Rational approximations to values of E-functions

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Abstract

We solve a long standing problem in the theory of Siegel's E-functions, initiated by Lang for Bessel's function J_0 in the 60's and considered in full generality by G. Chudnovsky in the 80's: we prove that irrational values taken at rational points by E-functions with rational Taylor coefficients have irrationality exponent equal to 2. This result had been obtained before by Zudilin under stronger assumptions on algebraic independence of E-functions, satisfied by J_0 but not by all hypergeometric E-functions for instance.

1 Introduction

In rational approximation, a landmark result is Roth's theorem [28], proved in 1955: if $\xi \in \mathbb{R} \setminus \mathbb{Q}$ is an algebraic number, then

$$\forall \varepsilon > 0 \quad \exists c > 0 \quad \forall (p,q) \in \mathbb{Z} \times \mathbb{N}^* \qquad \left| \xi - \frac{p}{q} \right| \geq \frac{c}{q^{2+\varepsilon}}. \tag{1.1}$$

This measure of irrationality is optimal in the following sense: it would be false with $\varepsilon = 0$ and $c = 1$, as can be proved using continued fractions or Dirichlet's pigeonhole principle. With respect to Lebesgue measure, almost all $\xi \in \mathbb{R} \setminus \mathbb{Q}$ satisfy (1.1). A folklore belief is that classical transcendental constants coming from analysis should satisfy (1.1); but this is known for very few of them, even amongst the most important ones. For instance, (1.1) is known for $\xi = \pi$ only in a weaker form, with 2 in the exponent of q replaced with 7.1033 (Zeilberger-Zudilin [32]); in other words, the irrationality exponent of π is at most 7.1033. For log(2) the situation is analogous: its irrationality exponent is at most 3.5746 (Marcovecchio [25]). In both cases, this exponent is currently best known and it comes after many successive improvements.

The situation is different for the values of the exponential function: it is well known that (1.1) holds with $\xi = e^r$ for any $r \in \mathbb{Q}^*$ (see the introduction of [4] or [22]). This result opens the way to a possible generalization to E-functions, a class of functions defined by Siegel [30] in 1929 to extend Diophantine properties of the exponential function. In this paper $\overline{\mathbb{Q}}$ is seen as a subset of $\mathbb{C}$.

Definition 1. *A power series $F(z)=\sum_{n=0}^{\infty} a_n z^n/n! \in \overline{\mathbb{Q}}[[z]]$ is an E-function if*
(i) F is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$, and there exists $C>0$ such that:
(ii) for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.
(iii) there exists a sequence of integers $(d_n)_{n\geq 0}$, with $1 \leq d_n \leq C^{n+1}$ for any $n \geq 0$, such that $d_n a_m$ are algebraic integers for all $0 \leq m \leq n$.

Siegel's original definition is more general: in (ii) and (iii), he required bounds of the form "for all $\varepsilon>0$, $\ldots \leq n!^{\varepsilon}$ for any $n \geq N(\varepsilon)$" instead of "$\ldots \leq C^{n+1}$ for any $n \geq 0$". E-functions in the sense of Definition 1 shall be called E-functions in the strict sense. Note that (i) implies that the a_n's all lie in a certain number field $\mathbb{K}$. An E-function is an entire function; it is transcendental unless it is a polynomial.

Amongst the simplest examples of E-functions, we mention $\exp(z)=\sum_{n=0}^{\infty} z^n/n!$ and Bessel's function $J_0(z):=\sum_{n=0}^{\infty}(iz/2)^{2n}/n!^2$. Both are specializations of the generalized hypergeometric function

$$ {}_pF_q\begin{bmatrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{bmatrix}; z\Bigg] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n $$

where $q \geq p \geq 0$ and we define the Pochhammer symbol $(a)_n := a(a+1)\cdots(a+n-1)$ for $n \geq 1$, $(a)_0 := 1$. Siegel has proved that when $q \geq p \geq 0$, the parameters a_j and b_j are in $\mathbb{Q}$ (with the restriction that $b_j \notin \mathbb{Z}_{\leq 0}$ so that $(b_j)_n \neq 0$ for all $n \geq 0$) and $c \in \overline{\mathbb{Q}}$, then ${}_pF_q[a_1,\ldots,a_p;b_1,\ldots b_q;cz^{q-p+1}]$ is an E-function. However, the $\overline{\mathbb{Q}}$-algebra generated by such hypergeometric E-functions is not large enough to contain all E-functions, as recently shown by Fresán-Jossen [18]. We also point out that values of E-functions at algebraic points are closely related to exponential periods, see [19].

The Diophantine theory of the values taken at algebraic points by E-functions has a long history with classical results due, in chronological order, to Siegel, Shidlovskii, Nesterenko, André and Beukers in particular. We refer to their original works [30, 29, 26, 2, 8] as well as to [9, 16, 33] for precise statements of these results and others.

Our purpose is to prove the following result, namely: (1.1) holds for all irrational values of E-functions with rational Taylor coefficients at rational points.

Theorem 1. *Let $f \in \mathbb{Q}[[z]]$ be an E-function and let $r \in \mathbb{Q}$. Then either $f(r) \in \mathbb{Q}$, or for any $\varepsilon>0$, there exists a constant $c>0$ depending on f, r, ε such that for any $p \in \mathbb{Z}$ and any $q \in \mathbb{N}^*$, we have*

$$ \left| f(r) - \frac{p}{q} \right| \geq \frac{c}{q^{2+\varepsilon}}. \tag{1.2} $$

We shall first give the proof of Theorem 1 for E-functions in the strict sense, using in particular various lemmas in Zudilin's paper [33] and results by André and Beukers in the theory of E-operators for E-functions in the strict sense. Then we shall explain in §3.4 the changes that must be made to the proof to obtain Theorem 1 for E-functions in Siegel's sense.

Theorem 1 was announced in 1984 by Chudnovsky [10, p. 1926, Theorem 1 and Corollary] for E-functions in $\mathbb{Q}[[z]]$ in Siegel's original sense and in a stronger form, namely a measure of linear independence. However, as explained in [11, p. 245], the proof contains a gap in the zero estimate. Zudilin [33] succeeded in filling this gap, thereby proving Theorem 1 (in an even more precise form, namely with a decreasing function of q instead of ε), but under additional assumptions on f and r and for E-functions in the strict sense. Precisely, let m be the minimal order of a non-trivial inhomogeneous differential equation with coefficients in $\mathbb{Q}(z)$ satisfied by f. Then Zudilin assumes that either $m \leq 2$ (Corollary 1 on page 583) or that $f, f', \dots, f^{(m-1)}$ are algebraically independent over $\mathbb{Q}(z)$ (Corollary 1 on page 557). In both cases, he also assumes that r is not a singularity of the equation. In the present paper, we use a different approach to zero estimates (see below), which enables us to prove Theorem 1 without any additional assumption on f.

As just explained, the interest of Theorem 1 is that it applies to all E-functions with rational coefficients, for instance to all generalized hypergeometric E-functions of the form ${}_pF_q[a_1, \dots, a_p; b_1, \dots b_q; cz^{q-p+1}]$ with $q \geq p \geq 0$, $c \in \mathbb{Q}$ and rational parameters a_j and b_j, whereas Zudilin applies his results to them only under some assumptions on the parameters, in particular to $J_0(z) = {}_1F_2[1; 1, 1; (iz/2)^2]$. The E-function $g(z) := {}_1F_2[1/2; 1/3, 2/3; z^2]$ does not fall under the scope of Zudilin's results because a minimal inhomogeneous differential equation satisfied by g is $9z^2g'''(z) + 9zg''(z) - (36z^2 + 1)g'(z) - 36zg(z) = 0$ of order $m = 3$ while $g(z)^2 - g'(z)^2/4 + 9z^2(4g(z) - g''(z))^2/4 = 1$; on the other hand, $g(r) \notin \mathbb{Q}$ for all $r \in \mathbb{Q}^*$ (see [17, §7] for details) so that (1.2) holds for all these values. See below for non-trivial examples of transcendental ${}_1F_1$ series with rational parameters taking a rational value at a non-zero rational point, showing that it is not possible to exclude *a priori* the "$f(r) \in \mathbb{Q}$" possibility even in the hypergeometric case.

Another interesting example is the generating E-function of Apéry numbers $\mathcal{A}_{2,2}(z) := \sum_{n=0}^{\infty} (\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2) z^n/n!$. We proved in [16] that for any $r \in \mathbb{Q}^*$, $\mathcal{A}_{2,2}(r)$ has irrationality exponent at most 5 (because we are not able to check that the additional conditions of Zudilin's result are met). Using Theorem 1 we deduce the optimal measure (1.2) immediately for $\mathcal{A}_{2,2}(r)$.

A different problem, very interesting but not studied in this paper, is to know in the setting of Theorem 1 whether $f(r)$ is rational or not when $f \in \mathbb{Q}[[z]]$ is transcendental (if f is a polynomial, obviously $f(r) \in \mathbb{Q}$ for all $r \in \mathbb{Q}$). Beukers' linear independence theorem [8, Corollary 1.4] provides a sufficient condition: given $\sum_{j=0}^{m} p_j(z) f^{(j)}(z) = q(z)$ a non-trivial inhomogeneous equation satisfied by f transcendental of minimal order $m \geq 1$ with polynomial coefficients, if $r \in \mathbb{Q}^*$ is such that $p_m(r) \neq 0$ then $f(r) \notin \mathbb{Q}$. If $p_m(r) = 0$, it can be decided algorithmically whether $f(r) \in \mathbb{Q}$ or not, see [1, 9]. Of course there are trivial examples of transcendental E-functions such that $f(r) \in \mathbb{Q}$ for some $r \in \mathbb{Q}^*$, for instance $(z - 1)e^z$ at $z = 1$. But there are also non-trivial examples such as ${}_1F_1[5; 7/3; -2/3] = 5/27$ and ${}_1F_1[6; -2/5; -12/5] = 1309/625$; see [9, §4.2] for more "exotic" evaluations of hypergeometric E-functions at rational points.

As a consequence of [14, Theorem 4], if f is an E-fonction with rational Taylor coefficients such that $f(1) = e^\alpha$ for some $\alpha \in \overline{\mathbb{Q}}$, then $\alpha \in \mathbb{Q}$. (The proof of this result is an immediate adaptation of that of [13, Proposition 2], which is due to the referee of that paper.) In particular, Theorem 1 does not apply directly to numbers such as $e^{\sqrt{2}}$, for which (1.2) is conjectural. ([1])

It would be very interesting to generalize Theorem 1 to any E-function f in $\mathbb{K}[[z]]$ evaluated at any $\alpha \in \mathbb{K}^*$, where $\mathbb{K}$ is a given fixed number field of degree d over $\mathbb{Q}$. To our knowledge, as a consequence of [16, Theorem 1], the current best upper bound in this generality for the irrationality exponent of $f(\alpha)$ (when it is irrational) is $d(m+1)^d$ where $m \geq 1$ is the minimal order of a non-trivial inhomogeneous differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ satisfied by f. Note however that if $\mathbb{K}$ is an imaginary quadratic number field, this exponent is 2 because all our arguments go through using the modulus of complex numbers instead of the absolute value of real numbers in the Diophantine construction in §§3.2, 3.3 and 3.4. Moreover, under further assumptions on f and α (similar to Zudilin's), this bound can be largely improved using the Lang-Galochkin transcendence measure; see [11, Theorem 5.29 and remarks]. Even more specifically, Kappe [22] has obtained the upper bound $4d^2 - 2d$ for the irrationality exponent of e^α for all $\alpha \in \overline{\mathbb{Q}}^*$ of degree d.

The proof of Theorem 1 is based on Chudnovsky's construction, i.e. on graded Padé approximation. Actually Zudilin used the same construction as Chudnovsky (and he provided all necessary details); so do we. The first new feature of Theorem 1 is that we have do not have to exclude the case where r is a singularity of a differential system, because we use (variants of) results of André and Beukers on E-operators and E-functions. The second, and main, new feature is that no assumption is needed on f in Theorem 1. Actually Zudilin had to make very strong assumptions in order to prove the zero estimate (namely, that a matrix consisting of values of polynomials has maximal rank – see Proposition 3 in §3.2 below). In the present paper, we prove this result using the multiplicity estimate of [15], which relies on the approach of Bertrand-Beukers [6] and Bertrand [5], generalized in [12]. The important feature of this multiplicity estimate is that it takes into account the possibility that some exceptional solutions of the underlying differential system have identically zero remainders. Since we make no assumption on f in Theorem 1, this could *a priori* happen and it would make it impossible to prove that the matrix we are interested in has maximal rank. We prove in §3.6 that there is no such exceptional solution, using both the multiplicity estimate of [15] and an independent technical result (see §4 below). Let us also mention that all the non-explicit constants in the proofs of these intermediate results are effective and could in principle be computed, and consequently the same can be said of the constant c in Theorem 1.

The structure of this paper is as follows. We gather in §2 the auxiliary results we shall need: a property of the minimal inhomogeneous differential equation of an E-function and

[1]Note however that Zudilin's theorem can be applied to the E-function $f(x) := e^{\sqrt{2}x} + e^{-\sqrt{2}x} \in \mathbb{Q}[[x]]$: the number $f(1)$ has irrationality exponent 2, so that $e^{\sqrt{2}}$ has irrationality exponent at most 4, which seems to be the best known upper bound (see [22] and [16] for proofs of the upper bounds 12 and 8 respectively).

a desingularization lemma, adapted from results of André and Beukers respectively. We also state in §2.3 the multiplicity estimate proved in [15]. Then §3 is devoted to the proof of Theorem 1, using a technical result stated and proved in §4. The case of E-functions in Siegel's sense is dealt with in §3.3.

