

Palindromic prefixes and episturmian words

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Abstract

Let w be an infinite word on an alphabet \mathcal{A} . We denote by $(n_i)_{i \geq 1}$ the increasing sequence (assumed to be infinite) of all lengths of palindromic prefixes of w . In this text, we give an explicit construction of all words w such that $n_{i+1} \leq 2n_i + 1$ for all i , and study these words. Special examples include characteristic Sturmian words, and more generally standard episturmian words. As an application, we study the values taken by the quantity $\limsup n_{i+1}/n_i$, and prove that it is minimal (among all nonperiodic words) for the Fibonacci word.

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1. Introduction

The purpose of this text is to study infinite words (on an arbitrary, not necessarily finite, alphabet \mathcal{A}) which have “sufficiently many” palindromic prefixes. The motivation comes from diophantine approximation (see below), though this question is also related to physics, namely to the spectral theory of discrete one-dimensional Schrödinger operators. Words with many palindromic factors can be used in this setting [11], corresponding to the combinatorial notion of “palindrome complexity” (see, for instance, [1]). On the other hand, replacing “whole-line methods” by “half-line methods” in connection with this problem leads [4] to the use of words with many palindromic prefixes, like the ones studied below.

In precise terms, given an infinite word w , we shall denote (in this Introduction) by $(n_i)_{i \geq 1}$ the increasing sequence of all lengths of palindromic prefixes of w , with $n_1 = 0$ corresponding

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to the empty prefix. The words studied here always have infinitely many palindromic prefixes, so we assume the sequence $(n_i)_{i \geq 1}$ to be infinite.

A trivial example of such a word w is any periodic word with a palindromic period. A more interesting example is the Fibonacci word $w = babbababbabba \dots$ on the two-letter alphabet $\{a, b\}$, for which the sequence $(n_i) = (0, 1, 3, 6, 11, \dots)$ is given by $n_i = F_{i+1} - 2$ (where F_i is the i th Fibonacci number); this follows from [6, Theorem 5]. More generally, any characteristic Sturmian word satisfies $n_{i+1} \leq 2n_i + 1$ for any i , and denoting by $[0, s_1, s_2, \dots,]$ the continued fraction expansion of its slope we have (see Section 3.1):

$$\limsup \frac{n_{i+1}}{n_i} = \limsup [1, 1, s_k, s_{k-1}, \dots, s_1]. \tag{1}$$

In particular, if w is the Fibonacci word then $\limsup n_{i+1}/n_i$ is the golden ratio $\gamma = (1 + \sqrt{5})/2$.

A generalization of characteristic Sturmian words to an arbitrary alphabet has been given by Droubay, Justin and Pirillo [8]: these are standard episturmian words. They also satisfy $n_{i+1} \leq 2n_i + 1$ for any i , but there is no easy equation like (1) to compute $\limsup n_{i+1}/n_i$.

In this text, we study the words w with abundant palindromic prefixes in the following sense:

Definition 1.1. An infinite word w is said to have *abundant palindromic prefixes* if the sequence $(n_i)_{i \geq 1}$ of all lengths of its palindromic prefixes is infinite and satisfies $n_{i+1} \leq 2n_i + 1$ for any $i \geq 1$.

A completely explicit construction of all words with abundant palindromic prefixes is given, which generalizes one of the constructions [12] of standard episturmian words. This is a strict generalization, i.e., there are words with abundant palindromic prefixes which are not standard episturmian. Moreover, our results extend to words such that $n_{i+1} \leq 2n_i + 1$ for any sufficiently large integer i ; in particular, a general construction of all such words is given.

For any word w , we let

$$\delta(w) = \limsup \frac{n_{i+1}}{n_i}$$

if w admits infinitely many palindromic prefixes, and $\delta(w) = \infty$ otherwise. Then $1/\delta(w)$ measures the “density” of palindromic prefixes in w . We let \mathcal{D} be the set of real numbers that can be written $\delta(w)$ for some word w (on a suitable alphabet). Moreover, we let \mathcal{D}_0 be the set of all numbers $\delta(w)$ obtained from words w with abundant palindromic prefixes. The inclusion $\mathcal{D}_0 \subset \mathcal{D} \cap [1, 2]$ trivially holds, and it is not difficult to prove (see Section 2.2) that $(2, +\infty] \subset \mathcal{D}$. Denoting by \sqcup the union of two disjoint sets, the following result holds.

Theorem 1.2. We have $\mathcal{D} = \mathcal{D}_0 \sqcup (2, +\infty]$.

Actually, for any word w such that $\delta(w) < 2$, there is a word w' with abundant palindromic prefixes such that the palindromic prefixes of w satisfy the same recurrence relation as those of w' (see Proposition 6.1) and, therefore, $\delta(w) = \delta(w')$.

The easiest examples of words with abundant palindromic prefixes are periodic words (with a palindromic period) and characteristic Sturmian words (for which $\delta(w)$ can be computed thanks to Eq. (1)). Denote by \mathcal{D}' the set of numbers $\delta(w)$, for these words w . Obviously we have $\mathcal{D}' \subset \mathcal{D}_0$, and the following theorem shows that this inclusion is an equality if we restrict to words with “sufficiently many” palindromic prefixes.

Theorem 1.3. *We have $\mathcal{D}' \cap [1, \sqrt{3}] = \mathcal{D}_0 \cap [1, \sqrt{3}] = \mathcal{D} \cap [1, \sqrt{3}]$.*

For a periodic word w with a palindromic period, we have trivially $\delta(w) = 1$. For a characteristic Sturmian word w with slope $[0, s_1, s_2, \dots]$, Eq. (1) allows one to compute $\delta(w)$. From this it is easy to deduce that the characteristic Sturmian word w with minimal value of $\delta(w)$ is the Fibonacci word. This shows that 1 and the golden ratio $\gamma = (1 + \sqrt{5})/2$ are the two smallest elements in \mathcal{D}' . Cassaigne studied [3, Corollary 1 and Theorem 2] the next elements, and his result (together with Theorem 1.3) yields:

Theorem 1.4. *The smallest elements in \mathcal{D}_0 (respectively in \mathcal{D}) make up an increasing sequence $(\sigma_n)_{n \geq 0}$ with $\sigma_0 = 1$ and $\sigma_1 = \gamma$, converging to the smallest accumulation point σ_∞ of \mathcal{D}_0 (respectively of \mathcal{D}).*

In more precise terms, this statement means that $\mathcal{D}_0 \cap [1, \sigma_\infty) = \{\sigma_n, n \geq 0\}$. Moreover, all σ_n , and $\sigma_\infty = 1.721\dots$, are given in an explicit way in terms of their continued fraction expansion. For instance, writing \bar{m} for the periodic repetition $mmm\dots = m^\omega$ of a finite sequence m , we have:

$$\sigma_2 = 1 + \sqrt{2}/2 = 1.707\dots = [1, 1, \bar{2}]$$

and

$$\sigma_3 = (2 + \sqrt{10})/3 = 1.720\dots = [1, 1, \bar{2}, \bar{1}, \bar{1}].$$

As a corollary, we see that the Fibonacci word has maximal “palindromic prefix density” among nonperiodic words.

Corollary 1.5. *Let w be an infinite word with $\delta(w) < \gamma$. Then w is periodic.*

For a characteristic Sturmian word w with slope $[0, s_1, s_2, \dots]$, Morse and Hedlund have computed [14] the recurrence function of w . This gives (see [3, Corollary 1]) a formula for the recurrence quotient $\varrho(w)$ of w , namely $\varrho(w) = 2 + \limsup[s_k, s_{k-1}, \dots, s_1]$. Therefore Eq. (1) gives in this case:

$$\delta(w) = \frac{2\varrho(w) - 3}{\varrho(w) - 1},$$

hence (as above) the Fibonacci word has minimal recurrence quotient (equal to $(5 + \sqrt{5})/2$) among all characteristic Sturmian words. Rauzy has conjectured [16] that it has minimal recurrence quotient among all nonperiodic words. Corollary 1.5 is an analogue of this conjecture.

