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Restricted rational approximation and Apéry-type constructions

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ABSTRACT

Let ξ be a real irrational number, and φ be a function (satisfying some assumptions). In this text we study the *φ-exponent of irrationality* of ξ, defined as the supremum of the set of μ for which there are infinitely many $q \geqslant 1$ such that q is a multiple of $\varphi(q)$ and $|\xi - \frac{p}{q}| \leqslant q^{-\mu}$ for some $p \in \mathbb{Z}$. We obtain general results on this exponent (a lower bound, the Haussdorff dimension of the set where it is large, ...) and connect it to sequences of small linear forms in 1 and ξ with integer coefficients, with geometric behaviour and a divisibility property of the coefficients. Using Apéry's proof that $\zeta(3)$ is irrational, we obtain an upper bound for the φ-exponent of irrationality of $\zeta(3)$, for a given φ.

1. INTRODUCTION

Apéry has proved [2] (see also [9] for a survey) that for $\xi = \zeta(3)$, $\alpha = e^3(1+\sqrt{2})^{-4} < 1$, and $\beta = e^3(1+\sqrt{2})^4 > 1$, the following holds:

$$(1.1)\qquad \begin{cases} \text{There exist two integer sequences } (u_n)_{n\geqslant 1} \text{ and } (v_n)_{n\geqslant 1} \text{ such that} \\ u_n \geqslant 0,\ |u_n\xi - v_n|^{1/n} \to \alpha \text{ and } u_n^{1/n} \to \beta \end{cases}$$

and also, with $\delta_n = d_n^3$ where $d_n = \mathrm{lcm}(1, 2, \ldots, n)$:

$$(1.2)\qquad \delta_n \text{ divides } u_n \text{ for any } n \geqslant 1.$$

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Since $\alpha < 1$, (1.1) implies the irrationality of $\zeta(3)$. It is well known that

$$\text{(1.3)} \qquad \text{(1.1) implies } \mu(\xi) \leqslant 1 - \frac{\log \beta}{\log \alpha},$$

where $\mu(\xi)$ is the exponent of irrationality of ξ, so that $\mu(\zeta(3)) \leqslant 13.4178202\ldots$. It is proved in [11] (together with more precise results connected to [10]) that, conversely, for $\xi \in \mathbb{R} \setminus \mathbb{Q}$,

$$\text{(1.4)} \qquad \text{if } \mu(\xi) < 1 - \frac{\log \beta}{\log \alpha} \text{ with } 0 < \alpha < 1 < \beta, \text{ then (1.1) holds.}$$

In this paper, we generalize both implications (1.3) and (1.4), to take into account the divisibility property (1.2), under the assumption that δ_n divides δ_{n+1} for any $n \geqslant 1$ (which is the case in Apéry's construction since $\delta_n = d_n^3$). For instance, Apéry's construction implies the following result (the analogue of which, where $\zeta(3)$ is replaced with $\log(2)$, has been proved by Dubitskas [6] in a stronger version, see further):

Theorem 1. *For any $\varepsilon > 0$, there are only finitely many integers $q \geqslant 1$ satisfying both*

$$\left|\zeta(3) - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}} \quad \textit{for some } p \in \mathbb{Z}$$

and

$$d_n^3 \textit{ divides } q, \quad \textit{with } n = \left[\frac{\log q}{\log((1+\sqrt{2})^4)}\right].$$

To state our results more precisely, let us denote by $\mathcal{E}$ the set of all functions $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ (with $\mathbb{N}^* = \{1, 2, 3, \ldots\}$) such that:

- For any $q \geqslant 1$, $\varphi(q+1)$ is a multiple of $\varphi(q)$.
- The limit $\gamma_\varphi := \lim_{q\to\infty} \frac{\log \varphi(q)}{\log q}$ exists and satisfies $0 \leqslant \gamma_\varphi < 1$.

The following definition generalizes that of the usual exponent of irrationality $\mu(\xi)$ (which is obtained as a special case when φ is the function $\mathbf{1}$ defined by $\mathbf{1}(q) = 1$ for any q).

Definition 1. For $\varphi \in \mathcal{E}$ and $\xi \in \mathbb{R} \setminus \mathbb{Q}$, the *$\varphi$-exponent of irrationality* of ξ is the supremum, denoted by $\mu_\varphi(\xi)$, of the set of real numbers μ for which there are infinitely many $q \geqslant 1$ such that

$$q \text{ is a multiple of } \varphi(q) \quad \text{and} \quad \left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{q^\mu} \quad \text{for some } p \in \mathbb{Z}.$$

Of course, when this set is $\mathbb{R}$, we have $\mu_\varphi(\xi) = +\infty$.

If we let $\varphi(q) = d_n^3$ where $n = [\frac{\log q}{\log((1+\sqrt{2})^4)}]$, then $\varphi \in \mathcal{E}$ and Theorem 1 means that $\mu_\varphi(\zeta(3)) \leqslant 2$.

In the case of $\log(2)$, Dubitskas' result implies $\mu_\varphi(\log(2)) \leqslant 2$ where $\varphi(q) = d_n$ with $n = [\frac{\log q}{\log(3+2\sqrt{2})}]$. On the other hand, Rivoal has proved [21] that

$$\left|\log(2) - \frac{p}{2^n d_n}\right| \geqslant \frac{1}{(2^n d_n)^{1.948967}} \quad \text{for any } p \in \mathbb{Z} \text{ and } n \text{ sufficiently large,}$$

so that only finitely many convergents in the continued fraction expansion of $\log(2)$ have a denominator of the form $2^n d_n$. Maybe Rivoal's methods (which apply also to $\log(r)$ for other positive rational numbers r) can lead to upper bounds less than 2 for $\mu_\varphi(\log(r))$, for suitable $\varphi \in \mathcal{E}$ and $r \in \mathbb{Q}$, $r > 0$.