2 Prerequisites

To begin with, we recall that the Nilsson class at 0 is the set of finite sums

$$f(z) = \sum_{e\in\mathbb{C}} \sum_{i\in\mathbb{N}} \lambda_{i,e}\, h_{i,e}(z) z^e (\log z)^i$$

where $\lambda_{i,e} \in \mathbb{C}$, and $h_{i,e}$ is holomorphic at 0. If such a function $f(z)$ is not identically zero, we may assume that $h_{i,e}(0) \neq 0$ for any i, e; then the generalized order of f at 0, denoted by $\mathrm{ord}_0 f$, is the minimal real part of an exponent e such that $\lambda_{i,e} \neq 0$ for some i. When $Y : \mathbb{C} \to \mathbb{C}^q$ is a vector-valued fonction, it is said to be Nilsson at 0 if its q coordinates are.

Thoughout the paper, we shall denote by $M_n(F)$ and $M_{n,m}(F)$ the sets of $n\times n$ matrices and of $n \times m$ matrices over a given field F, typically $\mathbb{C}$ or $\mathbb{C}(z)$.

2.1 Inhomogeneous differential equation of minimal order satisfied by an E-function

Let f be a transcendental E-function with coefficients in a number field $\mathbb{K}$. Consider $f_j(z) = f^{(j-1)}(z)$ for any $j \geq 1$, and denote by $m \geq 1$ the largest integer such that 1, $f_1(z)$, ..., $f_m(z)$ are linearly independent over $\mathbb{K}(z)$. Then $f_{m+1}(z) = f^{(m)}(z)$ is a $\mathbb{K}(z)$-linear combination of these functions: f is solution of a inhomogeneous linear differential equation of order m. This equation has minimal order amongst all inhomogeneous linear differential equations satisfied by f. Note that we consider homogeneous equations to be special cases of inhomogeneous equations for which the constant term is equal to 0, and it may happen that an inhomogeneous equation of minimal order for one of its solutions is in fact homogeneous.

Proposition 1. *The point 0 is either a regular point, or a regular singularity, of any inhomogeneous linear differential equation of minimal order satisfied by a transcendental E-function.*

André has proved this result for the homogeneous linear differential equation of minimal order (and even a more precise one, see [2, Théorème de pureté, p. 706]). In this proposition, it is understood that an inhomogeneous equation is regular or regular singular at 0 if the companion differential system $Y' = AY$ with solution ${}^t(1, f, f', \dots, f^{(m-1)})$ is.

Proof. Let f be a transcendental E-function solution of a non-trivial homogeneous differential equation $Ly = 0$ of minimal order $r \geq 1$, where $L \in \mathbb{C}(z)[d/dz]$. A minimality argument implies that a non-trivial minimal inhomogeneous equation satisfied by f has order r or $r-1$ (see [1, §4]).

In the former case, this inhomogeneous equation is then simply equal to L up a non-zero rational function factor, and the claim follows because, by André's above-mentioned theorem, 0 is a regular point or a regular singularity of L.

We now deal with the case of order $r-1$ which is more complicated. In this situation, there exists $R \in \mathbb{C}(z) \setminus \{0\}$ such that $L^*R = 0$, where L^* is the adjoint of L ([2]) and a minimal inhomogeneous equation satisfied by f is of the form

$$\sum_{j=0}^{r-1} p_j(z) f^{(j)}(z) = c \tag{2.1}$$

where

$$\frac{d}{dz}\left(\sum_{j=0}^{r-1} p_j(z)\frac{d^j}{dz^j}\right) = RL. \tag{2.2}$$

The rational function R can be explicitly determined from L, and then Eq. (2.2) enables to determine suitable $p_j \in \mathbb{C}(z)$ from R and L (first p_{r-1}, then p_{r-2}, etc). Finally, the constant c is computed by determining the constant term in the Laurent series expansion of the left-hand side of (2.1). See [9, §2.4] for more details. By minimality of L, $c \neq 0$ and without loss of generality we can assume that $c = 1$. Since f is transcendental, we have $r \geq 2$. Let $f_2, \dots, f_r$ be other local solutions of $Ly = 0$ at $z = 0$ such that $f_1 := f, f_2, \dots, f_r$ make up a $\mathbb{C}$-basis of the vector space of solutions of $Ly = 0$: by André's theorem, each f_k is in the Nilsson class at $z = 0$ because 0 is at worst a regular singularity of L. From Eq. (2.2), we observe that

$$\frac{d}{dz}\left(\sum_{j=0}^{r-1} p_j(z) f_k^{(j)}(z)\right) = R(z)Lf_k(z) = 0 \quad \text{for any } k \in [\![1, r]\!]$$

so that

$$\sum_{j=0}^{r-1} p_j(z) f_k^{(j)}(z) = c_k$$

for some $c_k \in \mathbb{C}$ (and $c_1 = 1$). Up to reordering the basis $f_1, f_2, \dots, f_r$ and multiplying each f_k by a non-zero constant, we can and shall assume without loss of generality that $f_1, \dots, f_s$ are such that $c_k = 1$ for $k = 1, \dots, s$ and $f_{s+1}, \dots, f_r$ are such that $c_k = 0$ for $k = s+1, \dots, r$, for some $s \in [\![1, r]\!]$. We now write the inhomogeneous equation $\sum_{j=0}^{r-1} p_j(z) y^{(j)}(z) = 1$ satisfied by f as a companion differential system $Y' = AY$ where $A \in M_r(\mathbb{C}(z))$, with the vector solution ${}^t(1, f, \dots, f^{(r-2)})$. From what precedes, it turns

[2]Given a differential operator $L = \sum_{j=0}^{r} q_j (d/dz)^j \in \mathbb{C}(z)[d/dz]$, its adjoint $L^* \in \mathbb{C}(z)[d/dz]$ is defined by $L^*y = \sum_{j=0}^{r} (-1)^j (q_j y)^{(j)}$; see [27, p. 38].

out that in fact

$$U := \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ f_1 & \cdots & f_s & f_{s+1} & \cdots & f_r \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\ f_1^{(r-2)} & \cdots & f_s^{(r-2)} & f_{s+1}^{(r-2)} & \cdots & f_r^{(r-2)} \end{pmatrix}$$

is a fundamental matrix solution of $Y' = AY$. (By definition, if $s = r$, there are only 1's on the first line of U). Indeed, the columns of U are solutions of $Y' = AY$ and they are $\mathbb{C}$-linearly independent because on the second line $f_1, \dots, f_r$ are $\mathbb{C}$-linearly independent, so that U is invertible. Since the entries of U are in the Nilsson class at $z = 0$, it follows from [31, p. 81] that 0 is a regular point or a regular singular point of $Y' = AY$. □

2.2 A version of Beukers' desingularization lemma

Beukers' desingularization lemma [8, Theorem 1.5] is very useful when dealing with E-functions, since it enables one to get rid of all non-zero singularities of the underlying (homogeneous) differential equation. In this section we state and prove a non-homogeneous version of this result that incorporates several additional features: the coefficients lie in a fixed number field (as in [16, Proposition 2]), and two useful properties are preserved (the value at 1 of the first E-function, and the property that 0 is a regular singularity). These properties will be very important in the proof of Theorem 1.

Proposition 2. *Let $g_1, \dots, g_m$ be E-functions with coefficients in a number field $\mathbb{K}$, such that $1, g_1, \dots, g_m$ are linearly independent over $\mathbb{C}(z)$. Assume that the column vector $(1, g_1, \dots, g_m)$ is solution of a first-order differential system $Y' = SY$ with $S \in M_{m+1}(\mathbb{K}(z))$.*

Then there exist E-functions $f_1, \dots, f_m$ with coefficients in $\mathbb{K}$ such that:

- *The functions 1, $f_1, \dots, f_m$ are linearly independent over $\mathbb{C}(z)$.*
- *There exist polynomials $Q_{j,l}(z) \in \mathbb{K}[z]$ such that $g_j(z) = \sum_{l=0}^{m} Q_{j,l}(z) f_l(z)$ for any $j \in [\![1, m]\!]$, where we let $f_0(z) = 1$.*
- *The column vector $(1, f_1, \dots, f_m)$ is solution of a first-order differential system $Y' = \widetilde{S}Y$ with $\widetilde{S} \in M_{m+1}(\mathbb{K}[z, 1/z])$.*
- *If $g_1(1)$ is transcendental then $g_1(1) = f_1(1)$.*
- *If 0 is a regular singularity of the system $Y' = SY$, then it is also a regular singularity of the system $Y' = \widetilde{S}Y$.*

Proof. It follows closely that of Beukers, or more precisely the version over a number field $\mathbb{K}$ given in [16, Proposition 2] (see also [9]). This approach consists in finitely many steps. At each step, one obtains a $\mathbb{K}$-linear combination of $1, g_1, \dots, g_m$ that vanishes at some non-zero algebraic point α; then one replaces one of the functions by this linear combination, and divides by $z-\alpha$ if $\alpha \in \mathbb{K}$ (by the minimal polynomial of α over $\mathbb{K}$ in the general case). The point is that the above-mentioned linear combination is never just the function 1, since 1 does not vanish at α. Therefore the function 1 can be preserved at each step. Moreover it is clear from the proof that 0 being a regular singularity holds throughout the procedure. This proves Proposition 2, except for the property $g_1(1) = f_1(1)$. However, if $g_1(1) = \sum_{l=0}^m Q_{1,l}(1) f_l(1)$ is transcendental then there exists $l_0 \in [\![1, m]\!]$ such that $Q_{1,l_0}(1) \neq 0$. Replacing $f_{l_0}(z)$ with $\sum_{l=0}^m Q_{1,l}(1) f_l(z)$ does not change the other properties of the functions $f_1, \ldots, f_m$, and provides $g_1(1) = f_{l_0}(1)$. Up to a permutation of $f_1, \ldots, f_m$ this concludes the proof of Proposition 2. □

2.3 A multiplicity estimate with possibly zero remainders

In this section we state the multiplicity estimate our proof relies on, namely Theorem A. It is a special case of [15, Theorem 3] because the vanishing orders are considered at only one point (namely 0), and the polynomials are evaluated at some $\alpha \neq 0$ which is not a singularity of the differential system $Y' = AY$.

Let $q \geq 1$, $A \in M_q(\mathbb{C}(z))$, $n \geq 0$, and $P_1, \ldots, P_q \in \mathbb{C}[z]$ be such that $\deg P_i \leq n$ for any i. We identify tuples in $\mathbb{C}^q$ with column matrices in $M_{q,1}(\mathbb{C})$. Then with any solution $Y = (y_1, \ldots, y_q)$ of the differential system $Y' = AY$ is associated a remainder $R(Y)$ defined by

$$R(Y)(z) = \sum_{i=1}^{q} P_i(z) y_i(z).$$

The derivatives of such a remainder can be written as [29, Chapter 3, §4]

$$R(Y)^{(k-1)}(z) = \sum_{i=1}^{q} P_{k,i}(z) y_i(z), \tag{2.3}$$

where the rational functions $P_{k,i} \in \mathbb{C}(z)$ are defined for $k \geq 1$ and $1 \leq i \leq q$ by

$$\begin{pmatrix} P_{k,1} \\ \vdots \\ P_{k,q} \end{pmatrix} = \left(\frac{\mathrm{d}}{\mathrm{d}z} + {}^t A \right)^{k-1} \begin{pmatrix} P_1 \\ \vdots \\ P_q \end{pmatrix}. \tag{2.4}$$

We consider the matrix $M(z) = (P_{k,i}(z))_{1 \leq i,k \leq q} \in M_q(\mathbb{C}(z))$; obviously the poles of the coefficients $P_{k,i}$ of M are amongst those of A.

The main new feature of the multiplicity estimate proved in [15] is that it takes into account the possibility that $R(Y)(z)$ is identically zero for some non-zero solutions $Y(z)$ of the differential system $Y' = AY$. To state this result, we denote by $\varrho \geq 0$ the dimension of the $\mathbb{C}$-vector space of solutions Y such that $R(Y)(z)$ is identically zero.

Theorem A. *There exists a positive constant c_1, which depends only on A, with the following property. Let $(Y_j)_{j\in J}$ be a family of solutions of $Y' = AY$ such that the functions $R(Y_j)$, $j \in J$, are $\mathbb{C}$-linearly independent and belong to the Nilsson class at 0. Assume that*

$$\sum_{j\in J} \mathrm{ord}_0(R(Y_j)) \geq (n+1)(q-\varrho) - \tau \tag{2.5}$$

for some $\tau \in \mathbb{Z}$. Then:

- *We have $\tau \geq -c_1$.*
- *If $0 \leq \tau \leq n - c_1$ then for any $\alpha \in \mathbb{C}^*$ which is not a singularity of the differential system $Y' = AY$, the matrix $(P_{k,i}(\alpha))_{1\leq i\leq q, 1\leq k<\tau+c_1} \in M_{q,\tau+c_1-1}(\mathbb{C})$ has rank at least $q-\varrho$.*

In this setting, under the assumptions of (ii), the matrix $(P_{k,i}(\alpha))$ has rank equal to $q-\varrho$. Indeed, there exist ϱ $\mathbb{C}$-linearly independent solutions Y such that $R(Y)$ is identically zero. For each of them, we have $\sum_{i=1}^{q} P_{k,i}(\alpha)y_i(\alpha) = R(Y)^{k-1}(\alpha) = 0$ for any $k \geq 1$: since α is not a singularity, this provides ϱ linearly independent linear relations between the columns of the matrix $(P_{k,i}(\alpha))$.