The motivation for this text comes from diophantine approximation. Actually $\mathcal{D}_0 \setminus \{1\}$ is equal [9] to the set denoted by $\mathcal{S}_0 \cap [1, 2]$ in [10], defined in terms of an exponent that measures the simultaneous approximation to a real number and its square by rational numbers with the same denominator. In particular, Theorem 2.1 in [10] follows from this equality and Theorem 1.3 stated above.

This connection between palindromic prefixes and diophantine approximation is due to Roy [17]. It allows one to get a purely number-theoretical proof of Corollary 1.5 stated above, by applying Davenport–Schmidt’s theorem [5] on simultaneous approximation to ξ and ξ^2 to the real number ξ obtained (as in [17]) from an infinite word w .

The structure of this text is as follows. We first explain the notation (Section 2.1), and prove that for any $\alpha > 2$ there is a word w such that $\delta(w) = \alpha$ (Section 2.2). This explains why the rest of the text is devoted only to words w such that $\delta(w) \leq 2$.

Then we recall how characteristic Sturmian words (Section 3.1) and standard episturmian words (Section 3.2) are constructed, with a special emphasis on their palindromic prefixes. In Section 4, we construct all words with abundant palindromic prefixes (Section 4.1). To study these words, the key definition is the one of reduced functions, which allows us to state (Section 4.2) the main results on words with abundant palindromic prefixes. Moreover, we explain (Section 4.3) how to compute $\delta(w)$ for such a word w , using the associated reduced function ψ . The proof of the results stated in Section 4 is given in Section 5, using general lemmas (Sections 5.1 and 5.2) that might be of independent interest.

Next we briefly explain how to generalize the results of Section 4 to words that satisfy $n_{i+1} \leq 2n_i + 1$ for any sufficiently large i (Section 6.1). This allows to prove (in Section 6.2) Theorem 1.2 stated above.

Theorem 1.3 is proved in Section 7.1, and the set \mathcal{D}_0 (respectively \mathcal{D}') is studied near $\sqrt{3}$ in Section 7.3 (respectively in Section 7.4); this implies that Theorem 1.3 is optimal. We also define \mathcal{A} -strict words with abundant palindromic prefixes in Section 7.2, and prove that any \mathcal{A} -strict standard episturmian word w such that $\delta(w) < \sqrt{3}$ is either periodic or characteristic Sturmian.

Section 8 contains questions and open problems about words with abundant palindromic prefixes. At last, Appendix A is devoted to the proof of two technical results: Proposition 6.2 (stated in Section 6.2) and Lemma 7.1 (stated in Section 7.1). These statements concern asymptotic properties of the sequence (n_i) associated with a word w such that $\delta(w) < 2$. They are also useful for the diophantine analogue [9] of this text.

2. Notation and a peculiar construction

2.1. Notation

Throughout the text, we consider a (finite or infinite) alphabet \mathcal{A} , which we assume to be disjoint from $\mathbb{N}^* = \{1, 2, 3, \dots\}$. Of course, this is not a serious restriction; it allows us to consider $\mathcal{A} \sqcup \mathbb{N}^*$ as a disjoint union.

We denote by $|u|$ the length of a finite word u , that is the number of letters in u , and by ε the empty word (which has length zero). Given a finite word $u = u_1 \dots u_p$ with $u_i \in \mathcal{A}$ for any $i \in \{1, \dots, p\}$, we denote by \tilde{u} its mirror image $u_p \dots u_1$, in such a way that u is a palindrome if, and only if, $u = \tilde{u}$. We set $\tilde{\varepsilon} = \varepsilon$, so that ε is considered a palindrome. We say that a word $u' = u'_1 \dots u'_{p'}$ is a prefix of u if $p' \leq p$ and $u_j = u'_j$ for any $j \leq p'$, that is if there is a word u'' such that $u = u'u''$. We extend this definition to the case where u is an infinite¹ word $u_1 u_2 u_3 \dots$. In particular, ε is a palindromic prefix of any (finite or infinite) word.

In the same way, a word u'' is a suffix of u if, and only if, there is a finite word u' such that $u = u'u''$. If this happens then either both u and u'' are finite, or both u and u'' are infinite.

If w and w' are finite words such that w' is a prefix of w , we denote by $w'^{-1}w$ the word w'' such that $w = w'w''$. In the same way, if $w = w'w''$, we write $w' = ww''^{-1}$. An important special case is the following: if w and w' are palindromes and w' is a prefix of w , then w' is also

¹ In this text, we consider only right infinite words. In particular, all palindromes are assumed to be finite.

a suffix of w and $ww'^{-1}w$ is again a palindrome (of which w is a prefix). In this situation, if $w = w'w''$ then we have $ww'^{-1}w = w'w''^2$ (see Lemma 5.1 below).

Remark 2.1. Let w be a word on the (finite or infinite) alphabet \mathcal{A} , such that $n_{i+1} \leq 2n_i$ for any i sufficiently large (with the sequence $(n_i)_{i \geq 1}$ defined in the Introduction). Then only finitely many letters of \mathcal{A} occur in w ; this follows from Proposition 6.1 proved below. Therefore the interesting case, throughout this paper, is when \mathcal{A} is finite.

2.2. Words with scarce palindromic prefixes

In this section, we prove that $(2, +\infty] \subset \mathcal{D}$. This result explains why all words w studied in the rest of this text are such that $\delta(w) \leq 2$.

Obviously there are words w with only a finite number of palindromic prefixes; they satisfy $\delta(w) = \infty$ hence $\infty \in \mathcal{D}$. Now let α be a real number greater than 2, and choose $\varepsilon > 0$ such that $2 + \varepsilon < \alpha$. Denote by $(p_k)_{k \geq 0}$ a sequence of positive integers such that $\frac{p_k}{10^k}$ tends to α , with $\frac{p_k}{10^k} > 2$ for any k . We define an increasing sequence $(n_i)_{i \geq 1}$ in the following way. We let $n_1 = 0$, $n_2 = 1$ and if $i \geq 2$ is even we let v_{i+1} be the maximal integer such that there exists a multiple of $10^{v_{i+1}}$, denoted by n_{i+1} , with $2n_i < n_{i+1} < (2 + \varepsilon)n_i + 1$. If $i \geq 3$ is odd, we let $n_{i+1} = p_{v_i} \frac{n_i}{10^{v_i}}$. With this definition, we have $n_{i+1} \geq 2n_i + 1$ for any $i \geq 1$, and v_i (which is defined only when i is odd) tends to infinity as i tends to infinity. This implies $\limsup \frac{n_{i+1}}{n_i} = \alpha$.

