Multiplicity Estimates and Degeneration

Stéphane Fischler and Michael Nakamaye

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Abstract: In this paper we prove a multiplicity estimate, which is best possible up to a multiplicative constant, in which the set of points is connected to an action of $\mathrm{SL}_2(\mathbf{Z})$ on the torus $\mathbf{G}_m^2(\mathbf{C})$. This result is motivated by the construction, due to Roy, of a non-trivial auxiliary function that could be used to study the points on the Grassmannian whose coordinates are logarithms of algebraic numbers: to make use of this construction, only a zero estimate connected to the action of $\mathrm{GL}_m(\mathbf{Z})$ on $\Lambda^k\mathbf{C}^m$ is missing. The result we prove is essentially analogous to it.

The proof is based on the fact that a zero (or multiplicity) estimate can be derived from a lower bound for a Seshadri constant. Then a degeneration argument is used: inside a family any such lower bound holds on an open subset, so proving it for sufficiently many special cases yields it for almost all cases.

Introduction

The goal of this article is twofold. First we will use the technique of degeneration to obtain lower bounds for the Seshadri constant associated to a set of points S and a line bundle L on a complex projective variety X: this leads immediately to an associated multiplicity estimate bounding the possible multiplicity of a non–zero section of L along S. Secondly we would like to apply this method to study the action of a non–commutative group, $\mathrm{SL}_2(\mathbf{Z})$, on a projective variety. The larger goal of this work is to develop the techniques necessary to prove multiplicity estimates on non–commutative algebraic groups and thereby open the door to transcendence proofs in a broader setting.

Suppose X is a smooth projective variety defined over the complex numbers and $S \subset X$ a finite set of points. Suppose L is a line bundle on X and that $0 \neq \sigma \in H^0(X, L)$ satisfies $\mathrm{mult}_x(\sigma) \geq M$ for all $x \in S$. If S' is a small "perturbation" of S one may ask whether or not there exists a section $\sigma' \in H^0(X, L)$ such that $\mathrm{mult}_{x'}(\sigma') \geq M$ for all $x' \in S'$. The answer can be no, as is seen in the following simple example. Suppose $\pi : Y \to \mathbf{P}^2$ is the blow–up of $\mathbf{P}^2$ at a point P with exceptional divisor E. Let $L = \pi^*\mathcal{O}_{\mathbf{P}^2}(1)(E)$ and let S consist of a single point $\{x\}$. If $x \in E$ then there will be a section of multiplicity 2 at x while if x is not contained in E then the maximum possible multiplicity is one. Thus at special points the multiplicity can get larger.

In the theory of multiplicity estimates, the goal is to provide an upper bound for the multiplicity of *any* non–zero section of $H^0(X, L)$ along S. Recall that the multiplicity of a section $\sigma \in H^0(X, L)$ along S is given by

$$\mathrm{mult}_S(\sigma) = \min_{x \in S}\{\mathrm{mult}_x(\sigma)\}.$$

Suppose that T is a "simpler" set of points for which one can readily show that $\mathrm{mult}_T(\tau) \leq M$ for all $0 \neq \tau \in H^0(X, L)$. The conclusion will be that for any set of points T' "sufficiently close" to T (indeed, we will see that in fact what is involved is a Zariski open subset) $\mathrm{mult}_{T'}(\tau') \leq M$ for all $0 \neq \tau' \in H^0(X, L)$. Now it could of course happen that this open set does not contain the original S in which one is interested: indeed this is the inevitable downside of the degeneration technique. On the other hand, in the applications to transcendence theory, the set of points S is rarely completely rigid as one can take, for example, multiples or linear combinations of the points in S without changing their transcendence properties. Thus our hope is that the degeneration technique, combined with a judicious choice of set S, will allow for interesting applications in transcendence theory.

To apply this degeneration technique, the setting we consider is inspired by Roy's approach [11] to the conjecture of algebraic independence of logarithms of algebraic numbers. Letting $\mathbf{L} = \{\lambda \in \mathbf{C} : \exp(\lambda) \in \overline{\mathbf{Q}}^\star\}$ denote the $\mathbf{Q}$-vector space of logarithms of algebraic numbers, this conjecture states that elements of $\mathbf{L}$ are algebraically independent if and only if they are linearly independent over $\mathbf{Q}$ (see Chapter 3 of [4]). This is equivalent to the following conjecture: if X is a proper subvariety of affine space $\mathbf{A}^n_{\mathbf{Q}}$ and $x \in X \cap \mathbf{L}^n$ is a point of X whose coordinates belong to $\mathbf{L}$, then the coordinates of x are $\mathbf{Q}$-linearly dependent.

If, in addition, X is required to have linear geometric components, then this conjecture is equivalent to Baker's theorem. There are a few other examples of varieties X for which this conjecture is known (see for instance [2]), but for most varieties it is a very difficult open problem. For most of the known cases, the strategy of proof relies on applying existing transcendence results, such as the linear subgroup Theorem (Theorem 2.1 of [13]), which is known not to be sufficient to prove the general conjecture (see Proposition 2 of [10]). Therefore it would be very interesting to apply transcendence techniques directly on X.