We remark that Theorem A would not hold if the linear independence assumption were on the Y_j rather than on the $R(Y_j)$. Indeed, for instance if $R(Y_j)$ were identically zero for some $j \in J$, then $\mathrm{ord}_0(R(Y_j))$ would be infinite and Eq. (2.5) would hold for any $\tau \in \mathbb{Z}$. Moreover, solutions Y such that $R(Y)$ is identically zero are taken advantage of in Theorem A: each such solution (in a linearly independent family) provides the same benefit in Eq. (2.5) as an additional function Y_j such that $R(Y_j)$ would vanish to order $n+1$. This property will be used in a crucial way in the proof of Theorem 1 (see §3.6).

Remark 1. Following the proof [15] of Theorem A and using the results of [7] shows that c_1 can be effectively computed in terms of A.

3 Proof of the main result

This section is devoted to the proof of Theorem 1, using a technical result that will proved later in §4.

In §3.1 we introduce the notation and setting of the proof, applying the results of §§2.1 and 2.2. Then in §3.2 we give construction and properties of graded Padé approximants, including the zero estimate (namely Proposition 3). Admitting this result, we prove Theorem 1 stated in the introduction in §3.3, and its generalization to an E-function in Siegel's orginal sense, with arbitrary Taylor coefficients in $\overline{\mathbb{Q}}$, in §3.4 (namely Theorem **??** stated where).

The rest of the paper is devoted to the proof of Proposition 3, carried out in §3.6 using the differential system considered in §3.5 and the technical result proved in §4.

3.1 Setting, notations and parameters

In this section we describe the setting of the proof of Theorem 1. We use the results of §§2.1 and 2.2 to obtain a family of linearly independent E-functions, solution of a first order differential system with no non-zero finite singularity. Then we introduce (essentially) the same notation and parameters as in Zudilin's proof.

To prove Theorem 1 we start with an E-function $g(z)$ with coefficients in $\mathbb{Q}$, and $r \in \mathbb{Q}^*$. We assume that $g(r)$ is irrational; then $g(r)$ is transcendental by [14, Theorem 4]. Considering $g(rz)$ instead of g, we may assume that $r = 1$.

As in §2.1 we consider a (possibly) inhomogeneous linear differential equation of minimal order satisfied by the transcendental E-function g. Proposition 1 asserts that 0 is (at worst) a regular singularity of this equation. Viewing this equation as a differential system of order one satisfied by the column vector $(1, g, g', \ldots, g^{(m-1)})$, we apply Proposition 2 and obtain E-functions 1, f_1, …, f_m with coefficients in $\mathbb{Q}$, linearly independent over $\mathbb{C}(z)$, such that $f_1(1) = g(1)$ is the number we are interested in to prove Theorem 1. The important point is that $(f_1, \ldots, f_m)$ is a solution of a first-order inhomogeneous differential system

$$f_l'(z) = S_{l,0}(z) + \sum_{j=1}^{m} S_{l,j}(z) f_j(z) \text{ for any } l \in [\![1, m]\!] \tag{3.1}$$

with $S_{l,j}(z) \in \mathbb{Q}[z, 1/z]$: the only possible finite singularity of this system is zero, and it is regular (in case it is a singularity).

We consider multi-indices $\kappa \in \mathbb{N}^m$ and sums over such multi-indices. In such a sum, whenever an index κ belongs to $\mathbb{Z}^m$ but not to $\mathbb{N}^m$ (i.e., has at least one negative component), the term corresponding to this index will be considered as 0. For $\kappa = (\kappa_1, \ldots, \kappa_m) \in \mathbb{N}^m$, we write $|\kappa| = \kappa_1 + \ldots + \kappa_m$.

We denote by $(e_1, \ldots, e_m)$ the canonical basis of $\mathbb{Z}^m$, i.e. $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the i-th coordinate is equal to 1. We let

$$\Omega = \{\kappa \in \mathbb{N}^m,\, N - 1 \le |\kappa| \le N\}, \quad \Theta = \{\kappa \in \mathbb{N}^m,\, |\kappa| = N\},$$

and

$$\omega = \operatorname{Card} \Omega = \binom{N+m-2}{m-1} + \binom{N+m-1}{m-1}, \quad \theta = \operatorname{Card} \Theta = \binom{N+m-1}{m-1}.$$

We remark, for future reference, that

$$\frac{\omega}{\theta} = 2 - \frac{m-1}{N+m-1} = 1 + \frac{N}{N+m-1}.$$

We also fix a bijective map $[\![1, \omega]\!] \to \Omega$ so that indices $\kappa \in \Omega$ can be seen as integers between 1 and ω; for instance a family $(x_\omega)_{\omega \in \Omega} \in \mathbb{C}^\Omega$ can also be seen as a tuple in $\mathbb{C}^\omega$, or as a column matrix in $M_{\omega,1}(\mathbb{C})$.

Let us introduce now the parameters that will be used in the construction. Let N be sufficiently large with respect to $f_1, \ldots, f_m$, and let $\eta > 0$ be a real number such that

$$\eta \leq \frac{1}{3(N+m-1)}. \tag{3.2}$$

This number η plays the role of the real denoted by ε in [33]. Let M be sufficiently large with respect to $f_1, \ldots, f_m$, N, and η; consider

$$K = \left\lfloor \frac{(\omega - \eta)M}{\theta} \right\rfloor. \tag{3.3}$$

Remark 2. These parameters are the same as the ones used by Zudilin, except that he assumes that equality holds in Eq. (3.2), and gives an explicit value for M in terms of N. The reason for this difference is that we have not computed explicitly the constant C_4 in Proposition 3 (which depends on N). This is useless for our purpose but it could be done (see Remark 3); it would lead to explicit values of η and M, and then to an explicit value of the constant c in Theorem 1.

3.2 Construction and properties of graded Padé approximants

In this section, we state the construction and properties of graded Padé approximants, sketched by Chudnovsky [10] and proved in detail by Zudilin [33]. Apart from the zero estimate (namely Proposition 3 below), the results are exactly the same as in Zudilin's paper.

To motivate this construction, let us explain it with different notations in the case $m = 2$. We shall construct polynomials $A_0, \ldots, A_N, B_0, \ldots, B_{N-1}$ such that $A_i(z) + B_{i-1}(z)f_1(z) + B_i(z)f_2(z)$ vanishes with high multiplicity at 0, for any $i \in [\![0, N]\!]$. The point here is that B_{-1} and B_N are considered to be identically zero, so that for $i = N$ the function $A_N(z) + B_{N-1}(z)f_1(z)$ vanishes to high order at 0, and is therefore presumably small at $z = 1$.

Let us come back to our general setting now. Recall that the parameters are given by Eqs. (3.2) and (3.3); they are the same as in [33], except that η and M are not fixed in terms of N. The following construction is exactly [33, Lemma 1.1].

Lemma 1. *There exist polynomials $P_\kappa(z)$ of degree less than M, for $\kappa \in \Omega$, not all zero, such that*

$$\mathrm{ord}_0\Big(P_\kappa(z) + \sum_{j=1}^{m} P_{\kappa - e_j}(z) f_j(z)\Big) \geq K \quad \text{for any } \kappa \in \Theta$$

and

$$\pi_{\kappa,\nu} \in \mathbb{Z}, \quad |\pi_{\kappa,\nu}| \leq C_0^{\omega M/\eta}$$

for any $\kappa \in \Omega$ and any $\nu \in [\![0, M-1]\!]$, where the coefficients $\pi_{\kappa,\nu}$ are defined by

$$P_\kappa(z) = \sum_{\nu=0}^{M-1} \frac{\pi_{\kappa,\nu}}{\nu!} z^\nu \quad \text{for any } \kappa \in \Omega.$$

Here and below, we denote by C_0, C_1, …, C_4 positive constants that depend only on f_1, …, f_m (except that C_4 depends also on N).

Now recall that all coefficients $S_{l,j}(z)$ of the differential system (3.1) belong to $\mathbb{Q}[z, 1/z]$, i.e. that this system has no non-zero finite singularity. Therefore denoting by $T(z)$ the least common denominator of the $S_{l,j}(z)$, we have

$$T(z) = z^i \text{ for some } i \in \mathbb{N}, \text{ and } T(z)S_{l,j}(z) \in \mathbb{Q}[z] \text{ for any } l, j. \tag{3.4}$$

As in [33, Eq. (1.8)] we define recursively polynomials $P_\kappa^{[k]}(z)$, for $k \geq 1$ and $\kappa \in \Omega$, by letting $P_\kappa^{[1]}(z) = P_\kappa(z)$ and for any $k \geq 1$ and any $\kappa \in \Omega$,

$$\begin{aligned} P_\kappa^{[k+1]}(z) \quad &= \quad T(z)\Big(\frac{\mathrm{d}}{\mathrm{d}z}P_\kappa^{[k]}(z) + (|\kappa| + 1 - N)\sum_{l=1}^m S_{l,0}(z)P_{\kappa-e_l}^{[k]}(z) \\ &\qquad - \sum_{l=1}^m \sum_{j=1}^m (\kappa_j - \delta_{l,j} + 1)S_{l,j}(z)P_{\kappa-e_l+e_j}^{[k]}(z)\Big). \end{aligned} \tag{3.5}$$

We recall that $\delta_{l,j}$ is Kronecker's symbol, and whenever $\kappa - e_l \in \mathbb{Z}^m$ (resp. $\kappa - e_l + e_j$) has a negative coefficient, the corresponding term is omitted. The only difference with [33, Eq. (1.8)] is a shift in the index k: our $P_\kappa^{[k]}(z)$ is denoted by $P_\kappa^{[k-1]}(z)$ in [33]. The connection of this definition of $P_\kappa^{[k]}(z)$ with a differential system will be explained in §3.5 below.

The following result is part (a) of [33, Lemma 1.3], with coefficients $\pi_{k,\kappa,\nu}$ defined by

$$P_\kappa^{[k]}(z) = \sum_{\nu=0}^{M+t(k-1)-1} \frac{\pi_{k,\kappa,\nu}}{\nu!} z^\nu \text{ for any } \kappa \in \Omega \text{ and any } k \geq 1,$$

and

$$t = \max\Big(\deg T(z), \max_{1 \leq l \leq m} \max_{0 \leq j \leq m} \deg(T(z)S_{l,j}(z))\Big).$$

Lemma 2. *For any* $\kappa \in \Omega$*, any* $k \geq 1$ *and any* $\nu \in [\![0, M+t(k-1)-1]\!]$*, we have* $\pi_{k,\kappa,\nu} \in \mathbb{Z}$*, and*

$$|\pi_{k,\kappa,\nu}| \leq C_0^{\omega M/\eta} M^{C_2 \eta M} \qquad \text{if } k < C_1 \eta M.$$

The following result is the special case $l^* = 1$ and $\alpha = 1$ of part (b) of [33, Lemma 1.3].

Lemma 3. *For any* $k \geq 1$ *such that* $k < C_1 \eta M$*, we have*

$$\Big|P_{(N,0,0,\ldots,0)}^{[k]}(1) + P_{(N-1,0,0,\ldots,0)}^{[k]}(1)f_1(1)\Big| \leq C_0^{\omega M/\eta} M^{C_2 \eta M} C_3^M M^{-K}.$$

The main point of the present paper is the following result. It is proved in [33, Lemma 3.5] with an explicit value of C_4, namely ω, under the assumption that f_1, …, f_m are algebraically independent.

Proposition 3. *There exists a constant C_4, which depends on $f_1, \ldots, f_m$ and on N (but not on M or η), such that the matrix $(P_\kappa^{[k]}(1))_{\kappa\in\Omega, 1\le k\le \lfloor 2\eta M\rfloor + C_4}$ has rank ω.*

Remark 3. We shall prove that C_4 can (in principle) be computed effectively in terms of $f_1, \ldots, f_m$ and N.

Proposition 3 will be proved in §3.6, using the differential system considered in §3.5 and Theorem 2 proved in §4. In the next two sections, we admit Proposition 3 and deduce from it the results announced in the introduction.

3.3 Proof of Theorem 1 for E-functions in the strict sense

Let us now prove Theorem 1 for E-functions in the strict sense, following [33, pp. 581–583] but without explicit expressions for η and M.

Starting wih an E-function $g \in \mathbb{Q}[[z]]$ and $r \in \mathbb{Q}$ such that $g(r) \notin \mathbb{Q}$, we construct $f_1, \ldots, f_m$ as in §3.1 so that $f_1(1) = g(r)$. Let $\varepsilon > 0$; we may assume that ε is sufficiently small in terms of $f_1, \ldots, f_m$. We choose $N = \lfloor m/\varepsilon \rfloor + 1$ so that

$$N \ge m/\varepsilon$$

and N can be made sufficiently large in terms of $f_1, \ldots, f_m$. We recall that $\omega/\theta = 1 + \frac{N}{N+m-1}$, so that

$$\Big(1 + \frac{m}{N}\Big)\Big(1 - \frac{\omega}{\theta}\Big) < -1.$$

Using this bound and the fact that C_2, t, ω and θ depend only on N and $f_1, \ldots, f_m$, we may choose $\eta > 0$ sufficiently small (with respect to N, $f_1, \ldots, f_m$) so that Eq. (3.2) holds and

$$\Big(1 + \frac{m}{N}\Big)\Big(1 + 2t\eta + C_2\eta - \frac{\omega - \eta}{\theta}\Big) < -(1 + 2t\eta + C_2\eta). \tag{3.6}$$

In what follows, we assume M to be sufficiently large in terms on η, N, $f_1, \ldots, f_m$; we shall denote by $C_j(\eta, N)$ positive constants that depend on η, N, $f_1, \ldots, f_m$.

