Zeros of E-functions and of exponential polynomials defined over $\overline{\mathbb{Q}}$

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Abstract

Zeros of Bessel functions J_α play an important role in physics. They are a motivation for studying zeros of exponential polynomials defined over $\overline{\mathbb{Q}}$, and more generally of E-functions. In this paper we partially characterize E-functions with zeros of the same multiplicity, and prove a special case of a conjecture of Jossen on entire quotients of E-functions, related to Ritt's theorem and Shapiro's conjecture on exponential polynomials. We also deduce from Schanuel's conjecture many results on zeros of exponential polynomials over $\overline{\mathbb{Q}}$, including π, logarithms of algebraic numbers, and zeros of J_α when 2α is an odd integer. For the latter we define (if $\alpha \neq \pm 1/2$) an analogue of the minimal polynomial and Galois conjugates of algebraic numbers. At last, we study conjectural generalizations to factorization and zeros of E-functions.

1 Introduction

We recall the definition of E-functions. As usual, we embed $\overline{\mathbb{Q}}$ into $\mathbb{C}$. A power series $f(x) = \sum_{n=0}^{\infty} a_n x^n/n! \in \overline{\mathbb{Q}}[[x]]$ is said to be an E-function if

(i) $f(x)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(x)$.

(ii) There exists $C_1 > 0$ such that all Galois conjugates of a_n have modulus $\leq C_1^{n+1}$, for $n \geq 0$.

(iii) There exists $C_2 > 0$ and a sequence of positive integers d_n, with $d_n \leq C_2^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.

If $a_n \in \mathbb{Q}$, (ii) and (iii) read $|a_n| \leq C_1^{n+1}$ and $d_n a_m \in \mathbb{Z}$; in (i), there exists such a differential equation with coefficients in $\mathbb{Q}(x)$, and the normalized one of minimal order also has coefficients in $\mathbb{Q}(x)$. The set of E-functions is a subring of $\overline{\mathbb{Q}}[[x]]$, whose units are of the form $\alpha e^{\beta x}$ for some $\alpha, \beta \in \overline{\mathbb{Q}}$, $\alpha \neq 0$. Abusing a standard terminology for G-functions, and because no confusion will be possible here, we shall say that an E-function is *globally bounded* if there exists an integer D such that $D^{n+1} a_n$ is an algebraic integer for all $n \geq 0$ (*i.e.*, the associated G-function $\sum_{n\geq 0} a_n x^n$ is globally bounded in the usual sense). Finally, when some function F is solution of a linear differential equation with polynomial coefficents, we shall often say that F is holonomic.

Siegel [26] defined and studied E-functions in 1929, and this in particular enabled him to prove the Bourget hypothesis, *i.e.*, that J_n and J_m share no common zero $\xi \in \mathbb{C}^*$, where m and n are distinct non-negative integers. Here J_α is one of the Bessel functions defined by $J_\alpha(x) := \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}(\frac{x}{2})^{2m+\alpha}$. As this formula shows, $\Gamma(\alpha+1)x^{-\alpha}J_\alpha(x)$ is an E-function for $\alpha \in \mathbb{Q} \setminus (-\mathbb{N}^*)$. Zeros of Bessel functions play an important role in many areas of physics; they are a motivation for studying zeros of E-functions.

During an online talk in 2021, Jossen [15] stated the following conjecture on the zeros of E-functions.

Conjecture 1.1 (Jossen). (*i*) *If f and g are two E-functions such that f/g is an entire function, then f/g is an E-function.*

(*ii*) *If two E-functions f and g share at least one common root, then there exists a non-unit E-function h such that f/h and g/h are E-functions.*

His conjecture is inspired by Shapiro's conjecture [24] (see below) and by Ritt-type division theorems [20, 22] (see also [25]). Part (*i*) holds if f and g are exponential polynomials over $\overline{\mathbb{Q}}$ [20, 22]. Using [6, Proposition 4.1], part (*i*) holds if $g \in \overline{\mathbb{Q}}[X]$ and part (*ii*) does if the common root is algebraic. As far as we know, these are the only known results on Conjecture 1.1.

Using asymptotic expansions and the indicator function of a holomorphic function, we are able to prove the following.

Theorem 1.2. *Let g be an entire function such that g^m is an E-function for some $m \geq 1$. Let $L \in \overline{\mathbb{Q}}(x)[d/dx]$ be a differential operator such that $L(g)/g$ is an entire function. Then $L(g)/g \in \overline{\mathbb{Q}}[x]$.*

Corollary 1.3. *Part (i) of Jossen's conjecture holds true if f is of the form $L(g)$ for some differential operator $L \in \overline{\mathbb{Q}}(x)[d/dx]$, and in this case $L(g)/g$ is a polynomial.*

The present paper is devoted to a better understanding of zeros of E-functions, at least from a conjectural point of view. We conjecture that multiple zeros of an E-function f always occur for a trivial reason: the zeros of multiplicity j are exactly the zeros of an E-function g_j such that g_j^j divides f. A precise statement is the following.

Conjecture 1.4. *Any non-zero E-function f can be written as $\prod_{j\in J} g_j(x)^j$ where $J \subset \mathbb{N}^*$ is finite, and for each $j \in J$, g_j is an E-function with zeros of multiplicity 1 such that g_j and $g_{j'}$ have no common zero for $j \neq j'$. This representation is unique up to multiplication of the g_j's by units.*

If f is not a unit, the set J can be chosen as the set of multiplicities of zeros of f. As Jossen pointed out to us, it is then finite since except for singularities, all zeros of f have multiplicity less than the order of a linear differential equation it satisfies.

In direction of this conjecture, we shall prove the following result using Theorem 1.2 and the Hadamard factorization theorem.

Theorem 1.5. *Let $f \in \mathbb{Q}[[x]]$ be a non-zero globally bounded E-function with zeros each of the same multiplicity $m \geq 2$. Assume that f is solution of a non-trivial differential operator in $\mathbb{Q}(x)[d/dx]$ of order $\leq m+1$.*

Then there exists a non-zero globally bounded E-function g with zeros of multiplicity 1 such that $f = g^m$, and $g(x) \in \alpha\mathbb{Q}[[x]]$ where $\alpha^m \in \mathbb{Q}^*$. Moreover g is solution of a non-trivial differential operator in $\mathbb{Q}(x)[d/dx]$ of order ≤ 2.

The function g is unique up to the multiplication of α by any m-th root of unity. The proof of Theorem 1.5 shows more precisely that, under the same assumptions, if f is of minimal (differential) order $m+1$ then g is exactly of order 2, while if f is of order $\leq m$ then both g and f are of order 1, *i.e.* both of the form $p(x)e^{\xi x}$ with $\xi \in \mathbb{Q}$ and $p \in \mathbb{Q}[x]$.

In the special case where the E-functions under consideration are exponential polynomials over $\overline{\mathbb{Q}}$, we are going to prove that Conjectures 1.1 and 1.4, and even much more precise statements, follow from Schanuel's conjecture. To stay in the framework of E-functions (and because it changes completely the situation), all exponential polynomials considered in the sequel will be over $\overline{\mathbb{Q}}$, namely functions

$$f(x) = \sum_{i=1}^{N} P_i(x) e^{\beta_i x} \tag{1.1}$$

with $N \geq 1$, $P_1, \ldots, P_N \in \overline{\mathbb{Q}}[X]$, $\beta_1, \ldots, \beta_N \in \overline{\mathbb{Q}}$. We point out that we allow polynomial coefficients $P_i(x)$, instead of only constants in [22] for instance. However, we *always* restrict ourselves to algebraic β_i and polynomials P_i with algebraic coefficients, instead of $\beta_i \in \mathbb{C}$ and $P_i \in \mathbb{C}[X]$ in the literature on exponential polynomials.

Units of the ring $\mathcal{P}$ of exponential polynomials over $\overline{\mathbb{Q}}$ are functions of the form $\lambda e^{\alpha x}$ with $\lambda, \alpha \in \overline{\mathbb{Q}}$ and $\lambda \neq 0$. Irreducible elements are those non-units that cannot be written as a product of two non-units. Following Ritt, simple elements are those of the form $e^{\alpha x} \sum_{i=1}^{N} \lambda_i e^{\beta r_i x}$ with $\alpha, \beta \in \overline{\mathbb{Q}}$, $\beta \neq 0$, $N \geq 2$, $\lambda_1, \ldots, \lambda_N \in \overline{\mathbb{Q}}^*$, and pairwise distinct rational numbers $r_1, \ldots, r_N$. The *support* of such a simple function is the 1-dimensional $\mathbb{Q}$-vector space spanned by β.

With these notations, any $f \in \mathcal{P}$ can be written in a unique way (up to units) as a product of irreducible exponential polynomials over $\overline{\mathbb{Q}}$, and simple ones with pairwise distinct supports. This result is due to Ritt [21] when the coefficients P_i in Eq. (1.1) are constant, and to MacColl [18] in the general setting (see also [12]). This factorisation result enables one to define gcd's in the ring of exponential polynomials over $\overline{\mathbb{Q}}$ (see [5, Theorem 3.1.18]).

These results were first proved over $\mathbb{C}$, not $\overline{\mathbb{Q}}$. Many papers are devoted to the study of common zeros of two exponential polynomials with complex coefficients (namely, $\beta_i \in \mathbb{C}$ and $P_i \in \mathbb{C}[X]$ in Eq. (1.1)), for instance towards Shapiro's conjecture [24] (see also [10]): if two exponential polynomials (with constant coefficients, over $\mathbb{C}$) have infinitely many common zeros, then both are divisible by an exponential polynomial with infinitely many zeros. In this conjecture, one common zero is not enough: for instance, $e^x - e$ and $e^{x\sqrt{2}} - e^{\sqrt{2}}$ have a common zero at $x = 1$, but no non-unit common factor since the gcd of these exponential polynomials is equal to 1. On the contrary, part (ii) of Jossen's Conjecture 1.1 suggests that restricting to exponential polynomials over $\overline{\mathbb{Q}}$ (which are E-functions) prevents this kind of behaviour: any common zero should be explained by a common factor. We prove this in Theorem 1.7 below, assuming the following widely-believed very powerful conjecture, due to Schanuel [16].

