 4.4 (§4.5) that they are integers, thanks to the factor δ_n.

4.4. Application of the zero estimate In this section we deduce from Theorem 1.2 the following lemma, used at the end of §4.2. It provides an invertible matrix which enables us to apply Siegel's linear independence criterion (see §4.6).

Lemma 4.2. *In the setting of §4.2, let $i_0 = 1$ if $z_0 = e^{i\pi/N}$ and $i_0 = 2$ if $z_0 = 1$; let $s_{k,i}$ be defined by Eq. (4.16). Then there exist a positive constant c_2 (which depends only on a and N) and integers $1 \le k_{i_0} < k_{i_0+1} < \ldots < k_{a+N} \le c_2$ (which depend on a, N, r, and n) such that the matrix $[s_{k_j,i}]_{i_0 \le i,j \le a+N}$ is invertible.*

To begin with, let us recall from §4.2 that $\omega = e^{2i\pi/N}$, a, r, N, n are positive integers such that $1 \le r < \frac{a}{2N}$, n is a multiple of N, and

$$F(t) = (n/N)!^{a-2rN} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{\prod_{h=0}^{n/N}(t+Nh)^a}.$$

We have

$$S_0(z) = \sum_{t=n+1}^{\infty} F(-t) z^t = U(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(z)$$

$$\text{and } S_\infty(z) = \sum_{t=1}^{\infty} F(t) z^{-t} = V(z) + \sum_{j=1}^{a} P_j(z)\mathrm{Li}_j(1/z).$$

Since $P_j \in \mathbb{C}[z^N]$ for any $j \in \{1, \ldots, a\}$, we have $P_j(\omega^\ell z) = P_j(z)$ for any $\ell \in \mathbb{Z}$. Therefore letting

$$R_{0,\ell}(z) = S_0(\omega^\ell z), \qquad R_{\infty,\ell}(z) = S_\infty(\omega^\ell z), \qquad \overline{P}_{0,\ell}(z) = U(\omega^\ell z), \qquad \overline{P}_{\infty,\ell}(z) = V(\omega^\ell z) \tag{4.17}$$

for any $\ell \in \{1, \dots, N\}$, we have

$$R_{0,\ell}(z) = \overline{P}_{0,\ell}(z) + \sum_{j=1}^{a} P_j(z)(-1)^j \mathrm{Li}_j(\omega^\ell z) = O(z^{(r+1)n+1}), \quad z \to 0, \tag{4.18}$$

$$\text{and } R_{\infty,\ell}(z) = \overline{P}_{\infty,\ell}(z) + \sum_{j=1}^{a} P_j(z) \mathrm{Li}_j(\frac{1}{\omega^\ell z}) = O(z^{-rn-1}), \quad z \to \infty. \tag{4.19}$$

Moreover, recall that $d_0 = -\deg F = a(\frac{n}{N} + 1) - 2rn$; Lemma 3 of [10] shows that

$$\sum_{j=1}^{a} P_j(z)(-1)^{j-1} \frac{(\log z)^{j-1}}{(j-1)!} = O((z-1)^{d_0 - 1}), \quad z \to 1.$$

Using again the fact that $P_j(\omega^{-\ell} z) = P_j(z)$, we obtain for any $\ell \in \{1, \dots, N\}$:

$$R_{\omega^\ell}(z) := \sum_{j=1}^{a} P_j(z)(-1)^{j-1} \frac{(\log(\omega^{-\ell} z))^{j-1}}{(j-1)!} = O((z - \omega^\ell)^{d_0 - 1}), \quad z \to \omega^\ell. \tag{4.20}$$