The main result of this paper is the following generalization of (1.3) and (1.4), which allows one to deduce Theorem 1 from Apéry's construction:

Theorem 2. *Let $\xi \in \mathbb{R}\setminus\mathbb{Q}$, $0 < \alpha < 1$, $\beta > 1$, and $(\delta_n)_{n\geqslant 1}$ be a sequence of positive integers such that δ_n divides δ_{n+1} for any $n \geqslant 1$, and $\delta_n^{1/n}$ tends to δ as $n \to \infty$.*

Define a function $\varphi \in \mathcal{E}$ by

$$\varphi(q) = \delta_n \quad \text{with } n = \left[\frac{\log q}{\log(\delta/\alpha)}\right].$$

Then we have the following implications:

(i) *If* (1.1) *and* (1.2) *hold then* $\mu_\varphi(\xi) \leqslant \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}$.
(ii) *If* $\mu_\varphi(\xi) < \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}$ *then* (1.1) *and* (1.2) *hold.*

We also prove various results (of independent interest) on the φ-exponent of irrationality $\mu_\varphi(\xi)$, namely:

- For any $\xi \in \mathbb{R}\setminus\mathbb{Q}$, we have $\mu_\varphi(\xi) \geqslant 2 - \gamma_\varphi$, and equality holds for almost any ξ with respect to Lebesgue measure.
- For $\mu > 2 - \gamma_\varphi$, the set of real numbers $\xi \in \mathbb{R}\setminus\mathbb{Q}$ such that $\mu_\varphi(\xi) \geqslant \mu$ has Hausdorff dimension $\frac{2-\gamma_\varphi}{\mu}$.
- For any $\xi \in \mathbb{R}\setminus\mathbb{Q}$, we have $\mu_\varphi(\xi) = +\infty$ if, and only if, $\mu(\xi) = +\infty$ (that is, if and only if ξ is a Liouville number).

In the case of $\zeta(3)$, we obtain the following result as a consequence of Theorem 1 and this Hausdorff dimension computation.

Corollary 1. *For any $q \geqslant 1$, let $\varphi(q) = d_n^3$ where $n = [\frac{\log q}{\log((1+\sqrt{2})^4)}]$. Let $\mathcal{S}$ denote the set of all $\xi \in \mathbb{R}\setminus\mathbb{Q}$ such that $\mu_\varphi(\xi) > 2$. Then $\zeta(3) \notin \mathcal{S}$ and $\mathcal{S}$ has Hausdorff dimension* $0.5745\ldots$.

As far as we know, this is the largest known Hausdorff dimension for a subset of $\mathbb{R}$, defined by Diophantine conditions, which does not contain $\zeta(3)$. It is worthwile noticing that variants of Apéry's construction (due to Hata, and Rhin and Viola, ...) give better bounds for the (usual) irrationality exponent $\mu(\zeta(3))$, but do not seem to allow any improvement on Corollary 1 (see Section 4).

In this text, we consider asymptotic estimates like (1.1) since these can be easily used to work with exponents like $\mu_\varphi(\xi)$. However, in the case of $\zeta(3)$ (and also $\zeta(2)$ and $\log 2$), more precise estimates are known. They enable us to prove the following result, which refines Theorem 1 and is analogous to Dubitskas' result [6] for $\log 2$.

Theorem 3. *There exists $c > 0$ such that for any $q \geqslant 1$ and any $p \in \mathbb{Z}$ we have*

$$\left|\zeta(3) - \frac{p}{q}\right| \geqslant \frac{c(\log q)^3}{q^2}$$

provided that

$$d_n^3 \text{ divides } q, \quad \text{with } n = \left[\frac{\log q}{\log((1+\sqrt{2})^4)}\right].$$

Corollary 2. *Only finitely many convergents p/q in the continued fraction expansion of $\zeta(3)$ are such that d_n^3 divides q, with $n = [\frac{\log q}{\log((1+\sqrt{2})^4)}]$.*

The structure of this text is as follows. In Section 2, we prove the general results stated above (and some others) about the φ-exponent of irrationality $\mu_\varphi(\xi)$. In Section 3, we give a proof of Theorem 2. Finally, in Section 4 we apply our results to particular numbers ξ, especially $\zeta(3)$, $\zeta(2)$, and $\log 2$, and in Section 5 we prove Theorem 3 and Corollary 2.

2. GENERAL PROPERTIES

The definition of $\mu_\varphi(\xi)$ makes sense because for any $\varphi \in \mathcal{E}$ there are infinitely many integers q such that q is a multiple of $\varphi(q)$. More precise statements are given in the proofs of Lemmas 1 and 2 and also in the statement of Proposition 1.

Let us start with examples of functions in $\mathcal{E}$. Let $b_1, \ldots, b_r \geqslant 2$ be pairwise distinct integers, and $\varepsilon_1, \ldots, \varepsilon_r > 0$ be such that $\sum_{i=1}^r \varepsilon_i \log b_i < 1$. Then the function defined by $\varphi(q) = \prod_{i=1}^r b_i^{[\varepsilon_i \log q]}$ belongs to $\mathcal{E}$, and satisfies $\gamma_\varphi = \sum_{i=1}^r \varepsilon_i \log b_i$.

2.1. A lower bound

The following lemma generalizes the lower bound $\mu(\xi) \geqslant 2$ which holds for any $\xi \in \mathbb{R} \setminus \mathbb{Q}$. The proof uses the same tool as Dirichlet's proof, namely the pigeonhole principle.

Lemma 1. *For any $\varphi \in \mathcal{E}$ and any $\xi \in \mathbb{R} \setminus \mathbb{Q}$, we have $\mu_\varphi(\xi) \geqslant 2 - \gamma_\varphi$.*

Proof. Let $\varepsilon > 0$, and Q be sufficiently large in terms of ε. Consider, for $0 \leqslant n \leqslant [Q/\varphi(Q)]$, the fractional part of $n\varphi(Q)\xi$. Since $\xi \notin \mathbb{Q}$, this gives $[Q/\varphi(Q)] + 1$ pairwise distinct points in [0, 1]. Thanks to the pigeonhole principle, two of them lie within a distance less than or equal to $[Q/\varphi(Q)]^{-1}$. The difference of the corresponding integers n yields an integer m, with $1 \leqslant m \leqslant Q/\varphi(Q)$, such that $|m\varphi(Q)\xi - p| \leqslant [Q/\varphi(Q)]^{-1}$ for some $p \in \mathbb{Z}$. Now let $q = m\varphi(Q)$. Then $q \leqslant Q$ so that $\varphi(q)$ divides $\varphi(Q)$, and also $\varphi(Q)$ divides q by definition of q. Finally $\varphi(q)$ divides q, and since Q is sufficiently large in terms of ε we have:

$$\left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{q}\left[\frac{Q}{\varphi(Q)}\right]^{-1} \leqslant \frac{1}{qQ^{1-\gamma_\varphi-\varepsilon}} \leqslant \frac{1}{q^{2-\gamma_\varphi-\varepsilon}}.$$

This concludes the proof of Lemma 1. □

2.2. Comparisons between $\mu_\varphi(\xi)$ for various φ

In this section, we show how $\mu_\varphi(\xi)$ and $\mu_{\varphi'}(\xi)$ are connected for $\varphi, \varphi' \in \mathcal{E}$. It is specially interesting when $\varphi' = \mathbf{1}$ since in this case $\mu_{\varphi'}(\xi)$ is the classical exponent of irrationality $\mu(\xi)$.