Conjecture 1.6 (Schanuel). *Let $x_1, \dots, x_n$ be complex numbers, linearly independent over $\mathbb{Q}$. Then $\overline{\mathbb{Q}}(x_1, \dots, x_n, e^{x_1}, \dots, e^{x_n})$ has transcendence degree at least n over $\overline{\mathbb{Q}}$.*

Theorem 1.7. *Assume Schanuel's conjecture holds. Let f_1, f_2 be non-zero exponential polynomials over $\overline{\mathbb{Q}}$. Then $\frac{f_1}{\gcd(f_1,f_2)}$ and $\frac{f_2}{\gcd(f_1,f_2)}$ have no common zero in $\mathbb{C}^*$.*

In other words, common zeros of f_1 and f_2 in $\mathbb{C}^*$ are exactly the zeros of $\gcd(f_1, f_2)$, and for any such ξ, the order of vanishing of $\gcd(f_1, f_2)$ at ξ is the least of the orders of vanishing of f_1 and f_2 at ξ.

We point out that 0 may be a common zero of f_1 and f_2 even in the case they are coprime, for instance if $f_1(x) = e^x - 1$ and $f_2(x) = e^{x\sqrt{2}} - 1$ (see the remark after Theorem 4.1 in §4).

Theorem 1.7 enables us to define a kind of analogue of the minimal polynomial of an algebraic number for zeros of exponential polynomials over $\overline{\mathbb{Q}}$. Let $\xi \in \mathbb{C}^*$ be a zero of some $f \in \mathcal{P} \setminus \{0\}$. Using Ritt's theorem, ξ is a zero of either a simple or an irreducible exponential polynomial over $\overline{\mathbb{Q}}$. The former case means that $e^{\beta\xi}$ is algebraic for some $\beta \in \overline{\mathbb{Q}}^*$; these numbers ξ include π, logarithms of algebraic numbers, etc; they are periods. As noted by André [4, §§2.1 and 2.3] (see also [14, §3.3]), it is not clear how to define an analogue of the minimal polynomial for these numbers, in particular π. In this direction we prove the following.

Theorem 1.8. *Assume Schanuel's conjecture holds. Let f be an exponential polynomial over $\overline{\mathbb{Q}}$ such that $f(\pi) = 0$. Then there exists $N \geq 1$ such that*

$$\frac{f(x)}{e^{2ix/N} - e^{2i\pi/N}}$$

is an exponential polynomial over $\overline{\mathbb{Q}}$.

In some sense, the family of functions $e^{2ix/N} - e^{2i\pi/N}$ would be an analogue of the minimal polynomial for π. Note that the function $e^{2ix/N} - e^{2i\pi/N}$ vanishes exactly at the points $(1 + kN)\pi$, $k \in \mathbb{Z}$. Therefore π is the only complex number at which all these functions vanish. This approach does not yield any definition for a "conjugate" of π (recall that conjugates of an algebraic number are the complex roots of its minimal polynomial).

The same holds, for instance, with $\log(2)$ and the functions $e^{x/N} - e^{\log(2)/N}$.

If ξ is a zero of a some $f \in \mathcal{P} \setminus \{0\}$, and if $e^{\beta\xi}$ is algebraic for no $\beta \in \overline{\mathbb{Q}}^*$, then $h(\xi) = 0$ for some irreducible exponential polynomial h over $\overline{\mathbb{Q}}$. We suggest h as an analogue of the minimal polynomial for ξ, and the zeros of h as analogues of the conjugates of ξ, in view of the following result.

Theorem 1.9. *Assume Schanuel's conjecture holds. Let $\xi \in \mathbb{C}^*$, and h be an irreducible exponential polynomial over $\overline{\mathbb{Q}}$ such that $h(\xi) = 0$. Then for any exponential polynomial f over $\overline{\mathbb{Q}}$ such that $f(\xi) = 0$, f/h is an exponential polynomial over $\overline{\mathbb{Q}}$.*

For any integer $n \in \mathbb{Z}$, $\sqrt{\pi}x^{|n+1/2|}J_{n+1/2}(x)$ is an exponential polynomial over $\overline{\mathbb{Q}}$. If $n \in \{-1, 0\}$, it is $\sqrt{2}\cos x$ or $\sqrt{2}\sin x$ and the situation is similar to that of Theorem 1.8. Now for $n \in \mathbb{Z}\setminus\{-1, 0\}$ this exponential polynomial over $\overline{\mathbb{Q}}$ is irreducible (see the beginning of §4): Theorem 1.9 applies to all zeros $\xi \in \mathbb{C}^*$ of the Bessel function $J_{n+1/2}$. This suggests that $J_{n+1/2}$ can then be considered as an analogue of the minimal polynomial of ξ, and its zeros as analogues of its conjugates (with a slight abuse: to be precise, we mean $\sqrt{\pi}x^{-n-1/2}J_{n+1/2}(x)$ instead of $J_{n+1/2}$ itself). We believe that ξ is neither a period nor an exponential period. Accordingly this is probably the first attempt to define these notions for ξ. We would like to point out also that it would be very interesting to have a transitive group action on the set of conjugates of ξ, as the Galois action in the algebraic setting. The situation is the same for values $\xi = W(c)$ of the Lambert W function at non-zero algebraic points c (see the beginning of §4).

In order to deal with zeros of Bessel functions J_α of any rational order α, we propose conjectures to extend the above properties to the ring $\mathcal{E}$ of E-functions. We start with a generalization of Ritt's theorem, that enables us to define the gcd of two E-functions so that common zeros of E-functions should be the zeros of their gcd (as in Theorem 1.7). Then we adapt Theorems 1.8 and 1.9. For Bessel functions, we deduce from our general conjectures the following one (recall that $J_{\pm 1/2}(x)$ is a simple exponential polynomial over $\overline{\mathbb{Q}}$).

Conjecture 1.10. *Let $\alpha \in \mathbb{Q}\setminus(\{\pm 1/2\} \cup (-\mathbb{N}^*))$, and f be an E-function such that f and J_α share a common zero in $\mathbb{C}^*$. Then $f(x) = g(x)\Gamma(\alpha+1)x^{-\alpha}J_\alpha(x)$ for some E-function g.*

This conjecture suggests $\Gamma(\alpha + 1)x^{-\alpha}J_\alpha(x)$ and its zeros as analogues of the minimal polynomial and the Galois conjugates of ξ, when ξ is a zero of J_α (as above when $\alpha - 1/2 \in \mathbb{Z}$).

The structure of this paper is as follows. In §2, we prove Theorem 1.2, which implies a special case of Jossen's conjecture and will be used in §3 to prove Theorem 1.5. Then we move in §4 to zeros of exponential polynomials over $\overline{\mathbb{Q}}$, and conclude in §§5 and 6 with factorization and zeros of E-functions.

2 A special case of Jossen's conjecture

In this section, we prove Theorem 1.2 stated in the introduction. We shall use the indicator function h associated with an holomorphic function f of exponential type in a sector

$\alpha \leq \arg(x) \leq \beta$. We recall that f is said to be of exponential type in such a sector if there exists $\tau > 0$ such that, for any $x \in \mathbb{C}^*$ with $\alpha \leq \arg(x) \leq \beta$, we have $|f(x)| \leq \exp(\tau|x|)$. Then for any $\theta \in [\alpha, \beta]$, $h(\theta)$ is defined by

$$h(\theta) = \limsup_{r \to +\infty} \frac{\log|f(re^{i\theta})|}{r}.$$

We refer to [7, Ch. 5] for the general properties of the indicator function. We shall use the following consequence of [7, Theorem 6.2.4, Ch. 6, p. 82]:

Proposition 2.1. *If f is holomorphic and of exponential type in the (closed) upper half-plane, bounded on $\mathbb{R}$ and such that $h(\pi/2) \leq 0$, then it is bounded on the upper half-plane.*

Proof of Theorem 1.2. For any $\theta \in [-\pi, \pi]$ outside a finite set, the E-function g^m has an asymptotic expansion in a large sector bisected by θ of the form $\sum_{\rho \in \Sigma} f_\rho(z) e^{\rho z}$ with $f_\rho(x) \in \mathrm{NGA}\{1/x\}_1$ (see [3, Théorème de dualité] and [13, §4.1]). Up to changing z to $e^{i\alpha}x$ for a suitable α, we may assume that such an expansion holds in a large sector bisected by $\pi/2$. Shrinking Σ if necessary, we also assume that $f_\rho \neq 0$ for any $\rho \in \Sigma$.

Now we fix $\theta \in [0, \pi]$ and consider $g^m(x)$ as $|x| \to +\infty$ with $\arg(x) = \theta$; notice that in this direction, $|\exp(\rho x)| = \exp(|x|\mathrm{Re}(\rho e^{i\theta}))$ for any $\rho \in \Sigma$. Except for finitely many values of θ (related to Stokes' phenomenon), the maximum of $\mathrm{Re}(\rho e^{i\theta})$ as ρ ranges through Σ is obtained for only one value ρ_θ. Then, as $|x| \to +\infty$ in this direction, $\exp(\rho x)$ is exponentially smaller than $\exp(\rho_\theta x)$, for any $\rho \in \Sigma \setminus \{\rho_\theta\}$. Consequently, there exist $a_\theta \in \mathbb{Q}$, $j_\theta \in \mathbb{N}$, and $c_\theta \in \mathbb{C}^*$ such that

$$g^m(x) \sim c_\theta x^{a_\theta} (\log x)^{j_\theta} e^{\rho_\theta x}$$

as $|x| \to +\infty$ with $\arg(x) = \theta$. Taking m-th roots provides $d_\theta \in \mathbb{C}^*$ such that

$$g(x) \sim d_\theta x^{a_\theta/m} (\log x)^{j_\theta/m} e^{\rho_\theta x/m}.$$

Moreover, the same argument shows that either $(Lg)(x) \sim d_\theta x^{a'_\theta} (\log x)^{j'_\theta} e^{\rho_\theta x/m}$ for some $a'_\theta, j'_\theta \in \mathbb{Q}$, and $d_\theta \in \mathbb{C}^*$, or $(Lg)(x) = o(e^{\rho_\theta x/m})$. The latter happens if the part relative to $e^{\rho_\theta x/m}$ in the asymptotic expansion of g is annihilated by L. In both cases, we obtain $\frac{L(g)}{g}(x) = \mathcal{O}(x^{A_\theta})$ for some $A_\theta \in \mathbb{N}$, as $|x| \to +\infty$ with $\arg(x) = \theta$.