An important step in this direction is due to Roy. Let k, m be integers with $m \geq k + 2 \geq 4$ and let X be the affine cone over the grassmannian variety $G(k, \mathbf{C}^m)$, so that X is embedded into $\mathbf{A}^n_{\mathbf{Q}}$, with $n = \binom{m}{k}$, via a choice of basis for $\Lambda^k\mathbf{Q}^m$. Assume there exists $x = v_1 \wedge \ldots \wedge v_k \in X \cap \mathbf{L}^n$ with coordinates linearly independent over $\mathbf{Q}$. Let N be a very large integer, and denote by $S_N \subset X \cap \mathbf{L}^n$ the finite set of all points $(\Lambda^k A)(x) = Av_1 \wedge \ldots \wedge Av_k$ with $A \in \mathrm{GL}_m(\mathbf{Z})$ having entries in $\{0, 1, \ldots, N\}$. In Section 4 of [11], Roy makes (if $m \geq k(k+2)$) a non-trivial construction of a polynomial $P_N \in \mathbf{Z}[Y_1, \ldots, Y_n]$, of degree less than some explicit power of N, such that $P_N(\exp(x_1), \ldots, \exp(x_n)) = 0$ for any point $(x_1, \ldots, x_n) \in S_N$. To conclude the transcendence proof, it would suffice to have a zero estimate. However, zero estimates are known only when commutative linear algebraic groups are involved (see for instance [9], [1], [8], [6]); they generalize to non-commutative groups [7], but with a huge loss in the accuracy of the estimate, so that in the non-commutative setting it seems very difficult to make use of them.

In the present paper, we prove a sharp zero estimate analogous to the one needed to derive a contradiction from Roy's construction. Apart from the set of exceptions coming from the degeneration technique, the main difference is that instead of the action of $\mathrm{GL}_m(\mathbf{Z})$ on $\Lambda^k\mathbf{C}^m$ used to define the set of points S_N, we consider the natural action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{C}^2$.

In order to state this application of the degeneration technique, we require several preliminary definitions. Let $\mathbf{G}_m = \mathbf{C}^*$ be the multiplicative group. We will work on $\mathbf{G}_m \times \mathbf{G}_m$ which we will compactify as $\mathbf{P}^1 \times \mathbf{P}^1$. We will consider the following action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{G}_m \times \mathbf{G}_m$. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$

and let $P = (x, y) \in \mathbf{G}_m \times \mathbf{G}_m$. Then we define

$$M \cdot P = (x^a y^b, x^c y^d).$$

It is easy to check that this defines an action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{G}_m \times \mathbf{G}_m$. Moreover in this text we will consider only matrices M such that $a, b, c, d \geq 0$. Then the action extends to $\mathbf{A}^1 \times \mathbf{A}^1$, and in fact to $\mathbf{P}^1 \times \mathbf{P}^1$ except for the two points $(1, 0) \times (0, 1)$ and $(0, 1) \times (1, 0)$.

Let $\Gamma[N] = \{M \in \mathrm{SL}_2(\mathbf{Z}) : 0 \leq a, b, c, d \leq N\}$ and let $T[N]$ denote the set of all $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma[N]$ such that $a, b, c, d > 0$ and the pair b, d is maximal among those such that $b, d \leq N$ and $ad - bc = 1$.

Theorem 1 *For any $N \geq 1$ there exists a finite set $\Sigma_N \subset \mathbf{P}^1 \times \mathbf{P}^1 \backslash \{(1, 0) \times (0, 1), (0, 1) \times (1, 0)\}$ with the following property. Suppose $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a curve of bi–degree d_1, d_2, not necessarily irreducible, such that $T[N] \cdot x \subset C$ for some x in $\mathbf{P}^1 \times \mathbf{P}^1 \backslash \{(1, 0) \times (0, 1), (0, 1) \times (1, 0)\}$ with $x \notin \Sigma_N$. Then*

$$d_1 + d_2 \geq \frac{|T[N] \cdot x|}{N + 1}. \tag{1}$$

Moreover, if $\mathrm{mult}_y(C) \geq M$ for all $y \in T[N] \cdot x$ then $d_1 + d_2 \geq \frac{M|T[N]\cdot x|}{N+1}$.

As noted above, the possible finite number of exceptional cases in Theorem 1 comes from the degeneration argument used in the proof. It is not clear, however, that the inequality ever fails.

Even though Roy's construction suggests to study $\Gamma[N] \cdot x$, to make the degeneration technique work we have to restrict ourselves to $T[N] \cdot x$ (see the remark before Lemma 17). However this is not an important loss, since $T[N] \subset \Gamma[N]$ and both have the same cardinality up to a factor 2 as $N \to \infty$ (see §2).

Theorem 1 shows that the points of $T[N] \cdot x$ are well distributed for most x, relative to bi–homogeneous forms of bi–degree (d, d). Indeed, since $\dim\left(H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d, d))\right) = (d + 1)^2$

it follows that there exists a non-zero homogeneous polynomial of bi–degree (d, d) vanishing along $T[N] \cdot x$ as long as

$$d > \sqrt{|T[N] \cdot x|} - 1. \tag{2}$$

If the points of $T[N] \cdot x$ were in general position this would be the smallest degree possible. According to Theorem 1, if such a polynomial exists then we have

$$d \geq \frac{|T[N] \cdot x|}{2(N+1)}. \tag{3}$$

We shall prove in §2 that $|T[N]|$ grows like $\frac{6}{\pi^2}N^2$. Since for general x we have $|T[N] \cdot x| = |T[N]|$, plugging this into (2) and (3) shows that the points of $T[N] \cdot x$ are very well distributed, at least for most values of x. In the case where $|T[N] \cdot x|$ is (much) smaller than $|T[N]|$, Theorem 1 holds but it is (much) further from being optimal.