Let us start with the following remark.

Remark 1. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $\varphi \in \mathcal{E}$. Then $\frac{\mu_\varphi(\xi)}{1-\gamma_\varphi}$ is the supremum of the set of μ for which there are infinitely many q such that

$$q \text{ is a multiple of } \varphi(q) \quad \text{and} \quad \left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{(q/\varphi(q))^\mu} \quad \text{for some } p \in \mathbb{Z}.$$

The proof of this fact is easy, since for any $\varepsilon > 0$ and any q sufficiently large in terms of ε we have $q^{1-\gamma_\varphi-\varepsilon} \leqslant q/\varphi(q) \leqslant q^{1-\gamma_\varphi+\varepsilon}$.

This remark is crucial in the proof (given further) of the following lemma.

Lemma 2. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, and $\varphi, \varphi' \in \mathcal{E}$ be such that $\varphi'(q)$ divides $\varphi(q)$ for any $q \geqslant 1$. Then $\mu_{\varphi'}(\xi)$ is finite if, and only if, $\mu_\varphi(\xi)$ is finite, and in this case we have:*

$$\frac{1-\gamma_\varphi}{1-\gamma_{\varphi'}}\mu_{\varphi'}(\xi) \leqslant \mu_\varphi(\xi) \leqslant \mu_{\varphi'}(\xi).$$

Letting $\varphi' = \mathbf{1}$, we obtain the following corollary.

Corollary 3. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ and $\varphi \in \mathcal{E}$. Then $\mu_\varphi(\xi)$ is infinite if, and only if, $\mu(\xi)$ is infinite (that is, if and only if ξ is a Liouville number). Otherwise we have*

$$(1-\gamma_\varphi)\mu(\xi) \leqslant \mu_\varphi(\xi) \leqslant \mu(\xi).$$

Remark 2. Let $\varphi \in \mathcal{E}$ be such that $\gamma_\varphi = 0$. Then for any ξ we have $\mu_\varphi(\xi) = \mu(\xi)$ so that μ_φ is nothing but the usual exponent of irrationality. More generally, if $\varphi, \varphi' \in \mathcal{E}$

are such that $\varphi'(q)$ divides $\varphi(q)$ for any $q \geqslant 1$, and $\gamma_\varphi = \gamma_{\varphi'}$, then Lemma 2 shows that $\mu_\varphi(\xi) = \mu_{\varphi'}(\xi)$ for any ξ.

Proof of Lemma 2. If q is a multiple of $\varphi(q)$ then q is a multiple of $\varphi'(q)$, so the second inequality is trivial. Let us prove the first one, that is $\frac{\mu_{\varphi'}(\xi)}{1-\gamma_{\varphi'}} \leqslant \frac{\mu_\varphi(\xi)}{1-\gamma_\varphi}$. This follows immediately from Remark 1 and the following fact. *If $\varphi, \varphi' \in \mathcal{E}$ are such that $\varphi'(q)$ divides $\varphi(q)$ for any $q \geqslant 1$, and if $q' \geqslant 1$ is such that $\varphi'(q')$ divides q', then there exists an integer multiple q of q' such that $\varphi(q)$ divides q and $q/\varphi(q) = q'/\varphi'(q')$.* To prove this fact, we let q be the least integer such that $q \geqslant q'$ and $\frac{q'}{\varphi'(q')}\varphi(q) \leqslant q$. Such an integer exists since $\varphi \in \mathcal{E}$. If $q = q'$ then $\varphi(q') \leqslant \varphi'(q')$ so that $\varphi(q') = \varphi'(q')$, and the conclusion holds with $q = q'$. Otherwise, we have $q > q'$ and, by minimality, $q \geqslant \frac{q'}{\varphi'(q')}\varphi(q) \geqslant \frac{q'}{\varphi'(q')}\varphi(q-1) > q-1$ so that $q = \frac{q'}{\varphi'(q')}\varphi(q)$. Now $q' < q$ implies that $\varphi(q')$ divides $\varphi(q)$; since $\varphi'(q')$ divides $\varphi(q')$, we obtain that $q = q'\frac{\varphi(q)}{\varphi'(q')}$ is a multiple of q'. This concludes the proof of the fact, and that of Lemma 2. □

2.3. A special set of functions

In this subsection, we focus on specific functions $\varphi \in \mathcal{E}$, of major importance to us since they are the ones involved in Theorem 2. Actually, since our interest lies only on the exponents of irrationality $\mu_\varphi(\xi)$, Lemma 4 below shows that we do not lose anything by considering only these functions (and even only a part of them). Let us start by the following lemma, in which these functions φ are defined. We omit the proof, since it is very easy.

Lemma 3. *Let $(\delta_n)_{n\geqslant 1}$ be a sequence of positive integers such that δ_n divides δ_{n+1} for any $n \geqslant 1$, and $\delta_n^{1/n}$ tends to δ as $n \to \infty$. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < \delta$. Define a function φ by*

$$\varphi(q) = \delta_n \quad \text{with } n = \left[\frac{\log q}{\log(\delta/\alpha)}\right].$$

Then we have $\varphi \in \mathcal{E}$ and $\gamma_\varphi = \frac{\log\delta}{\log(\delta/\alpha)}$.