Now let us construct a word w such that $(n_i)_{i \geq 1}$ is exactly the sequence of all lengths of palindromic prefixes of w . We consider an alphabet $\mathcal{A} = \{\delta_k, k \in \mathbb{N}\}$ with $\delta_i \neq \delta_j$ when $i \neq j$. We define finite palindromes π_i , of length n_i , by $\pi_1 = \varepsilon$ and, for $i \geq 1$:

$$\begin{cases} \pi_{i+1} = \pi_i \delta_i \delta_0^{n_{i+1}-2n_i-2} \delta_i \pi_i & \text{if } n_{i+1} \geq 2n_i + 2, \\ \pi_{i+1} = \pi_i \delta_i \pi_i & \text{if } n_{i+1} = 2n_i + 1. \end{cases}$$

Then for any $i \geq 1$, π_i is a palindrome written on the alphabet $\{\delta_0, \dots, \delta_{i-1}\}$. It is also a prefix of π_{i+1} , and all palindromic prefixes of π_{i+1} (except π_{i+1} itself) are prefixes of π_i . The infinite word w defined as the limit of π_i as i tends to infinity satisfies the required property: its palindromic prefixes are exactly the π_i 's, with $n_i = |\pi_i|$. Therefore $\alpha = \limsup \frac{n_{i+1}}{n_i} = \delta(w) \in \mathcal{D}$.

This proves the desired result, namely $(2, +\infty] \subset \mathcal{D}$.

3. Sturmian and episturmian words

In this section, we recall how to construct characteristic, or standard, Sturmian (Section 3.1) and standard episturmian (Section 3.2) words, with a focus on the properties of their palindromic prefixes.

3.1. Characteristic Sturmian words

In this section, we recall a construction of characteristic Sturmian words (see [13, Chapter 2]) and properties of their palindromic prefixes.

We consider the two-letter alphabet $\mathcal{A} = \{a, b\}$. Let s_1, s_2, \dots , be an infinite sequence of positive integers. Define $\sigma_0 = a$, $\sigma_1 = a^{s_1-1}b$ and, by induction, $\sigma_n = \sigma_{n-1}^{s_n} \sigma_{n-2}$ for any $n \geq 2$. In the terminology of [13, p. 75], (σ_n) is the standard sequence associated with $(s_1 - 1, s_2, s_3, \dots)$. For any $n \geq 1$, σ_n is a prefix of σ_{n+1} ; therefore the words σ_n tend to an infinite word c_α , called the *characteristic Sturmian word* with slope $\alpha = [0, s_1, s_2, \dots]$.

For $n \geq 2$ and $1 \leq p \leq s_n$, the word $\sigma_{n-1}^p \sigma_{n-2}$ is a prefix of $\sigma_n = \sigma_{n-1}^{s_n} \sigma_{n-2}$ (since σ_{n-2} is a prefix of σ_{n-1}), hence of c_α . Moreover, it ends with ba if n is even, and with ab if n is odd. As it is a standard word, there exists a palindrome $\hat{\pi}_{n,p}$ such that

$$\begin{cases} \sigma_{n-1}^p \sigma_{n-2} = \hat{\pi}_{n,p} ba & \text{if } n \text{ is even,} \\ \sigma_{n-1}^p \sigma_{n-2} = \hat{\pi}_{n,p} ab & \text{if } n \text{ is odd.} \end{cases} \tag{2}$$

Actually $\hat{\pi}_{n,p}$ is even a central word, so it can be written $\pi ab \pi'$ for some palindromes π, π' (see [7]). However, in what follows, we shall use only the fact that the words $\hat{\pi}_{n,p}$ defined in this way are palindromes. This fact can be proved directly (see, for instance, [2, Lemma 5.3]).

We shall now define a sequence $(\pi_i)_{i \geq 1}$ of palindromic prefixes of c_α . First, for any $k \geq 1$ we let $t_k = s_1 + \dots + s_k$. Now observe that for any $i \geq s_1$ there is exactly one pair (n, p) with $n \geq 2$ and $1 \leq p \leq s_n$ such that $i = t_{n-1} + p - 1$. Therefore the equality

$$\hat{\pi}_{n,p} = \pi_{t_{n-1}+p-1} \quad \text{for } n \geq 2 \text{ and } 1 \leq p \leq s_n$$

defines π_i in a unique way for $i \geq s_1$ (and $\pi_{t_k-1} = \hat{\pi}_{k,s_k}$ is obtained from σ_k by removing the last two letters). If $s_1 \geq 2$, we let $\pi_i = a^{i-1}$ for any $i \in \{1, \dots, s_1 - 1\}$. Then π_i is defined for any $i \geq 1$; we have $\pi_1 = \varepsilon$ and each π_i is a prefix of π_{i+1} . Moreover, all π_i 's are palindromic prefixes of c_α .

Actually the π_i 's are the only palindromic prefixes of c_α . This follows from de Luca's result ([6, Theorem 5]; see also [8, §3]) that π_{i+1} is the right palindromic closure of $\pi_i \delta_i$, where $\delta_i \in \mathcal{A}$ is the letter in π_{i+1} that comes right after π_i (see Section 3.2 below). Another proof of this result can be obtained by applying Theorem 4.12 proved in this text (see Example 4.6).

Since the π_i 's are exactly the palindromic prefixes of c_α , we have the equality $\delta(c_\alpha) = \limsup |\pi_{i+1}|/|\pi_i|$. It is not difficult to deduce Eq. (1) from this (see [2, Proposition 7.1]).

Let $k \geq 3$. It is not difficult to prove the relation

$$\pi_{t_k+\ell} = \sigma_k^{\ell+1} \pi_{t_{k-1}-1} \quad \text{for any } \ell \in \{0, \dots, s_{k+1}\} \tag{3}$$

using (for the case $\ell = s_{k+1}$) the identity $\sigma_{k-1} \pi_{t_k-1} = \sigma_k \pi_{t_{k-1}-1}$ (see, for instance, [2, Lemma 5.1]). From Eq. (3) immediately follows

$$\begin{cases} \pi_{t_{k+1}} = \pi_{t_k} \pi_{t_{k-1}-1}^{-1} \pi_{t_k}, \\ \pi_{t_k+\ell+1} = \pi_{t_k+\ell} \pi_{t_k+\ell-1}^{-1} \pi_{t_k+\ell} \quad \text{for any } \ell \in \{1, \dots, s_{k+1} - 1\}. \end{cases} \tag{4}$$

We are going now to define a map $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ in such a way that, for any $i \geq 1$:

$$\begin{cases} \pi_{i+1} = \pi_i \pi_{\psi(i)}^{-1} \pi_i & \text{if } \psi(i) \in \mathbb{N}^*, \\ \pi_{i+1} = \pi_i \psi(i) \pi_i & \text{if } \psi(i) \in \mathcal{A}. \end{cases} \tag{5}$$

The possibility to define inductively, in this way, the palindromic prefixes of c_α using ψ will be the crucial point in the construction of Section 4.

For $k \geq 3$ we let $\psi(t_k) = t_{k-1} - 1$, and if $i > t_3$ is not among the t_k 's we let $\psi(i) = i - 1$. Then Eq. (4) shows that (5) holds for any $i \geq t_3$. To define the values $\psi(i)$ for $1 \leq i < t_3$, we distinguish between two cases.

First, let us assume $s_1 = 1$. Then $\pi_\ell = b^{\ell-1}$ for $1 \leq \ell \leq t_2$ and $\pi_{t_2+\ell} = (b^{s_2} a)^\ell b^{s_2}$ for any $0 \leq \ell \leq s_3$. We let $\psi(1) = b$, $\psi(t_2) = a$ and $\psi(i) = i - 1$ for $i \in \{2, \dots, t_3 - 1\} \setminus \{t_2\}$. Then Eq. (5) holds for any $i \geq 1$.

Now let us assume $s_1 \geq 2$. Then $\pi_\ell = a^{\ell-1}$ for $1 \leq \ell \leq t_1$ and $\pi_{t_1+\ell} = (a^{s_1-1} b)^\ell a^{s_1-1}$ for any $0 \leq \ell \leq s_2$. Moreover, Eq. (3) holds also for $k = 2$. We let $\psi(1) = a$, $\psi(t_1) = b$, $\psi(t_2) = t_1 - 1$ and $\psi(i) = i - 1$ for $i \in \{2, \dots, t_3 - 1\} \setminus \{t_1, t_2\}$. Then Eq. (5) holds for any $i \geq 1$.