Combining Eqns. (4.18), (4.19), and (4.20) with $1 \le \ell \le N$, we have solved a simultaneous Padé approximation problem. The $(n+1)(a+2N)$ unknowns are the coefficients of $P_1, \dots, P_a, \overline{P}_{0,1}, \dots, \overline{P}_{0,N}, \overline{P}_{\infty,1}, \dots, \overline{P}_{\infty,N}$, which are polynomials of degree less than or equal to n. There are

$$2N((r+1)n+1) + N(d_0 - 1) = n(a+2N) + (a+1)N$$

linear equations, since a priori we have $R_{\infty,\ell}(z) = O(z^n)$ as $z \to \infty$. The difference between the number of unknowns and the number of equations is equal to $N - a(N-1)$. If $N = 1$ this is equal to 1: the Padé approximation problem is exactly (2.2), i.e. the one of [10, Théorème 1], which has a unique solution up to proportionality. Whenever $N \ge 2$ we have $N - a(N-1) < 0$: the problem we have solved has more equations than unknowns. This is due to the fact that we always assume n to be an integer multiple of N. Anyway to complete the proof, it is sufficient to bound from above the difference between the number of unknowns and the number of equations by a constant independent from n; we do not need to study whether the Padé approximation problem has a unique solution or not.

Let $q = a + 2N$, and $A \in M_q(\mathbb{C}(z))$ be the matrix of which the coefficients $A_{i,j}$ are given by:

$$\begin{cases} A_{i,i-1}(z) = \frac{-1}{z} \text{ for any } i \in \{2, \dots, a\} \\ A_{1,a+\ell}(z) = \frac{\omega^\ell}{\omega^\ell z - 1} \text{ for any } \ell \in \{1, \dots, N\} \\ A_{1,a+N+\ell}(z) = \frac{1}{z(1-\omega^\ell z)} \text{ for any } \ell \in \{1, \dots, N\} \end{cases}$$

and all other coefficients are zero. We consider the following solutions of the differential system $Y' = AY$, with $1 \le \ell \le N$:

$$Y_{0,\ell}(z) = {}^t\Big(-\mathrm{Li}_1(\omega^\ell z), \mathrm{Li}_2(\omega^\ell z), \dots, (-1)^a \mathrm{Li}_a(\omega^\ell z), 0, \dots, 0, 1, 0, \dots, 0 \Big),$$

$$Y_{\infty,\ell}(z) = {}^t\Big(\mathrm{Li}_1(\frac{1}{\omega^\ell z}), \mathrm{Li}_2(\frac{1}{\omega^\ell z}), \dots, \mathrm{Li}_a(\frac{1}{\omega^\ell z}), 0, \dots, 0, 1, 0, \dots, 0 \Big),$$

$$Y_{\omega^\ell}(z) = {}^t\Big(1, -\log(\omega^{-\ell}z), \frac{(\log(\omega^{-\ell}z))^2}{2!}, \ldots, (-1)^{a-1}\frac{(\log(\omega^{-\ell}z))^{a-1}}{(a-1)!}, 0, \ldots, 0\Big)$$

where the coefficient 1 in $Y_{0,\ell}(z)$ (resp. $Y_{\infty,\ell}(z)$) is in position $a+\ell$ (resp. $a+N+\ell$).

We let $J_0 = \{(0,1),(0,2),\ldots,(0,N)\}$, $J_\infty = \{(\infty,1),(\infty,2),\ldots,(\infty,N)\}$, $J_{\omega^\ell} = \{\omega^\ell\}$ for $1 \le \ell \le N$, and $\Sigma = \{0,\infty\} \cup \{\omega^\ell, 1 \le \ell \le N\}$. We also let $P_{a+\ell}(z) = \overline{P}_{0,\ell}(z) = U(\omega^\ell z)$ and $P_{a+N+\ell} = \overline{P}_{\infty,\ell}(z) = V(\omega^\ell z)$ for any $\ell \in \{1,\ldots,N\}$. Then with the notation of the introduction we have $R(Y_{0,\ell}) = R_{0,\ell}(z)$, $R(Y_{\infty,\ell}) = R_{\infty,\ell}(z)$, and $R(Y_{\omega^\ell}) = R_{\omega^\ell}(z)$ for any $\ell \in \{1,\ldots,N\}$.