When $\alpha = \delta/2$, the definition of φ in this lemma means

$$\varphi(q) = \delta_n = \varphi(2^n) \quad \text{when } 2^n \leqslant q < 2^{n+1}.$$

The following lemma shows that we would not lose too much by considering only functions φ obtained in this way. The number 2 in 2^n is not important, it could be replaced with any other number greater than one.

Lemma 4. *Let $\varphi \in \mathcal{E}$. Define a function $\widetilde{\varphi}$ by letting, for any integers $q \geqslant 1$ and $n \geqslant 0$:*

$$\widetilde{\varphi}(q) = \varphi(2^n) \quad \text{when } 2^n \leqslant q < 2^{n+1}.$$

Then we have $\widetilde{\varphi} \in \mathcal{E}$, $\gamma_{\widetilde{\varphi}} = \gamma_\varphi$, and $\mu_\varphi(\xi) = \mu_{\widetilde{\varphi}}(\xi)$ for any ξ.

Proof. The properties $\widetilde{\varphi} \in \mathcal{E}$ and $\gamma_{\widetilde{\varphi}} = \gamma_\varphi$ are obvious, and $\mu_\varphi(\xi) = \mu_{\widetilde{\varphi}}(\xi)$ follows from Remark 2 since $\widetilde{\varphi}(q)$ divides $\varphi(q)$ for any $q \geqslant 1$. $\square$

The function $\widetilde{\varphi}$ of Lemma 4 is useful to prove the following statement, which shows "how many" integers q are multiples of $\varphi(q)$. This statement will be helpful in the proof of metric results in Section 2.4.

Proposition 1. *Let $\varphi \in \mathcal{E}$, and denote by $\mathcal{Q}_\varphi$ the set of all $q \geqslant 1$ such that $\varphi(q)$ divides q. Then for any $\varepsilon > 0$, the series $\sum_{q \in \mathcal{Q}_\varphi} q^{-1+\gamma_\varphi-\varepsilon}$ is convergent and the series $\sum_{q \in \mathcal{Q}_\varphi} q^{-1+\gamma_\varphi+\varepsilon}$ is divergent.*

Proof. Let $\varepsilon > 0$; we may assume $\varepsilon < 1 - \gamma_\varphi$. For any $n \geqslant 0$ and q such that $2^n \leqslant q < 2^{n+1}$, we let $\varphi_1(q) = \varphi(2^n)$ and $\varphi_2(q) = \varphi(2^{n+1})$. As in Lemma 4, we have $\varphi_1, \varphi_2 \in \mathcal{E}$ with $\gamma_{\varphi_1} = \gamma_{\varphi_2} = \gamma_\varphi$, and $\mathcal{Q}_{\varphi_2} \subset \mathcal{Q}_\varphi \subset \mathcal{Q}_{\varphi_1}$. This implies

$$\begin{aligned}
\sum_{q \in \mathcal{Q}_\varphi} q^{-1+\gamma_\varphi-\varepsilon} &\leqslant \sum_{q \in \mathcal{Q}_{\varphi_1}} q^{-1+\gamma_\varphi-\varepsilon} \\
&= \sum_{n \geqslant 0} \sum_{\substack{q \in \mathcal{Q}_{\varphi_1} \\ 2^n \leqslant q < 2^{n+1}}} q^{-1+\gamma_\varphi-\varepsilon} \\
&\leqslant \sum_{n \geqslant 0} \left(2^n\right)^{-1+\gamma_\varphi-\varepsilon} \frac{2^n}{\varphi(2^n)} < +\infty
\end{aligned}$$

since, for n sufficiently large, $(2^n)^{\gamma_\varphi-\varepsilon} < \frac{1}{n^2}\varphi(2^n)$. In the same way,

$$\begin{aligned}
\sum_{q \in \mathcal{Q}_\varphi} q^{-1+\gamma_\varphi+\varepsilon} &\geqslant \sum_{q \in \mathcal{Q}_{\varphi_2}} q^{-1+\gamma_\varphi+\varepsilon} \\
&= \sum_{n \geqslant 0} \sum_{\substack{q \in \mathcal{Q}_{\varphi_2} \\ 2^n \leqslant q < 2^{n+1}}} q^{-1+\gamma_\varphi+\varepsilon} \\
&\geqslant \sum_{n \geqslant 0} \left(2^{n+1}\right)^{-1+\gamma_\varphi+\varepsilon} \frac{2^n}{\varphi(2^{n+1})} = +\infty
\end{aligned}$$

since $(2^{n+1})^{\gamma_\varphi+\varepsilon} \geqslant \varphi(2^{n+1})$ for n sufficiently large. This concludes the proof of Proposition 1. $\square$

2.4. Metric results

Proposition 2. *Let $\varphi \in \mathcal{E}$. For almost any $\xi \in \mathbb{R}$ in the sense of Lebesgue measure, we have $\mu_\varphi(\xi) = 2 - \gamma_\varphi$.*

Proof. Thanks to Lemma 1, it is enough to prove that for any $\varepsilon > 0$ the set of all $\xi \in [0, 1]$ with $\mu_\varphi(\xi) > 2 - \gamma_\varphi + \varepsilon$ has Lebesgue measure 0. Now this set is contained, for any $q_0 \geqslant 1$, in the union of $[\frac{p}{q} - \frac{1}{q^{2-\gamma_\varphi+\varepsilon}}, \frac{p}{q} + \frac{1}{q^{2-\gamma_\varphi+\varepsilon}}]$ with $0 \leqslant p \leqslant q$ and $q \in \mathcal{Q}_\varphi$, $q \geqslant q_0$ (where $\mathcal{Q}_\varphi$ is defined in the statement of Proposition 1), so that is has measure less than or equal to

$$\sum_{\substack{q\in\mathcal{Q}_\varphi\\ q\geqslant q_0}} \frac{2(q+1)}{q^{2-\gamma_\varphi+\varepsilon}} \leqslant 4 \sum_{\substack{q\in\mathcal{Q}_\varphi\\ q\geqslant q_0}} \frac{1}{q^{1-\gamma_\varphi+\varepsilon}}.$$

Now Proposition 1 proves that this upper bound is finite, and tends to 0 as q_0 tends to infinity. This concludes the proof of Proposition 2. □

Proposition 3. *Let $\varphi \in \mathcal{E}$ and $\mu > 2 - \gamma_\varphi$. Then the set of all real numbers $\xi \in \mathbb{R}$ such that $\mu_\varphi(\xi) \geqslant \mu$ has Lebesgue measure zero and Hausdorff dimension $\frac{2-\gamma_\varphi}{\mu}$.*

In the special case $\varphi = \mathbf{1}$, we obtain the classical theorem of Jarník [15] and Besicovitch [3] stating that the set of $\xi \in \mathbb{R}$ such that $\mu(\xi) \geqslant \mu$ has Hausdorff dimension $2/\mu$ (see for instance [5], p. 104, Theorem 5.2, or Chapter 10 of [8]).