3.2. Standard episturmian words

Denote by $w^{(+)}$ the (right) palindromic closure of a finite word w , that is the shortest palindrome of which w is a prefix. Let $\Delta = \delta_1\delta_2\dots$ be an infinite word on an alphabet \mathcal{A} . Droubay, Justin and Pirillo gave [8] the following definition (see [12, Corollary 2.2]):

Definition 3.1. The standard episturmian word with directive word Δ is the limit of the sequence $(\pi_i)_{i \geq 1}$ defined by $\pi_1 = \varepsilon$ and $\pi_{i+1} = (\pi_i \delta_i)^{+}$ for $i \geq 1$.

The important point here (which will be generalized in Section 4.1) is that a standard episturmian word can be constructed as a limit of an infinite sequence of its palindromic prefixes.

Given Δ , define a function $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ as follows. For $n \geq 1$, let $\psi(n) = \delta_n$ if the letter δ_n occurs for the first time in Δ at the n th position. Otherwise, let $\psi(n) = n'$ where n' is the greatest integer such that $1 \leq n' \leq n - 1$ and $\delta_{n'} = \delta_n$. Then for any $i \geq 1$ we have [12, p. 287]:

$$\pi_{i+1} = \pi_i \pi_{\psi(i)}^{-1} \pi_i \quad \text{if } \psi(i) \in \mathbb{N}^*$$

and

$$\pi_{i+1} = \pi_i \psi(i) \pi_i \quad \text{if } \psi(i) \in \mathcal{A}.$$

The crucial remark in what follows is that these equalities could have been taken as a definition of the sequence (π_i) , and therefore of standard episturmian words.

Example 3.2. Let s_1, s_2, \dots be a sequence of positive integers, and $\mathcal{A} = \{a, b\}$ be a two-letter alphabet. The standard episturmian word with directive word $\Delta = a^{s_1-1} b^{s_2} a^{s_3} b^{s_4} \dots$ is the characteristic Sturmian word with slope $[0, s_1, s_2, \dots]$. This follows from Section 3.1 (see also [6, proof of Theorem 5]).

Example 3.3. Let $\mathcal{A} = \{a, b, c\}$ and $\Delta = (abc)^\omega = abcabcabc \dots$. Then the standard episturmian word w with directive word Δ is [12, Example 2.1] the Tribonacci (or Rauzy [15]) word (that is, the fixed point $abacabaabacabab \dots$ of the morphism defined by $a \mapsto ab, b \mapsto ac$ and $c \mapsto a$). The corresponding function ψ is given by $\psi(n) = n - 3$ for $n \geq 4$, and $\psi(n) = \delta_n$ for $1 \leq n \leq 3$.

4. Words with abundant palindromic prefixes

In this section, we give a general construction (Section 4.1) of all words with abundant palindromic prefixes, using functions ψ . Then we define (Section 4.2) *reduced* functions ψ ; this definition allows us to state the main results about words with abundant palindromic prefixes, namely Theorems 4.12 and 4.14. At last, we explain in Section 4.3 how to compute $\delta(w)$ (for a word w with abundant palindromic prefixes) using the associated reduced function ψ .

4.1. A general construction

Let $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ be any map such that, for each $n \geq 1$:

$$\text{either } \psi(n) \in \mathcal{A} \quad \text{or} \quad 1 \leq \psi(n) \leq n - 1.$$

Define $\pi_1 = \varepsilon$ and, for $i \geq 1$:

$$\pi_{i+1} = \pi_i \pi_{\psi(i)}^{-1} \pi_i \quad \text{if } \psi(i) \in \mathbb{N}^*$$

and

$$\pi_{i+1} = \pi_i \psi(i) \pi_i \quad \text{if } \psi(i) \in \mathcal{A}.$$

It is not difficult to prove by induction that all π_i 's are palindromes, and that π_i is a prefix of π_{i+1} (for instance, if $\psi(i) \in \mathbb{N}^*$, writing $\pi_i = \pi_{\psi(i)} b_i = \tilde{b}_i \pi_{\psi(i)}$ yields $\pi_{i+1} = \tilde{b}_i \pi_{\psi(i)} b_i = \pi_{\psi(i)} b_i^2$; the easy Lemma 5.1 stated below can also be used). However, in general there is no letter $\delta_i \in \mathcal{A}$ such that π_{i+1} be the palindromic closure of $\pi_i \delta_i$.

Definition 4.1. We call *word with abundant palindromic prefixes* associated with ψ , and denote by w_ψ , the limit of the sequence (π_i) .

This definition is consistent with the one given in the Introduction since the following result holds (it is proved in Section 5 as a consequence of Theorem 4.14 stated below).

Theorem 4.2. Let w be an infinite word, and $(n_i)_{i \geq 1}$ be the increasing sequence (assumed to be infinite) of the lengths of its palindromic prefixes (with $n_1 = 0$). Then the following statements are equivalent:

- (i) We have $n_{i+1} \leq 2n_i + 1$ for any $i \geq 1$ (i.e., w has abundant palindromic prefixes).
- (ii) For some function ψ , we have $w = w_\psi$ (i.e., w is the word with abundant palindromic prefixes associated with ψ).

Let us study in more details the word with abundant palindromic prefixes associated with a map ψ . First, let us consider the letter δ_i in π_{i+1} that comes right after π_i . This is the first letter of $\pi_i^{-1} \pi_{i+1}$, the one such that $\pi_i \delta_i$ is a prefix of π_{i+1} . We have $\delta_i = \psi(i)$ if $\psi(i) \in \mathcal{A}$, and $\delta_i = \delta_{\psi(i)}$ otherwise. This explains the following definition.

Definition 4.3. We call *word of first letters* associated with ψ the word $\Delta = \delta_1 \delta_2 \dots$ defined (for each $n \geq 1$) by $\delta_n = \psi(n)$ if $\psi(n) \in \mathcal{A}$, and $\delta_n = \delta_{\psi(n)}$ otherwise.

The assumptions on ψ imply $\psi(1) \in \mathcal{A}$ and $\pi_2 = \psi(1) = \delta_1$. For $\psi(2)$ there are two possibilities: either $\psi(2) \in \mathcal{A}$ (then $\pi_3 = \psi(1)\psi(2)\psi(1)$ and $\delta_2 = \psi(2)$), or $\psi(2) = 1$ (then $\pi_3 = \psi(1)\psi(1)$ and $\delta_2 = \psi(1)$).

Already from this example we can see that several functions ψ may lead to the same word of first letters Δ : for instance, taking $\psi(2) = \psi(1) \in \mathcal{A}$ yields the same value of δ_2 as taking $\psi(2) = 1 \in \mathbb{N}^*$, but not the same value of π_3 . Using this example it is not difficult to produce functions ψ and ψ' with the same word of first letters but such that $w_\psi \neq w_{\psi'}$. Therefore a word w_ψ with abundant palindromic prefixes is not given just by its word of first letters Δ , but by a richer structure: the function² ψ . To be precise, ψ is given exactly by the word $\Delta = \delta_1 \delta_2 \dots$ together with the choice, for any $n \geq 1$, of an integer $n' \in \{0, \dots, n - 1\}$ that satisfies either $n' = 0$ or $\delta_{n'} = \delta_n$. If we fix Δ , then a special choice of ψ is obtained by taking for n' the greatest

² Actually one may restrict to reduced functions, see Section 4.2 below.

integer $n' < n$ such that $\delta_{n'} = \delta_n$ (and $n' = 0$ if there is no such integer, i.e., if the letter δ_n occurs for the first time in Δ at the n th position). For this function ψ , the word w_ψ is the standard episturmian word with directive word Δ (see Section 3.2). Therefore Definitions 4.1 and 4.3 generalize Definition 3.1 of standard episturmian words.