Since P_a is not the zero polynomial, we have $R_{\omega^\ell}(z) \neq 0$ for any ℓ; the $\mathbb{C}$-linear independence of $R_{0,1}(z)$, …, $R_{0,N}(z)$ (resp. of $R_{\infty,1}(z)$, …, $R_{\infty,N}(z)$) follows directly (resp. up to changing z to $1/z$) from the following lemma, which is not difficult to prove using monodromy (see [28]).

LEMMA 4.3. *The functions 1 and* $\mathrm{Li}_j(\omega^\ell z)$, *for* $j \ge 1$ *and* $1 \le \ell \le N$, *are linearly independent over* $\mathbb{C}(z)$.

Eqns. (4.18), (4.19), and (4.20) yield $\mathrm{ord}_0(R_{0,\ell}(z)) \ge (r+1)n+1$, $\mathrm{ord}_\infty(R_{\infty,\ell}(z)) \ge rn+1$, and $\mathrm{ord}_{\omega^\ell}(R_{\omega^\ell}(z)) \ge d_0 - 1$ for any $\ell \in \{1,\ldots,N\}$, so that

$$\sum_{\sigma\in\Sigma}\sum_{j\in J_\sigma} \mathrm{ord}_\sigma R_j(z) \ge (2r+1)Nn + N(d_0+1) = (n+1)q - nN - \tau \text{ with } \tau = N - a(N-1);$$

here $q = a+2N$, and we recall that $d_0 = -\deg F = a(\frac{n}{N}+1) - 2rn$. This number τ is exactly the difference between the number of unknowns and the number of equations computed after Eq. (4.20).

Now for any $k \ge 1$ and any $\ell \in \{1,\ldots,N\}$ we let

$$\overline{P}_{k,0,\ell} = \omega^{\ell(k-1)}U_k(\omega^\ell z) \text{ and } \overline{P}_{k,\infty,\ell} = \omega^{\ell(k-1)}V_k(\omega^\ell z), \tag{4.21}$$

and

$$\mathbf{P}_k = {}^t\Big(P_{k,1}, P_{k,2}, \ldots, P_{k,a}, \overline{P}_{k,0,1}, \ldots, \overline{P}_{k,0,N}, \overline{P}_{k,\infty,1}, \ldots, \overline{P}_{k,\infty,N}\Big) \in M_{q,1}(\mathbb{C}(z)),$$

so that $\mathbf{P}_1 = {}^t(P_1, \ldots, P_{a+2N})$. Then it is not difficult to check that

$$\mathbf{P}_k = \Big(\frac{\mathrm{d}}{\mathrm{d}z} + {}^tA\Big)^{k-1}\mathbf{P}_1.$$

To illustrate this equality, we notice that Eq. (4.17) yields

$$R_{0,\ell}^{(k-1)} = \overline{P}_{k,0,\ell}(z) + \sum_{j=1}^{a} P_{k,j}(z)(-1)^j \mathrm{Li}_j(\omega^\ell z)$$

$$\text{and } R_{\infty,\ell}^{(k-1)} = \overline{P}_{k,\infty,\ell}(z) + \sum_{j=1}^{a} P_{k,j}(z)\mathrm{Li}_j(\frac{1}{\omega^\ell z})$$

since (as in [25, Chapter 3, §4])

$$S_0^{(k-1)} = U_k(z) + \sum_{j=1}^{a} P_{k,j}(z)(-1)^j \mathrm{Li}_j(z)$$

$$\text{and } S_\infty^{(k-1)} = V_k(z) + \sum_{j=1}^{a} P_{k,j}(z)\mathrm{Li}_j(1/z).$$