Since the matrix $(P_\kappa^{[k]}(1))_{\kappa\in\Omega, 1\le k\le \lfloor 2\eta M\rfloor + C_4}$ of Proposition 3 has rank ω, the submatrix consisting in the columns indexed by $\kappa = (N, 0, \ldots, 0)$ and $\kappa = (N-1, 0, \ldots, 0)$ has rank 2. This provides positive integers $k_1, k_2 < 2\eta M + C_4$ such that

$$\det \begin{pmatrix} P^{[k_1]}_{(N,0,0,\ldots,0)}(1) & P^{[k_1]}_{(N-1,0,0,\ldots,0)}(1) \\ P^{[k_2]}_{(N,0,0,\ldots,0)}(1) & P^{[k_2]}_{(N-1,0,0,\ldots,0)}(1) \end{pmatrix} \neq 0. \tag{3.7}$$

For any $j \in \{1, 2\}$, let

$$p_j = -(M + tk_j)!\, P^{[k_j]}_{(N,0,0,\ldots,0)}(1) \quad \text{and} \quad q_j = (M + tk_j)!\, P^{[k_j]}_{(N-1,0,0,\ldots,0)}(1).$$

Lemma 2 yields $p_j, q_j \in \mathbb{Z}$, and also

$$\begin{aligned} |q_j| &\leq (M + t(2\eta M + C_4))!\, e C_0^{\omega M/\eta} M^{C_2 \eta M} \\ &\leq (M(1+2t\eta) + tC_4))^{tC_4}\, (M(1+2t\eta))!\, e C_0^{\omega M/\eta} M^{C_2\eta M} \\ &\leq C_5(\eta, N)^M\, M^{(1+2t\eta + C_2\eta)M} \end{aligned} \tag{3.8}$$

provided $M \geq C_6(\eta, N)$, and Lemma 3 yields in the same way (using Eq. (3.3))

$$\begin{aligned} |q_j f_1(1) - p_j| &\leq (M + t(2\eta M + C_4))!\, C_0^{\omega M/\eta} M^{C_2\eta M} C_3^M M^{-K} \\ &\leq C_7(\eta, N)^M\, M^{(1+2t\eta+C_2\eta - \frac{\omega-\eta}{\theta})M} \end{aligned} \tag{3.9}$$

if $M \geq C_8(\eta, N)$.

Now let $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$; upon changing the constant c in Theorem 1 (since $f_1(1) \notin \mathbb{Q}$), we may assume that $|p|$ and q are sufficiently large (with respect to η, N, f_1, …, f_m, since these quantities have been chosen in terms of $f_1(1)$ and ε only). We choose for M the least integer such that

$$C_7(\eta, N)^M\, M^{(1+2t\eta+C_2\eta-\frac{\omega-\eta}{\theta})M} \leq \frac{1}{2q}. \tag{3.10}$$

This integer exists because we have assumed $\eta > 0$ sufficiently small in terms of N, f_1, …, f_m, so that $1 + 2t\eta + C_2\eta - \frac{\omega-\eta}{\theta} < 0$; moreover M can be made large enough (in terms of η, N, f_1, …, f_m) by assuming that q is. Then Eq. (3.9) yields

$$q\,|q_j f_1(1) - p_j| \leq 1/2 \text{ for any } j \in \{1, 2\}.$$

Now Eq. (3.7) yields $\det \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \neq 0$, so that (p, q) is non-collinear to at least one of the (p_j, q_j), $j \in \{1, 2\}$. For this index j we have

$$\det \begin{pmatrix} p & p_j \\ q & q_j \end{pmatrix} \in \mathbb{Z} \setminus \{0\}. \tag{3.11}$$

This determinant is also equal to

$$\det \begin{pmatrix} p - qf_1(1) & p_j - q_j f_1(1) \\ q & q_j \end{pmatrix},$$

so that

$$\begin{aligned} |q_j|\,|qf_1(1) - p| &= \left| \det \begin{pmatrix} p & p_j \\ q & q_j \end{pmatrix} - q\,(q_j f_1(1) - p_j) \right| \\ &\geq 1 - q\,|q_j f_1(1) - p_j| \geq 1 - 1/2 = 1/2 \end{aligned}$$

and, using Eqs. (3.8) and (3.6):

$$|qf_1(1) - p| \geq \frac{1}{2|q_j|} \geq \frac{1}{2} C_5(\eta, N)^{-M} M^{-(1+2t\eta+C_2\eta)M}$$
$$\geq \left[C_7(\eta, N)^{M-1} (M-1)^{(1+2t\eta+C_2\eta-\frac{\omega-\eta}{\theta})(M-1)} \right]^{1+m/N}$$

provided $M \geq C_9(\eta, N)$. Since M is the least integer such that Eq. (3.10) holds, we deduce that

$$|qf_1(1) - p| > \left(\frac{1}{2q}\right)^{1+m/N}.$$

Since $m/N \leq \varepsilon$ this concludes the proof of Theorem 1 for E-functions in the strict sense.

3.4 Proof of Theorem 1 for E-functions in Siegel's sense

In this section, we explain the changes that must be done to obtain Theorem 1 to any E-function f in Siegel's original sense.

• Firstly, in the proof of Theorem 1 for E-functions in the strict sense, we use various results of André and Beukers that they have proved only for E-functions in the strict sense (using the theory of E-operators due to the former). Since then, all these results have been proved to hold *verbatim* for E-functions in Siegel's sense by Lepetit [24], completing the results already given in [3, pp. 746–747].

• Secondly, in §3.2 we use *verbatim* Zudilin's estimates that he has also proved only for E-functions in $\mathbb{Q}[[z]]$ in the strict sense. Let us mention the changes that must be made to his lemmas to deal with Siegel's E-functions in $\mathbb{Q}[[z]]$. We recall that the archimedean and non-archimedean bounds on the Taylor coefficients of Siegel's E-functions are of the form "for all $\varepsilon' > 0, \ldots \leq n^{\varepsilon' n}$ for all $n \geq N(\varepsilon')$". In Lemma 1, this changes the quantity $C_0^{\omega M/\eta}$ by $M^{\omega\varepsilon' M/\eta}$, where $\varepsilon' > 0$ is fixed and independent of the other parameters but arbitrarily small, and $M \geq M_0(\varepsilon')$. The same remark applies in Lemma 2, where $C_0^{\omega M/\eta} M^{C_2\eta M}$ becomes $M^{\omega\varepsilon' M/\eta + C_2\eta M}$, and in Lemma 3, where M^{-K} reads $M^{-K(1-\varepsilon')}$ and the constant C_3 is also possibly changed but it still does not depend on M. With these estimates, we conclude the proof as that of Theorem 1 for E-functions in the strict sense because ε' can be taken arbitrarily small provided M is assumed to be large enough, which can be assumed as in §3.3.

The rest of the present paper is devoted to a proof of Proposition 3, which has been admitted in §§3.3 and 3.4.

3.5 Differential system

In this section we define a matrix $A \in M_\omega(\mathbb{Q}(z))$ and consider the differential system $Y' = AY$, of which solutions will be constructed in Proposition 4. As stated in §3.1 a bijective map $[\![1, \omega]\!] \to \Omega$ is fixed, so that a solution Y is a vector $(y_\kappa(z))$ indexed by $\kappa \in \Omega$. Here and below, we identify tuples in $\mathbb{C}^q$ with column matrices in $M_{q,1}(\mathbb{C})$.

We shall also relate the notation $P_\kappa^{[k]}$ of §3.2 to the $P_{k,\kappa}$ of §2.3; in what follows we will use mostly the notation of §2.3, including

$$R(Y)(z) = \sum_{\kappa\in\Omega} P_\kappa(z) y_\kappa(z) \tag{3.12}$$

when $Y = (y_\kappa(z))$ is a solution of the differential system $Y' = AY$.

The matrix $A = (A_{\lambda,\kappa}(z))_{\lambda,\kappa\in\Omega} \in M_\omega(\mathbb{Q}(z))$ that we consider is defined (in terms of the coefficients $S_{l,j}(z)$ of the differential system (3.1)) by

$$A_{\lambda,\kappa}(z) = \begin{cases} -\lambda_j S_{l,j}(z) \;\text{ if } \kappa = \lambda - e_j + e_l \text{ for some } j,l \in [\![1,m]\!] \text{ with } j \neq l, \\ -\sum_{j=1}^m \lambda_j S_{j,j}(z) \;\text{ if } \kappa = \lambda, \\ S_{j,0}(z) \;\text{ if } \kappa = \lambda + e_j \text{ and } |\lambda| = N-1, \\ 0 \;\text{ otherwise,} \end{cases} \tag{3.13}$$

as in [33, Eq. (3.2)]. We recall from §3.1 that all rational functions $S_{l,j}(z)$, and therefore all coefficients of A, belong to $\mathbb{Q}[z, 1/z]$. With this definition, Eq. (3.5) reads

$$\begin{pmatrix} P_1^{[k+1]}(z) \\ \vdots \\ P_\omega^{[k+1]}(z) \end{pmatrix} = T(z)\left(\frac{\mathrm{d}}{\mathrm{d}z} + {}^t A(z)\right) \begin{pmatrix} P_1^{[k]}(z) \\ \vdots \\ P_\omega^{[k]}(z) \end{pmatrix}.$$

Except for the multiplicative factor $T(z)$ (used to ensure that all $P_\kappa^{[k]}(z)$ are polynomials), this is the same recurrence relation as the one used in §2.3 to define the rational functions $P_{k,\kappa}(z)$ (see Eq. (2.4)); notice that $P_{k,\kappa}(z) \in \mathbb{Q}[z, 1/z]$ since $A \in M_\omega(\mathbb{Q}[z, 1/z])$. Using the fact that $P_{1,\kappa} = P_\kappa^{[1]} = P_\kappa$ by definition, we obtain by induction that

$$\begin{pmatrix} P_1^{[k]}(z) \\ \vdots \\ P_\omega^{[k]}(z) \end{pmatrix} = T(z)^{k-1} \begin{pmatrix} P_{k,1}(z) \\ \vdots \\ P_{k,\omega}(z) \end{pmatrix} + \sum_{k'=1}^{k-1} U_{k,k'}(z) \begin{pmatrix} P_{k',1}(z) \\ \vdots \\ P_{k',\omega}(z) \end{pmatrix}$$

with rational functions $U_{k,k'}(z) \in \mathbb{Q}[z, 1/z]$, since $T(z)$ and all coefficients of $A(z)$ belong to $\mathbb{Q}[z, 1/z]$. Now recall from Eq. (3.4) that $T(z) = z^i$ for some $i \in \mathbb{N}$, so that $T(1) = 1$ and

$$\mathrm{rk}(P_\kappa^{[k]}(1))_{\kappa\in\Omega, 1\leq k\leq k_0} = \mathrm{rk}(P_{k,\kappa}(1))_{\kappa\in\Omega, 1\leq k\leq k_0} \tag{3.14}$$

for any $k_0 \geq 1$. This equality will be used at the end of §3.6 to prove Proposition 3, since Theorem A yields a lower bound on $\mathrm{rk}(P_{k,\kappa}(1))_{\kappa\in\Omega, 1\leq k\leq k_0}$.

The end of this section is devoted to the proof of the following result; notice that parts (*i*) and (*ii*) are essentially proved in [33, pp. 575–576]. For $\kappa \in \Theta$ we define $Z_\kappa = (z_{\kappa,\lambda})_{\lambda\in\Omega} \in \mathbb{C}^\omega$ by

$$z_{\kappa,\lambda} = \begin{cases} 1 \text{ if } \lambda = \kappa, \\ f_j(1) \text{ if } \lambda = \kappa - e_j \text{ for some } j \in [\![1,m]\!], \\ 0 \text{ otherwise.} \end{cases}$$

Proposition 4. *There exist solutions $Y_\kappa(z)$ of the differential system $Y' = AY$, for $\kappa \in \Theta$, such that:*

(*i*) *For any $\kappa \in \Theta$, $Y_\kappa(1) = Z_\kappa$.*

(*ii*) *The functions $Y_\kappa(z)$, $\kappa \in \Theta$, are linearly independent over $\mathbb{C}$.*

(*iii*) *For any $\kappa \in \Theta$, we have $R(Y_\kappa)(z) = O(z^{K-cN})$ as $z \to 0$, where $c > 0$ is a constant that depends only on $f_1, \ldots, f_m$.*

(*iv*) *For any $\kappa \in \Theta$, the function $R(Y_\kappa)(z)$ belongs to the Nilsson class at 0.*

Remark 4. The constant c in part (*iii*) can be made effective, using the results of [7].

Proof. Consider the differential system

$$a'_k(z) = -S_{1,k}(z)a_1(z) - \ldots - S_{m,k}(z)a_m(z) \text{ for any } k \in [\![1, m]\!]. \tag{3.15}$$

Since the system (3.1) of §3.1 has no non-zero finite singularity, all rational functions $S_{\ell,k}(z)$ belong to $\mathbb{Q}[z, 1/z]$ and the system (3.15) has no non-zero finite singularity. In particular, there exists a fundamental matrix of solutions $(\varphi_{k,l}(z))_{1\le k,l\le m}$ such that $\varphi_{k,l}(1)$ is equal to the Kronecker symbol $\delta_{k,l}$. Let $\varrho_1, \ldots, \varrho_m$ be independent variables, and put $a_k(z) = \sum_{l=1}^m \varrho_l \varphi_{k,l}(z)$ for $k \in [\![1, m]\!]$. Then $(a_1(z), \ldots, a_m(z))$ is a solution of the system (3.15).