Changing again x to $e^{i\alpha}x$ if necessary, we may assume that for any $\theta \in \{0, \pi/2, \pi\}$ the above-mentioned maximum of $\mathrm{Re}(\rho e^{i\theta})$ is obtained for only one value ρ_θ. Then we have $\frac{Lg}{g}(x) = \mathcal{O}(x^{A_\theta})$ as $|x| \to +\infty$ with $\arg(x) = \theta$. Now let $A = \max(A_0, A_\pi)$ and $f(x) = (x+i)^{-A} \frac{L(g)}{g}(x)$. Then f is holomorphic and of exponential type in the closed upper half-plane, bounded on $\mathbb{R}$. Its indicator function satisfies $h(\pi/2) \leq 0$. Using Proposition 2.1, it is bounded, so that $|\frac{L(g)}{g}(x)| \leq |x+i|^A$ for any $x \in \mathbb{C}$ with $\mathrm{Im}(x) \geq 0$. The same proof yields $|\frac{L(g)}{g}(x)| \leq |x-i|^A$ when $\mathrm{Im}(x) \leq 0$. Therefore $L(g)/g$ has (at most) a polynomial growth at infinity: it is a polynomial by Liouville's theorem. Now g^m has algebraic Taylor coefficients at 0: so do g, $L(g)$ and $L(g)/g$. Finally, $L(g)/g \in \overline{\mathbb{Q}}[x]$: this concludes the proof of Theorem 1.2. □

3 Proof of Theorem 1.5

We do not repeat the assumptions of Theorem 1.5 which are assumed throughout this section. If 0 is a zero of f (of multiplicity m), $\widetilde{f}(x) := f(x)/x^m \neq 0$ is still an E-function in $\mathbb{Q}[[x]]$ with zeros all of multiplicity m, all different from 0, and $\widetilde{f}(0) \in \mathbb{Q}^*$. It is then clearly enough to prove the theorem for $\widetilde{f}(x)/\widetilde{f}(0)$, which is globally bounded and solution of a differential operator in $\overline{\mathbb{Q}}(x)[d/dx]$ of order $\leq m+1$. Therefore, without loss of generality, we assume that $f(0) = 1$ from now on.

Before going further and because this will appear below, we recall that E-functions have been defined by Siegel [26] in a more general way, *i.e.*, the two bounds $(\cdots) \leq C_i^{n+1}$ in the definition in the introduction of E-functions (which are often said to be *in the strict sense*) are replaced by: for all $\varepsilon > 0$, $(\cdots) \leq n!^{\varepsilon}$ for all $n \geq N(\varepsilon)$. A globally bounded E-function as defined in the introduction is automatically a strict E-function by a theorem of Perron [19]; see the details in [3, p. 715]. We have decided to state our conjectures for E-functions in the strict sense, but we could formulate the same conjectures for E-functions in Siegel's sense *mutatis mutandis*. However, it is believed that an E-function in Siegel's sense is automatically a strict E-function; see again [3, p. 715] for a discussion.

3.1 Existence of g entire of order ≤ 1 such that $f = g^m$ and $g(0) = 1$

We recall that the order of an entire function h is the infimum of the set of all c such that $h(x) = O(\exp(|x|^c))$ as x tends to infinity in the complex plane. Since f is of order $\rho(f) \leq 1$, by the Hadamard factorization theorem [7, Chapter 2], we have

$$f(x) = e^{\beta x} \prod_{\zeta \in Z(f)} \Big(1 - \frac{x}{\zeta}\Big)^m e^{mx/\zeta},$$

with $\beta = f'(0) \in \mathbb{Q}$, where $Z(f)$ is the set of zeros of f. Then

$$g(x) := e^{\beta x/m} \prod_{\zeta \in Z(f)} \Big(1 - \frac{x}{\zeta}\Big) e^{x/\zeta}$$

is also an entire function of order $\rho(g) \leq 1$, and such that $g(0) = 1$ and $g^m = f$.

Let

$$g(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n, \quad b_0 = 1.$$

By [7, p. 9], we have

$$\limsup_{n \to +\infty} \frac{n \log(n)}{\log(n!/b_n)} = \rho(g) \leq 1,$$

so that for all $\varepsilon > 0$, we have $b_n = \mathcal{O}(n!^{\varepsilon})$ for all $n \geq N(\varepsilon)$.

This is the growth requested on the sequence $(b_n)_n$ for g to be an E-function in Siegel's sense. Moreover, if g can be proved to be holonomic, then by above mentioned theorem of Perron [19] this bound automatically implies that $b_n = \mathcal{O}(C^n)$ for some $C > 0$ which is the growth requested in (ii) for g to be an E-function in the strict sense.

3.2 Denominators of the Taylor coefficients of g at the origin

Let $f(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ with $a_0 = 1$. We consider again the function g defined in the previous section. Locally around $x = 0$, we have (because $g(0) = 1$):

$$\begin{aligned} g(x) = f(x)^{1/m} &= \Big(1 + \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n\Big)^{1/m} \\ &= \sum_{k=0}^{\infty} \binom{1/m}{k} \Big(\sum_{n=1}^{\infty} \frac{a_n}{n!} x^n\Big)^k \\ &= \sum_{\ell=0}^{\infty} \frac{b_\ell}{\ell!} x^\ell, \end{aligned}$$

so that for any $\ell \geq 1$,

$$b_\ell = \sum_{k=1}^{\ell} (-1)^k \frac{(-1/m)_k}{k!} \sum_{\substack{n_1+\cdots+n_k=\ell \\ n_j \geq 1}} \binom{\ell}{n_1, \ldots, n_k} a_{n_1} \cdots a_{n_k} \in \mathbb{Q}. \tag{3.1}$$

Since f is assumed to be globally bounded and $a_0 = 1$, consider an integer $D \geq 1$ such that $D^n a_n \in \mathbb{Z}$ for all $n \geq 1$. It is well-known that $m^{2k} \frac{(-1/m)_k}{k!} \in \mathbb{Z}$ for all $k \geq 1$ (see for instance [11, Theorem 4]). Therefore, we deduce from (3.1) that $(m^2 D)^\ell b_\ell \in \mathbb{Z}$ for all $\ell \geq 0$. This estimate is what is requested in (iii) for g to be an E-function, and even a globally bounded one.

However, if f is not assumed to be globally bounded, it does not seem possible to deduce from (3.1) that the least positive common denominator of $b_0, \ldots, b_n$ is bounded by $n!^\varepsilon$, resp by C^{n+1}, if the least positive common denominator of $a_0, \ldots, a_n$ is bounded by $n!^\varepsilon$, resp by D^{n+1}.

In summary, the results proven so far show that if g is also holonomic, then it is a globally bounded E-function.

3.3 Holonomicity of g

Let $m \geq 2$. By Leibniz formula, we have for all $k \geq 0$:

$$f^{(k)} = (g^m)^{(k)} = \sum_{\ell_1+\cdots+\ell_m=k} \binom{k}{\ell_1, \ldots, \ell_m} g^{(\ell_1)} \cdots g^{(\ell_m)}$$

where the sum runs over the integers $\ell_j \geq 0$ such that $\ell_1 + \cdots + \ell_m = k$.

We claim that there exist $P_{k,m} \in \mathbb{Z}[X_1, \ldots, X_{k+1}] \setminus \{0\}$ and $c_{k,m} \in \mathbb{N}^*$ such that for $k \leq m - 1$,

$$(g^m)^{(k)} = g P_{k,m}(g, g', \ldots, g^{(k)})$$

and for $k = m$ and $k = m + 1$,

$$(g^m)^{(k)} = gP_{k,m}(g, g', \dots, g^{(k)}) + c_{k,m} g^{(k-m+1)} (g')^{m-1}.$$

Let $k \leq m - 1$. We see that in Leibniz formula, given an m-tuple $(\ell_1, \dots, \ell_m)$ such $\ell_1 + \dots + \ell_m = k$, we obviously cannot have $\ell_j \geq 1$ for each j, hence g always appears in the products $g^{(\ell_1)} \cdots g^{(\ell_m)}$ and the claim follows in this case.

Let $k = m$. If one of the $\ell_j = 0$, the corresponding term $g^{(\ell_1)} \cdots g^{(\ell_m)}$ contributes to $gP_{k,m}(g, g', \dots, g^{(k)})$. If none of the $\ell_j = 0$, *i.e.*, all are ≥ 1, then in fact $\ell_1 = \dots = \ell_m = 1$ because otherwise $\ell_1 + \dots + \ell_m \geq m + 1 > k$; hence in that case, we have a unique term $m! \cdot (g')^m$. This proves the claim in this case.

Let $k = m + 1$. Again if one of the $\ell_j = 0$, the corresponding term $g^{(\ell_1)} \cdots g^{(\ell_m)}$ contributes to $gP_{k,m}(g, g', \dots, g^{(k)})$. If none of the $\ell_j = 0$, *i.e.*, all are ≥ 1, then exactly one must be equal to 2 and the others must be equal to 1. Indeed, if at least two are ≥ 2, then $\ell_1 + \dots + \ell_m \geq m + 2 > k$ while if they are all equal to 1, we have $\ell_1 + \dots + \ell_m = m < k$; hence in that case we have a (non-empty) sum of terms all of the form $c \cdot g''(g')^{m-1}$, $c \in \mathbb{N}^*$. This proves the claim in this case as well.