Provided n is large enough, we have checked all assumptions of Theorem 1.2 (using, among others, Eq. (4.11)). We apply this result with $\alpha = z_0$; recall that $z_0 \in \{1, e^{i\pi/N}\}$. In the case $z_0 = 1$, we obtain positive integers $k_2 < \ldots < k_q < \tau + c_2 \le c_2 + 1$ such that the matrix with columns $\mathbf{P}_{k_2}(1), \ldots, \mathbf{P}_{k_q}(1)$ has rank $q-1$. Now $P_{k,1}(1) = 0$ for any $k \le c_2$ (using Eq. (4.11) since n is large enough) so that the first row of this matrix is identically zero. Removing this row yields the following invertible matrix (with $z_0 = 1$ and $i_0 = 2$):

$$\left[\begin{array}{c} [z_0^{k_j-1} P_{k_j,i}(z_0)]_{i_0\le i\le a, i_0\le j\le q} \\ \hline [z_0^{k_j-1} \overline{P}_{k_j,0,i}(z_0)]_{1\le i\le N, i_0\le j\le q} \\ \hline [z_0^{k_j-1} \overline{P}_{k_j,\infty,i}(z_0)]_{1\le i\le N, i_0\le j\le q} \end{array}\right]. \tag{4.22}$$

If $z_0 = e^{i\pi/N} \notin \Sigma$ then Theorem 1.2 provides directly $k_1 < \ldots < k_q \le c_2$ such that the matrix (4.22) with $i_0 = 1$ is invertible.

Now Eq. (4.12) with $z = \omega^\ell z_0$ yields, since $U_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$:

$$\omega^{(k-1)\ell} z_0^{k-1} U_k(\omega^\ell z_0) = \sum_{\lambda=0}^{N-1} \omega^{\ell\lambda} z_0^\lambda U_{k,\lambda}(z_0) \text{ for any } \ell \in \{1, \ldots, N\}.$$

Therefore we have for any $\lambda \in \{0, \ldots, N-1\}$:

$$U_{k,\lambda}(z_0) = \frac{z_0^{k-1-\lambda}}{N} \sum_{\ell=1}^{N} \omega^{(k-1-\lambda)\ell} U_k(\omega^\ell z_0) = \frac{z_0^{k-1-\lambda}}{N} \sum_{\ell=1}^{N} \omega^{-\lambda\ell} \overline{P}_{k,0,\ell}(z_0) \tag{4.23}$$

using Eq. (4.21). Moreover the same relation holds with $V_{k,\lambda}$ and $\overline{P}_{k,\infty,\ell}$ for $\lambda \in \{0, \ldots, N-1\}$. We recall that $s_{k,i}$ was defined in Eq. (4.16) (§4.3) by

$$\begin{aligned} s_{k,i} = {} & \delta_n z_0^{k-1} P_{k,i}(z_0) \text{ for } 1 \le i \le a, \\ \text{and } s_{k,a+1+\lambda} = {} & \delta_n (U_{k,\lambda}(z_0) + (-1)^p V_{k,N-\lambda}(z_0)) \text{ for } 0 \le \lambda \le N-1. \end{aligned}$$

For any $\lambda \in \{0, \ldots, N-1\}$ we deduce that

$$s_{k,a+1+\lambda} = \frac{\delta_n}{N} z_0^{-\lambda} \sum_{\ell=1}^{N} \omega^{-\lambda\ell} z_0^{k-1} \overline{P}_{k,0,\ell}(z_0) \pm (-1)^p \frac{\delta_n}{N} z_0^{-N+\lambda} \sum_{\ell=1}^{N} \omega^{\lambda\ell} z_0^{k-1} \overline{P}_{k,\infty,\ell}(z_0)$$

where $\pm$ is $+$ if $1 \le \lambda \le N-1$, and z_0^N if $\lambda = 0$; indeed $V_{k,N} = V_{k,0}$ satisfies the equation analogous to Eq. (4.23) with $\lambda = 0$, but not with $\lambda = N$ if $z_0 = e^{i\pi/N}$.