Suppose that $\phi : \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ is the natural embedding using the complete linear series $|\mathcal{O}_{\mathbf{P}^1 \times \mathbf{P}^1}(1,1)|$. Applying Theorem 1 to $\mathbf{P}^1 \times \mathbf{P}^1$ within this embedding gives the following: if $D \subset \mathbf{P}^3$ is a divisor of degree d and $D \cap \phi(\mathbf{P}^1 \times \mathbf{P}^1)$ is a proper intersection which contains $\phi(T[N] \cdot x)$ then

$$2d \geq \frac{|T[N] \cdot x|}{N+1}. \tag{4}$$

Now we have $|T[N]| \sim \frac{6}{\pi^2}N^2$ as $N \to \infty$ (see §2). Thus, assuming $|T[N] \cdot x| = |T[N]|$, the right hand side of (4) is equivalent to $\frac{6}{\pi^2}N$. Hence the minimal degree of a polynomial on $\phi(\mathbf{P}^1 \times \mathbf{P}^1)$ vanishing on $T[N] \cdot x$ is at least equivalent to $\frac{3}{\pi^2}N$: this is only a small constant away from saying that the points of $T[N] \cdot x$ are in general position in $\phi(\mathbf{P}^1 \times \mathbf{P}^1)$. The methods used to prove Theorem 1 also allow for a generalization to other embeddings of $\mathbf{P}^1 \times \mathbf{P}^1$ in projective space but the language of Seshadri constants is no longer sufficiently general to describe the situation and a more appropriate method of attack is the intersection theoretic approach of [8].

Finally, Theorem 1 is not stated as a multiplicity estimate but rearranging the terms gives the desired formulation: if $0 \neq \sigma \in H^0\left(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(d_1, d_2)\right)$ vanishes along $T[N] \cdot x$ to multiplicity M then, for all but finitely many values of x's we have

$$M \leq \frac{(d_1 + d_2)(N+1)}{|T[N] \cdot x|}.$$

The organization of the paper is as follows. In §1 we review the basic definitions and behavior of positivity of line bundle in families, including the notion of Seshadri constants. The main result, that Seshadri constants can only *decrease* under specialization, is stated precisely and we explain how to apply it to deduce results like Theorem 1. In §2 we study a few special properties of $\mathrm{SL}_2(\mathbf{Z})$ which will be used in §3 where we prove Theorem 1.

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1 Positivity in families

Here we recall the basic definitions concerning positivity of line bundles on algebraic varieties as well as their behavior when considered variationally in families. Suppose X is a smooth projective variety and L a line bundle on X. Recall that L is called *ample* if there is a positive integer n and an embedding $\phi : X \to \mathbf{P}^N$ in projective space so that $\phi^*\mathcal{O}_{\mathbf{P}^N}(1) = L^{\otimes n}$. A line bundle M is called *nef* if $M \cdot C \geq 0$ for all curves $C \subset X$. Suppose $\pi : \mathcal{F} \to B$ is a family of varieties, that is π is a surjective proper morphism and we will assume, for simplicity, that B is an integral scheme. For each $b \in B$ let $\mathcal{F}_b = \pi^{-1}(b)$. Suppose $\mathcal{L}$ is a line bundle on $\mathcal{F}$ and for $b \in B$ let $\mathcal{L}_b = \mathcal{L}|\mathcal{F}_b$. The first result we will need is [5] Theorem 1.2.17:

Theorem 2 *The set of points $b \in B$ such that $\mathcal{L}_b$ is ample on $\mathcal{F}_b$ is open.*

The open set in Theorem 2 can of course be empty.

Unlike ampleness, the behavior of nefness in families is a little bit more complicated. This can be viewed as a result of the fact that a nef line bundle is a limit of ample line bundles and consequently the set of exceptional points for nefness in a family can be a countable union of proper subvarieties associated to the limiting sequence of ample bundles (see [5] Proposition 1.4.14):

Theorem 3 *The set of points $b \in B$ such that $\mathcal{L}_b$ is nef on $\mathcal{F}_b$ is the complement of a countable union of closed subvarieties.*

Theorem 3 will be applied to the variational study of Seshadri constants which we define now; strictly speaking, this is not necessary for the proof of our main theorem but it is precisely this behavior of Seshadri constants in families, namely their ability to become *smaller* at special points, which motivates the proof of Theorem 1 and so we have included this discussion. Let X be a smooth projective variety and S a finite set of points on X. Let $\pi : Y \to X$ be the blow–up of X along S with exceptional divisor E. Note that E will be reducible in general with one irreducible component for each point in S. Let L be an ample line bundle on X.

Definition 4 *The* Seshadri constant *of L relative to S is defined by*

$$\epsilon(S, L) = \sup_{\alpha \in \mathbf{Q}^+} \{\pi^*(L)(-\alpha E) \ \textit{is ample}\}.$$

Note that the same definition, [5] Definition 5.4.1, applies to a possibly non–reduced subscheme $S \subset X$. This will be critical in what follows at the end of this section. The bundle $\pi^*(L)(-\epsilon(S,L)E)$ is never ample, but it is nef since it is a limit of ample line bundles: this is a corollary of Kleiman's theorem, [5] Theorem 1.4.23. Hence, for $\alpha \in \mathbf{Q}^+$ the line bundle $\pi^*(L)(-\alpha E)$ is ample if and only if $0 < \alpha < \epsilon(S,L)$.

Combining Theorem 3 and Definition 4 we can see that if the set S is allowed to vary in an algebraic family then the Seshadri constant $\epsilon(S,L)$ can get smaller at special points and the collection of special points in general can be a countable union of proper subvarieties. We shall make this remark more precise in Corollaries 11 and 12 below.

A subvariety $V \subset X$ is called *Seshadri exceptional* relative to L if

$$\deg_{\pi^* L(-\epsilon(S,L))}(\tilde{V}) = 0,$$

where $\tilde{V}$ is the strict transform of V in Y, and if V is not properly contained in any other subvariety having this property. To build an intuition for the behavior of these Seshadri constants and their associated exceptional subvarieties when the set S is allowed to vary we consider a few simple examples.