Proposition 3 follows immediately from Proposition 1 and the following theorem due to Borosh and Fraenkel [4].

Theorem 4 (Borosh–Fraenkel). *Let $\nu \in [0, 1]$, and $\mathcal{Q}$ be a subset of $\mathbb{N}^*$ such that, for any $\varepsilon > 0$, the series $\sum_{q\in\mathcal{Q}} q^{-\nu-\varepsilon}$ is convergent and the series $\sum_{q\in\mathcal{Q}} q^{-\nu+\varepsilon}$ is divergent. Let $\mu > \nu + 1$. Then the set of all $\xi \in \mathbb{R}$ for which there are infinitely many $q \in \mathcal{Q}$ such that*

$$\left|\xi - \frac{p}{q}\right| \leqslant \frac{1}{q^\mu} \qquad \textit{for some } p \in \mathbb{Z}$$

has Hausdorff dimension $\frac{\nu+1}{\mu}$.

Actually in [4] it is assumed that the series $\sum_{q\in\mathcal{Q}} q^{-\nu}$ is divergent, but this assumption is not necessary (see the remark before Lemma 2.1 of [22], p. 72).

3. A TRANSFERENCE THEOREM

In this section, we prove Theorem 2 stated in the Introduction. Let us recall the following from Lemma 3 stated in Section 2.3. Let $(\delta_n)_{n\geqslant1}$ be a sequence of positive integers such that δ_n divides δ_{n+1} for any $n \geqslant 1$, and $\delta_n^{1/n}$ tends to δ as $n \to \infty$. Let $\alpha \in \mathbb{R}$ be such that $0 < \alpha < \delta$. Define a function φ by

$$\varphi(q) = \delta_n \quad \text{with } n = \left[\frac{\log q}{\log(\delta/\alpha)}\right]. \tag{3.1}$$

Then we have $\varphi \in \mathcal{E}$ and $\gamma_\varphi = \frac{\log\delta}{\log(\delta/\alpha)}$.

We can now re-state Theorem 2 as follows.

Theorem 5. *Let $\xi \in \mathbb{R}\setminus\mathbb{Q}$, $0 < \alpha < 1$, $\beta > 1$, and $(\delta_n)_{n\geqslant 1}$ be a sequence of positive integers such that δ_n divides δ_{n+1} for any $n \geqslant 1$, and $\delta_n^{1/n}$ tends to δ as $n \to \infty$. Let $\varphi \in \mathcal{E}$ be the function defined by* (3.1). *Then the following implications hold.*

(i) *If there exist two integer sequences (u_n) and (v_n) such that $u_n \geqslant 0$, $|u_n\xi - v_n|^{1/n} \to \alpha$, $u_n^{1/n} \to \beta$ and δ_n divides u_n for any n, then we have*

$$\mu_\varphi(\xi) \leqslant \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}. \tag{3.2}$$

(ii) *If we have*

$$\mu_\varphi(\xi) < \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}$$

then there exist two integer sequences (u_n) and (v_n) such that $u_n \geqslant 0$, $|u_n\xi - v_n|^{1/n} \to \alpha$, $u_n^{1/n} \to \beta$ and δ_n divides u_n for any n.

In the special case where $\beta = \delta\beta_0$ and $\alpha = \delta/\beta_0$, we have $\frac{\log\beta-\log\alpha}{\log\delta-\log\alpha} = 2$. This is the situation with Apéry's construction for $\xi = \zeta(3)$ and $\xi = \zeta(2)$, and also with Alladi and Robinson's [1] for $\xi = \log 2$ (see Section 4 for more details). In this case, thanks to Lemma 3 and Proposition 3, the upper bound (3.2) means that ξ lies outside a set of Hausdorff dimension $1 - \frac{\log\delta}{2\log\beta_0}$ whereas the usual bound $\mu(\xi) \leqslant 1 - \frac{\log\beta}{\log\alpha}$ means that ξ lies outside a set of Hausdorff dimension $1 - \frac{\log\delta}{\log\beta_0}$. This means this Hausdorff dimension has come twice closer to 1.

Putting part (i) of Theorem 5 with the lower bound of Lemma 1, we obtain the following corollary which is a special case of the linear independence criteria of [12].

Corollary 4. *Let $\xi \in \mathbb{R}$, $0 < \alpha < 1$, $\beta > 1$, and $(\delta_n)_{n\geqslant 1}$ be a sequence of positive integers such that δ_n divides δ_{n+1} for any $n \geqslant 1$, and $\delta_n^{1/n}$ tends to δ as $n \to \infty$.*

If there exist two integer sequences (u_n) and (v_n) such that $u_n \geqslant 0$, $|u_n\xi - v_n|^{1/n} \to \alpha$, $u_n^{1/n} \to \beta$ and δ_n divides u_n for any n, then we have

$$\delta \leqslant \alpha\beta.$$

Proof of (i) of Theorem 5. If $\delta = 1$, we have $\gamma_\varphi = 0$ thanks to Lemma 3, so that $\mu_\varphi(\xi) = \mu(\xi)$ using Corollary 3, and assertion (i) follows from the classical implication (1.3). So we may assume $\delta > 1$.