Remark 4.4. Two distinct functions ψ and ψ' always lead to distinct sequences (π_i) and (π'_i) , but may lead to the same word $w_\psi = w_{\psi'}$ (see Example 4.8 below).

Example 4.5. If $\psi(n) = n - 1$ for any $n \geq N$ then $\pi_{N+\ell} = \pi_N \omega^\ell$ for any $\ell \geq 0$, with $\omega = \pi_{N-1}^{-1} \pi_N$. Therefore in this case w_ψ is ultimately periodic, hence periodic with a palindromic period (see Lemma 5.6 below).

Example 4.6. Let $\mathcal{A} = \{a, b\}$ be a two-letter alphabet, and $(s_k)_{k \geq 1}$ be a sequence of positive integers. For any $k \geq 1$, let $t_k = s_1 + \dots + s_k$ if $s_1 \geq 2$ and $t_k = s_1 + \dots + s_{k+1}$ if $s_1 = 1$. In both cases, let $t_0 = 1$. Moreover, let $\psi(i) = i - 1$ if $i \geq 1$ is not among t_0, t_1, t_2, \dots , and $\psi(t_k) = t_{k-1} - 1$ for any $k \geq 2$. If $s_1 \geq 2$, let $\psi(1) = a$ and $\psi(t_1) = b$; if $s_1 = 1$, let $\psi(1) = b$ and $\psi(t_1) = a$. Then the word w_ψ associated with ψ is the characteristic Sturmian word with slope $[0, s_1, s_2, \dots]$. The function ψ , the palindromes π_i and the sequence (t_k) are exactly the same as in Section 3.1 (except that the index k in t_k is shifted if $s_1 = 1$).

Example 4.7. In the previous example, if $s_k = 1$ for any $k \geq 1$ then $\psi(1) = b$, $\psi(2) = a$ and $\psi(i) = i - 2$ for any $i \geq 3$. The word $w_\psi = babbab \dots$ is the Fibonacci word.

4.2. Reduced functions

Two problems immediately arise from the construction of words with abundant palindromic prefixes. First, are there other palindromic prefixes of w_ψ than the π_i 's? Second, can two distinct functions ψ and ψ' lead to the same word w ?

In general, the answers to both questions are positive, as shown in the following example. This is the reason why *reduced* functions are studied below.

Example 4.8. Let ψ be a function, and $i \geq 2$ be an integer, such that $\psi(i + 1) = \psi(i) = i - 1$. Let b_i be the finite nonempty word such that $\pi_i = \pi_{i-1} b_i$. Then $\pi_{i+1} = \pi_{i-1} b_i^2$ and $\pi_{i+2} = \pi_{i-1} b_i^4$. Now Lemma 5.1 stated below shows that $\pi_{i-1} b_i^3$ is a palindromic prefix of π_{i+2} (hence of w_ψ), of length strictly between those of π_{i+1} and π_{i+2} . This gives a palindromic prefix of w_ψ which is not among the π_n 's constructed from ψ . To avoid this problem, consider a function ψ' such that $\psi'(n) = \psi(n)$ for $n \leq i$, $\psi'(i + 1) = i$ and $\psi'(i + 2) = i + 1$. Denoting by (π'_n) the sequence of finite palindromes associated with ψ' , we have $\pi'_n = \pi_n$ for $n \leq i + 1$, $\pi'_{i+2} = \pi_{i-1} b_i^3$ and $\pi'_{i+3} = \pi_{i-1} b_i^4$. For $n \geq i + 3$, we let $\psi'(n) = \psi(n - 1)$ if $\psi(n - 1) \leq i + 1$, and $\psi'(n) = \psi(n - 1) + 1$ otherwise. Then we have $\pi'_n = \pi_{n-1}$ for any $n \geq i + 3$, and $w_{\psi'} = w_\psi$. In this way the functions ψ and ψ' define the same word, but the family of finite palindromes associated with ψ' contains the “missing” palindrome $\pi_{i-1} b_i^3$.

Let $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ be any function (in the sequel we always assume that, for each $n \geq 1$, either $\psi(n) \in \mathcal{A}$ or $1 \leq \psi(n) \leq n - 1$).

Denote by $(t_k)_{k \geq 0}$ the family of all indexes n (in increasing order) such that either $1 \leq \psi(n) \leq n - 2$ or $\psi(n) \in \mathcal{A}$. This family can be either finite or infinite. We always have $t_0 = 1$, since $\psi(1) \in \mathcal{A}$.

Definition 4.9. A function ψ is said to be *reduced* if the associated sequence (t_k) satisfies, for any $k \geq 1$, the following two conditions:

- $\psi(t_k) \neq \psi(t_{k-1})$.
- Either $\psi(t_k) \in \mathcal{A}$ or $\psi(t_k) < t_{k-1}$.

In the special case where the family (t_k) is finite (i.e., $\psi(n) = n - 1$ for n sufficiently large, see Example 4.5), we assume in this definition that both properties hold for any k such that t_k exists.

Remark 4.10. The function ψ in Example 4.6 is reduced, and the definition of (t_k) given there is consistent with the one introduced here.

Remark 4.11. The function ψ in Example 4.8 is not reduced. In fact there is an integer k such that $i + 1 = t_k$, and we have $t_{k-1} \leq i - 1 = \psi(t_k)$.

In the situation of Example 4.8, we have seen that ψ is not reduced, and that the π_i 's are not the only palindromic prefixes of w_ψ . Actually both phenomena are equivalent:

Theorem 4.12. Let $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ be a function such that, for each $n \geq 1$, either $\psi(n) \in \mathcal{A}$ or $1 \leq \psi(n) \leq n - 1$. Then the following assertions are equivalent:

- The function ψ is reduced.
- The palindromic prefixes of w_ψ are exactly the π_i 's constructed from ψ .

This theorem will be proved in the next section (Section 5.3). It is not difficult to deduce the following corollary (see Example 4.5 and Lemma 5.6).

Corollary 4.13. Let ψ be a reduced function. Then w_ψ is periodic if, and only if, $\psi(n) = n - 1$ for any sufficiently large integer n .

Let ψ be a reduced function, and (π_i) be the associated sequence of finite palindromes (that is, thanks to Theorem 4.12, the sequence of all palindromic prefixes of w_ψ). Then the following assertions are easily seen to be equivalent:

- For any sufficiently large i we have $\psi(i) \in \mathbb{N}^*$.
- For any sufficiently large i we have $|\pi_{i+1}| \leq 2|\pi_i|$.

If these assertions hold then w_ψ can be written on a finite alphabet.

In addition to Theorem 4.12, another important property of reduced functions is the following generalization of Theorem 4.2, proved in Section 5 below.

Theorem 4.14. Let w be an infinite word, and $(n_i)_{i \geq 1}$ be the increasing sequence (assumed to be infinite) of the lengths of its palindromic prefixes (with $n_1 = 0$). Then the following statements are equivalent:

- (i) We have $n_{i+1} \leq 2n_i + 1$ for any $i \geq 1$ (i.e., w has abundant palindromic prefixes).
- (ii) There exists a function ψ such that $w = w_\psi$.
- (iii) There exists a reduced function ψ such that $w = w_\psi$.

Moreover, the reduced function ψ in (iii) is unique.