Consider the vector $\overline{Y}(z) = (\overline{y}_\lambda(z))_{\lambda\in\Omega}$ defined by:

$$\overline{y}_\lambda(z) = \begin{cases} a_1(z)^{\lambda_1} \ldots a_m(z)^{\lambda_m} \text{ if } |\lambda| = N, \\ a_1(z)^{\lambda_1} \ldots a_m(z)^{\lambda_m}(1 + a_1(z)f_1(z) + \ldots + a_m(z)f_m(z)) \text{ if } |\lambda| = N - 1. \end{cases} \tag{3.16}$$

Each of these functions is a polynomial in the variables $\varrho_1, \ldots, \varrho_m$, with coefficients that depend on z; all monomials that appear in this expression have total degree $N - 1$ or N. Therefore we have

$$\overline{Y}(z) = \sum_{\kappa\in\Omega} \varrho_1^{\kappa_1} \ldots \varrho_m^{\kappa_m} Y_\kappa(z) \tag{3.17}$$

and this expression defines functions $Y_\kappa(z)$ independent from $\varrho_1, \ldots, \varrho_m$. We shall be interested in these functions only when $\kappa \in \Theta$, i.e. $|\kappa| = N$.

To prove part (*i*), we deduce from $\varphi_{k,l}(1) = \delta_{k,l}$ that $a_k(1) = \varrho_k$, and Eq. (3.16) yields

$$\overline{y}_\lambda(1) = \begin{cases} \varrho_1^{\lambda_1} \ldots \varrho_m^{\lambda_m} \text{ if } |\lambda| = N, \\ \varrho_1^{\lambda_1} \ldots \varrho_m^{\lambda_m}(1 + \varrho_1 f_1(1) + \ldots + \varrho_m f_m(1)) \text{ if } |\lambda| = N - 1. \end{cases} \tag{3.18}$$

Given $\kappa \in \Theta$, we write $Y_\kappa(1) = (z_{\kappa,\lambda})_{\lambda\in\Omega}$. Then Eq. (3.17) shows that $z_{\kappa,\lambda}$ is the coefficient of $\varrho_1^{\kappa_1} \ldots \varrho_m^{\kappa_m}$ in the expression of $\overline{y}_\lambda(1)$. Using Eq. (3.18), we obtain

$$z_{\kappa,\lambda} = \begin{cases} 1 \text{ if } \lambda = \kappa, \\ f_j(1) \text{ if } \lambda = \kappa - e_j \text{ for some } j \in [\![1, m]\!], \\ 0 \text{ otherwise.} \end{cases}$$

By definition of Z_κ, this means $Y_\kappa(1) = Z_\kappa$ and concludes the proof of part (i).

Part (ii) follows easily from part (i). Indeed we consider the matrix $M \in M_{\omega,\theta}(\mathbb{C})$ with columns Z_κ, $\kappa \in \Theta$. We may assume that the bijective map $[\![1,\omega]\!] \to \Omega$ we have chosen in §3.1 maps $[\![1,\theta]\!]$ to Θ: it allows us to identify Θ and $[\![1,\theta]\!]$. Then by definition on the Z_κ, we have $M = \begin{pmatrix} I \\ M' \end{pmatrix}$ for some matrix $M' \in M_{\omega-\theta,\theta}(\mathbb{C})$, where $I \in M_\theta(\mathbb{C})$ is the identity matrix. Therefore M has rank θ, and the vectors Z_κ are linearly independent over $\mathbb{C}$. Using part (i), this concludes the proof of (ii).

Let us prove parts (iii) and (iv) now. For brevity we let $\rho^\kappa = \rho_1^{\kappa_1} \dots \rho_m^{\kappa_m}$ and define $a(z)^\lambda$ in an analogous way. We have:

$$\begin{aligned}
\sum_{\kappa\in\Omega} \rho^\kappa R(Y_\kappa)(z) = R(\overline{Y})(z) &= \sum_{\lambda\in\Omega} P_\lambda(z)\overline{y}_\lambda(z) \text{ using Eqns. (3.12) and (3.17)} \\
&= \sum_{\lambda\in\Theta} P_\lambda(z)a(z)^\lambda + \sum_{\lambda\in\Omega\setminus\Theta} P_\lambda(z)a(z)^\lambda\Big(1 + \sum_{j=1}^m a_j(z)f_j(z)\Big) \text{ by Eq. (3.16)} \\
&= \sum_{\lambda\in\Theta} a(z)^\lambda\Big(P_\lambda(z) + \sum_{j=1}^m P_{\lambda-e_j}(z)f_j(z)\Big) + \sum_{\lambda\in\Omega\setminus\Theta} P_\lambda(z)a(z)^\lambda.
\end{aligned}$$

Now recall that $a_k(z) = \sum_{l=1}^m \varrho_l \varphi_{k,l}(z)$. In the previous expression, we fix $\kappa \in \Theta$ and identify the coefficients of ρ^κ in both sides. Since the second term of the right hand side is homogeneous of degree $N-1$, whereas $|\kappa| = N$, it does not contribute and we have

$$R(Y_\kappa)(z) = \sum_{\lambda\in\Theta} b_{\lambda,\kappa}(z)\Big(P_\lambda(z) + \sum_{j=1}^m P_{\lambda-e_j}(z)f_j(z)\Big) \tag{3.19}$$

where $b_{\lambda,\kappa}(z)$ is the coefficient of ρ^κ in the expansion of $a(z)^\lambda$. This coefficient $b_{\lambda,\kappa}(z)$ is an explicit homogeneous polynomial of degree N (with constant integer coefficients) in the functions $\varphi_{k,l}(z)$.

Now denote by $Z' = SZ$ the differential system (3.1) of §3.1. It has at worst a regular singularity at 0, and therefore admits a fundamental matrix of solutions $M(z)$ with coefficients in the Nilsson class at 0. Then ${}^tM(z)^{-1}$ is a fundamental matrix of solutions of the (dual) differential system $Y' = -{}^tSY$. Removing the first coordinate of the solutions yields a fundamental matrix of solutions of the differential system (3.15), all of which coefficients are in the Nilsson class at 0. Accordingly all $\varphi_{k,l}$ and all $b_{\lambda,\kappa}$ belong to the Nilsson class at 0, and so does $R(Y_\kappa)$ using Eq. (3.19): this proves part (iv). Moreover, there exists a constant c, which depends only on this system, such that $\varphi_{k,l}(z) = O(z^{-c})$ as $z \to 0$. Therefore we have $b_{\lambda,\kappa}(z) = O(z^{-cN})$ for any $\lambda, \kappa \in \Theta$. Using Eq. (3.19) and Lemma 1 this concludes the proof of part (iii).

To conclude the proof of Proposition 4, let us prove that $Y'_\kappa = AY_\kappa$ for any $\kappa \in \Theta$. To begin with, Eq. (3.16) yields for any $\lambda \in \Theta$:

$$\begin{aligned}
\overline{y}'_\lambda(z) &= -\sum_{j=1}^m \lambda_j a(z)^{\lambda - e_j} \sum_{l=1}^m S_{l,j}(z) a_l(z) \text{ using Eq. (3.15)} \\
&= -\sum_{j=1}^m \sum_{l=1}^m \lambda_j S_{l,j}(z) a(z)^{\lambda - e_j + e_l} \\
&= \sum_{\kappa \in \Omega} A_{\lambda,\kappa}(z) \overline{y}_\kappa(z) \text{ using Eq. (3.13).}
\end{aligned}$$

To prove the same formula for $\lambda \in \Omega \setminus \Theta$, we notice that, using Eqns. (3.15) and (3.1):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\Big(1+\sum_{p=1}^m a_p(z) f_p(z)\Big) &= \sum_{p=1}^m a'_p(z) f_p(z) + \sum_{l=1}^m a_l(z) f'_l(z) \\
&= -\sum_{p=1}^m \sum_{l=1}^m S_{l,p}(z) a_l(z) f_p(z) + \sum_{l=1}^m a_l(z)(S_{l,0}(z) + \sum_{p=1}^m S_{l,p}(z) f_p(z)) \\
&= \sum_{l=1}^m S_{l,0}(z) a_l(z).
\end{aligned}$$

This gives for any $\lambda \in \Omega \setminus \Theta$, using Eqns. (3.16) and (3.15):

$$\begin{aligned}
\overline{y}'_\lambda(z) &= a(z)^\lambda \sum_{p=1}^m S_{p,0}(z) a_p(z) + \sum_{j=1}^m \lambda_j a(z)^{\lambda - e_j} a'_j(z)(1 + \sum_{p=1}^m a_p(z) f_p(z)) \\
&= \sum_{p=1}^m S_{p,0}(z) a(z)^{\lambda + e_p} - \sum_{j=1}^m \lambda_j a(z)^{\lambda - e_j} \sum_{l=1}^m S_{l,j}(z) a_l(z)(1 + \sum_{p=1}^m a_p(z) f_p(z)) \\
&= \sum_{p=1}^m S_{p,0}(z) \overline{y}_{\lambda + e_p}(z) - \sum_{j=1}^m \sum_{l=1}^m \lambda_j S_{l,j}(z) \overline{y}_{\lambda - e_j + e_l}(z) \\
&= \sum_{\kappa \in \Omega} A_{\lambda,\kappa}(z) \overline{y}_\kappa(z) \text{ using Eq. (3.13).}
\end{aligned}$$

This concludes the proof of Proposition 4. □

3.6 Application of the multiplicity estimate: proof of Proposition 3

In this section we prove Proposition 3 stated in §3.2, as consequence of Theorem A stated in §2.3 (using also Theorem 2 that will be stated and proved in §4). We apply Theorem A in the setting of §3.5, namely with $q = \omega$, the differential system $Y' = AY$ where A is defined by Eq. (3.13), $n = M - 1$ and the polynomials $P_\kappa(z)$, $z \in \Omega$, defined in Lemma 1. All vector spaces, dimensions and other notions of linear algebra are over $\mathbb{C}$.

We denote by $\mathcal{R}$ the space of solutions Y of $Y' = AY$ such that $R(Y)(z)$ is identically zero, so that $\varrho = \dim \mathcal{R}$ with the notation of Theorem A. The difficult part is to prove that $\mathcal{R} = \{0\}$; this is a necessary condition for the matrix of Proposition 3 to be of maximal rank (see the remark after Theorem A). Before going into details, let us explain our strategy by studying the influence on Eq. (2.5) of Theorem A of a non-zero solution $Y \in \mathcal{R}$: it changes ϱ to $\varrho + 1$, and therefore decreases the right hand side by $n+1$. Accordingly it has the same effect on Eq. (2.5) as an additional solution Y_j that would vanish at 0 to order $n+1$. This is optimal in the general setting, and it was sufficient in the application we had in [15], but it isn't for our purposes. Indeed, if a non-zero solution $Y \in \mathcal{R}$ is amongst the solutions Y_κ of Proposition 4, then it cannot be included in the family (Y_j) of Theorem A because in that theorem the functions $R(Y_j)$ have to be linearly independent over $\mathbb{C}$. In Eq. (2.5) of Theorem A this results in a loss of $\mathrm{ord}_0 R(Y) \geq K - cN$ in the left hand side; as explained above we gain $n+1$ because $Y \in \mathcal{R}$, but the gain does not compensate the loss because $K - cN$ is much bigger than $n + 1 = M$ (recall from Eq. (3.3) that K/M is very close to $\omega/\theta = 2 - \frac{m-1}{N+m-1} > 1$). The bound on τ such that Eq. (2.5) holds is not good enough, and we cannot complete the proof unless we manage to gain more. Such a gain is provided by Theorem 2 proved in §4: the solutions $Y \in \mathrm{Span}\{Y_\kappa\}$ such that $R(Y) = 0$ are balanced by additional solutions $Y \notin \mathrm{Span}\{Y_\kappa\}$ such that $R(Y) = 0$, that count for $n+1$ in Eq. (2.5). If $\mathcal{R} \neq \{0\}$, the overall gain is even slightly bigger than the loss, so that we deduce a too good estimate on τ, in contradiction with part (i) of Theorem A. Therefore $\mathcal{R} = \{0\}$, and part (ii) of Theorem A enables us to deduce Proposition 3.

To work out the proof of Proposition 3, we let $\mathcal{F} = \mathrm{Span}\{Y_\kappa,\ \kappa \in \Theta\}$ where the solutions Y_κ are given by Proposition 4. They are in the Nilsson class at 0, and are linearly independent over $\mathbb{C}$ so that $\dim \mathcal{F} = \theta$.

We denote by J a maximal subset of Θ such that $\mathcal{R} \cap \mathrm{Span}\{Y_\kappa,\ \kappa \in J\} = \{0\}$. Then we have $\mathcal{F} = (\mathcal{F} \cap \mathcal{R}) \oplus \mathrm{Span}\{Y_\kappa,\ \kappa \in J\}$ so that $\mathrm{Card}\, J = \theta - \dim(\mathcal{F} \cap \mathcal{R})$.

With this definition, the functions $R(Y_\kappa)$, $\kappa \in J$, are linearly independent over $\mathbb{C}$: if $\sum_{\kappa \in J} \lambda_\kappa R(Y_\kappa) = 0$ with $\lambda_\kappa \in \mathbb{C}$, then $\sum_{\kappa \in J} \lambda_\kappa Y_\kappa \in \mathcal{R} \cap \mathrm{Span}\{Y_\kappa,\ \kappa \in J\} = \{0\}$ so that $\lambda_\kappa = 0$ for any κ.