Let $\mu > \frac{\log\beta-\log\alpha}{\log\delta-\log\alpha}$, $\varepsilon > 0$ be sufficiently small, and q be sufficiently large such that $\varphi(q)$ divides q and $|\xi - p/q| < q^{-\mu}$ for some $p \in \mathbb{Z}$. Let

$$n = \left[(\log q)\frac{\log(\delta-\varepsilon) - \log\delta + \log\alpha}{\log(\alpha+\varepsilon)\log(\delta/\alpha)}\right] + 1$$

and $m = [\frac{\log q}{\log(\delta/\alpha)}]$. We have $\log(\delta-\varepsilon) - \log\delta + \log\alpha < \log(\alpha+\varepsilon) < 0$ so that $m \leqslant n$, and $\varphi(q) = \delta_m$ divides δ_n. Therefore δ_m is a common divisor of q and u_n, and the determinant $\begin{vmatrix} q & p \\ u_n & v_n \end{vmatrix}$ is an integer multiple of δ_m. Now this determinant is equal (up to a sign) to $\begin{vmatrix} q & q\xi-p \\ u_n & u_n\xi-v_n \end{vmatrix}$; we shall prove it has absolute value less than δ_m, so that it is zero.

Since $\log(\alpha+\varepsilon)<0$, the lower bound $n \geqslant (\log q)\frac{\log(\delta-\varepsilon)-\log\delta+\log\alpha}{\log(\alpha+\varepsilon)\log(\delta/\alpha)}$ yields

$$\frac{n}{\log q}\log(\alpha+\varepsilon) < \frac{\log(\delta-\varepsilon)}{\log(\delta/\alpha)} - 1,$$

so that $\log q + n\log(\alpha+\varepsilon) < (m+1)\log(\delta-\varepsilon)$ and $q(\alpha+\varepsilon)^n < (\delta-\varepsilon)^{m+1}$. On the other hand, with this choice of n we have $n \leqslant (\log q)\frac{\log(\delta-\varepsilon)-\log\delta+\log\alpha}{\log(\alpha+\varepsilon)\log(\delta/\alpha)} + 1$ so that

$$\begin{aligned}&\frac{n}{\log q}\log(\beta+\varepsilon) - \frac{\log(\delta-\varepsilon)}{\log(\delta/\alpha)}\\ &\quad\leqslant \frac{1}{\log(\delta/\alpha)}\left[\frac{\log(\beta+\varepsilon)(\log(\delta-\varepsilon)-\log(\delta/\alpha))}{\log(\alpha+\varepsilon)} - \log(\delta-\varepsilon)\right].\end{aligned}$$

If ε is sufficiently small, the right-hand side is close enough to $\frac{\log\beta-\log\delta}{\log(\delta/\alpha)}$ to ensure that it is less than $\mu-1$. Therefore we have

$$\mu-1 > \frac{n}{\log q}\log(\beta+\varepsilon) - \frac{\log(\delta-\varepsilon)}{\log(\delta/\alpha)},$$

so that $(\mu-1)\log q > n\log(\beta+\varepsilon) - (m+1)\log(\delta-\varepsilon)$ and $(\beta+\varepsilon)^n < q^{\mu-1}(\delta-\varepsilon)^{m+1}$.

Using these two estimates, we obtain the following upper bound for the absolute value of $\begin{vmatrix} q & q\xi-p \\ u_n & u_n\xi-v_n \end{vmatrix}$:

$$q(\alpha+\varepsilon)^n + \frac{(\beta+\varepsilon)^n}{q^{\mu-1}} < 2(\delta-\varepsilon)^{m+1} < \delta_m.$$

Therefore this determinant is zero, and

$$\frac{1}{q^{\mu-1}} > |q\xi-p| = \frac{q}{u_n}|u_n\xi-v_n| > q\left(\frac{\alpha-\varepsilon}{\beta+\varepsilon}\right)^n$$

hence $q^\mu < ((\beta+\varepsilon)/(\alpha-\varepsilon))^n$, therefore

$$\begin{aligned}\mu\log q &< n\big(\log(\beta+\varepsilon)-\log(\alpha-\varepsilon)\big)\\ &< (\log q)\big(\log(\beta+\varepsilon)-\log(\alpha-\varepsilon)\big)\frac{\log(\delta-\varepsilon)-\log\delta+\log\alpha}{\log(\alpha+2\varepsilon)\log(\delta/\alpha)}\end{aligned}$$

which contradicts the assumption on μ for ε sufficiently small. This concludes the proof of (i) of Theorem 5. □

The following lemma is essentially a special case of the one proved in [10] (in the proof of Lemma 7.3, on p. 39). We give the proof since (as announced in [10]) it is really easier than the one of [10].

Lemma 5. *Let ε and Q be real numbers such that $0 < \varepsilon < 1$ and $Q > 1$. Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$ be such that $0 < \xi < 1$. Then at least one of the following two assertions holds:*

(i) *There exist integers p and q such that $1 \leqslant q < \frac{2}{\varepsilon}$ and*

$$\left|\xi - \frac{p}{q}\right| < \frac{3}{qQ}. \tag{3.3}$$

(ii) *There exist integers p and q such that $Q \leqslant q \leqslant 2Q$ and*

$$\frac{p+\varepsilon}{q} \leqslant \xi \leqslant \frac{p+3\varepsilon}{q}.$$

This lemma is useful when ε is really bigger than $1/Q$. In this case, p/q is a very precise approximation to ξ in case (i), whereas in case (ii) it is not precise at all but we have a very good control upon the exact size of q and $q\xi - p$ (namely, not only upper bounds as usual, but also lower bounds).

Proof of Lemma 5. Let $\mathcal{F}$ be the set of all fractions p/q with $0 \leqslant p \leqslant q$ and $1 \leqslant q \leqslant 2Q$. For $f \in \mathcal{F}$, we write $f = \tilde{p}/\tilde{q}$ as a fraction in its lowest terms, and also $f = p/q$ where q is the least denominator of f such that $Q \leqslant q \leqslant 2Q$. With this convention, for $f \in \mathcal{F}$ we denote by $\mathcal{I}_f$ the interval $[\frac{p+\varepsilon}{q}, \frac{p+3\varepsilon}{q}]$. Let us assume that (ii) does not hold, i.e. ξ does not belong to any of these intervals $\mathcal{I}_f$. Assertion (i) holds with $q = 1$ and $p = 0$ if $\xi \leqslant \varepsilon/Q = \min \mathcal{I}_0$, so we can assume $\xi > \min \mathcal{I}_0$ and therefore $\xi > \max \mathcal{I}_0$. Let f be the greatest fraction in $\mathcal{F}$ such that $\max \mathcal{I}_f < \xi$. Since $\xi < 1$, there is a least element $f' \in \mathcal{F}$ such that $f' > f$. Thanks to our assumption on ξ, we have $\xi \notin \mathcal{I}_{f'}$ so that $\xi < \min \mathcal{I}_{f'}$. Letting $f = p/q$ and $f' = p'/q'$ with the same convention as above, we have

$$\frac{p+3\varepsilon}{q} < \xi < \frac{p'+\varepsilon}{q'}. \tag{3.4}$$

Since $Q \leqslant q, q' \leqslant 2Q$, this gives

$$\frac{p'}{q'} - \frac{p}{q} > \frac{3\varepsilon}{q} - \frac{\varepsilon}{q'} \geqslant \frac{\varepsilon}{2Q}. \tag{3.5}$$