Let $M = [m_{i,j}]_{i_0\le i\le a+N, i_0\le j\le a+2N}$ be the matrix defined by:

$$\begin{aligned} m_{i,i} &= \delta_n \text{ for any } i \in \{i_0, \ldots, a\}, \\ m_{a+1+\lambda,a+\ell} &= \frac{\delta_n}{N} z_0^{-\lambda} \omega^{-\lambda\ell} \text{ for any } \lambda \in \{0, \ldots, N-1\} \text{ and any } \ell \in \{1, \ldots, N\}, \\ m_{a+1+\lambda,a+N+\ell} &= \pm(-1)^p \frac{\delta_n}{N} z_0^{-N+\lambda} \omega^{\lambda\ell} \text{ for any } \lambda \in \{0, \ldots, N-1\} \text{ and any } \ell \in \{1, \ldots, N\}, \end{aligned}$$

and all other coefficients are zero. Then M has rank $a + N + 1 - i_0$; denoting by $P \in \mathrm{GL}_{a+2N+1-i_0}(\mathbb{C})$ the matrix (4.22), the matrix MP has rank $a + N + 1 - i_0$. Now MP is exactly the matrix $[s_{k_j,i}]_{i_0\le i,j\le a+N}$. This concludes the proof of Lemma 4.2.

4.5. Arithmetic and Asymptotic Properties In this section we prove the following result, used in the proof of Theorem 4.1; see §4.2 for the notation.

PROPOSITION 4.4. *Let*

$$\alpha = e^a 4^{a/N-r}(N+1)^{2r+2} r^{-a/N+4r+2} \text{ and } \beta = (2e^N)^{a/N}(rN+1)^{2r+2}. \tag{4.24}$$

Then we have $s_{k,i} \in \mathbb{Z}$ *for any* $i \in \{1, \dots, a+N\}$ *and any* $k \le d_0 - 1$, *and as* $n \to \infty$:

$$\Big| \sum_{i=i_0}^{a+N} s_{k,i}\xi_i' \Big| \le \alpha^{n(1+o(1))}, \qquad \max_{1 \le i \le a+N} |s_{k,i}| \le \beta^{n(1+o(1))}.$$

In this proposition and throughout this section, we denote by $o(1)$ any sequence that tends to 0 as $n \to \infty$; it usually depends also on a, r, N, and k. When Proposition 4.4 is applied in the proof of Theorem 4.1 (see §4.2), this dependence is not a problem since a, r, N are fixed parameters and k is bounded from above by c_2. At last we recall that d_n is the least common multiple of 1, 2, ..., n, and that

$$\delta_n = (Nd_n)^a N^{an/N}.$$

Let us start with a lemma, in which (as in §4.3)

$$F(t) = (n/N)!^{a-2rN} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{\prod_{h=0}^{n/N}(t+Nh)^a} = \sum_{h=0}^{n/N}\sum_{j=1}^{a} \frac{p_{j,h}}{(t+Nh)^j}.$$

LEMMA 4.5. *For any* $j \in \{1, \dots, a\}$ *and any* $h \in \{0, \dots, n/N\}$ *we have*

$$(Nd_{n/N})^{a-j} N^{an/N} p_{j,h} \in \mathbb{Z} \tag{4.25}$$

$$\text{and } |p_{j,h}| \le \Big((2/N)^{a/N}(rN+1)^{2r+2} \Big)^{n(1+o(1))} \tag{4.26}$$

where $o(1)$ *is a sequence that tends to 0 as* $n \to \infty$ *and may depend also on* N, a, *and* r.