Let us now assume that f is solution of a differential operator in $\mathbb{Q}(x)[d/dx]$ of order $m + 1$, *i.e.*, we have

$$\sum_{k=0}^{m+1} a_k(x) f^{(k)}(x) = 0, \quad a_{m+1} \neq 0,\ a_k \in \mathbb{Z}[x].$$

Using the above expressions for $(g^m)^{(k)}$, we then obtain an algebraic differential equation for g of the form

$$\big(c_{m+1,m} a_{m+1} g'' + c_{m,m} a_m g'\big) g'^{m-1} = gQ\big(x, g, g', \dots, g^{(m+1)}\big), \quad Q \in \mathbb{Z}[X_0, X_1, \dots X_{m+2}].$$

Since g has simple zeros, it has no common zeros with g'. Since $Q(x, g, g', \dots, g^{(m+1)})$ is an entire function, all these simple zeros are zeros of $c_{m+1,m} a_{m+1} g'' + c_{m,m} a_m g'$ so that

$$h := \frac{c_{m+1,m} a_{m+1} g'' + c_{m,m} a_m g'}{g} \tag{3.2}$$

is an entire function. Hence by Theorem 1.2, h is in $\overline{\mathbb{Q}}[x]$. Moreover, as the right-hand side of (3.2) is in $\mathbb{Q}((x))$, we deduce that $h \in \mathbb{Q}[x]$. The equation

$$c_{m+1,m} a_{m+1} g'' + c_{m,m} a_m g' - hg = 0$$

then shows that g is solution of a differential operator in $\mathbb{Q}(x)[d/dx]$ of order 2. Note that if f is of minimal differential order $m+1$, then g is of minimal differential order 2: indeed, by what precedes, g would otherwise be of minimal order 1 and thus f as well, which is not possible.

Finally, let us assume that f is solution of a differential operator in $\mathbb{Q}(x)[d/dx]$ of order $\leq m$. We distinguish two cases.

First case. Let us assume that f has infinitely many zeros of multiplicity m. Then at one such zero ξ which is not one of the finitely many singularities of the differential equation of f, we have $f(\xi) = f'(\xi) = \cdots = f^{(m-1)}(\xi) = 0$. Since ξ is an ordinary point of this equation of order $\leq m$, by the Cauchy-Lipschitz theorem, f must be identically zero, which is excluded.

Second case. Let us assume that f has finitely many zeros. Then $f \in \mathbb{Q}[[x]]$ is of the form $p(x)e^{\xi x}$ by the Hadamard factorization formula, with $\xi \in \mathbb{C}$ and $p \in \mathbb{C}[x]$. Since f is an E-function, we know that $\xi \in \overline{\mathbb{Q}}$ hence $p \in \overline{\mathbb{Q}}[x]$. If $\xi = 0$, then $p = f$ is in $\mathbb{Q}[x]$. Let us now assume that f is transcendental, *i.e.*, $\xi \neq 0$. By [8, Theorem 3.4], there exist unique $u, v \in \overline{\mathbb{Q}}[x]$ and $h \in \overline{\mathbb{Q}}[[x]]$ a purely transcendental E-function ([1]) such that v is monic, $v(0) \neq 0$, $\deg(u) < \deg(v)$ and $f = u + vh$ (canonical decomposition of f); moreover u, v, h have rational Taylor coefficients because $f \in \mathbb{Q}[[x]]$. Let $p_\mu \in \overline{\mathbb{Q}}^*$ denote the leading coefficient of p: since $p_\mu e^{\xi x}$ is purely transcendental, $(p(x)/p_\mu) \cdot p_\mu e^{\xi x}$ is the canonical decomposition of f and thus $(p(x)/p_\mu) \in \mathbb{Q}[x]$ and $p_\mu e^{\xi x} \in \mathbb{Q}[[x]]$. Hence $p_\mu \in \mathbb{Q}$ and consequently $f \in \mathbb{Q}[x]e^{\mathbb{Q}x}$ with roots of multiplicity m. Therefore g is in $\mathbb{Q}[x]e^{\mathbb{Q}x}$, is solution of a differential operator in $\mathbb{Q}(x)[d/dx]$ of order 1, and is globally bounded because this is the case of all E-functions in $\overline{\mathbb{Q}}[x]e^{\overline{\mathbb{Q}}x}$. This case is thus possible.

4 Zeros of exponential polynomials over $\overline{\mathbb{Q}}$

In this section we consider exponential polynomials over $\overline{\mathbb{Q}}$, defined as functions

$$f(x) = \sum_{i=1}^{N} P_i(x)e^{\beta_i x} \tag{4.1}$$

with $N \geq 1$, $P_1, \ldots, P_N \in \overline{\mathbb{Q}}[X]$, $\beta_1, \ldots, \beta_N \in \overline{\mathbb{Q}}$. We emphasize that we allow polynomial coefficients $P_i(x)$, instead of only constants in [22] for instance. However we restrict to algebraic β_i and polynomials with algebraic coefficients, instead of $\beta_i \in \mathbb{C}$ and $P_i \in \mathbb{C}[X]$ in the literature on exponential polynomials. This changes the situation completely. This restriction is necessary for exponential polynomials to be E-functions. *All exponential polynomials considered in this section are over $\overline{\mathbb{Q}}$, unless stated otherwise.*

Units and irreducible elements of the ring $\mathcal{P}$ of exponential polynomials have been defined in the introduction. For instance, given $c \in \overline{\mathbb{Q}}^*$, the function $h_c(x) = xe^x - c$ is irreducible in $\mathcal{P}$. Indeed, using Ritt's theory this follows from the irreducibility of $X_0X_1^n - c$ in $\overline{\mathbb{Q}}[X_0, X_1]$ for any $n \geq 1$ (which is a consequence of Eisenstein's criterion applied to $X_1^n - c/X_0$, seen as a polynomial in X_1 with coefficients in the factorial ring $\overline{\mathbb{Q}}[1/X_0]$).

For any $n \in \mathbb{Z}$, we have $\sqrt{\pi/2}x^{|n+1/2|}J_{n+1/2}(x) = e^{-ix}(A_n(x)e^{2ix} + B_n(x))$ with $A_n, B_n \in \mathbb{Q}[X]$. Ritt's theory shows that the irreducibility of this exponential polynomial in $\mathcal{P}$

[1]By definition, a *purely transcendental* E-function h is such that $h(\eta) \notin \overline{\mathbb{Q}}$ for all $\eta \in \overline{\mathbb{Q}}^*$.

follows from that of $A_n(X)Y^k + B_n(X)$ in $\overline{\mathbb{Q}}[X,Y]$ for any $k \geq 1$; we shall prove it now for any $n \in \mathbb{Z} \setminus \{-1, 0\}$ using Eisenstein's criterion. To begin with, we notice that $x^{-|n+1/2|}A_n(x)e^{ix}$ and $x^{-|n+1/2|}B_n(x)e^{-ix}$ are solution of the Bessel differential operator of order 2 that annihilates $J_{n+1/2}$, of which 0 is the only finite singularity. Therefore $A_n(x)$ and $B_n(x)$ are also solutions of differential equations of order 2 with no non-zero finite singularities. The Cauchy-Lipschitz theorem shows that they have only simple roots (except possibly 0). Since $n \notin \{-1, 0\}$, at least one of A_n and B_n has a simple root ξ. It cannot be a common root of A_n and B_n, since otherwise $J_{n+1/2}$ would have a non-zero algebraic zero, in contradiction with Siegel's theorem. Therefore Eisenstein's criterion applies with the irreducible polynomial $X - \xi$, and proves that $A_n(X)Y^k + B_n(X)$ is irreducible in $\overline{\mathbb{Q}}(X)[Y]$ and then in $\overline{\mathbb{Q}}[X,Y]$ since A_n and B_n are coprime. This concludes the proof that $\sqrt{\pi/2}x^{|n+1/2|}J_{n+1/2}(x)$ is irreducible in $\mathcal{P}$.

Theorem 1.7 stated in the introduction is an immediate consequence of the following result, since units of $\mathcal{P}$ have no zeros.

Theorem 4.1. *Assume that Schanuel's conjecture holds. Let f_1, f_2 be non-zero exponential polynomials with (at least) a common zero $\xi \in \mathbb{C}^*$. Then there exists $f \in \mathcal{P} \setminus \{0\}$, which vanishes at ξ, such that f_1/f and f_2/f are exponential polynomials.*

We point out that Theorem 4.1 would be false with $\xi = 0$; for instance $f_1(x) = e^x - 1$ and $f_2(x) = e^{x\sqrt{2}} - 1$ both vanish at 0, but are not multiple of an exponential polynomial vanishing at 0 because they are simple with distinct supports (see Lemma 4.4 below). Another remark is that Theorem 4.1 is remniscent of the Shapiro conjecture: if f_1, f_2 are exponential polynomials with constant complex coefficients (*i.e.*, of the form (4.1) with $\beta_i, P_i \in \mathbb{C}$) with infinitely many common zeros, then the conclusion of Theorem 4.1 holds. In the complex setting, one common zero is not enough: for instance $e^x - e$ and $e^{x\sqrt{2}} - e^{\sqrt{2}}$ have a common zero at 1, but are not multiple of an exponential polynomial vanishing at 1. This example shows that restricting to $\beta_i \in \overline{\mathbb{Q}}$ and $P_i \in \overline{\mathbb{Q}}[X]$ in Eq. (4.1) changes completely the situation.

Corollary 4.2. *If Schanuel's conjecture holds then irreducible exponential polynomials have only simple zeros.*

Proof of Corollary 4.2. Let $h \in \mathcal{P}$ be irreducible. If ξ is a multiple zero of h, then it is also a zero of h'. Using Theorem 4.1 it is a zero of $\gcd(h, h')$ so this gcd is not a unit. Since h is irreducible, this gcd is equal to h (up to a unit), and h divides h' in $\mathcal{P}$. Denoting by ω the multiplicity of ξ as a zero of h, we obtain $\omega \leq \omega - 1$ since all exponential polynomials are holomorphic at ξ: this is a contradiction, and Corollary 4.2 is proved. □

Let us deduce the following consequence of Theorem 4.1; it contains Theorems 1.8 and 1.9 stated in the introduction.