Now we write $\frac{p}{q} = \frac{\tilde{p}}{\tilde{q}}$ and $\frac{p'}{q'} = \frac{\widetilde{p}'}{\widetilde{q}'}$ as fractions in their lowest terms. Since they are consecutive Farey fractions, it is well known (see for instance [13]) that $\tilde{q} + \widetilde{q}' > 2Q$ and $\frac{p'}{q'} - \frac{p}{q} = \frac{1}{\tilde{q}\widetilde{q}'}$. Let $m = \min(\tilde{q}, \widetilde{q}')$ and $M = \max(\tilde{q}, \widetilde{q}')$. Then $M > Q$ so that $\frac{p'}{q'} - \frac{p}{q} = \frac{1}{\tilde{q}\widetilde{q}'} < \frac{1}{mQ}$. Thanks to (3.5), this implies $m < 2/\varepsilon$. Now we use Equation (3.4) to bound from above the distance of ξ to the fraction (either $\frac{\tilde{p}}{\tilde{q}}$ or $\frac{\widetilde{p}'}{\widetilde{q}'}$) with denominator m. If $m = \tilde{q}$ we obtain

$$\left|\xi - \frac{\tilde{p}}{\tilde{q}}\right| < \frac{p'}{q'} - \frac{p}{q} + \frac{\varepsilon}{q'} < \frac{1}{mQ} + \frac{2}{mQ} = \frac{3}{mQ}$$

whereas if $m = \widetilde{q}'$ we obtain

$$\left|\xi - \frac{\widetilde{p}'}{\widetilde{q}'}\right| < \max\left(\frac{\varepsilon}{q'},\ \ \frac{p'}{q'} - \frac{p}{q} - \frac{3\varepsilon}{q}\right) < \frac{2}{mQ}.$$

So in both cases assertion (i) holds. This concludes the proof of Lemma 5. □

Proof of (ii) of Theorem 5. First of all, let us notice that the assumptions of (ii) imply $\delta < \beta$, since $\mu_\varphi(\xi) \geqslant 1$ thanks to Lemma 1. Let μ be such that $\mu_\varphi(\xi) < \mu < \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}$. Let n be sufficiently large. We denote by ξ_n be the fractional part of $\delta_n\xi$,

$$Q_n = \frac{3\beta^n}{\delta^n} \quad \text{and} \quad \varepsilon_n = 3\alpha^n\frac{\delta_n}{\delta^n}$$

so that $\varepsilon_n < 1$ and $Q_n > 1$. Let us apply Lemma 5 to ξ_n, ε_n and Q_n. In the first case, we obtain integers u_n and v_n such that $u_n < \frac{2}{3}\alpha^{-n}\frac{\delta^n}{\delta_n}$ and

$$|u_n\xi_n - v_n| \leqslant \frac{3}{Q_n} = \frac{\delta^n}{\beta^n} < \frac{1}{(\alpha^{-n}\delta^n)^{\mu-1}} < \frac{1}{(u_n\delta_n)^{\mu-1}}$$

since $\mu - 1 < \frac{\log(\beta/\delta)}{\log(\delta/\alpha)}$.

By definition of ξ_n, there is an integer $\tilde{v}_n$ such that $u_n\xi_n - v_n = u_n\delta_n\xi - \tilde{v}_n$. So we have

$$\left|\xi - \frac{\tilde{v}_n}{u_n\delta_n}\right| = \frac{1}{u_n\delta_n}|u_n\xi - v_n| < \frac{1}{(u_n\delta_n)^\mu}.$$

Moreover we have $\varphi(u_n\delta_n) = \delta_k$ with

$$k = \left[\frac{\log(u_n\delta_n)}{\log(\delta/\alpha)}\right] \leqslant \left[\frac{\log(\frac{2}{3}\alpha^{-n}\delta^n)}{\log(\delta/\alpha)}\right] \leqslant n$$

so that δ_k divides δ_n, and finally $\delta_k = \varphi(u_n\delta_n)$ divides $u_n\delta_n$. Since $\mu_\varphi(\xi) < \mu$, this is possible only for a finite number of values of n. Therefore, as soon as n is sufficiently large, statement (ii) in Lemma 5 holds; let p_n and q_n be the integers provided by this lemma. We have

$$Q_n \leqslant q_n \leqslant 2Q_n \quad \text{and} \quad \varepsilon_n \leqslant q_n\xi_n - p_n \leqslant 3\varepsilon_n,$$

so that $\lim q_n^{1/n} = \beta/\delta$ and $\lim |q_n\xi_n - p_n|^{1/n} = \alpha$. As above, there is an integer $\tilde{p}_n$ such that $q_n\xi_n - p_n = q_n\delta_n\xi - \tilde{p}_n$. Letting $u_n = \delta_n q_n$ and $v_n = \tilde{p}_n$ concludes the proof of (ii) of Theorem 5. □

4. APPLICATION TO PARTICULAR NUMBERS ξ

Let us start by summarizing Theorem 2 and Proposition 3 in the following corollary (which contains Corollary 1 as a special case).