Proof of Lemma 4.5: We follow the approach of [12] and [7] by letting

$$\begin{aligned}
F_0(t) &= \tfrac{(n/N)!}{\prod_{h=0}^{n/N}(t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N}\binom{n/N}{h}}{t+Nh}, \\
G_i(t) &= \tfrac{(t-in/N)_{n/N}}{\prod_{h=0}^{n/N}(t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^{h+n/N} N^{-n/N}\binom{n/N}{h}\binom{Nh+in/N}{n/N}}{t+Nh} \text{ for } 1 \le i \le rN, \\
H_i(t) &= \tfrac{(t+1+in/N)_{n/N}}{\prod_{h=0}^{n/N}(t+Nh)} = \sum_{h=0}^{n/N} \frac{(-1)^h N^{-n/N}\binom{n/N}{h}\binom{-Nh+(i+1)n/N}{n/N}}{t+Nh} \text{ for } N \le i \le (r+1)N-1.
\end{aligned}$$

Then the partial fraction expansion of $F = F_0^{a-2rN} G_1 \dots G_{rN} H_N \dots H_{(r+1)N-1}$ can be obtained my multiplying those of F_0, G_i and H_i using repeatedly the formula

$$\frac{1}{(t+Nh)(t+Nh')^\ell} = \frac{1}{N^\ell (h'-h)^\ell (t+Nh)} - \sum_{i=1}^{\ell} \frac{1}{N^{\ell+1-i}(h'-h)^{\ell+1-i}(t+Nh')^i} \tag{4.27}$$

with $h \neq h'$. The denominator of $p_{j,h}$ comes both from this formula (and this contribution divides $(Nd_{n/N})^{a-j}$) and from the denominators of the coefficients in the partial fraction expansions of F_0, G_i, H_i (which belong to $N^{-n/N}\mathbb{Z}$, so that $N^{an/N}$ accounts for this contribution). This concludes the proof of (4.25).

On the other hand, bounding from above the coefficients of the partial fraction expansions of F_0, G_i, H_i yields

$$|p_{j,h}| \leq n^{O(1)} N^{-an/N} 2^{an/N} \prod_{i=1}^{rN} \frac{(n+in/N)!}{(n/N)!(n+(i-1)n/N)!} \prod_{i=N}^{(r+1)N-1} \frac{((i+1)n/N)!}{(n/N)!(in/N)!}$$

where $O(1)$ is a constant depending only on a, r, N which can be made explicit (see [7] for details). Simplifying the products and using the bound $\frac{m!}{m_1!\ldots m_c!} \leq c^m$ valid when $m_1 + \ldots + m_c = m$, one obtains

$$|p_{j,h}| \leq n^{O(1)} (2/N)^{an/N} \Big(\frac{((r+1)n)!}{n!(n/N)!^{rN}} \Big)^2 \leq n^{O(1)} (2/N)^{an/N} (rN+1)^{2(r+1)n}.$$

This concludes the proof of Lemma 4.5.

Proof of Proposition 4.4: Let $H(P)$ denote the exponential height of a polynomial $P \in \mathbb{C}[X]$, that is the maximum modulus of a coefficient of P. Recall that $P_j(z) = \sum_{h=0}^{n/N} p_{j,h} z^{Nh}$, $U(z) = -\sum_{t=1}^{n} z^t \sum_{j=1}^{a} \sum_{h=0}^{\lfloor (t-1)/N \rfloor} \frac{p_{j,h}}{(t-Nh)^j}$ and $V(z) = -\sum_{t=0}^{n-1} z^t \sum_{j=1}^{a} \sum_{h=\lceil (t+1)/N \rceil}^{n/N} \frac{p_{j,h}}{(Nh-t)^j}$. Using Lemma 4.5 we see that these polynomials have coefficients in $\delta_n^{-1}\mathbb{Z}$ and height less than H_n for some $H_n \leq \Big((2/N)^{a/N} (rN+1)^{2r+2} \Big)^{n(1+o(1))}$. Now let $\widetilde{P}_{k,j} = z^{k-1} P_{k,j}$ for any k, j. Then the recurrence relation (4.6) yields

$$\widetilde{P}_{k,j} = z\widetilde{P}'_{k-1,j} - (k-2)\widetilde{P}_{k-1,j} - \widetilde{P}_{k-1,j+1}$$

where $\widetilde{P}_{k-1,j+1} = 0$ if $j = a$, so that $\widetilde{P}_{k,j}$ is a polynomial of degree at most n, with coefficients in $\delta_n^{-1}\mathbb{Z}$ and height $H(\widetilde{P}_{k,j}) \leq (n+1)_{k-1} H_n$, by induction on k.