Example 5 *Suppose $\pi : X \to \mathbf{P}^2$ is the blow up of $\mathbf{P}^2$ at a point P with exceptional divisor E. Let $L = \pi^*\mathcal{O}(1)(-\alpha E)$ where $0 < \alpha < 1$. Let $S = \{x\}$. Then*

$$\epsilon(S,L) = \begin{cases} 1-\alpha & \text{if } x \notin E \\ \min\{\alpha, 1-\alpha\} & \text{if } x \in E. \end{cases}$$

The Seshadri exceptional subvarietes are either the strict transform of the line joining x and P, in the first case, or the exceptional divisor E in the second case.

Example 6 *Suppose $S = \{x,y,z\} \subset X = \mathbf{P}^2$ where we will assume, for simplicity, that x,y,z are distinct. Here we find*

$$\epsilon(S, \mathcal{O}_{\mathbf{P}^2}(1)) = \begin{cases} 1/3 & \text{if } x,y,z \text{ are collinear} \\ 1/2 & \text{if } x,y,z \text{ are not collinear.} \end{cases}$$

In the first case, the Seshadri exceptional subvariety is the line containing x, y and z. In the second case, the Seshadri exceptional subvarieties are the three lines joining pairs of the points $\{x,y,z\}$.

Example 7 *Suppose $X = \mathbf{P}^2$, $L = \mathcal{O}_{\mathbf{P}^2}(1)$, and $S = \{x_1, \dots, x_n\}$ where $n \geq 10$. Except when n is a perfect square or when the points $\{x_i\}$ are in special position, the exact value of $\epsilon(S,L)$ is unknown. The famous conjecture of Nagata predicts that the value should be maximal for very general points $x_1, \dots, x_n$:*

$$\epsilon(S,L) = \frac{1}{\sqrt{n}}.$$

If true, this would mean that the variety X itself is Seshadri exceptional.

Example 8 *Continuing Example 7, if $n = a^2$ is a perfect square, then it is known that*

$$\epsilon(S, L) = \frac{1}{a}.$$

This can be seen by choosing the points $x_1, \ldots, x_{a^2}$ to be the points of intersection of two general hypersurfaces of degree a. Because $\epsilon(S, L)$ can only become smaller at special points and because $\frac{1}{a}$ is the maximum possible value for $\epsilon(S, L)$, it follows that $\epsilon(S, L) = \frac{1}{a}$ for any sufficiently general collection of points S. The Seshadri constant formulation of Nagata's Conjecture, in this setting, is a little weaker than another version of Nagata's conjecture which states that for sufficiently general points $x_1, \ldots, x_{a^2}$ there is no hypersurface of degree less than or equal to ma with multiplicity at least m at these points.

The logic of Example 8 is precisely that which guides our proof of Theorem 1. We would now like to state the general result of which Theorem 1 is a corollary. Suppose Y is a smooth projective variety. The easiest way to parametrize finite collections of points in Y is via a subvariety $V \subset Y \times B$ where B is an algebraic variety and, if $p_2 : Y \times B \to B$ is the projection to the second factor, then $p_2 : V \to B$ is finite. Thus B can be viewed as a parameter space where to each point $b \in B$ corresponds the preimage $V \cap p_2^{-1}(b)$ which will be a scheme supported on a finite set of points in Y. We do *not* ask that V be irreducible as we are interested in parametrizing finite subsets of points in Y. Let $\pi : Z \to Y \times B$ be the blow–up of $Y \times B$ along V with exceptional divisor E. We would like to apply Theorems 2 and 3 to the study of the Seshadri constants associated to the subschemes $V \cap p_2^{-1}(b)$ as a function of b. Let $p_1 : Y \times B \to Y$ be the projection to the first factor. For $b \in B$, we denote by Y_b the fibre $p_2^{-1}(b)$ and by $\tilde{Y}_b$ its strict transform in Z, which in this case is simply the set theoretic inverse image $\pi^{-1}(Y_b)$.

Proposition 9 *For any $\mathbf{Q}$–divisor D on Z, the set of points $U_D \subset B$ defined by*

$$U_D = \{b \in B : D|\tilde{Y}_b \text{ is ample}\}$$

is an open, possibly empty, subset of B.

Proof of Proposition 9. We may assume without loss of generality that D is a divisor as ampleness of $D|\tilde{Y}_b$ is unchanged when replacing D with any positive multiple. Let $f = p_2\pi : Z \to B$, making Z a family of varieties parametrized by B. Then we can apply Theorem 2 to the divisor D on Z to conclude that the set U_D of $b \in B$ such that D_b, the restriction of D to $\tilde{Y}_b$, is ample is an open subset of B.

Proposition 10 *For any $\mathbf{Q}$–divisor D on Z, the set of points $U'_D \subset B$ defined by*

$$U'_D = \{b \in B : D|\tilde{Y}_b \text{ is nef}\}$$

is the complement, in B, of countably many proper subvarieties.

Proof of Proposition 10. As in the previous proof we may assume that D is a divisor. Let $f = p_2\pi : Z \to B$, making Z a family of varieties parametrized by B. Then we can apply Corollary 3 to the divisor D on Z to conclude that the set U'_D of $b \in B$ such that D_b, the restriction of D to $\tilde{Y}_b$, is nef is the complement of a countable union of closed subsets of B.

Returning to the notation introduced before the statement of Proposition 9, we will be specifically interested in the case where $D = (p_1\pi)^*(A)(-\alpha E)$ where A is an ample line bundle on Y and α is a rational number. In this case, the set U_D is the collection of points $b \in B$ for which the Seshadri constant of A along the subscheme $p_2^{-1}(b)$ is larger than α, as we shall prove now.