Corollary 5. *Let $\xi \in \mathbb{R} \setminus \mathbb{Q}$, $0 < \alpha < 1$, $\beta > 1$ and $\delta \geqslant 1$. Assume there exist integer sequences $(u_n)_{n\geqslant 1}$, $(v_n)_{n\geqslant 1}$ and $(\delta_n)_{n\geqslant 1}$ such that δ_n divides u_n and δ_{n+1} for any n, and*

$$u_n \geqslant 0, \qquad |u_n\xi - v_n|^{1/n} \to \alpha, \qquad u_n^{1/n} \to \beta \quad \text{and} \quad \delta_n^{1/n} \to \delta.$$

Let $\varphi \in \mathcal{E}$ be defined by $\varphi(q) = \delta_n$ where $n = [\frac{\log q}{\log(\delta/\alpha)}]$. Then we have

$$\mu_\varphi(\xi) \leqslant \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}.$$

Moreover, the set $\mathcal{S}$ of all $\xi' \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mu_\varphi(\xi') > \frac{\log\beta - \log\alpha}{\log\delta - \log\alpha}$, which does not contain ξ, has Hausdorff dimension $\frac{\log\delta - 2\log\alpha}{\log\beta - \log\alpha}$.

When $\beta = \delta\beta_0$ and $\alpha = \delta/\beta_0$, the Hausdorff dimension of $\mathcal{S}$ is $1 - \frac{\log\delta}{2\log\beta_0}$ (see the remark after the statement of Theorem 5). The linear forms constructed by Apéry for $\xi = \zeta(3)$ satisfy the assumptions of Corollary 5 with $\delta = e^3$ and $\beta_0 = (1+\sqrt{2})^4$ so that $\mathcal{S}$ has Hausdorff dimension $1 - \frac{3}{2\log(1+\sqrt{2})^4} = 0.5745\dots$. This Hausdorff dimension is larger than what we have been able to deduce from other constructions of linear forms in 1 and $\zeta(3)$ (due to Dvornicich and Viola [7], Hata [14], Rhin and Viola [19]), eventhough these constructions yield better upper bounds for $\mu(\zeta(3))$. It would be pleasant to have a precise statement showing that Apéry's linear forms are the ones, among a given set, that give the largest Hausdorff dimension for $\mathcal{S}$. Trying to find a point of view from which Apéry's linear forms would be "better" than its further refinements was the starting point of [10] (see also [11]).

For $\xi = \zeta(2)$, the situation is similar. Apéry's linear forms correspond to $\delta = e^2$ and $\beta_0 = ((\sqrt{5}-1)/2)^5$, so that $\mathcal{S}$ has Hausdorff dimension $1 - \frac{2}{2\log(((\sqrt{5}+1)/2)^5)} = 0.5843\dots$.

For $\xi = \log(2)$, Alladi–Robinson's linear forms [1] give $\delta = e$ and $\beta_0 = 3 + 2\sqrt{2}$, so that $\mathcal{S}$ has Hausdorff dimension $0.7163\dots$.

Let ξ be an algebraic irrational number, and $\varphi \in \mathcal{E}$. Assume there exists a finite set S of primes such that, for any $q \geqslant 1$, all prime factors of $\varphi(q)$ belong to S (this is the case for instance when φ is constructed as in the beginning of Section 2). Then Ridout's theorem [20] implies $\mu_\varphi(\xi) = 2 - \gamma_\varphi$. It would be interesting to generalize this result to other functions φ (for instance the one of Corollary 1). Indeed, when ξ is an algebraic irrational number or $\xi \in \{\log 2, \zeta(2), \zeta(3)\}$, it seems natural to imagine that $\mu_\varphi(\xi) = 2 - \gamma_\varphi$ for any $\varphi \in \mathcal{E}$.

5. REFINED RESULTS FOR $\zeta(3)$

In this section, we prove Theorem 3 stated in the introduction (of which Corollary 2 is an immediate consequence). We follow Dubitskas' proof [6] in the case of $\log 2$.

It is known that Apéry's linear forms are such that, for some $c_1, c_2 > 0$,

$$(5.1) \qquad u_n \sim c_1 d_n^3 \frac{(\sqrt{2}+1)^{4n}}{n^{3/2}} \quad \text{and} \quad u_n\zeta(3) - v_n \sim c_2 d_n^3 \frac{(\sqrt{2}-1)^{4n}}{n^{3/2}}$$

(for u_n this is due to Cohen [18], see also Example 3.2 of [23] or [16]; for $u_n\zeta(3) - v_n$ see [17]).

Let q be a sufficiently large positive integer. Let n be such that

$$3c_2\frac{(\sqrt{2}-1)^{4n}}{n^{3/2}} \leqslant \frac{1}{q} < 3c_2\frac{(\sqrt{2}-1)^{4(n-1)}}{(n-1)^{3/2}}.$$

Then we have $q > (3c_2)^{-1}(n-1)^{3/2}(\sqrt{2}+1)^{4(n-1)} > (\sqrt{2}+1)^{4(n+1)}$ since n is sufficiently large, so that $[\frac{\log q}{\log((1+\sqrt{2})^4)}] \geqslant n+1$ and d_{n+1}^3 divides q.

Now the determinant $\begin{vmatrix} u_n & u_{n+1} \\ v_n & v_{n+1} \end{vmatrix}$ is non-zero (this classical fact [18] can be deduced from the estimates (5.1)), so that at least one among $\begin{vmatrix} u_n & q \\ v_n & p \end{vmatrix}$ and $\begin{vmatrix} u_{n+1} & q \\ v_{n+1} & p \end{vmatrix}$ is non-zero. Let us assume that $\begin{vmatrix} u_n & q \\ v_n & p \end{vmatrix} \neq 0$ (otherwise the proof is similar). Since d_n^3 divides the coefficients in the first row, this determinant has absolute value greater than or equal to d_n^3. We obtain in this way (since n is large enough)

$$\begin{aligned} d_n^3 &\leqslant u_n|q\zeta(3) - p| + q|u_n\zeta(3) - v_n| \\ &\leqslant 2c_1 d_n^3\frac{(\sqrt{2}+1)^{4n}}{n^{3/2}}|q\zeta(3) - p| + \frac{2}{3}d_n^3 \end{aligned}$$

so that

$$\begin{aligned} |q\zeta(3) - p| &\geqslant \frac{1}{6c_1}n^{3/2}\big(\sqrt{2}-1\big)^{4n} \\ &\gg n^3 \times 3c_2\frac{(\sqrt{2}-1)^{4(n-1)}}{(n-1)^{3/2}} \gg \frac{n^3}{q} \gg \frac{(\log q)^3}{q}, \end{aligned}$$

thereby concluding the proof.

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