In the same way, letting $\widetilde{U}_k = z^{k-1} U_k$, Eq. (4.7) yields

$$\widetilde{U}_k = z\widetilde{U}'_{k-1} - (k-2)\widetilde{U}_{k-1} - zQ_{k-1}$$

where $Q_{k-1} = \frac{1}{1-z}\widetilde{P}_{k-1,1}$. Provided $k \leq d_0 - 1$, Eq. (4.11) asserts that $P_{k-1,1}(1) = 0$ so that Q_{k-1} is a polynomial and $H(Q_{k-1}) \leq nH(P_{k-1,1}) \leq (n)_{k-1} H_n$. By induction on $k \leq d_0 - 1$, we deduce that $\widetilde{U}_k$ is a polynomial of degree at most n, with coefficients in $\delta_n^{-1}\mathbb{Z}$ and height $H(\widetilde{U}_k) \leq k(n)_{k-1} H_n$. Now Eq. (4.12) reads $\widetilde{U}_k(z) = \sum_{\lambda=1}^{N} z^{\lambda-1} U_{k,\lambda}(z)$ with $U_{k,\lambda} \in \mathbb{Q}[z^N, z^{-N}]$. If $k \leq d_0 - 1$ then $U_{k,\lambda}$ belongs to $\mathbb{Q}[z^N]$, has degree at most n (as a polynomial in z), coefficients in $\delta_n^{-1}\mathbb{Z}$ and height $H(U_{k,\lambda}) \leq k(n)_{k-1} H_n$.

Proceeding in the same way, it is not difficult to prove that the same properties hold for $V_{k,\lambda}$. This implies immediately that $s_{k,i} \in \mathbb{Z}$, and (since $d_n = e^{n(1+o(1))}$) the upper bound on $|s_{k,i}|$ in Proposition 4.4.

To prove that $\left| \sum_{i=i_0}^{a+N} s_{k,i}\xi'_i \right| \leq \alpha^{n(1+o(1))}$, we recall that $d_0 = -\deg F$ and write, as $|t| \to \infty$:

$$F(t) = \sum_{d=d_0} \frac{\mathfrak{A}_d}{t^d} \text{ where } \mathfrak{A}_d = \sum_{j=1}^{a} \sum_{h=0}^{n/N} (-Nh)^{d-j} \binom{d-1}{d-j} p_{j,h}$$

since $(t+Nh)^{-j} = \sum_{\ell=0}^{\infty} \binom{\ell+j-1}{\ell}(-Nh)^{\ell} t^{-j-\ell}$ (see [10, p. 1378]). Lemma 4.5 provides a positive real number $A_n \leq \Big((2/N)^{a/N}(rN+1)^{2r+2}\Big)^{n(1+o(1))}$ such that $|\mathfrak{A}_d| \leq (2n)^d A_n$ for any $d \geq d_0$. Then we have for any $t \in \mathbb{Z}$ such that $|t| \geq 2n+1$:

$$|F(t)| \leq A_n \sum_{d=d_0}^{\infty} (2n/t)^d \leq (2n+1)A_n(2n/t)^{d_0}. \tag{4.28}$$

For any $z \in \mathbb{C}$ such that $|z| \leq 1$, and any $k \leq d_0 - 1$, we obtain

$$\begin{aligned} |S_0^{(k-1)}(z)| &= \Big| \sum_{t=(r+1)n+1}^{\infty} F(-t)(t-k+2)_{k-1} z^{t-k+1} \Big| \leq (2n+1)A_n(2n)^{d_0} \sum_{t=(r+1)n+1}^{\infty} t^{k-1-d_0} \\ &\leq (2n+1)A_n(2n)^{d_0} \int_{(r+1)n}^{\infty} t^{k-1-d_0} \mathrm{d}t \leq (2n+1)A_n 2^{d_0} n^k r^{k-d_0}. \end{aligned}$$