Therefore we may apply Theorem A with

$$\tau = (\omega - \varrho)M - \sum_{\kappa \in J} \mathrm{ord}_0(R(Y_\kappa)) \leq (\omega - \varrho)M - (K - cN)(\theta - \dim(\mathcal{F} \cap \mathcal{R})) \qquad (3.20)$$

using assertion (iii) of Proposition 4 and the equality $\mathrm{Card}\, J = \theta - \dim(\mathcal{F} \cap \mathcal{R})$.

Now consider the map $\mathrm{ev}_1 : Y \mapsto Y(1)$, from the space of solutions of $Y' = AY$ to $\mathbb{C}^\omega$. It is bijective because 1 is not a singularity of this differential system, defined by (3.13) with $S_{l,j}(z) \in \mathbb{Q}[z, 1/z]$ for any l, j. We let $F = \mathrm{ev}_1(\mathcal{F})$; this space is spanned by the vectors Z_κ, $\kappa \in \Theta$, by Proposition 4 (i).

We claim that $R = \mathrm{ev}_1(\mathcal{R})$ is defined over $\mathbb{Q}$. Indeed given a solution $Y = (y_\kappa(z))_{\kappa \in \Omega}$ of $Y' = AY$, we have $Y(1) \in R$ if, and only if, the function $\sum_{\kappa \in \Omega} P_\kappa(z) y_\kappa(z)$ is identically zero. This means that all derivatives of this function vanish at 1, i.e. $\sum_{\kappa \in \Omega} P_{k,\kappa}(1) y_k(1) = 0$

for any $k \geq 1$ using Eq. (2.3). This set of linear equations with rational coefficients $P_{k,\kappa}(1)$ defines the subspace R.

By construction (see §3.1) the functions 1, $f_1(z)$, ..., $f_m(z)$ are linearly independent over $\mathbb{C}(z)$, and they make up a vector solution of a differential system of order 1 with coefficients in $\mathbb{Q}[z, 1/z]$, given by Eq. (3.1). Since 1 is not a singularity of this system, Beukers' refinement [8, Corollary 1.4] of the Siegel-Shidlovskii theorem shows that the values at 1 of these functions are linearly independent over $\overline{\mathbb{Q}}$. Therefore Theorem 2 applies with $\xi_1 = f_1(1)$, ..., $\xi_m = f_m(1)$.

To begin with, assume that $R \neq \{0\}$ and $R \neq \mathbb{C}^\Omega$. Then Theorem 2 yields

$$\dim R > \Big(2 - \frac{m-1}{N+m-1}\Big) \dim(F \cap R).$$

Since both sides of the inequality are integer multiples of $\frac{1}{N+m-1}$, we obtain

$$\begin{aligned}\dim R &\geq \Big(2 - \frac{m-1}{N+m-1}\Big) \dim(F \cap R) + \frac{1}{N+m-1} \\ &= \frac{\omega}{\theta} \dim(F \cap R) + \frac{1}{N+m-1}.\end{aligned}$$

We have $\dim R = \dim \mathcal{R} = \varrho$ and $\dim(F \cap R) = \dim(\mathcal{F} \cap \mathcal{R})$, so that Eq. (3.20) yields

$$\tau \leq \Big(1 - \frac{\varrho}{\omega}\Big)\omega M - (K - cN)\theta\Big(1 - \frac{\varrho}{\omega} + \frac{1}{\omega(N+m-1)}\Big).$$

Now recall that $\omega M - K\theta \leq 2\eta M$ using Eq. (3.3), and $\varrho, \theta \leq \omega$, so that

$$\begin{aligned}\tau &\leq \Big(1 - \frac{\varrho}{\omega}\Big)(2\eta M + cN\theta) - \frac{(\omega - 2\eta)M - cN\theta}{\omega(N+m-1)} \\ &\leq 2\eta M + cN\theta - \frac{\Big(1 - \frac{2\eta}{\omega}\Big)M}{N+m-1} + \frac{cN}{N+m-1} \\ &\leq -\frac{\Big(\frac{1}{3} - \frac{2\eta}{\omega}\Big)M}{N+m-1} + c(1 + N\theta) < -c_1\end{aligned}$$

since $\eta \leq \frac{1}{3(N+m-1)}$ and M has been chosen large enough with respect to N, m and η; the crucial point here is that the constants c of Proposition 4 and c_1 of Theorem A depend only on N and f_1, ..., f_m but not on M. This upper bound on τ contradicts conclusion (i) of Theorem A.

Therefore we have $R = \{0\}$ or $R = \mathbb{C}^\Omega$. The latter means that for any solution $Y = (y_\kappa(z))_{\kappa\in\Omega}$ of the differential system $Y' = AY$, the function $R(Y)(z) = \sum_{\kappa\in\Omega} P_\kappa(z) y_\kappa(z)$ is identically zero. This is impossible: there exist $\kappa_0 \in \Omega$ and $z_0 \in \mathbb{C}^*$ such that $P_{\kappa_0}(z_0) \neq 0$, and since z_0 is not a singularity there exists a solution Y such that $y_{\kappa_0}(z_0) = 1$ and $y_\kappa(z_0) = 0$ for any $\kappa \neq \kappa_0$.

Accordingly we have proved that $R = \{0\}$, so that $\mathcal{F} \cap \mathcal{R} = \mathcal{R} = \{0\}$ and Eq. (3.20) yields

$$\tau \leq \omega M - (K - cN)\theta \leq 2\eta M + cN\theta$$

using Eq. (3.3). Therefore $\tau \leq \lfloor 3\eta M \rfloor \leq M - c_1$ since M has been chosen sufficiently large with respect to N, $f_1, \ldots, f_m$ and η. Increasing τ if necessary, we may assume $\tau \geq 0$ so that Theorem A yields $\mathrm{rk}(P_{k,\kappa}(1))_{\kappa\in\Omega, 1\leq k\leq \lfloor 2\eta M\rfloor + c_1 - 1} = \omega$. Using Eq. (3.14) proved in §3.5, this concludes the proof of Proposition 3.

4 The key technical result

In this section we state and prove Theorem 2, a key ingredient in the proof of Proposition 3 given in §3.6. The motivation and application of this result is explained at the beginning of §3.6. We believe that analogous auxiliary results might be useful when the multiplicity estimate of [15] is applied in settings where $R(Y)$ may vanish identically for non-zero solutions Y.

Theorem 2 is an independent result, for which we need only the following notations. We consider integers $m, N \geq 1$ and complex numbers $\xi_1, \ldots, \xi_m$; we assume that $1, \xi_1, \ldots, \xi_m$ are linearly independent over $\overline{\mathbb{Q}}$. As in §3.1 we let $\Omega = \{\kappa \in \mathbb{N}^m,\ N-1 \leq |\kappa| \leq N\}$ where $|\kappa| = \kappa_1 + \ldots + \kappa_m$, and $\omega = \mathrm{Card}\,\Omega = \binom{N+m-2}{m-1} + \binom{N+m-1}{m-1}$; we denote by $(e_j)_{1\leq j\leq m}$ the canonical basis of $\mathbb{Z}^m$, and by $(E_\kappa)_{\kappa\in\Omega}$ that of $\mathbb{C}^\Omega$. In other words, we have $E_\kappa = (\delta_{\kappa,\kappa'})_{\kappa'\in\Omega}$ where $\delta_{\kappa,\kappa'}$ is the Kronecker's symbol. We consider

$$Z_\kappa = E_\kappa + \sum_{j=1}^{m} \xi_j E_{\kappa - e_j} \text{ for any } \kappa \in \Theta = \{\kappa \in \mathbb{N}^m,\ |\kappa| = N\}$$

with the convention that $E_{\kappa-e_j} = 0$ if $\kappa - e_j \in \mathbb{Z}^m$ has at least one negative component (namely, if $\kappa_j = 0$); with $\xi_j = f_j(1)$ these are exactly the vectors Z_κ defined before Proposition 4 in §3.5. We denote by F the subspace of $\mathbb{C}^\Omega$ generated by these vectors, namely

$$F = \mathrm{Span}\{Z_\kappa,\ \kappa \in \Theta\}.$$

It is not difficult (see the proof of (*ii*) in Proposition 4 above) to see that the Z_κ are linearly independent, so that $\dim F = \binom{N+m-1}{m-1}$, denoted by θ.

Theorem 2. *Let R be a vector subspace of $\mathbb{C}^\Omega$ defined over $\overline{\mathbb{Q}}$. Then we have:*

$$\dim R \geq \left(2 - \frac{m-1}{N+m-1}\right) \dim(F \cap R),$$

and equality holds if and only if $R = \{0\}$ or $R = \mathbb{C}^\Omega$.

By *defined over* $\overline{\mathbb{Q}}$, or *rational over* $\overline{\mathbb{Q}}$, we mean that R has a $\mathbb{C}$-basis consisting in vectors in $\overline{\mathbb{Q}}^\Omega$. This is equivalent to the existence of a system of linear equations with coefficients in $\overline{\mathbb{Q}}$ that defines R. We point out that an inequality on dimensions, such as the one of Theorem 2, is reminiscent of the notion of (e, j)-irrationality introduced in [20] and [21].

4.1 Two lemmas

The data of Theorem 2 depend only on $m, N \geq 1$ and on $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$; in the proof we shall often deduce Theorem 2 for some triples (m, N, ξ) from the same statement for other triples. It will always be assumed, implicitly or explicitly, that 1, ξ_1, ..., ξ_m are linearly independent over $\overline{\mathbb{Q}}$. We first prove two lemmas; recall that we identify tuples in $\mathbb{C}^m$ to column matrices in $M_{m,1}(\mathbb{C})$.

Lemma 4. *Let $m, N \geq 1$ and $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{C}^m$, with 1, ξ_1, ..., ξ_m linearly independent over $\overline{\mathbb{Q}}$. Let $A \in \mathrm{GL}_m(\overline{\mathbb{Q}})$; define $\xi' = (\xi'_1, \dots, \xi'_m)$ by $\xi' = A\,\xi$.*

If Theorem 2 holds for (m, N, ξ) with any subspace R defined over $\overline{\mathbb{Q}}$ of a given dimension ϱ, then it also does for (m, N, ξ').

Proof. Decomposing A into a product of simpler matrices, we may restrict to the following 3 cases.

• Case 1: $\xi'_j = \xi_{\sigma(j)}$ for any $j \in [\![1, m]\!]$, with $\sigma \in \mathfrak{S}_m$. In this case Lemma 4 is obvious, by permuting the coordinates in $\mathbb{N}^m$ and accordingly in $\mathbb{C}^\Omega$.

• Case 2: $\xi'_{j_0} = \lambda \xi_{j_0}$ for some $\lambda \in \overline{\mathbb{Q}}^*$ and $j_0 \in [\![1, m]\!]$, and $\xi'_j = \xi_j$ for any $j \in [\![1, m]\!] \setminus \{j_0\}$. Let $f : \mathbb{C}^\Omega \to \mathbb{C}^\Omega$ be the (bijective) linear map defined by $f(E_\kappa) = \lambda^{\kappa_{j_0}} E_\kappa$ for any $\kappa \in \Omega$. We denote by (Z_κ) and F (resp. (Z'_κ) and F') the data associated with ξ (resp. ξ'). Then we have for any $\kappa \in \Theta$:

$$f(Z'_\kappa) = f(E_\kappa) + \sum_{j=1}^{m} \xi'_j f(E_{\kappa - e_j})$$
$$= \lambda^{\kappa_{j_0}} E_\kappa + \lambda \xi_{j_0} \lambda^{\kappa_{j_0} - 1} E_{\kappa - e_{j_0}} + \sum_{j \neq j_0} \xi_j \lambda^{\kappa_{j_0}} E_{\kappa - e_j} = \lambda^{\kappa_{j_0}} f(Z_\kappa)$$

so that $f(F') = F$. Let R' be a subspace of $\mathbb{C}^\Omega$ defined over $\overline{\mathbb{Q}}$, with $\dim R' = \varrho$. Taking $R = f(R')$ we have $\dim R = \dim R'$, $\dim(F \cap R) = \dim(F' \cap R')$ and R is defined over $\overline{\mathbb{Q}}$. Therefore Theorem 2 applied to (m, N, ξ) with R shows that Theorem 2 holds for (m, N, ξ') with R'.