Let $i : Y_b \to Y \times B$ be the natural inclusion and $g : \tilde{Y}_b \to Y_b$ be the blow–up of Y_b along the subscheme $V \cap Y_b$. We obtain a commutative diagram

$$\begin{tikzcd} \tilde{Y}_b \arrow[r, "j"] \arrow[d, "g"'] & Z \arrow[d, "\pi"] \\ Y_b \arrow[r, "i"] & Y \times B \end{tikzcd}$$

By [3] Corollary 7.15, $j(\tilde{Y}_b) = \pi^{-1}(Y_b)$. In particular, $(p_2\pi) : Z \to B$ is a family of blow–ups whose fiber over b is the blow–up of Y_b along the scheme $Y_b \cap V$. Thus we may apply Propositions 9 and 10 respectively to deduce the following corollaries, in which the Seshadri constants are considered on Y_b.

Corollary 11 *Let A be an ample line bundle on Y. For any $\delta > 0$ the set of all $b \in B$ such that*

$$\epsilon(V \cap p_2^{-1}(b), A) > \delta$$

is the complement in B of a Zariski closed subset Z_δ.

Corollary 12 *Let A be an ample line bundle on Y. For any $\delta > 0$ the set of all $b \in B$ such that*

$$\epsilon(V \cap p_2^{-1}(b), A) \geq \delta$$

is the complement in B of the union of a countable list of subvarieties $Z_{i,\delta}$.

The strength of these results comes from the fact that one can choose for b a point, or a collection of points, for which the set $V \cap p_2^{-1}(b)$ is as simple as possible allowing an easy estimate or calculation of $\epsilon(V \cap p_2^{-1}(b), A)$. The weakness comes from the subvarieties Z_δ or $Z_{i,\delta}$ over which there is very little control. In the application to $\mathrm{SL}_2(\mathbf{Z})$ below we will be able to obtain some detailed knowledge about Z_δ by choosing an entire collection of b's for which we can estimate the Seshadri constant. What should be clear from this method is that it can be very powerful in situations where one can allow S to vary while it will provide little insight if S is rigid.

We close this section with a brief discussion of relative amplitude and other notions of positivity which will be critical in Section 3. Suppose $f : V \to W$ is a surjective map of varieties and L a line bundle on V. Then L is called *relatively ample* for f, or f–ample, if $L|f^{-1}(w)$ is ample for all $w \in W$. Note that this is *not* the standard definition of relative amplitude, given for example in [5] §1.7, although it is equivalent by [5] Theorem 1.7.8. The standard and most important example is the following: suppose $\pi : Y \to X$ is the blow up of an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ with exceptional divisor E. Then $\mathcal{O}_Y(-E)$ is π–ample (see [3] Chapter II, Proposition 7.13). We will also require, in the proof of Lemma 17, the Nakai-Moishezon for ampleness which states that a line bundle L on a variety V is ample if and only if $\deg_L(W) > 0$ for every subvariety $W \subset V$. In addition, Lemma 16 uses the fact that if $f : Y \to X$ is a morphism of varieties and L a line bundle on Y then we can conclude that L is nef provided it is f-ample and each irreducible component of its base locus is contained in a fibre of f.

2 Points in $\mathrm{SL}_2(\mathbf{Z})$

In this section we gather together the results about the sets $\Gamma[N]$ and $T[N]$ defined in the introduction. Strictly speaking, we do not need to study $\Gamma[N]$ as it is $T[N]$ which is relevant for Theorem 1. But we have included results about $\Gamma[N]$ (provided to us by Michel Laurent) because it is in some sense a more "natural" subset of $\mathrm{SL}_2(\mathbf{Z})$.

Lemma 13 *As $N \to \infty$, we have*

$$|T[N]| = \left(\frac{6}{\pi^2} + o(1)\right) N^2 \text{ and } |\Gamma[N]| = \left(\frac{12}{\pi^2} + o(1)\right) N^2.$$

Proof of Lemma 13. Let A_N denote the set of all pairs (x, y) such that $1 \leq x, y \leq N$ and $\gcd(x, y) = 1$. Then it is well known (see for instance Theorem 4 of [12], p. 45) that $|A_N| = \left(\frac{6}{\pi^2} + o(1)\right) N^2$ as $N \to \infty$. Moreover, if $(x, y) \in A_N$ and $y \geq 2$ then there is a unique pair (u, v) such that $0 \leq u < y$, $0 \leq v < x$, and $ux - vy = 1$. It is straightforward to check that $T[N] \to A_N$, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto (a, c)$ is a bijective map; this proves the result for $T[N]$.

Now let $\Gamma_1[N]$ (resp. $\Gamma_2[N]$) be the set of all matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma[N]$ such that $\max(a, c)$ is greater than (resp. less than) $\max(b, d)$. For any $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1[N]$ with $c \geq 2$, let (u, v) be the pair such that $0 \leq u < c$, $0 \leq v < a$, and $ua - vc = 1$. Then there exists a non-negative integer k such that $d = u + kc$ and $b = v + ka$; since $\max(a, c) > \max(b, d)$ this implies $k = 0$ and $(b, d) = (v, u)$. Now if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma_1[N]$ is such that $c = 1$, then $ad = b + 1 \leq a$ so that $d = 1$: there are $N - 1$ such matrices. Therefore we have $|\Gamma_1[N]| = \left(\frac{6}{\pi^2} + o(1)\right) N^2$ as

$N \to \infty$; the same result can be proved for $\Gamma_2[N]$ in a similar way. Since $\Gamma[N] = \Gamma_1[N] \cup \Gamma_2[N]$ this concludes the proof of Lemma 13.

In the proof of Theorem 1 we shall also make use of the following result.

Lemma 14 *For any pair (b, d) with $1 \leq b, d \leq N$, there is at most one pair (a, c) such that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in T[N]$.*

Proof of Lemma 14. Assume that $\left[\begin{smallmatrix} a_1 & b \\ c_1 & d \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a_2 & b \\ c_2 & d \end{smallmatrix}\right]$ belong to $T[N]$, with $1 \leq a_1 < a_2 \leq N$, $1 \leq c_1 < c_2 \leq N$ and $1 \leq b, d \leq N$. Then we have $a_1 b - c_1 d = 1$, $a_2 b - c_2 d = 1$, and there exists a positive integer r such that $a_2 = a_1 + rd$ and $c_2 = c_1 + rb$. But then $1 \leq d + a_1, b + c_1 \leq N$ and, contrary to hypothesis, (b, d) was not the largest solution to $a_1 d - bc_1 = 1$. This concludes the proof.