Moreover the same upper bound holds for $S_\infty(z) = \sum_{t=rn+1}^{\infty} F(t) z^{-t}$ provided $|z| \geq 1$. Since

$$S(z) = \sum_{\ell=1}^{N} \omega^{\ell} \mu_{\ell} S_0(\omega^{\ell} z) + \omega^{\ell} \nu_{N-\ell} S_\infty(\omega^{\ell} z)$$

and $d_0 = a(n/N+1) - 2rn$, we obtain $|\delta_n S^{(k-1)}(1)| \leq \alpha^{n(1+o(1))}$ for any $z \in \mathbb{C}$ such that $|z| = 1$, and any $k \leq d_0 - 1$; here the constant implied in $o(1)$ may depend on k (but not on n). This concludes the proof of Proposition 4.4.

4.6. Siegel's linear independence criterion The proofs of all linear independence results in this paper rely on the following criterion, which is based on Siegel's ideas (see for instance [8, p. 81–82 and 215–216], [16, §3] or [15, Proposition 4.1]).

Proposition 4.6. *Let $\theta_1, \ldots, \theta_p$ be real numbers, not all zero. Let $\tau > 0$, and (Q_n) be a sequence of real numbers with limit $+\infty$. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$, and for any $n \in \mathcal{N}$ let $L^{(n)} = [\ell_{i,j}^{(n)}]_{1 \leq i,j \leq p}$ be a matrix with integer coefficients and non-zero determinant, such that as $n \to \infty$ with $n \in \mathcal{N}$:*

$$\max_{1 \leq i,j \leq p} |\ell_{i,j}^{(n)}| \leq Q_n^{1+o(1)}$$

$$\textit{and} \max_{1 \leq j \leq p} |\ell_{1,j}^{(n)} \theta_1 + \ldots + \ell_{p,j}^{(n)} \theta_p| \leq Q_n^{-\tau+o(1)}.$$

Then we have

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_1, \ldots, \theta_p) \geq \tau + 1.$$

In the proof of Theorem 4.1 we apply this proposition with $Q_n = \beta^n$ and $\tau = -\frac{\log \alpha}{\log \beta}$ (so that $Q_n^{-\tau} = \alpha^n$), where α and β are defined in §4.2; $\mathcal{N}$ is the set of integer multiples of N.

Eventhough it is a classical result, let us recall the proof of Proposition 4.6. Let $d = \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_1, \ldots, \theta_p)$, and F be a subspace of $\mathbb{R}^p$ defined over $\mathbb{Q}$, of dimension d, which contains the point $(\theta_1, \ldots, \theta_p)$. Let $n \in \mathcal{N}$ be sufficiently large, and denote by $L_j^{(n)}$ the linear form $\ell_{1,j}^{(n)} X_1 + \ldots + \ell_{p,j}^{(n)} X_p$ on $\mathbb{R}^p$. Up to reordering $L_1^{(n)}, \ldots, L_p^{(n)}$, we may assume the restrictions of $L_1^{(n)}, \ldots, L_d^{(n)}$ to F to be linearly independent linear forms on F. Denoting by $(u_1, \ldots, u_d)$ an $\mathbb{R}$-basis of F consisting in vectors of $\mathbb{Z}^p$, the matrix $[L_j^{(n)}(u_t)]_{1\leq j,t\leq d}$ has a non-zero integer determinant. Now $(\theta_1, \ldots, \theta_p)$ is a linear combination of $u_1, \ldots, u_d$; the same linear combination of the columns has coefficients less than $Q_n^{-\tau+o(1)}$ in absolute value. Therefore $Q_n^{d-1-\tau+o(1)}$ is an upper bound on this non-zero integer determinant: this concludes the proof of Proposition 4.6.

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