It is possible to write down a “reduction” algorithm (generalizing Example 4.8) that allows one to obtain, from any function ψ , the reduced function ψ' such that $w_\psi = w_{\psi'}$. In this situation, the construction of Section 4.1 applied with ψ gives a sequence (π_i) of palindromic prefixes of w_ψ ; with ψ' , it gives another sequence (π'_i) . Theorem 4.12 shows that (π_i) is a sub-sequence of (π'_i) . Again, the “reduction” algorithm allows one to obtain explicitly the full sequence (π'_i) from the sub-sequence (π_i) . This algorithm is partly used in [9], but in the present text we shall not need it; the crucial point here is just the uniqueness of the reduced function ψ' corresponding to ψ .

Definition 4.15. Let w be a word with abundant palindromic prefixes. The reduced function ψ in Theorem 4.14 is called the *directive function* of w .

Remark 4.16. The uniqueness assertion in Theorem 4.14 immediately follows from Theorem 4.12 and Remark 4.4.

Now we can put Definitions 4.3 and 4.15 together in the following way:

Definition 4.17. Let w be a word with abundant palindromic prefixes. We call *word of first letters* associated with w the word of first letters associated with the directive function of w .

The following property holds: if (π_i) is the sequence of all palindromic prefixes of a word w with abundant palindromic prefixes, and $\Delta = \delta_1 \delta_2 \dots$ is the associated word of first letters, then $\pi_i \delta_i$ is a prefix of π_{i+1} for any $i \geq 1$.

4.3. Computation of $\delta(w)$ using reduced functions

Definition 4.18. With any reduced function ψ we associate the increasing sequence of nonnegative integers $(n_i)_{i \geq 1}$ defined by $n_1 = 0$ and, for all $i \geq 1$:

$$n_{i+1} = 2n_i - n_{\psi(i)} \quad \text{if } \psi(i) \in \mathbb{N}^*$$

and

$$n_{i+1} = 2n_i + 1 \quad \text{if } \psi(i) \in \mathcal{A}.$$

Theorem 4.12 shows that n_i is the length of the i th palindromic prefix of w_ψ . In the same way, we introduce the following definition so that $\delta(\psi) = \delta(w_\psi)$:

Definition 4.19. For any reduced function ψ we let $\delta(\psi) = \limsup \frac{n_{i+1}}{n_i}$, where (n_i) is associated with ψ as in Definition 4.18.

This definition of $\delta(\psi)$ is completely elementary. It is useful because of the following fact: for a word w with abundant palindromic prefixes, we have $\delta(w) = \delta(\psi)$ where ψ is the directive function of w (see Definition 4.15).

5. Proof of the main results

5.1. General lemmas about palindromic prefixes

The first lemma is very easy, and sufficient to prove half of Theorem 4.12 (see Section 5.3 below).

Lemma 5.1. *Let p and u be two words, such that p and pu are palindromes. Then pu^2 is a palindrome (and so is, by induction, the word pu^n for any $n \geq 2$). Similarly, if p and up are palindromes then $u^n p$ is a palindrome for any $n \geq 0$.*

Proof. If p and pu are palindromes then we have $\tilde{p} = p$ and $\tilde{u}p = pu$ hence

$$\widetilde{pu^2} = \tilde{u}\tilde{u}p = \tilde{u}pu = pu^2.$$

The case where p and up are palindromes is analogous. This concludes the proof of Lemma 5.1. \square

In particular, in this situation pu and pu^2 are palindromes, one is a prefix of the other, and the quotient of their lengths is less than 2 (or equal to 2 when p is empty). The following lemma gives a kind of converse to this phenomenon (at least in the case $n' = n$).

Lemma 5.2. *Let w be an infinite word, and n, n', n'' be integers such that $n' \leq n'' \leq n + n'$. We assume that the prefixes of w with lengths n, n', n'' are palindromes, denoted by a, a' and a'' , respectively. Let a_0 be the prefix of w of length $n + n' - n''$. Then the following holds:*

- There is a word b such that $a = a_0b$ and $a'' = a'b$.
- If $n'' \geq n - n'$ then a_0 is a palindrome.

Remark 5.3. This lemma will be used only when $n'' \geq n - n'$, and in this case the first property will be written

$$a'' = a'a_0^{-1}a$$

since a_0 is both a suffix of a' and a prefix of a . Moreover, an important special case is when $n = n'$. The lemma then reads: if a and a' are palindromes, with $n \leq n'' \leq 2n$, then a_0 is a palindrome and we have $a = a_0b$ and $a'' = a_0b^2$.

Proof of Lemma 5.2. As a' is a prefix of a'' , there exists a word b such that $a'' = a'b$. The word b is a suffix of a'' , therefore its mirror image \tilde{b} is a prefix of a'' (hence also of w) since a'' is a palindrome. Now \tilde{b} has length $n'' - n' \leq n$, therefore \tilde{b} is a prefix of a . As a is a palindrome, b is a suffix of a : there exists a word c such that $a = cb$. It is clear that $c = a_0$ is the prefix of w of length $n + n' - n''$.

Assume now $n'' \geq n - n'$, and let us show that a_0 is a palindrome. Let $1 \leq i \leq (n + n' - n'')/2$; then we have $i \leq n'$ hence:

$$w_{n+n'-n''+1-i} = w_{n''-n'+i} = w_{n'+1-i} = w_i,$$

by using successively that a , a'' and a' are palindromes. This concludes the proof of Lemma 5.2. \square

Lemma 5.4. *Let w be an infinite word. Let $n' < n''$ be two consecutive lengths of palindromic prefixes of w ; let us denote by π' and π'' the corresponding prefixes, with $\pi'' = \pi'\omega$ for some word ω . Then any palindromic prefix π of w such that $n' \leq |\pi| \leq n' + n''$ can be written $\pi'\omega^t$ with $t \geq 0$.*

Proof. Assume there is a prefix π of w , of length n , which contradicts the lemma and has minimal length. As n' and n'' are consecutive, we have $n > n''$. Lemma 5.2 gives a palindromic prefix π_0 of w of length $n - |\omega| > n'$, such that $\pi = \pi_0\omega$. This contradicts the minimality of π , and concludes the proof. \square

Lemma 5.5. *Let w be an infinite word. Let $n_0 < n_1 < n_2$ be three consecutive lengths of palindromic prefixes of w ; let us denote by π_0 , π_1 and π_2 the corresponding prefixes. Then:*

- either $\pi_2 = \pi_1\pi_0^{-1}\pi_1$,
- or $n_2 > n_0 + n_1$.

Proof. If $n_2 \leq n_0 + n_1$, one may apply Lemma 5.2 with $n = n_2$, $n' = n_0$ and $n'' = n_1$. Then $n_2 + n_0 - n_1$ is the length of a palindromic prefix of w ; but this length is strictly between n_0 and n_2 , therefore it is n_1 . We get in this way $\pi_2 = \pi_1\pi_0^{-1}\pi_1$, which concludes the proof of the lemma. \square

5.2. Ultimately periodic words

Lemma 5.6. *Let w be an infinite ultimately periodic word, infinitely many prefixes of which are palindromic. Then w is periodic with a palindromic period. Moreover, if d denotes the smallest length of a period of w then there exists $r \in \{1, \dots, d\}$ with the following property. For any $n \geq d$, the prefix of w of length n is a palindrome if, and only if, $n \equiv r \pmod d$.*

Proof. If w were ultimately periodic but not periodic, there would exist two nonempty words π_0 and π such that $w = \pi_0\pi\pi\pi \dots$, and such that the last letter of π_0 be different from that of π . But this contradicts the assumption that w has arbitrary long palindromic prefixes. In fact, if we denote by $z_1 \dots z_d$ the word π and by $z_0 \neq z_d$ the last letter of π_0 , then this assumption implies that the word $z_{d-1} \dots z_0$ appears infinitely many times in w , and is therefore a cyclic permutation of the period $z_1 \dots z_d$. As $z_0 \neq z_d$, this is impossible.