• Case 3: $\xi'_{j_0} = \xi_{j_0} + \xi_{j_1}$ with distinct $j_0, j_1 \in [\![1, m]\!]$, and $\xi'_j = \xi_j$ for any $j \in [\![1, m]\!] \setminus \{j_0\}$. We consider the linear map $f : \mathbb{C}^\Omega \to \mathbb{C}^\Omega$ given by

$$f(E_\kappa) = \sum_{t=0}^{\kappa_{j_0}} \binom{t + \kappa_{j_1}}{\kappa_{j_1}} E_{\kappa - t e_{j_0} + t e_{j_1}} \text{ for any } \kappa \in \Omega.$$

It is bijective because $f(E_\kappa) - E_\kappa$ is a linear combination of the $E_{\kappa'}$ with $\kappa'_{j_0} < \kappa_{j_0}$. For any $\kappa \in \Theta$ we have:

$$f(E_{\kappa-e_{j_1}}) + f(E_{\kappa-e_{j_0}}) = \sum_{t=0}^{\kappa_{j_0}} \binom{t+\kappa_{j_1}-1}{\kappa_{j_1}-1} E_{\kappa-te_{j_0}+(t-1)e_{j_1}} + \sum_{t=0}^{\kappa_{j_0}-1} \binom{t+\kappa_{j_1}}{\kappa_{j_1}} E_{\kappa-(t+1)e_{j_0}+te_{j_1}}$$

$$= \sum_{t=0}^{\kappa_{j_0}} \left(\binom{t+\kappa_{j_1}-1}{\kappa_{j_1}-1} + \binom{t+\kappa_{j_1}-1}{\kappa_{j_1}} \right) E_{\kappa-te_{j_0}+(t-1)e_{j_1}} \text{ } (t' = t+1 \text{ in the second sum})$$

$$= \sum_{t=0}^{\kappa_{j_0}} \binom{t+\kappa_{j_1}}{\kappa_{j_1}} E_{\kappa-te_{j_0}+(t-1)e_{j_1}} \qquad (4.1)$$

so that

$$f(Z'_\kappa) = f(E_\kappa) + \sum_{j=1}^{m} \xi_j f(E_{\kappa-e_j}) + \xi_{j_1} f(E_{\kappa-e_{j_0}})$$

$$= \sum_{t=0}^{\kappa_{j_0}} \binom{t+\kappa_{j_1}}{\kappa_{j_1}} \Big(E_{\kappa-te_{j_0}+te_{j_1}} + \sum_{j\neq j_1} \xi_j E_{\kappa-e_j-te_{j_0}+te_{j_1}} \Big) + \xi_{j_1} \Big(f(E_{\kappa-e_{j_1}}) + f(E_{\kappa-e_{j_0}}) \Big)$$

$$= \sum_{t=0}^{\kappa_{j_0}} \binom{t+\kappa_{j_1}}{\kappa_{j_1}} Z_{\kappa-te_{j_0}+te_{j_1}} \in F, \text{ using Eq. (4.1).}$$

Therefore $f(F') = F$, and we deduce the result as in Case 2. This concludes the proof of Lemma 4. □

Lemma 5. *Let $m, N \geq 1$. We consider the subspace*

$$K = \mathrm{Span}\{E_\kappa, \, \kappa \in \Omega, \, \kappa_m = 0\}.$$

Assume that Theorem 2 holds for any $\xi \in \mathbb{C}^m$ and any R defined over $\overline{\mathbb{Q}}$ such that

$$\dim(F \cap R \cap K) \geq 2\dim(F \cap R) - \dim(R).$$

Then Theorem 2 holds for any $\xi \in \mathbb{C}^m$ and any R defined over $\overline{\mathbb{Q}}$.

Proof. We start with any $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{C}^m$ such that $1, \xi_1, \ldots, \xi_m$ are linearly independent over $\overline{\mathbb{Q}}$. Let R be a subspace of $\mathbb{C}^\Omega$ defined over $\overline{\mathbb{Q}}$, and put $\varrho = \dim R$. We shall construct $g \in \mathrm{GL}(\mathbb{C}^m)$ defined over $\overline{\mathbb{Q}}$ (i.e., whose matrix in the canonical basis of $\mathbb{C}^m$ has coefficients in $\overline{\mathbb{Q}}$), such that $\xi' = g(\xi)$ satisfies $\dim(F' \cap R' \cap K) \geq 2\dim(F' \cap R') - \dim(R')$ for any subspace R' of $\mathbb{C}^\Omega$ defined over $\overline{\mathbb{Q}}$ of dimension ϱ, where F' is associated with $\xi' = g(\xi)$ as before the statement of Theorem 2. Then Lemma 4 shows that if Theorem 2 holds for ξ', then it does for ξ.

Denote by $\mathcal{V}_0$ the Zariski closure of $\{\xi\}$ in the affine space $\mathbb{A}^m_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$, i.e. the smallest subset of $\mathbb{C}^m$, defined by polynomial equations with coefficients in $\overline{\mathbb{Q}}$, that contains ξ. In more concrete terms, $\mathcal{V}_0$ is the set of all $(z_1, \ldots, z_m) \in \mathbb{C}^m$ such that $P(z_1, \ldots, z_m) = 0$ for any $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that $P(\xi_1, \ldots, \xi_m) = 0$. Since ξ_1 is transcendental, $\mathcal{V}_0$ has dimension at least 1. There exists an algebraic curve $\mathcal{C}$, defined by polynomial equations with coefficients in $\overline{\mathbb{Q}}$, contained in $\mathcal{V}_0$ but in no hypersurface defined over $\overline{\mathbb{Q}}$ of degree less than or equal to θ (with possible exceptions for such hypersurfaces that contain $\mathcal{V}_0$).

There exists $j_0 \in [\![1, m]\!]$ such that the j_0-th projection $\mathcal{C} \to \mathbb{C}$, $(\chi_1, \ldots, \chi_m) \mapsto \chi_{j_0}$, has infinite image; then this image contains all real numbers greater than some M_0. Parametrizing a branch of $\mathcal{C}$, we obtain algebraic functions over $\overline{\mathbb{Q}}(z)$, denoted by $\chi_1(z), \ldots, \chi_m(z)$, such that $(\chi_1(a), \ldots, \chi_m(a)) \in \mathcal{C}$ for any real $a \geq M_0$, and $|\chi_{j_0}(a)| \to \infty$ as $a \to +\infty$ with $a \in \mathbb{R}$. Their asymptotic behaviour as $a \to +\infty$ is given by $\chi_j(a) \sim \varpi_j^0 a^{d_j}$ with $\varpi_j^0 \in \overline{\mathbb{Q}}^*$ and $d_j \in \mathbb{Q}$; we have $d_{j_0} > 0$. Let $D = \max(d_1, \ldots, d_m) > 0$, and put $\varpi_j = \varpi_j^0$ for any $j \in [\![1, m]\!]$ such that $d_j = D$, and $\varpi_j = 0$ otherwise. In this way, for any $j \in [\![1, m]\!]$ we have

$$\varpi_j = \lim_{a \in \mathbb{R},\, a \to +\infty} a^{-D} \chi_j(a) \quad \text{with } D > 0. \tag{4.2}$$

We can now construct a bijective linear map $g : \mathbb{C}^m \to \mathbb{C}^m$ defined over $\overline{\mathbb{Q}}$ such that $g(\varpi_1, \ldots, \varpi_m) = (0, \ldots, 0, 1)$. We let $\xi' = (\xi'_1, \ldots, \xi'_m) = g(\xi)$; then 1, ξ'_1, …, ξ'_m are linearly independent over $\overline{\mathbb{Q}}$. We denote by F', $\mathcal{C}'$, ϖ'_j, …the objects defined as above, starting from ξ' instead of ξ. Then we may choose $\mathcal{C}' = g(\mathcal{C})$, $(\chi'_1(z), \ldots, \chi'_m(z)) = g(\chi_1(z), \ldots, \chi_m(z))$ so that $D' = D$ and $(\varpi'_1, \ldots, \varpi'_m) = g(\varpi_1, \ldots, \varpi_m) = (0, \ldots, 0, 1)$. As explained at the beginning of the proof, we shall prove that ξ' satisfies the additional property $\dim(F \cap R \cap K) \geq 2\dim(F \cap R) - \dim(R)$ for any subspace R of $\mathbb{C}^\Omega$ of dimension ϱ defined over $\overline{\mathbb{Q}}$; here and below (until the end of the proof), for simplicity we write ξ, $\mathcal{C}$, $\mathcal{V}_0$, F, R, ϖ_j, …instead of ξ', $\mathcal{C}'$, $\mathcal{V}'_0$, F', R', ϖ'_j, ….

We let R be a subspace of $\mathbb{C}^\Omega$ of dimension ϱ defined over $\overline{\mathbb{Q}}$, write

$$d = \dim(F \cap R), \qquad d' = \dim(F \cap R \cap K),$$

and assume (by contradiction) that $d' < 2d - \varrho$.

For any $\chi = (\chi_1, \ldots, \chi_m) \in \mathbb{C}^m$, we denote by F_χ the subspace defined exactly like F, except that χ_1, …, χ_m are used instead of ξ_1, …, ξ_m to define the Z_κ. We denote by $\mathcal{V}$ the set of all $\chi \in \mathbb{C}^m$ such that

$$\dim F_\chi = \theta, \quad \dim(F_\chi \cap R) = d, \quad \dim(F_\chi \cap R \cap K) = d',$$

i.e. such that these dimensions are the same as for $\chi = \xi$.

We claim that we have inclusions

$$(\mathbb{C}^m \setminus H_1) \cap (\mathbb{C}^m \setminus H_2) \cap (\mathbb{C}^m \setminus H_3) \cap H_4 \cap \ldots \cap H_v \subset \mathcal{V} \subset H_4 \cap \ldots \cap H_v \tag{4.3}$$

where $v \geq 3$ and H_1, …, H_v are hypersurfaces of $\mathbb{C}^m$ of degree at most θ defined over $\overline{\mathbb{Q}}$, such that $\mathcal{V}_0 \not\subset H_i$ for any $i \in [\![1, 3]\!]$. We recall that $\mathcal{V}_0$ is the Zariski closure of $\{\xi\}$ in $\mathbb{A}^m_{\overline{\mathbb{Q}}}$, so that $\mathcal{V}_0 \not\subset H_i$ is equivalent to $\xi \notin H_i$. Indeed we denote by $Z(X_1, \ldots, X_m) \in M_{\omega,\theta}(\overline{\mathbb{Q}}[X_1, \ldots, X_m])$ the matrix of which the columns are the coordinates of the Z_κ in the canonical basis of $\mathbb{C}^\Omega$ (for $\kappa \in \Theta$), in which ξ_j is replaced with X_j. Then $\dim F_\chi = \mathrm{rk}(Z(\chi))$

is equal to θ if, and only if, at least one minor of $Z(\chi)$ of size θ is non-zero. The coefficients of $Z(X_1, \ldots, X_m)$ are polynomials of total degree at most 1 in $X_1, \ldots, X_m$, so each minor of size θ has degree at most θ. We choose a minor which is non-zero at ξ, and denote by H_1 the hypersurface defined in $\mathbb{C}^m$ by the vanishing of this minor.

Now we consider the matrix $S(X_1, \ldots, X_m) \in M_{\omega,\theta+\varrho}(\overline{\mathbb{Q}}[X_1, \ldots, X_m])$ of which the first θ columns are those of $Z(X_1, \ldots, X_m)$, and the last ϱ columns belong to $\overline{\mathbb{Q}}^{\Omega}$ and make up a basis of R (which is possible since R is defined over $\overline{\mathbb{Q}}$). Assuming that $\dim F_\chi = \theta$, we have $\dim(F_\chi \cap R) = d$ if, and only if, $\dim(F_\chi + R) = \theta + \varrho - d$; this is equivalent to $\mathrm{rk}(S(\chi)) = \theta + \varrho - d$. This condition can be expressed as the vanishing of all minors of size $\theta + \varrho - d + 1$, and the non-vanishing of at least one minor of size $\theta + \varrho - d$. Again we choose such a minor of size $\theta + \varrho - d$ that does not vanish at ξ, and denote by H_2 the corresponding hypersurface (which has degree at most θ); we define H_4, H_5, … to be the hypersurfaces defined by the vanishing of the minors of size $\theta + \varrho - d + 1$. We proceed in the same way with $R \cap K$ instead of R to ensure that $\dim(F_\chi \cap R \cap K) = d'$. This concludes the proof of the claimed inclusions (4.3).

These inclusions imply that $\mathcal{C} \setminus \mathcal{C}_1 \subset \mathcal{V}$ for some finite set $\mathcal{C}_1$. Indeed, we have $\xi \in \mathcal{V} \subset H_i$ for any $i \in [\![4, v]\!]$, so that $\mathcal{C} \subset \mathcal{V}_0 \subset H_i$ by definition of $\mathcal{V}_0$, since H_i is defined over $\overline{\mathbb{Q}}$. For $i \in [\![1, 3]\!]$, we have $\mathcal{V}_0 \not\subset H_i$ and H_i is a hypersurface of degree at most θ, so that $\mathcal{C} \not\subset H_i$ (by construction of $\mathcal{C}$) and $\mathcal{C} \cap H_i$ is a finite set; taking for $\mathcal{C}_1$ the union of these finite sets, Eq. (4.3) yields $\mathcal{C} \setminus \mathcal{C}_1 \subset \mathcal{V}$.

Since $\mathcal{C}_1$ is finite, there exists a real number $M_1 \geq M_0$ such that for any real $a \geq M_1$, the point $\chi(a) = (\chi_1(a), \ldots, \chi_m(a))$ belongs to $\mathcal{C} \setminus \mathcal{C}_1 \subset \mathcal{V}$. We shall focus on real algebraic values of $a \geq M_1$; since $\chi(a) \in \mathcal{V} \cap \overline{\mathbb{Q}}^m$, the subspace $F_{\chi(a)}$ is then defined over $\overline{\mathbb{Q}}$ and we have $\dim F_{\chi(a)} = \theta$, $\dim(F_{\chi(a)} \cap R) = d$, $\dim(F_{\chi(a)} \cap R \cap K) = d'$. We fix such an a, denoted by a_0, and consider a subspace W defined over $\overline{\mathbb{Q}}$ such that

$$\big(F_{\chi(a_0)} \cap R \cap K\big) \oplus W = F_{\chi(a_0)} \cap R. \tag{4.4}$$

For any $a \in \overline{\mathbb{Q}} \cap \mathbb{R}$ such that $a \geq M_1$, we have $\chi(a) \in \mathcal{V}$ so that

$$\dim W + \dim\big(F_{\chi(a)} \cap R\big) = (d - d') + d > \varrho = \dim R$$

since we have assumed (by contradiction) that $d' < 2d - \varrho$. Accordingly these subspaces of R are not in direct sum: there exists $u_a \in W \cap F_{\chi(a)} \cap R$ with $u_a \neq 0$. Since $W \cap F_{\chi(a)} \cap R$ is defined over $\overline{\mathbb{Q}}$, we may assume that $u_a \in \overline{\mathbb{Q}}^{\Omega}$. We write $u_a = (u_{a,\kappa})_{\kappa\in\Omega}$ with $u_{a,\kappa} \in \overline{\mathbb{Q}}$. These coordinates satisfy

$$u_{a,\kappa} = \sum_{j=1}^{m} \chi_j(a) u_{a,\kappa+e_j} \text{ for any } \kappa \in \Omega \setminus \Theta \tag{4.5}$$

since all generators of $F_{\chi(a)}$ satisfy these linear equations.