Corollary 4.3. *Assume that Schanuel's conjecture holds, and let $\xi \in \mathbb{C}^*$ be a zero of an exponential polynomial. Then one, and only one, of the following holds:*

- *We have $e^{\beta\xi} \in \overline{\mathbb{Q}}$ for some $\beta \in \overline{\mathbb{Q}}^*$, and for any $f \in \mathcal{P}$ with $f(\xi) = 0$ there exists $N \geq 1$ such that f is divisible by $e^{\beta x/N} - e^{\beta\xi/N}$ in $\mathcal{P}$.*
- *We have $h(\xi) = 0$ for some irreducible $h \in \mathcal{P}$, and h divides in $\mathcal{P}$ any exponential polynomial that vanishes at ξ.*

Proof of Corollary 4.3. To begin with, let $\xi \in \mathbb{C}^*$ and $h, f \in \mathcal{P}$ be such that $h(\xi) = f(\xi) = 0$, with h irreducible. Using Theorem 1.7, $\gcd(h, f)$ vanishes at ξ so it is not a unit. Since h is irreducible, it is h (up to a unit) so that h divides f in $\mathcal{P}$. This proves the second part. Moreover, if $f(x) = e^{\beta x} - c$ with $\beta, c \in \overline{\mathbb{Q}}^*$ then f is simple so h cannot divide f (by uniqueness in Ritt's factorization theorem): ξ cannot be in both situations of Corollary 4.3.

At last, if $\xi \in \mathbb{C}^*$ is a zero of some $f_0 \in \mathcal{P} \setminus \{0\}$ but of no irreducible $h \in \mathcal{P}$, then ξ a zero of a simple factor g of f_0. Up to a unit, we can write $g(x) = P(e^{\beta x})$ for some $\beta \in \overline{\mathbb{Q}}^*$ and $P \in \overline{\mathbb{Q}}[X] \setminus \{0\}$. Therefore $e^{\beta\xi}$ is a zero of P: it is algebraic. If $f \in \mathcal{P}$ is such that $f(\xi) = 0$, there exists $\beta_1 \in \overline{\mathbb{Q}}^*$ and $P_1 \in \overline{\mathbb{Q}}[X] \setminus \{0\}$ such that $g_1(x) = P_1(e^{\beta_1 x})$ divides f in $\mathcal{P}$ and vanishes at ξ. Both $e^{\beta\xi}$ and $e^{\beta_1\xi} = (e^{\beta\xi})^{\beta_1/\beta}$ are algebraic, and $e^{\beta\xi} \neq 1$: the Gel'fond-Schneider theorem yields $\beta_1/\beta \in \mathbb{Q}$. This provides $\beta_2 \in \overline{\mathbb{Q}}^*$ such that $\beta, \beta_1 \in \beta_2\mathbb{Z}$. Up to multiplying g and g_1 with suitable units, we may assume that $\beta = n\beta_2$ and $\beta_1 = n_1\beta_2$ with $n, n_1 \in \mathbb{N}^*$. Letting $Q(X) = P(X^n)$ and $Q_1(X) = P_1(X^{n_1})$ we have $g(x) = Q(e^{\beta_2 x})$ and $g_1(x) = Q_1(e^{\beta_2 x})$. Since $g(\xi) = g_1(\xi) = 0$, the polynomials Q and Q_1 have $e^{\beta_2\xi}$ as a common root, so they are multiples of $X - e^{\beta_2\xi}$ in $\overline{\mathbb{Q}}[X]$. Finally the simple function $e^{\beta_2 x} - e^{\beta_2\xi}$, with $\beta_2 = \beta/n$, vanishes at ξ and divides f. This concludes the proof of Corollary 4.3. □

Proof of Theorem 4.1. We denote by $\beta_1, \ldots, \beta_N$ the exponents in an expression (4.1) of f_1, and by $\beta'_1, \ldots, \beta'_{N'}$ those for f_2. Let W be the $\mathbb{Z}$-module generated by $\beta_1, \ldots, \beta_N, \beta'_1, \ldots, \beta'_{N'}$. There exists a $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_p$ of W; all exponents β_i, β'_j can be written as $\mathbb{Z}$-linear combinations of $\alpha_1, \ldots, \alpha_p$. Multiplying f_1 and f_2 with $e^{\gamma x}$ for a suitable $\gamma \in W \subset \overline{\mathbb{Q}}$, we may assume that these linear combinations involve only non-negative coefficients. Then we have

$$f_1(x) = P_1(x, e^{\alpha_1 x}, \ldots, e^{\alpha_p x}) \quad \text{and} \quad f_2(x) = P_2(x, e^{\alpha_1 x}, \ldots, e^{\alpha_p x})$$

for some $P_1, P_2 \in \overline{\mathbb{Q}}[X_0, \ldots, X_p] \setminus \{0\}$. The complex numbers $\alpha_1\xi, \ldots, \alpha_p\xi$ are linearly independent over $\mathbb{Q}$ because $\alpha_1, \ldots, \alpha_p$ are, and $\xi \neq 0$; but they are linearly dependent over $\overline{\mathbb{Q}}$. Schanuel's conjecture implies that

$$\overline{\mathbb{Q}}(\alpha_1\xi, \ldots, \alpha_p\xi, e^{\alpha_1\xi}, \ldots, e^{\alpha_p\xi}) = \overline{\mathbb{Q}}(\xi, e^{\alpha_1\xi}, \ldots, e^{\alpha_p\xi})$$

has transcendence degree at least p over $\overline{\mathbb{Q}}$. In other words, letting

$$J = \{S \in \overline{\mathbb{Q}}[X_0, \ldots, X_p], \quad S(\xi, e^{\alpha_1\xi}, \ldots, e^{\alpha_p\xi}) = 0\},$$

the zero set of J in $\overline{\mathbb{Q}}^{p+1}$ has dimension at least p; since P_1 and P_2 are non-zero and belong to J, this dimension is equal to p. Therefore J is principal: there exists $P \in$

$\overline{\mathbb{Q}}[X_0, \dots, X_p] \setminus \{0\}$ such that J consists in all multiples of P, and accordingly there exist $T_1, T_2 \in \overline{\mathbb{Q}}[X_0, \dots, X_p]$ such that $P_1 = T_1 P$ and $P_2 = T_2 P$. Letting

$$f(x) = P(x, e^{\alpha_1 x}, \dots, e^{\alpha_p x}), \quad \widetilde{f}_1(x) = T_1(x, e^{\alpha_1 x}, \dots, e^{\alpha_p x}), \quad \widetilde{f}_2(x) = T_2(x, e^{\alpha_1 x}, \dots, e^{\alpha_p x}),$$

we have $f(\xi) = 0$, $f_1 = f\widetilde{f}_1$ and $f_2 = f\widetilde{f}_2$. This concludes the proof of Theorem 4.1. □

To conclude this section we state the following well-known result, valid also in the complex setting. It follows immediately from [5, Proposition 3.1.1].

Lemma 4.4. *Let f_1, f_2 be non-zero exponential polynomials such that $f_1 f_2$ is simple. Then f_1 and f_2 are simple and f_1, f_2, $f_1 f_2$ have the same support.*

This lemma shows that if f is simple, then all divisors of f in $\mathcal{P}$ are also simple with the same support.

5 Factorization of E-functions

In this section we describe the (conjectural) structure of the ring $\mathcal{E}$ of E-functions.

A *unit* $u \in \mathcal{E}$ is an E-function such that $uv = 1$ for some $v \in \mathcal{E}$; we denote by $\mathcal{E}^\times$ the set of units. Units are exactly non-vanishing E-functions, *i.e.*, the functions of the form $\lambda e^{\alpha x}$ with $\lambda, \alpha \in \overline{\mathbb{Q}}$ and $\lambda \neq 0$ (using the Hadamard factorization theorem).

An *irreducible* element is a non-unit $f \in \mathcal{E}$ such that $f = gh$ with $g, h \in \mathcal{E}$ implies that either g or h is a unit. We shall see in the proof of Proposition 5.7 below that if $f(x_0) = 0$ for some $x_0 \in \overline{\mathbb{Q}}$, then f is irreducible if and only if $f(x) = (x - x_0)u(x)$ for some unit u. In particular a polynomial $f \in \overline{\mathbb{Q}}[x]$ is irreducible in $\mathcal{E}$ if, and only if, it has degree one. We conjecture Bessel's E-functions $\Gamma(\alpha+1)x^{-\alpha}J_\alpha(x)$ to be irreducible when $\alpha \in \mathbb{Q} \setminus (\{\pm 1/2\} \cup (-\mathbb{N}^*))$.

The integral domain $\mathcal{E}$ is not a factorial ring. Indeed, for any $N \geq 1$ we have

$$e^x - 1 = \prod_{k=1}^{N} \left(e^{x/N} - e^{2i\pi k/N} \right) \tag{5.1}$$

whereas in a factorial ring, no non-zero element can be written as a product of arbitrarily many non-units. In the spirit of [23, Theorem 5] we conjecture that, as in the setting of exponential polynomials over $\overline{\mathbb{Q}}$, this problem happens only with simple functions, defined as follows.

Definition 5.1. *A* simple *element of $\mathcal{E}$ is an E-function of the form*

$$g(x) = x^{-\omega} e^{\alpha x} \sum_{i=1}^{N} \lambda_i e^{\beta r_i x} \tag{5.2}$$

with $\alpha, \beta \in \overline{\mathbb{Q}}$, $\beta \neq 0$, $N \geq 2$, $\lambda_1, \ldots, \lambda_N \in \overline{\mathbb{Q}}^*$, *and pairwise distinct rational numbers* $r_1, \ldots, r_N$*; the integer* ω *is chosen so that* f *is holomorphic at* 0 *and* $g(0) \neq 0$*. The* support *of* g*, denoted by* $\mathrm{supp}(g)$*, is the* 1*-dimensional vector space*

$$\mathrm{Span}_{\mathbb{Q}}(\beta) = \{\beta r,\, r \in \mathbb{Q}\} \subset \overline{\mathbb{Q}}$$

which depends only on g *and not on the choice of* α, β, $\lambda_1, \ldots, \lambda_N$, $r_1, \ldots, r_N$.