3 Application to $\mathrm{SL}_2(\mathbf{Z})$

In this section we address the problem of finding a lower bound for $\epsilon(S, A)$ in the special case where $S = S_x = T[N] \cdot x$ for a point $x \in \mathbf{P}^1 \times \mathbf{P}^1 \setminus \{(1,0) \times (0,1), (0,1) \times (1,0)\}$ and where $A = \mathcal{O}(1,1)$. As discussed in the introduction, our goal is to view this problem *variationally* with respect to x. Since $\epsilon(S_x, A)$ can get smaller at special points, it would be helpful to have a lower bound on $\epsilon(S_x, A)$ for as large a set of points x as possible. We will focus on those x's of the form $x = (1,1) \times (z,1)$ and $x = (z,1) \times (1,1)$ because $T[N] \cdot x$, in these cases, resembles a two dimensional lattice for which one can easily compute the Seshadri constant. Corollary 11 then allows us to extend this bound to an open set of $x \in \mathbf{P}^1 \times \mathbf{P}^1 \setminus \{(1,0) \times (0,1), (0,1) \times (1,0)\}$.

For the remainder of this paper we will use the following notation. We let $X = \mathbf{P}^1 \times \mathbf{P}^1$, $T = X \setminus \{(1,0) \times (0,1), (0,1) \times (1,0)\}$, and we denote by A the ample line bundle $\mathcal{O}(1,1)$ on X. For each $M \in \mathrm{SL}_2(\mathbf{Z})$ with non–negative entries, we let

$$\Gamma_M = \left\{ (x_0^a y_0^b, x_1^a y_1^b) \times (x_0^c y_0^d, x_1^c y_1^d) \times (x_0, x_1) \times (y_0, y_1) : (x_0, x_1) \times (y_0, y_1) \in T \right\}.$$

We write

$$V = \cup_{M \in T[N]} \Gamma_M.$$

Denote by $p_1 : X \times T \to X$ and $p_2 : X \times T \to T$ the projections to the corresponding factors. We will let X_t denote the fibre $p_2^{-1}(t)$ of the second projection. Let $\pi : Z \to X \times T$ be the blow–up of $X \times T$ along V with exceptional divisor E.

We will deduce Theorem 1 from Corollary 11 using the following result.

Proposition 15 *For any $t \in (1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$ the line bundle*

$$\pi^*(\mathcal{O}(N+1, N+1, 0, 0))(-E)|(p_2 \pi)^{-1}(t)$$

is ample.

Proof that Proposition 15 implies Theorem 1. Recall that $A = \mathcal{O}(1,1)$. Granting Proposition 15 we know that for any $t \in (1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$ the line bundle $(p_1\pi)^*(A)(-\frac{1}{N+1}E)|(p_2\pi)^{-1}(t)$ is ample. Using the commutative diagram given before the statement of Corollary 11, with X_t in place of Y_b and $(p_2\pi)^{-1}(t)$ in place of $\tilde{Y}_b$ we see, using the remark following Definition 4, that

$$\epsilon(V \cap p_2^{-1}(t), A) > \frac{1}{N+1} \tag{5}$$

for all $t \in (1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$: note that the Seshadri constant is computed on the variety X_t. By Corollary 11 applied with $Y = X$ and $B = T$, there is a Zariski closed subset $\Sigma_N \subset T$ which does not meet $(1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$ and such that (5) holds for all $t \in T \backslash \Sigma_N$. Since $(1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$ is an ample divisor on $\mathbf{P}^1 \times \mathbf{P}^1$ it follows that Σ_N is a finite set. Suppose, then, that $t \in T \backslash \Sigma_N$ and let $\tilde{X}_t = (p_2\pi)^{-1}(t)$ denote the strict transform of X_t in Z. By (5), $(p_1\pi)^*(A)\left(-\frac{1}{N+1}E\right)$ is ample on $\tilde{X}_t$ and in particular if D is any effective divisor on $\tilde{X}_t$ then

$$(p_1\pi)^*(A)\left(-\frac{1}{N+1}E\right) \cdot D > 0. \tag{6}$$

Suppose now that $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ is a curve containing $T[N] \cdot t$ where $t \in T \backslash \Sigma_N$. Viewing $\mathbf{P}^1 \times \mathbf{P}^1$ as the fibre X_t with strict transform $\tilde{X}_t$ in Z, we let $\tilde{C}$ denote the strict transform of C in $\tilde{X}_t$. Then we find

$$\begin{aligned}
(p_1\pi)^*(A)\left(-\frac{1}{N+1}E\right) \cdot \tilde{C} &= (p_1\pi)^*(A) \cdot \tilde{C} + \mathcal{O}_Z\left(-\frac{1}{N+1}E\right) \cdot \tilde{C} \\
&\leq \deg_A(C) - \frac{1}{N+1} \sum_{y \in T[N] \cdot t} \operatorname{mult}_y(C) \\
&\leq d_1 + d_2 - \frac{|T[N] \cdot t|}{N+1}
\end{aligned}$$

where the first inequality uses [3] Chapter V Propositions 3.1 and 3.6 and the second inequality uses the hypothesis that $T[N] \cdot t \subset C$. Applying (6) to $\tilde{C}$ yields Theorem 1. Note that the reason why the first inequality above is not necessarily an equality is that if $t \in T$ is a point for which $(p_2\pi)^{-1}(t) \cap V$ has cardinality strictly less than $|T[N]|$ then $E|(p_2\pi)^{-1}(t)$ will no longer be the exceptional divisor for the blow–up of X_t along the point set $(p_2\pi)^{-1}(t)$ but rather along a "thicker" subscheme with the same support. The impact of blowing up along a thicker subcheme will be that $E \cdot \tilde{C}$ will be potentially *larger* than $\operatorname{mult}_y(C)$.