Therefore w is periodic, and can be written $w = \pi\pi\pi \dots$ with a period π of minimal length d . Let $n \geq d$ be the length of a palindromic prefix of w . Then we have $w_i = w_{n+1-i}$ for all $i \in \{1, \dots, d\}$. If n' is another such integer, not congruent to $n \pmod d$, we obtain $w_i = w_{i+\varepsilon}$ for all $i \in \{1, \dots, d\}$ with $1 \leq \varepsilon \leq d - 1$; this contradicts the minimality of d . Therefore all

lengths of palindromic prefixes lie in the same congruence class mod d ; conversely it is clear that any $n \geq d$ that belongs to this class is the length of a palindromic prefix of w . \square

Example 5.7. For the word $a(abba)^\omega = (abb)^\omega$, we have $d = 4$ and $r = 2$.

5.3. Proof of Theorem 4.12

Throughout the proof, we fix a function ψ , and consider the palindromes π_i used to define w_ψ . For any $i \geq 1$ we let $n_i = |\pi_i|$.

First, let us prove the easier implication (using only Lemma 5.1). Assume ψ is not reduced, and all palindromic prefixes of w_ψ are among the π_i 's. There is an index $k \geq 1$ such that either $\psi(t_k) \geq t_{k-1}$ (with $\psi(t_k) \in \mathbb{N}^*$) or $\psi(t_k) = \psi(t_{k-1})$.

Let $j = t_{k-1}$ and $i = t_k$. There are nonempty words b and b' such that $\pi_{j+1} = \pi_j b$ and $\pi_{i+1} = \pi_i b'$. Since $\psi(j + 1) = \dots = \psi(i - 1) = 1$, we have $\pi_\ell = \pi_j b^{\ell-j}$ for any $\ell \in \{j, \dots, i\}$ and in particular $\pi_i = \pi_j b^{i-j}$. Applying Lemma 5.1 to the palindromes $\pi_j b^{i-j-1}$ and π_i proves that $\pi_i b$ is a palindrome. If $b' = b^n$ with $n \geq 2$, this is a palindromic prefix of π_{i+1} (hence of w_ψ), whose length is strictly between those of π_i and π_{i+1} ; this contradicts the assumption that all palindromic prefixes of w_ψ are among the π_i 's. Therefore b' cannot be a nontrivial power of b .

In the case where $\psi(i) \in \mathbb{N}^*$ and $\psi(i) \geq j$, we have $b' = b^{i-\psi(i)}$ with $\psi(i) \leq i - 2$, hence a contradiction.

Assume now $\psi(j) = \psi(i) \in \mathcal{A}$. Then $b = \psi(j)\pi_j$ hence $b' = \psi(j)\pi_i = \psi(j)\pi_j b^{i-j} = b^{i-j+1}$ with $i - j + 1 \geq 2$; this is again a contradiction.

At last, assume $\psi(j) = \psi(i) \in \mathbb{N}^*$. Then we have in the same way $b = \pi_{\psi(j)}^{-1}\pi_j$ hence $b' = \pi_{\psi(j)}^{-1}\pi_i = b^{i-j+1}$, hence a contradiction. This concludes the proof of the first implication in Theorem 4.12.

Let us prove the converse now. Assume ψ is reduced, and let $w = w_\psi$. Let π'' be the palindromic prefix of w of minimal length among those which are not π_i 's. Let i be the integer such that $|\pi_i| < |\pi''| < |\pi_{i+1}|$. Since $\pi_1 = \varepsilon$ and $\pi_2 = \psi(1) \in \mathcal{A}$, we have $i \geq 2$.

Let ω be such that $\pi'' = \pi_i \omega$. Then Lemma 5.4 gives an integer $t \geq 2$ such that $\pi_{i+1} = \pi_i \omega^t$; the definition of π_{i+1} shows that ω is a suffix of π_i . Now Lemma 5.2 (with $n = n' = n_i$ and $n'' = |\pi''|$) implies that $\pi_i \omega^{-1}$ is a palindromic prefix of w .

Let us prove that i is among the t_k 's. This is obvious if $\psi(i) \in \mathcal{A}$, so we may assume $\psi(i) \in \mathbb{N}^*$. Then $\pi_i = \pi_{\psi(i)} \omega^t$ and the palindromic prefix $\pi_i \omega^{-1}$ has length strictly between $|\pi_{\psi(i)}|$ and $|\pi_i|$ since $t \geq 2$. By minimality of π'' , this implies $\psi(i) \leq i - 2$, and concludes the proof that there exists $k \geq 1$ such that $i = t_k$.

Let $j = t_{k-1}$, and b be the word such that $\pi_{j+1} = \pi_j b$. Then we have $\pi_i = \pi_j b^{i-j}$, since $\psi(\ell) = \ell - 1$ for any $\ell \in \{j + 1, \dots, i - 1\}$. Now we have $t \geq 2$, hence $2|\omega| \leq n_i + 1$ and Lemma 5.5 yields:

$$n_i - |\omega| \geq \frac{n_i - 1}{2} \geq \frac{n_j + |b| - 1}{2} \geq \frac{n_j + n_{j-1}}{2} > n_{j-1}.$$

Therefore the palindromic prefix $\pi_i \omega^{-1}$ is among π_j, \dots, π_{i-1} . This shows that ω is a power of b , say $\omega = b^u$.

First, let us assume $\psi(i) \in \mathcal{A}$. If we also have $\psi(j) \in \mathcal{A}$ then $b = \psi(j)\pi_j$ and $\psi(i)\pi_i = \omega^t = b^{ut} = (\psi(j)\pi_j)^{ut}$ hence $\psi(i) = \psi(j) \in \mathcal{A}$, which is a contradiction. Now in the case $\psi(j) \notin \mathcal{A}$

we have $0 \leq n_{\psi(j)} < n_{j-1} \leq |b| - 1$ by Lemma 5.5. However $n_{\psi(j)} \equiv n_j \equiv n_i \equiv -1 \pmod{|b|}$ since $\psi(i)\pi_i = \omega^t$; this is again a contradiction.

To conclude the proof, we have to consider the case where $\psi(i) \in \mathbb{N}^*$. Since ψ is reduced, this gives $\psi(i) < t_{k-1} = j$. We see that $n_{\psi(i)} = n_i - t|\omega|$ belongs to $n_i + |b|\mathbb{Z} = n_j + |b|\mathbb{Z}$. Now $n_j \leq 2n_{j-1} + 1 < 2|b|$ thanks to Lemma 5.5, hence $n_{\psi(i)} = n_j - |b|$. This implies $|b| \leq n_j$, hence $\psi(j) \notin \mathcal{A}$. But we then have $n_{\psi(j)} = n_j - |b| = n_{\psi(i)}$, and this equality contradicts the assumption that ψ is reduced.

This concludes the proof of Theorem 4.12.

5.4. Proof of Theorem 4.14

The uniqueness statement at the end of Theorem 4.14 follows from Theorem 4.12 and Remark 4.4 (as noticed in Remark 4.16 above). Let us prove that (i)–(iii) in Theorem 4.14 are equivalent.

The implications (iii) \Rightarrow (ii) and (ii) \Rightarrow (i) are obvious; let us prove that (i) implies (iii). Assume that w satisfies $n_{i+1} \leq 2n_i + 1$ for all $i \geq 1$. We denote by π_i the palindromic prefix of w with length n_i .