We point out that this definition is slightly different from the usual one in the setting of exponential polynomials over $\overline{\mathbb{Q}}$ (which amounts to taking $\omega = 0$ in Eq. (5.2)). For instance $\frac{\sin(x)}{x}$ (which is an E-function but not an exponential polynomial) is simple, whereas $\sin(x)$ is not, with our definition. This modification is necessary for Conjecture 5.4 below to be reasonable (otherwise, conjecturally $\frac{\sin(x)}{x}$ would have no factorization).

An E-function is simple if, and only if, it is not a unit and it can be written as

$$g(x) = x^{-\omega} e^{\alpha x} \sum_{i=1}^{N} \lambda_i e^{\beta_i x}$$

with $N \geq 2$, $\alpha, \lambda_1, \ldots, \lambda_N \in \overline{\mathbb{Q}}$, $\beta_1, \ldots, \beta_N \in \overline{\mathbb{Q}}$ such that $\beta_i / \beta_j \in \mathbb{Q}$ for any i, j such that $\beta_j \neq 0$, and $\omega \in \mathbb{N}$ such that $g(0) \neq 0$. With this notation, the support of g is $\mathrm{Span}_{\mathbb{Q}}(\beta_i)$, for any i such that $\beta_i \neq 0$.

We shall use the following characterization very often; here $\mathrm{ord}_1 P$ is the order of multiplicity of 1 as a root of P.

Lemma 5.2. *A function* g *is a simple* E*-function if, and only if, it can be written as*

$$g(x) = x^{-\mathrm{ord}_1 P} u(x) P(e^{\beta x}) \tag{5.3}$$

with $\beta \in \overline{\mathbb{Q}}^*$*, a unit* $u \in \mathcal{E}^\times$*, and* $P \in \overline{\mathbb{Q}}[X]$ *such that* $P(0) \neq 0$*. Moreover, we have* $\mathrm{supp}(g) = \mathrm{Span}_{\mathbb{Q}}(\beta)$.

We point out that this expression is not unique: for instance $e^{\beta x} = (e^{\beta x/N})^N$ so that β and $P(X)$ can be replaced with β/N and $P(X^N)$, for any $N \geq 1$. Given finitely many simple functions with the same support, this remark enables one to write them as (5.3) with the same β. Indeed given finitely many elements $\beta_i \in V \setminus \{0\}$, where V is a 1-dimensional vector space over $\mathbb{Q}$, there exists $\beta \in V \setminus \{0\}$ such that all β_i are integer multiples of β (see the proof of Corollary 4.3 in §4).

Proof of Lemma 5.2. Let g be a simple function, written as (5.2). Modifying α if necessary, we may assume that $r_i \geq 0$ for any i, with equality for an index i. Let D be a common denominator of the rationals r_i; replacing β with β/D we may assume that $D = 1$, so that $r_i \in \mathbb{N}$ for any i. Then letting $u(x) = e^{\alpha x}$ and $P(X) = \sum_{i=1}^{N} \lambda_i X^{r_i} \in \overline{\mathbb{Q}}[X]$, we have $P(0) \neq 0$ (because $\lambda_i \neq 0$ for all i, and $r_i = 0$ for some i), and $g(x) = x^{-\omega} u(x) P(e^{\beta x})$. We may write $P(X) = (X-1)^{\mathrm{ord}_1 P} Q(X)$ with $Q(X) \in \overline{\mathbb{Q}}[X]$ such that $Q(1) \neq 0$. Then we have $g(x) = x^{-\omega} u(x)(e^{\beta x} - 1)^{\mathrm{ord}_1 P} Q(e^{\beta x})$, and the function $Q(e^{\beta x})$ does not vanish at $x = 0$. Now both $g(x)$ and $\frac{e^{\beta x}-1}{x}$ are holomorphic and do not vanish at 0, so that $\omega = \mathrm{ord}_1 P$ and g is of the form (5.3). The converse can be proved easily along the same lines. This concludes the proof of Lemma 5.2. □

Definition 5.3. *A non-zero element $f \in \mathcal{E}$ is said to be* normalized *if its Taylor expansion $\sum_{n=0}^{\infty} a_n x^n$ around the origin satisfies*

$$a_0 = \ldots = a_{p-1} = 0, \quad a_p = 1, \quad a_{p+1} = 0$$

for some $p \geq 0$.

In other words, f is normalized if, and only if, its first non-zero Taylor coefficient at the origin is equal to 1, and the next one vanishes. The point is that for any non-zero $f \in \mathcal{E}$, there exists a unique $u \in \mathcal{E}^\times$ such that fu is normalized. Moreover, any product of normalized E-functions is again normalized. This allows one to have really equalities between functions, not only "up to units". We conjecture that the following analogue of Ritt's factorization theorem holds.

Conjecture 5.4. *Let f be a non-zero E-function. Then there exist a unit $u \in \mathcal{E}^\times$, simple normalized E-functions $s_1, \ldots, s_p$ with pairwise distinct supports (where $p \geq 0$), and irreducible normalized E-functions $h_1, \ldots, h_n$ (with $n \geq 0$), such that*

$$f = u s_1 \ldots s_p h_1 \ldots h_n. \tag{5.4}$$

Moreover $u, s_1, \ldots, s_p, h_1, \ldots, h_n$ are unique.

To state Conjecture 5.4 in a different way, we denote by $\mathcal{V}$ the set of all $\mathbb{Q}$-vector spaces of dimension 1 contained in $\overline{\mathbb{Q}}$, and by $\mathcal{I}$ the set of all normalized irreducible E-functions.

Proposition 5.5. *Conjecture 5.4 is equivalent to the following statement.*

Let f be a non-zero E-function. There exist a unit u, a function s_V which is either equal to 1 or simple normalized with support V (for each $V \in \mathcal{V}$), and a non-negative integer n_h (for each $h \in \mathcal{I}$), such that

$$f = u \Big(\prod_{V \in \mathcal{V}} s_V \Big) \Big(\prod_{h \in \mathcal{I}} h^{n_h} \Big), \tag{5.5}$$

with $s_V = 1$ for all but finitely many $V \in \mathcal{V}$, and $n_h = 0$ for all but finitely many $h \in \mathcal{I}$.

Moreover u, $(s_V)_{V \in \mathcal{V}}$ and $(n_h)_{h \in \mathcal{I}}$ are uniquely determined by f.

We notice that in the products of Eq. (5.5), all factors are equal to the constant function 1 except finitely many of them.

The proof of Proposition 5.5 is straightforward since decompositions (5.4) and (5.5) are equivalent. Indeed for $h \in \mathcal{I}$, n_h is the number of i such that $h_i = h$; and for $V \in \mathcal{V}$, $s_V = s_i$ if there is a (necessarily unique) index i such that $\mathrm{supp}(s_i) = V$, and $s_V = 1$ otherwise.

Until the end of §5, we assume that Conjecture 5.4 holds.

Definition 5.6. *Let f be a non-zero E-function, and $h \in \mathcal{I}$. The h*-adic valuation *of f, denoted by $v_h(f)$, is the exponent n_h of h in the decomposition* (5.5) *of f.*

Let $x_0 \in \overline{\mathbb{Q}}$. Then $x - x_0$ is an irreducible E-function. Indeed, if $x - x_0 = f_1 f_2$ with $f_1, f_2 \in \mathcal{E}$, then up to swapping f_1 and f_2 we may assume that $f_1(x_0) = 0$; then [6, Proposition 4.1] yields $f_3 \in \mathcal{E}$ such that $f_1(x) = (x - x_0) f_3(x)$, and therefore $f_2 f_3 = 1$ so that f_2 is a unit. Now consider the E-function h_{x_0} defined by $h_0(x) = x$ and, if $x_0 \neq 0$, $h_{x_0}(x) = \frac{-1}{x_0} e^{x/x_0}(x - x_0) = \left(1 - \frac{x}{x_0}\right) e^{x/x_0}$. Then h_{x_0} is irreducible too, and it is normalized so that $h_{x_0} \in \mathcal{I}$.

Proposition 5.7. *Let $x_0 \in \overline{\mathbb{Q}}$, and f be a non-zero E-function. Then $v_{h_{x_0}}(f)$ is the order of vanishing of f at x_0, where $h_0(x) = x$ and, if $x_0 \neq 0$,*

$$h_{x_0}(x) = \left(1 - \frac{x}{x_0}\right) e^{x/x_0}.$$

We point out that Proposition 5.7 applies to any $x_0 \in \overline{\mathbb{Q}}$, including $x_0 = 0$, whereas the corresponding statement with exponential polynomials would be false for $x_0 = 0$: indeed $e^x - 1$ vanishes at 0 but is not divisible by x in the ring of exponential polynomials.

Proof of Proposition 5.7. Let x_0 be an algebraic number. First of all, we claim that if g is simple then $g(x_0) \neq 0$. This is part of the definition of a simple function if $x_0 = 0$. Otherwise, Lemma 5.2 yields $g(x) = x^{-\mathrm{ord}_1 P} u(x) P(e^{\beta x})$ for some non-zero $\beta \in \mathrm{supp}(g) \subset \overline{\mathbb{Q}}$, $u \in \mathcal{E}^\times$ and $P \in \overline{\mathbb{Q}}[X]$. Now $e^{\beta x_0}$ is transcendental due to the Hermite-Lindemann Theorem, and $P \neq 0$, so that $P(e^{\beta x_0})$ is non-zero and finally $g(x_0) \neq 0$.

Now let us prove that if $h \in \mathcal{I}$ is such that $h(x_0) = 0$, then $h = h_{x_0}$. Indeed, using [6, Proposition 4.1], h can be written as $(x - x_0) h_1(x)$ with $h_1 \in \mathcal{E}$. Since $x - x_0$ vanishes at x_0, it is not a unit. Now h is irreducible, so that h_1 is a unit: we have $h(x) = \lambda e^{\alpha x}(x - x_0)$ for some $\lambda, \alpha \in \overline{\mathbb{Q}}$ with $\lambda \neq 0$. To conclude we recall that h is normalized. If $x_0 = 0$ then $h(0) = 0$ and $h'(0) = \lambda$ so that $\lambda = 1$ and $\alpha = 0$: we have $h(x) = x = h_0(x)$. On the contrary, if $x_0 \neq 0$ then $h(0) = -\lambda x_0$ so that $\lambda = -1/x_0$ and $\alpha = 1/x_0$. This concludes the proof that if $h \in \mathcal{I}$ is such that $h(x_0) = 0$, then $h = h_{x_0}$.