To illustrate the ideas of the proof of Proposition 15, we begin by showing that Proposition 15 holds for most points $t \in (1,1) \times \mathbf{P}^1 \cup \mathbf{P}^1 \times (1,1)$. In particular, we consider points t such that $|T[N] \cdot t| = |T[N]|$.

Lemma 16 *Suppose z is non–zero and not a root of unity and let $x = (1,1) \times (z,1)$. Then*

$$\epsilon(S_x, A) \geq \frac{1}{N}.$$

Proof of Lemma 16. Note that

$$S_x = T[N] \cdot x \subset \{(z^b, 1) \times (z^d, 1) |\, \text{there exists } 0 < a, b, c, d \leq N \text{ with } ad - bc = 1\}.$$

In particular let

$$R_x = \{(z^i, 1) \times (z^j, 1) |\, 0 < i, j \leq N\}.$$

Then $S_x \subset R_x$. Consider the following sections:

$$\sigma_1 = \prod_{i=1}^{N} (X_0 - z^i X_1) \in H^0 \left(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(N, 0) \right),$$
$$\sigma_2 = \prod_{j=1}^{N} (Y_0 - z^j Y_1) \in H^0 \left(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(0, N) \right),$$

where X_0, X_1 are projective coordinates on the first copy of $\mathbf{P}^1$ and Y_0, Y_1 are projective coordinates on the second copy. Both σ_1 and σ_2 vanish to order 1 along R_x, have distinct tangent directions at each point of R_x, and have no common non–isolated zeroes outside of R_x. The same is true of the sections

$$\sigma_1 \otimes Y_0^N, \sigma_1 \otimes Y_1^N, X_0^N \otimes \sigma_2, X_1^N \otimes \sigma_2 \in H^0 \left(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(N, N) \right).$$

Let $\psi : W \to \mathbf{P}^1 \times \mathbf{P}^1$ be the blow–up of S_x with exceptional divisor E_x. Then

$$\psi^*(\sigma_1 \otimes Y_0^N)(-E_x), \psi^*(\sigma_1 \otimes Y_1^N)(-E_x), \psi^*(X_0^N \otimes \sigma_2)(-E_x), \psi^*(X_1^N \otimes \sigma_2)(-E_x)$$

are sections in $H^0 (W, \psi^* \mathcal{O}(N, N)(-E_x))$ whose only positive dimensional common zeroes lie on fibres of ψ. Since $\psi^*(\mathcal{O}(N, N))(-E_x)$ is ψ–ample it follows that $\psi^*(\mathcal{O}(N, N))(-E_x)$ is nef and hence, using Definition 4,

$$\epsilon(R_x, \mathcal{O}(N, N)) \geq 1.$$

Thus $\epsilon(R_x, \mathcal{O}(1, 1)) \geq \frac{1}{N}$ and, since $S_x \subset R_x$, this shows that $\epsilon(S_x, \mathcal{O}(1, 1)) \geq \frac{1}{N}$, concluding the proof of Lemma 16.

Note that Lemma 16 is definitely true for *any* $x \in (1, 1) \times \mathbf{P}^1$ or $x \in \mathbf{P}^1 \times (1, 1)$ but this is of no help in establishing Proposition 15 because if $t = (1, 1) \times (z, 1)$ where z is a root of unity then $(p_2 \pi)^{-1}(t)$ is the blow–up of X_t along $V \cap X_t$ but this scheme is reduced only when the cardinality of $V \cap X_t$ is $|T[N]|$. This non–reduced information can be encoded by blowing up X_t along $V \cap X_t$ within the entire family of $t \in (1, 1) \times \mathbf{P}^1$ or $t \in \mathbf{P}^1 \times (1, 1)$. Note that the reason why one can *not* simply work on the entire base T is that the explicit sections used in Lemma 16 do *not* extend well to sections over the whole base.

With this in mind, we let $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times (1, 1) \times \mathbf{P}^1$ and denote by $\pi_Y : Z_Y \to Y$ the blow–up of Y along $V \cap Y$, with exceptional divisor E_Y. For any pair (b, d) with $1 \leq b, d \leq N$, there is at most one pair (a, c) such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in T[N]$ (see Lemma 14), so that $V \cap Y$

is a reduced scheme. This would be false with $\Gamma[N]$ instead of $T[N]$ in the definition of V; this is the reason why we are not able to prove Theorem 1 with $\Gamma[N]$.

For $x \in \mathbf{P}^1$ let $\mathbf{P}_x = \mathbf{P}^1 \times \mathbf{P}^1 \times (1,1) \times \{x\} \subset Y$. Let $\tilde{\mathbf{P}}_x$ be the strict transform of $\mathbf{P}_x$ in Z_Y and $p_x : \tilde{\mathbf{P}}_x \to \mathbf{P}_x$ the induced map.

We will deduce Proposition 15 from the following lemma.