Let $(T_1, \ldots, T_w)$ be a basis of W consisting of vectors of $\overline{\mathbb{Q}}^{\Omega}$, with $w = \dim W = d - d'$. Since $u_a \in W \cap \overline{\mathbb{Q}}^{\Omega}$ there exist $\lambda_{a,1}, \ldots, \lambda_{a,w} \in \overline{\mathbb{Q}}$, not all zero, such that $u_a = \sum_{\ell=1}^{w} \lambda_{a,\ell} T_\ell$.

Writing $T_\ell = (t_{\ell,\kappa})_{\kappa\in\Omega}$ we have $u_{a,\kappa} = \sum_{\ell=1}^{w} \lambda_{a,\ell} t_{\ell,\kappa}$ for any $\kappa \in \Omega$. Using this into Eq. (4.5) yields

$$\sum_{\ell=1}^{w} \lambda_{a,\ell} P_{\ell,\kappa}(a) = 0 \text{ for any } \kappa \in \Omega \setminus \Theta, \tag{4.6}$$

where

$$P_{\ell,\kappa}(z) = -t_{\ell,\kappa} + \sum_{j=1}^{m} t_{\ell,\kappa+e_j} \chi_j(z)$$

is a function algebraic over $\overline{\mathbb{Q}}(z)$. Let $P(z)$ denote the matrix $(P_{\ell,\kappa}(z))_{\kappa\in\Omega\setminus\Theta, 1\le\ell\le w}$. For any $a \in \overline{\mathbb{Q}} \cap \mathbb{R}$ with $a \geq M_1$, Eq. (4.6) shows that $\mathrm{rk}(P(a)) < w$: all minors of size w of the matrix $P(z)$ vanish at a. Since these minors are functions algebraic over $\overline{\mathbb{Q}}(z)$, they are identically zero: $P(z)$ has rank at most $w-1$, as a matrix with coefficients in $\overline{\mathbb{Q}(z)}$. This provides algebraic functions $\mu_1(z), \ldots, \mu_w(z) \in \overline{\mathbb{Q}(z)}$, not all zero, such that $\sum_{\ell=1}^{w} \mu_\ell(z) P_{\ell,\kappa}(z) = 0$ for any $\kappa \in \Omega \setminus \Theta$. In other words,

$$\sum_{\ell=1}^{w} \mu_\ell(z) \Big(- t_{\ell,\kappa} + \sum_{j=1}^{m} t_{\ell,\kappa+e_j} \chi_j(z) \Big) = 0 \text{ for any } \kappa \in \Omega \setminus \Theta. \tag{4.7}$$

Now as $z \to +\infty$ with $z \in \mathbb{R}$, each non-zero function $\mu_\ell(z)$ has an asymptotic behavior given by $\mu_\ell(z) \sim \mu_{\ell,0} z^{e_\ell}$ with $\mu_{\ell,0} \in \overline{\mathbb{Q}}^*$ and $e_\ell \in \mathbb{Q}$; if $\mu_\ell(z)$ is identically zero we put $e_\ell = -\infty$. Let $e = \max(e_1, \ldots, e_m)$; for any ℓ, we let $\mu_{\ell,1} = \mu_{\ell,0}$ if $e_\ell = e$, and $\mu_{\ell,1} = 0$ otherwise, so that $\lim_{z\to+\infty} z^{-e}\mu_\ell(z) = \mu_{\ell,1}$. We recall that Eq. (4.2) has a similar flavour, and that in this equation we have $\varpi_1 = \ldots = \varpi_{m-1} = 0$, $\varpi_m = 1$, since ξ' (denoted now by ξ) has been constructed for this purpose. Combining these limits and letting $z \to +\infty$, with $z \in \mathbb{R}$, Eq. (4.7) yields (since $D > 0$)

$$\sum_{\ell=1}^{w} \mu_{\ell,1} t_{\ell,\kappa+e_m} = 0 \text{ for any } \kappa \in \Omega \setminus \Theta.$$

Let $T = \sum_{\ell=1}^{w} \mu_{\ell,1} T_\ell \in W \setminus \{0\}$, and write $T = (t_\kappa)_{\kappa\in\Omega}$. Then we have $t_{\kappa+e_m} = 0$ for any $\kappa \in \Omega \setminus \Theta$, i.e. $t_\lambda = 0$ for any $\lambda \in \Theta$ such that $\lambda_m \geq 1$. For any $\lambda \in \Omega \setminus \Theta$ such that $\lambda_m \geq 1$, we obtain $t_\lambda = \sum_{j=1}^{m} \chi_j(a_0) t_{\lambda+e_j} = 0$ since $T \in W \subset F_{\chi(a_0}$ (see Eq. (4.5)). Therefore $T \in K$: this is a contradiction with the definition (4.4) of W. This concludes the proof of Lemma 5. □

4.2 Proof of Theorem 2

We prove Theorem 2 by induction on $m + N$. Letting $m, N \geq 1$, if $m \geq 2$ (resp. $N \geq 2$) we may assume that Theorem 2 holds with $m-1$ instead of m (resp. $N-1$ instead of N). We shall apply this idea using the linear map $\pi : \mathbb{C}^\Omega \to \mathbb{C}^{\overline{\Omega}}$ defined by $\pi(E_\kappa) = E_{\kappa-e_m}$ for any $\kappa \in \Omega$; here we let

$$\overline{\Omega} = \{\kappa \in \mathbb{N}^m,\ N-2 \leq |\kappa| \leq N-1\}.$$

As explained at the beginning of §4, we have $E_{\kappa-e_m} = 0$ if, and only if, $\kappa - e_m$ has a negative coordinate, i.e. $\kappa_m = 0$. Therefore the kernel of π is

$$K = \ker \pi = \operatorname{Span}\{E_\kappa,\ \kappa \in \Omega,\ \kappa_m = 0\}$$

with the same notation K as in Lemma 5.

The sketch of the proof is the following. Using Theorem 2 with $m-1$ if $m \geq 2$, we shall prove that

$$\dim(R \cap K) \geq \Big(2 - \frac{m-2}{N+m-2}\Big) \dim(F \cap R \cap K). \tag{4.8}$$

Then we shall use Theorem 2 with $N-1$ (if $N \geq 2$) to prove that

$$\dim \pi(R) \geq \Big(2 - \frac{m-1}{N+m-2}\Big) \dim \pi(F \cap R). \tag{4.9}$$

At last we shall conclude the proof by combining these inequalities with the one provided by Lemma 5.

Let us start by proving Eq. (4.8). It holds trivially if $m = 1$ since in this case, $K = \{0\}$. Therefore we may assume $m \geq 2$ and let $\Omega' = \{\kappa \in \mathbb{N}^{m-1},\ N-1 \leq |\kappa| \leq N\}$, $\xi' = (\xi_1, \ldots, \xi_{m-1})$, and F' be the subspace defined from $(m-1, N, \xi')$ as explained before the statement of Theorem 2, spanned by vectors $Z'_\kappa \in \mathbb{C}^{\Omega'}$ for $\kappa \in \Theta' = \{\kappa \in \mathbb{N}^{m-1},\ |\kappa| = N\}$.

Let ι denote the injective linear map $\mathbb{C}^{\Omega'} \to \mathbb{C}^{\Omega}$ defined by $\iota(E_\kappa) = E_{(\kappa,0)}$ for any $\kappa = (\kappa_1, \ldots, \kappa_{m-1}) \in \Omega'$, where $(\kappa, 0)$ stands for $(\kappa_1, \ldots, \kappa_{m-1}, 0)$. We claim that $\iota(F') = F \cap K$. The inclusion $\iota(F') \subset F \cap K$ is obvious since $\iota(Z'_\kappa) = Z_{(\kappa,0)}$ for any $\kappa \in \Theta'$. Conversely, let $Z = \sum_{\kappa\in\Theta} \lambda_\kappa Z_\kappa \in F \cap K$, with complex numbers λ_κ. For any $\kappa \in \Theta$, the coordinate of Z on E_κ (in the canonical basis of $\mathbb{C}^\Omega$) is equal to λ_κ. Since $Z \in K$, we deduce that $\lambda_\kappa = 0$ for any $\kappa \in \Theta$ such that $\kappa_m \geq 1$. Therefore $Z = \iota(\sum_{\kappa\in\Theta'} \lambda_{(\kappa,0)} Z'_\kappa) \in \iota(F')$ and the claim follows.

We may apply Theorem 2 with $(m-1, N, \xi')$ to the subspace $\iota^{-1}(R)$ which is defined over $\overline{\mathbb{Q}}$. This yields

$$\dim \iota^{-1}(R) \geq \Big(2 - \frac{m-2}{N+m-2}\Big) \dim(F' \cap \iota^{-1}(R)). \tag{4.10}$$

Since $\iota(F') = F \cap K$ we have $F' \cap \iota^{-1}(R) = \iota^{-1}(F \cap R \cap K)$. Now the image of ι is K, so that $\dim(F' \cap \iota^{-1}(R)) = \dim(F \cap R \cap K)$ and $\dim \iota^{-1}(R) = \dim(R \cap K)$: Eq. (4.8) follows from Eq. (4.10).

We shall now prove Eq. (4.9). If $N = 1$ it reads $\dim \pi(R) \geq \dim \pi(F \cap R)$ and holds trivially. Let us assume $N \geq 2$. Recall that $\overline{\Omega} = \{\kappa \in \mathbb{N}^m,\ N-2 \leq |\kappa| \leq N-1\}$, and denote by $\overline{Z}_\kappa$, for $\kappa \in \overline{\Theta} = \{\kappa \in \mathbb{N}^m,\ |\kappa| = N-1\}$, the vectors constructed from ξ with respect to m and $N-1$. We write $\overline{F} = \operatorname{Span}\{\overline{Z}_\kappa,\ \kappa \in \overline{\Theta}\}$. It is clear that for any $\kappa \in \Theta$

we have $\pi(Z_\kappa) = \overline{Z}_{\kappa - e_m}$ if $\kappa_m \geq 1$, and $\pi(Z_\kappa) = 0$ otherwise. Therefore $\pi(F) = \overline{F}$, and Theorem 2 applied to $\pi(R)$ with $(m, N-1, \xi)$ yields

$$\dim \pi(R) \geq \Big(2 - \frac{m-1}{N+m-2}\Big) \dim(\pi(F) \cap \pi(R)). \tag{4.11}$$

Since $\pi(F \cap R) \subset \pi(F) \cap \pi(R)$ this implies Eq. (4.9).

We may now conclude the proof of Theorem 2, since using Lemma 5 we may assume that

$$\dim R \geq 2 \dim(F \cap R) - \dim(F \cap R \cap K). \tag{4.12}$$

The restriction π_R has kernel $R \cap K$ and image $\pi(R)$, so that $\dim R = \dim(R \cap K) + \dim \pi(R)$. Similarly, $\dim(F \cap R) = \dim(F \cap R \cap K) + \dim \pi(F \cap R)$. Therefore adding Eqs. (4.8) and (4.9) yields

$$\dim R \geq \Big(2 - \frac{m-1}{N+m-2}\Big) \dim(F \cap R) + \frac{1}{N+m-2} \dim(F \cap R \cap K).$$

Multiplying this equation by $\frac{N+m-2}{N+m-1}$, and adding Eq. (4.12) divided by $N+m-1$, yields

$$\dim R \geq \Big(2 - \frac{m-1}{N+m-1}\Big) \dim(F \cap R). \tag{4.13}$$

This concludes the proof of the inequality in Theorem 2.

Assume now that equality holds in Eq (4.13). If $m = 1$ then $\dim F = \theta = 1$, $\omega = 2$, and R is either $\{0\}$ or $\mathbb{C}^\Omega$ since F is not defined over $\overline{\mathbb{Q}}$ (because ξ_1 is transcendental). Assume now that $m \geq 2$, and notice that equality holds in Eqs. (4.8), (4.9), (4.11) and (4.12). Using Theorem 2 with $m-1$ instead of m, since equality holds in Eq. (4.8) we have either $R \cap K = \{0\}$ or $R \cap K = K$. In the former case, equality in Eq. (4.12) implies $\dim R = 2 \dim(F \cap R)$, which implies $\dim R = 0$ since we have assumed that equality holds in Eq (4.13). In the latter case, we use (if $N \geq 2$) Theorem 2 and the fact that equality holds in Eq. (4.11) to deduce a new alternative: either $\pi(R) = \mathbb{C}^{\overline{\Omega}}$ or $\pi(R) = \{0\}$. In the former case, since $R \cap K = K$ we obtain $R = \mathbb{C}^\Omega$. In the latter case, we have $R = K$ so that Eq. (4.8) reads $\dim R = \Big(2 - \frac{m-2}{N+m-2}\Big) \dim(F \cap R) > 0$. This contradiction with equality in Eq. (4.13) concludes the proof of Theorem 2.

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