To conclude the proof of Proposition 5.7, we notice that in Eq. (5.5) no factor on the right-hand side vanishes at x_0, except the one that corresponds to h_{x_0}. □

Recall that g *divides* f (with $f, g \in \mathcal{E}$) means that $f = g g_1$ for some $g_1 \in \mathcal{E}$; Jossen's conjecture asserts that this is equivalent to f/g being entire. We shall explain now how to translate the property that g divides f in terms of the decompositions (5.5) of f and g.

Proposition 5.7 shows that h_{x_0} divides a non-zero $f \in \mathcal{E}$ if, and only if, $f(x_0) = 0$. Indeed a normalized irreducible E-function h divides $f \in \mathcal{E} \setminus \{0\}$ if, and only if, $v_h(f) \geq 1$: this follows at once from the unique decomposition (5.5).

Given simple functions g_1, g_2 with the same support $V \in \mathcal{V}$, using Lemma 5.2 and the remark following it, we write

$$g_1(x) = x^{-\mathrm{ord}_1 P_1} u_1(x) P_1(e^{\beta x}) \quad \text{and} \quad g_2(x) = x^{-\mathrm{ord}_1 P_2} u_2(x) P_2(e^{\beta x})$$

with $\beta \in V \setminus \{0\} \subset \overline{\mathbb{Q}}^*$, units $u_1, u_2 \in \mathcal{E}^\times$, and $P_1, P_2 \in \overline{\mathbb{Q}}[X]$ such that $P_1(0) \neq 0$ and $P_2(0) \neq 0$. We claim that g_1 divides g_2 in $\mathcal{E}$ if, and only if, P_1 divides P_2 in $\overline{\mathbb{Q}}[X]$. If P_1 divides P_2 this is clear. Otherwise, upon dividing P_1 and P_2 by their gcd we may assume they are coprime, and that P_1 is non constant. Then P_1 has a root $y_1 \in \mathbb{C}$, which is not a root of P_2. Since $P_1(0) \neq 0$ we have $y_1 \neq 0$: there exists $x_1 \in \mathbb{C}$ such that $e^{\beta x_1} = y_1$. If $x_1 = 0$ then we may also choose $x_1 = 2i\pi/\beta$, so we assume $x_1 \neq 0$. Then $g_1(x_1) = 0$ and $g_2(x_1) \neq 0$, so that g_1 does not divide g_2 in $\mathcal{E}$ (recall that E-functions are entire, and therefore holomorphic at x_1). This enables us to understand divisibility amongst simple functions with the same support. To understand divisibility in $\mathcal{E}$ we need the following definition.

Definition 5.8. *Let f be a non-zero E-function, and $V \in \mathcal{V}$. The* simple part with support V *of f, denoted by $s_V(f)$, is the function s_V in the decomposition* (5.5) *of f.*

Notice there is a slight abuse in this terminology: if $s_V(f) = 1$, it is not simple and therefore has no support. We call it the simple part with support V of f anyway.

With this definition we have the following result.

Proposition 5.9. *Let f_1, f_2 be non-zero E-functions. Then f_1 divides f_2 in $\mathcal{E}$ if, and only if, the following properties hold:*

- *For any $V \in \mathcal{V}$, $s_V(f_1)$ divides $s_V(f_2)$.*
- *For any $h \in \mathcal{I}$, $v_h(f_1) \leq v_h(f_2)$.*

Proof of Proposition 5.9. If f_1 divides f_2, we have $f_2 = f_1 f$ for some $f \in \mathcal{E} \setminus \{0\}$. Comparing the decompositions (5.5) of f, f_1 and f_2 gives $s_V(f_2) = s_V(f_1)s_V(f)$ and $v_h(f_2) = v_h(f_1) + v_h(f)$ by unicity, and this concludes the proof. Conversely, let $V \in \mathcal{V}$ and assume that $s_V(f_1)$ divides $s_V(f_2)$. Then we have $s_V(f_2) = s_V(f_1) f_V$ for some $f_V \in \mathcal{E} \setminus \{0\}$. Decomposing f_V as in (5.5) yields a decomposition of $s_V(f_2)$. By unicity of the latter, we have $s_V(f_2) = s_V(f_1)s_V(f_V)$. Moreover, if $s_V(f_2) = s_V(f_1) = 1$ then $f_V = 1$ so that $s_V(f_V) = 1$. Finally, if we assume that for any $V \in \mathcal{V}$, $s_V(f_1)$ divides $s_V(f_2)$ and for any $h \in \mathcal{I}$, $v_h(f_1) \leq v_h(f_2)$, then letting

$$f = \Big(\prod_{V \in \mathcal{V}} s_V(f_V)\Big)\Big(\prod_{h \in \mathcal{I}} h^{v_h(f_2) - v_h(f_1)}\Big)$$

(where in each product, only finitely many factors are different from 1), we have $f_2 = f_1 f$. This concludes the proof of Proposition 5.9. □

Proposition 5.9 enables us to define gcd's in $\mathcal{E}$ as follows, starting with simple functions with the same support.

Definition 5.10. *Let g_1, g_2 be simple functions with the same support V. Using Lemma 5.2 we may write $g_1(x) = x^{-\mathrm{ord}_1 P_1} u_1(x) P_1(e^{\beta x})$ and $g_2(x) = x^{-\mathrm{ord}_1 P_2} u_2(x) P_2(e^{\beta x})$ where $u_1, u_2 \in \mathcal{E}^\times$, $P_1, P_2 \in \overline{\mathbb{Q}}[X]$ don't vanish at 0, and $\beta \in V \setminus \{0\} \subset \overline{\mathbb{Q}}^*$. The* gcd of g_1 and g_2, *denoted by* $\gcd(g_1, g_2)$, *is then $P(e^{\beta x})$ where P is the gcd of P_1 and P_2 in $\overline{\mathbb{Q}}[X]$.*

The following lemma shows that this gcd is independent (up to a unit of $\mathcal{E}$) of the choice of β, P_1, P_2, u_1, u_2.

Lemma 5.11. *Let g_1, g_2, g_3 be simple functions with the same support V. Then g_3 divides* $\gcd(g_1, g_2)$ *if, and only if, g_3 divides both g_1 and g_2.*

The proof of this lemma is straightforward upon writing $g_i(x) = x^{-\mathrm{ord}_1 P_i} u_i(x) P_i(e^{\beta x})$ where β is independent of $i \in \{1, 2, 3\}$, and using the remark before Definition 5.8.

We can now generalize the definition of gcd.

Definition 5.12. *Let f_1, f_2 be non-zero E-functions. The* gcd of f_1 and f_2, *denoted by* $\gcd(f_1, f_2)$, *is*

$$\Big(\prod_{V \in \mathcal{V}} \gcd(s_V(f_1), s_V(f_2))\Big)\Big(\prod_{h \in \mathcal{I}} h^{\min(v_h(f_1), v_h(f_2))}\Big).$$

This definition makes sense because of the following result, which is an immediate consequence of Proposition 5.9 and Lemma 5.11.

Proposition 5.13. *Let f_1, f_2, f_3 be non-zero E-functions. Then f_3 divides* $\gcd(f_1, f_2)$ *if, and only if, f_3 divides both f_1 and f_2.*

To sum up the results obtained in this section, we state the following (see [2, p. 4] for the definition and various other names of a gcd domain, including pseudo-Bezout ring [9, p. 280]).

Theorem 5.14. *The ring $\mathcal{E}$ is neither factorial nor noetherian. However, if Conjecture 5.4 holds, it is a gcd domain.*

The non-factoriality of $\mathcal{E}$ has been proved already using Eq. (5.1). To prove that $\mathcal{E}$ is not noetherian, we show that $(I_N)_{N \geq 0}$ is an increasing sequence of ideals, where

$$I_N = \{f \in \mathcal{E}, \quad \forall k \in \mathbb{Z} \quad f(k 2^N \pi) = 0\}.$$

Indeed, for any $N \geq 0$ we have $I_N \subset I_{N+1}$ and $f_N(x) = \sin(x/2^{N+1})$ belongs to $I_{N+1} \setminus I_N$.

At last, given $f_1, f_2 \in \mathcal{E}$, we conjecture that the ideal of $\mathcal{E}$ generated by f_1 and f_2 is not principal in general, so that there are no Bezout relations in $\mathcal{E}$.

6 Zeros of E-functions

In this section we study zeros of E-functions. Our point of view is to state two conjectures dealing with special E-functions, and then to deduce the general properties in Theorems 6.3 and 6.5.

Conjecture 6.1. *Let h_1, h_2 be irreducible E-functions with (at least) a common zero in $\mathbb{C}$. Then $h_1 = uh_2$ for some unit E-function u.*

This conjecture holds if $h_1(x) = x - x_0$ for some $x_0 \in \overline{\mathbb{Q}}$. Indeed, if $h_2(x_0) = 0$ then h_{x_0} divides h_2 (see Proposition 5.7 above). Now both h_{x_0} and h_2 are irreducible, so that $h_2 = u_1 h_{x_0} = u_2 h_1$ for some $u_1, u_2 \in \mathcal{E}^\times$.

Conjecture 6.2. *Let $\xi \in \mathbb{C}^*$ be such that $e^{\beta\xi}$ is algebraic for some $\beta \in \overline{\mathbb{Q}}^*$. Then $h(\xi) \neq 0$ for any irreducible E-function h.*

This result follows from the Hermite-Lindemann theorem if $h(x) = x - x_0$ for some $x_0 \in \overline{\mathbb{Q}}$. If h is an exponential polynomial over $\overline{\mathbb{Q}}$, the conclusion of Conjecture 6.2 follows from Schanuel's conjecture (see Corollary 4.3 in §4).

In particular Conjecture 6.2 implies that no irreducible E-function vanishes at π, $\log 2$, etc.