Lemma 17 *For any $x \in \mathbf{P}^1$, $\pi_Y^* \mathcal{O}(1,1,0)(-\frac{1}{N+1}E_Y)|\tilde{\mathbf{P}}_x$ is ample.*

Proof of Lemma 17. We will first show that the restriction of the line bundle $L = \pi_Y^* \mathcal{O}(1,1,0)(-\frac{1}{N}E_Y)$ to $\tilde{\mathbf{P}}_x$ is nef, or equivalently, that $\pi_Y^* \mathcal{O}(N,N,0)(-E_Y)|\tilde{\mathbf{P}}_x$ is nef. We will parametrize the sections σ_1 and σ_2 used in the proof of Lemma 16, allowing the point $(z,1)$ to vary. More specifically, on $Y = \mathbf{P}^1 \times \mathbf{P}^1 \times (1,1) \times \mathbf{P}^1$ let X_0, X_1 be coordinates on the first factor, Y_0, Y_1 coordinates on the second factor, and Z_0, Z_1 coordinates on the final factor. In place of σ_1 and σ_2 from Lemma 16, we consider

$$\tau_1 = \prod_{i=1}^{N} (Z_1^i X_0 - Z_0^i X_1) \in H^0\left(Y, \mathcal{O}(N,0,m)\right),$$
$$\tau_2 = \prod_{j=1}^{N} (Z_1^j Y_0 - Z_0^j Y_1) \in H^0\left(Y, \mathcal{O}(0,N,m)\right),$$

where $m = N(N+1)/2$. Note that $\tau_1|\mathbf{P}^1 \times \mathbf{P}^1 \times (1,1) \times (z,1) = \sigma_1$ and similarly for τ_2 and σ_2.

Let $\tilde{\tau}_{1,1}, \tilde{\tau}_{1,2}$ be the sections of $\pi_Y^* \mathcal{O}(N,N,m)(-E_Y)$ determined by the section τ_1 as in Lemma 16 and let $\tilde{\tau}_{2,1}$ and $\tilde{\tau}_{2,2}$ be the sections of $\pi_Y^* \mathcal{O}(N,N,m)(-E_Y)$ determined by τ_2. For any $x \in (1,1) \times \mathbf{P}^1$, $\tilde{\tau}_{1,1}$, $\tilde{\tau}_{1,2}$, $\tilde{\tau}_{2,1}$, and $\tilde{\tau}_{2,2}$, restricted to $\tilde{\mathbf{P}}_x$, have no non–isolated common zeroes away from the exceptional divisor of $p_x : \tilde{\mathbf{P}}_x \to \mathbf{P}_x$. Thus $\pi_Y^* \mathcal{O}_Y(N,N,m)(-E_Y)|\tilde{\mathbf{P}}_x$ is nef away from the exceptional divisor. We also know that $\pi_Y^* \mathcal{O}_Y(N,N,m)(-E_Y)$ is π_Y–ample. Since $\pi_Y^* \mathcal{O}_Y(0,0,1)|\tilde{\mathbf{P}}_x$ is trivial, it follows that $\pi_Y^* \mathcal{O}_Y(N,N,0)(-E_Y)$ is nef on $\tilde{\mathbf{P}}_x$ for all $x \in \mathbf{P}^1$.

Since $\mathcal{O}_{\mathbf{P}_x}(1,1)$ is ample it follows that $\mathcal{O}_{\tilde{\mathbf{P}}_x}(N+1,N+1,0)(-E_Y)$ is big (see [5] §2.2) and nef. Moreover, if $C \subset E_Y$ then $\pi_Y^* \mathcal{O}_Y(N+1,N+1,0)(-E_Y) \cdot C > 0$ since $\mathcal{O}_Y(-E_Y)$ is π_Y–ample. Also if $C \subset \tilde{\mathbf{P}}_x$ is not contained in E_Y then

$$\pi_Y^* \mathcal{O}_Y(1,1,0) \cdot C = \deg_{\mathcal{O}_Y(1,1,0)}(\pi_Y(C)) > 0$$

and thus again $\pi_Y^* \mathcal{O}_Y(N+1,N+1,0)(-E_Y) \cdot C > 0$. By the Nakai–Moishezon criterion for ampleness, [3] Theorem 1.10, $\mathcal{O}_{\tilde{\mathbf{P}}_x}(N+1,N+1,0)(-E_Y)$ is ample and this concludes the proof of Lemma 17.

Proof of Proposition 15. Recall that $\pi : Z \to X \times T$ is the blow–up of $X \times T$ along $V = \cup_{M \in T[N]} \Gamma_M$ with exceptional divisor E. Let $L = \pi^*(\mathcal{O}(N+1,N+1,0,0))(-E)$ and consider the following commutative diagram where $p_x : \tilde{\mathbf{P}}_x \to \mathbf{P}_x$ is the blow–up of $\mathbf{P}_x$

considered in Lemma 17:

$$\begin{tikzcd} \tilde{\mathbf{P}}_x \arrow[r, "j"] \arrow[d, "p_x"'] & Z \arrow[d, "\pi"] \\ \mathbf{P}_x \arrow[r, "i"] & X \times T \end{tikzcd}$$

By Lemma 17 $L|\tilde{\mathbf{P}}_x$ is ample. Since i and j are both inclusions it follows that $L|(p_2\pi)^{-1}(t)$ is ample thereby concluding the proof of Proposition 15 when $t = (1,1) \times x$. To deal with the case $t = x \times (1,1)$, it is enough to observe that Lemma 17 is also true if Y is replaced by $Y' = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times (1,1)$ and if $\mathbf{P}_x$ is replaced by $\mathbf{P}'_x = \mathbf{P}^1 \times \mathbf{P}^1 \times \{x\} \times (1,1)$. Indeed, the proof of Lemma 17 can be repeated verbatim in this situation, the only change being that the coordinates z_0, z_1 used to produce the section τ_1 and τ_2 need to be taken on the third factor of $\mathbf{P}^1$ instead of the fourth factor.

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S. Fischler, Univ Paris-Sud, Laboratoire de Mathématiques d'Orsay, Orsay Cedex, F-91405; CNRS, Orsay cedex, F-91405, France. Email: stephane.fischler@math.u-psud.fr

M. Nakamaye, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131. Email: nakamaye@math.unm.edu