We are now ready to provide an analogue of Theorem 1.7 for E-functions.

Theorem 6.3. *Assume that Conjectures 5.4, 6.1 and 6.2 hold. Let f_1, f_2 be non-zero E-functions. Then $\frac{f_1}{\gcd(f_1,f_2)}$ and $\frac{f_2}{\gcd(f_1,f_2)}$ have no common zero in $\mathbb{C}$. In other words, common zeros of f_1 and f_2 are exactly the zeros of $\gcd(f_1, f_2)$, and for any such ξ, the order of vanishing of $\gcd(f_1, f_2)$ at ξ is the least of the orders of vanishing of f_1 and f_2 at ξ.*

At last, this result holds unconditionally if f_1 and f_2 are simple.

Assuming that Conjecture 5.4 holds (*i.e.*, that E-functions can be factored as in Ritt's theorem, so that gcd's exist), part (*ii*) of Jossen's Conjecture 1.1 implies the conclusion of Theorem 6.3: coprime E-functions have no common zeros. In a converse way, we have the following.

Corollary 6.4. *If Conjectures 5.4, 6.1 and 6.2 hold then:*

- *Irreducible E-functions have only simple zeros.*
- *Conjecture 1.4 and Jossen's Conjecture 1.1 hold.*

Proof of Corollary 6.4. The first part can be proved exactly in the same way as Corollary 4.2.

To deduce Conjecture 1.4, we may therefore restrict (using the decomposition (5.5)) to a simple function g, and even (using Lemma 5.2) to $g(x) = P(e^{\beta x})$ with $P \in \overline{\mathbb{Q}}[X]$ and

$\beta \in \overline{\mathbb{Q}}^*$. Factoring P in $\overline{\mathbb{Q}}[X]$, what remains to consider is the case where $P(X) = X - y_0$ for some $y_0 \in \overline{\mathbb{Q}}$. Then g' does not vanish, so that g has only simple roots.

To deduce Conjecture 1.1, we consider $f, g \in \mathcal{E} \setminus \{0\}$ such that f/g is entire. Dividing f and g by their gcd if necessary, we may assume that $\gcd(f, g) = 1$. Then all zeros of g are zeros of f, but f and g have no common zeros due to Theorem 6.3. Therefore g does not vanish: it is a unit of $\mathcal{E}$, and f/g is an E-function. This proves the first part of Jossen's conjecture; the second one follows at once from Theorem 6.3. □

Proof of Theorem 6.3. Upon dividing f_1 and f_2 by their gcd (which exists unconditionally if f_1 and f_2 are simple), we may assume that $\gcd(f_1, f_2) = 1$. Let $\xi \in \mathbb{C}$ be a common zero of f_1 and f_2. Then in the decomposition (5.5) of f_1 (resp. of f_2), at least one factor vanishes at ξ. There are 3 possibilities.

First, let us consider the case where there exist $V_1, V_2 \in \mathcal{V}$ such that $g_1 = s_{V_1}(f_1)$ and $g_2 = s_{V_2}(f_2)$ vanish at ξ; in particular this happens if f_1 and f_2 are simple. Recall that $g_i(x)$ can be written as $x^{-\mathrm{ord}_1 P_i} u_i(x) P_i(e^{\beta_i x})$ with $u_i \in \mathcal{E}^\times$, $\beta_i \in \mathrm{supp}(g_i) \setminus \{0\}$, and $P_i \in \overline{\mathbb{Q}}[X] \setminus \{0\}$. Since $g_i(\xi) = 0$, we have $\xi \neq 0$ and $e^{\beta_i \xi}$ is a root of P_i and therefore an algebraic number. Now $e^{\beta_1 \xi}$ and $e^{\beta_2 \xi}$ are both algebraic, so that β_1/β_2 is rational using the Gel'fond-Schneider Theorem. Accordingly we may write $g_i(x) = x^{-\mathrm{ord}_1 P_i} u_i(x) P_i(e^{\beta x})$ with β independent from i; and $g = \gcd(g_1, g_2)$ is defined by $g(x) = P(e^{\beta x})$ where P is the gcd of P_1 and P_2 in $\overline{\mathbb{Q}}[X]$. If ξ is a common zero of g_1/g and g_2/g then $e^{\beta\xi}$ is a common zero of P_1/P and P_2/P, which is impossible since these polynomials are coprime.

If there exist $h_1, h_2 \in \mathcal{I}$ such that $v_{h_1}(f_1)$ and $v_{h_2}(f_2)$ are positive and $h_1(\xi) = h_2(\xi) = 0$, then Conjecture 6.1 implies $h_1 = h_2$, and this irreducible E-function divides $\gcd(f_1, f_2) = 1$: this is a contradiction.

The last possibility (up to swapping f_1 and f_2) is that there exist $V \in \mathcal{V}$ and $h \in \mathcal{I}$ such that both $s_V(f_1)$ and h vanish at ξ, with $v_h(f_2) \geq 1$. Writing $s_V(f_1)(x) = x^{-\mathrm{ord}_1 P} u(x) P(e^{\beta x})$ with $u \in \mathcal{E}^\times$, $P \in \overline{\mathbb{Q}}[X]$ and $\beta \in \overline{\mathbb{Q}}$, we obtain that $e^{\beta\xi}$ is algebraic. Since $h(\xi) = 0$ this contradicts Conjecture 6.2, and concludes the proof of Theorem 6.3. □

We can deduce now an analogue of Corollary 4.3 for E-functions.

Theorem 6.5. *Assume that Conjectures 5.4, 6.1 and 6.2 hold. Let $\xi \in \mathbb{C}$ be a zero of a non-zero E-function. Then one, and only one, of the following holds:*

- *We have $\xi \neq 0$, $e^{\beta\xi} \in \overline{\mathbb{Q}}$ for some $\beta \in \overline{\mathbb{Q}}^*$, and for any $f \in \mathcal{E}$ with $f(\xi) = 0$ there exists $N \geq 1$ such that f is divisible by $e^{\beta x/N} - e^{\beta\xi/N}$ in $\mathcal{E}$.*
- *We have $h(\xi) = 0$ for some irreducible $h \in \mathcal{E}$, and h divides in $\mathcal{E}$ any E-function that vanishes at ξ.*

This result shows that there are (conjecturally) two types of zeros of E-functions. The first one corresponds to $\xi = \frac{\log(\alpha)}{\beta}$ with $\alpha, \beta \in \overline{\mathbb{Q}}^*$ and any determination of $\log(\alpha)$; the second one includes (conjecturally) zeros of Bessel functions J_α, $\alpha \in \mathbb{Q} \setminus \{\pm 1/2\}$, and values at algebraic points of the Lambert W function. Moreover Theorem 6.5 gives a kind of generalization of minimal polynomials (resp. conjugates) of algebraic numbers; in the second case, it would be the irreducible function h, which is unique up to multiplication by a unit of $\mathcal{E}$ (resp. its zeros).

If ξ is algebraic then the conclusion of Theorem 6.5 holds inconditionally. Indeed the first case cannot occur due to the Hermite-Lindemann Theorem, and the second one holds with $h(x) = x - \xi$ using [6, Proposition 4.1]. We see that the minimal function of ξ would be $x - \xi$, which is reasonable: recall that $\overline{\mathbb{Q}}[X] \subset \mathcal{E}$. Maybe the minimal polynomial of ξ over $\mathbb{Q}$ could be recovered in this setting by restricting to E-functions with rational coefficients, but we did not try to do it.

Proof of Theorem 6.5. First of all, Conjecture 6.2 shows that both cases of Theorem 6.5 cannot occur simultaneously. Let $f_0 \in \mathcal{E} \setminus \{0\}$ and $\xi \in \mathbb{C}$ be such that $f_0(\xi) = 0$. Decomposing f_0 as in Conjecture 5.4, at least one factor vanishes at ξ: either a simple function or an irreducible one.

If a simple function $g(x) = x^{-\mathrm{ord}_1 P} u(x) P(e^{\beta x})$ vanishes at ξ, then $e^{\beta\xi}$ is algebraic. We write $V = \mathrm{Span}_{\mathbb{Q}}(\beta) = \mathrm{supp}(g)$ and consider any $f \in \mathcal{E} \setminus \{0\}$ such that $f(\xi) = 0$. Theorem 6.3 shows that ξ is a zero of $\gcd(f, g) = \gcd(s_V(f), g)$. We have $s_V(f)(x) = x^{-\mathrm{ord}_1 Q} v(x) Q(e^{\beta' x})$ for some $v \in \mathcal{E}^\times$, $Q \in \overline{\mathbb{Q}}[X]$ and $\beta' \in V \setminus \{0\}$. Since β and β' span the same $\mathbb{Q}$-vector space, there exists $N \geq 1$ such that β' is an integer multiple of β/N. Then $s_V(f)$, g and $\gcd(s_V(f), g)$ can be written (up to units) as polynomials in $e^{\beta x/N}$ multiplied by suitable powers of x. This provides $S \in \overline{\mathbb{Q}}[X]$ and $w \in \mathcal{E}^\times$ such that $\gcd(s_V(f), g)(x) = x^{\mathrm{ord}_1 S} w(x) S(e^{\beta x/N})$ vanishes at ξ. We have $S(e^{\beta\xi/N}) = 0$ so that $S(X) = (X - e^{\beta\xi/N})T(X)$ for some $T \in \overline{\mathbb{Q}}[X]$, since $e^{\beta\xi/N}$ is algebraic. Then $e^{\beta x/N} - e^{\beta\xi/N}$ divides $\gcd(s_V(f), g) = \gcd(f, g)$, and therefore f, in $\mathcal{E}$. This concludes the proof in the case where a simple function vanishes at ξ.

Let us assume now that $h(\xi) = 0$ for some irreducible E-function h; we assume h to be normalized. For any $f \in \mathcal{E} \setminus \{0\}$ such that $f(\xi) = 0$, Theorem 6.3 shows that $\gcd(f, h) = h^{\min(1, v_h(f))}$ vanishes at ξ. Therefore $v_h(f) \geq 1$ and h divides f. This concludes the proof of Theorem 6.5. □

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