For any $i \geq 1$, let us define $\psi(i)$ in the following way. If $n_{i+1} = 2n_i + 1$, we let $\psi(i) \in \mathcal{A}$ be the letter in π_{i+1} that comes right after π_i (that is, the central letter of the palindrome π_{i+1} which has odd length). Otherwise, we apply Lemma 5.2 with $n = n' = n_i$ and $n'' = n_{i+1}$. This gives an integer $\psi(i)$ between 1 and $i - 1$ such that $\pi_{i+1} = \pi_i \pi_{\psi(i)}^{-1} \pi_i$.

With this construction, it is clear that $w = w_\psi$. More precisely, the palindromes π_i are exactly those constructed using ψ in Section 4.1. Since they are (by hypothesis) the only palindromic prefixes of w , Theorem 4.12 proves that ψ is reduced.

This concludes the proof of Theorem 4.14.

6. Palindromic prefix density

In Section 6.1, we show how the results on words with abundant palindromic prefixes can be generalized to words such that $n_{i+1} \leq 2n_i + 1$ for any sufficiently large i . This enables us in Section 6.2 to describe \mathcal{D} in terms of $\delta(\psi)$ for reduced functions ψ , and to prove Theorem 1.2 stated in the Introduction. This description makes use of a technical statement (Proposition 6.2), the proof of which is postponed to Appendix A.

Given an infinite word w , we denote by (n_i) the increasing sequence of all lengths of palindromic prefixes of w (with $n_1 = 0$). We let π_i be the palindromic prefix of w with length n_i .

6.1. Words with asymptotically abundant palindromic prefixes

The following proposition is a generalization of results stated in Section 4. It enables one to construct all words satisfying $n_{i+1} \leq 2n_i + 1$ for any sufficiently large i .

Proposition 6.1. *Let w be a word with infinitely many palindromic prefixes, and i_0 be an integer. The following statements are equivalent:*

- (i) *We have $n_{i+1} \leq 2n_i + 1$ for any $i \geq i_0$.*
- (ii) *There exists a reduced function ψ such that, for any $i \geq i_0$:*
 - *either $\psi(i) \in \mathcal{A}$ and $\pi_{i+1} = \pi_i \psi(i) \pi_i$,*
 - *or $\psi(i) \in \mathbb{N}^*$ and $\pi_{i+1} = \pi_i \pi_{\psi(i)}^{-1} \pi_i$.*

This proposition means that the palindromic prefixes π_i of w satisfy (for i sufficiently large) the same recurrence relation as those of the word with abundant palindromic prefixes w_ψ . This recurrence relation is completely determined by the function ψ .

Let ψ be a reduced function, and π_{i_0} be a finite word with exactly i_0 palindromic prefixes (including $\pi_1 = \varepsilon$ and π_{i_0} itself), denoted by π_1, \dots, π_{i_0} . Then using the construction of Section 4.1 (which is the same as (ii) in Proposition 6.1) for $i \geq i_0$ one obtains an infinite word w with infinitely many palindromic prefixes, of which π_{i_0} is a palindromic prefix. For this word we have $n_{i+1} \leq 2n_i + 1$ for any $i \geq i_0$, but there is no reason why this relation would hold for $i < i_0$: for instance, if $i_0 = 2$, π_{i_0} may be any palindrome whose only palindromic prefixes are ε , its first letter and itself, so it might have length greater than 3. Moreover, Proposition 6.1 implies that any word w satisfying (i) can be obtained in this way.

Proof of Proposition 6.1. The implication (ii) \Rightarrow (i) is clear. To prove the converse, we define $\psi(i)$ for $i \geq i_0$ as in the proof of Theorem 4.14 in Section 5.4. For $i < i_0$, we let $\psi(i) = a$ if i is even, and $\psi(i) = b$ if i is odd, where $a \neq b$ are two elements in \mathcal{A} (the trivial case where the alphabet contains only one letter is easily dealt with). All assertions in Proposition 6.1 immediately follow, except the fact that ψ is reduced. To prove this fact, one follows exactly the same lines as in the proof of Theorem 4.12 in Section 5.3. \square

6.2. Elementary description of \mathcal{D} and \mathcal{D}_0

To establish a relationship between \mathcal{D} and \mathcal{D}_0 , we shall use the definitions and statements of Section 4.3, and the following (technical) proposition proved in Appendix A.3:

Proposition 6.2. *Let ψ be a reduced function such that $\delta(\psi) < 2$, (n_i) be the associated sequence, and $(n'_i)_{i \geq 1}$ be another increasing sequence of nonnegative integers such that $n'_{i+1} = 2n'_i - n'_{\psi(i)}$ for i sufficiently large.*

Then the quotient n'_i/n_i has a finite positive limit as i tends to infinity, and we have:

$$\delta(\psi) = \limsup \frac{n'_{i+1}}{n'_i}.$$

Let w be any word such that $\delta(w) < 2$, and $(n_i)_{i \geq 0}$ be the sequence of all lengths of palindromic prefixes of w . Proposition 6.1 yields a reduced function ψ such that $n_{i+1} = 2n_i - n_{\psi(i)}$ for any sufficiently large integer i . This proves that (n_i) satisfies the same recurrence relation as the sequence associated with ψ (in Definition 4.18), but the initial values may be distinct. Proposition 6.2 shows that these initial values have no influence on $\limsup \frac{n_{i+1}}{n_i}$, hence $\delta(w) = \delta(\psi)$. This proves that $\mathcal{D} \cap [1, 2)$ is contained in the set of values $\delta(\psi)$ for reduced functions ψ , which is exactly \mathcal{D}_0 (see Section 4.3). Since \mathcal{D}_0 is obviously contained in \mathcal{D} , this gives $\mathcal{D} \cap [1, 2) = \mathcal{D}_0 \cap [1, 2)$. Now considering a characteristic Sturmian word whose slope has unbounded partial quotients proves (thanks to Eq. (1)) that $2 \in \mathcal{D}_0 \subset \mathcal{D}$. Therefore we have proved the following result:

Theorem 6.3. *Both \mathcal{D}_0 and $\mathcal{D} \cap [1, 2]$ are exactly the set of values taken by $\delta(\psi)$ for reduced functions ψ .*

Together with the inclusion $(2, +\infty] \subset \mathcal{D}$ proved in Section 2.2, this statement implies Theorem 1.2.

7. Peculiar study around $\sqrt{3}$

In this section, we focus on the sets \mathcal{D}_0 and \mathcal{D}' around $\sqrt{3}$. First, we prove Theorem 1.3 stated in the Introduction, namely $\mathcal{D}_0 \cap [1, \sqrt{3}] = \mathcal{D}' \cap [1, \sqrt{3}]$ (Section 7.1). Next, we define and study \mathcal{A} -strict words with abundant palindromic prefixes (Section 7.2); this allows us to prove that any \mathcal{A} -strict word w with abundant palindromic prefixes such that $\delta(w) < \sqrt{3}$ is either periodic or characteristic Sturmian.

At last, we prove that Theorem 1.3 is optimal, since there is no gap in \mathcal{D}_0 right above $\sqrt{3}$ (Section 7.3) whereas there is one in \mathcal{D}' (Section 7.4).

As in the previous section, we shall use a technical result (Lemma 7.1, stated in Section 7.1) the proof of which is postponed to Appendix A.

Given an infinite word w , we still denote by (n_i) the increasing sequence of all lengths of palindromic prefixes of w (with $n_1 = 0$). We let π_i be the palindromic prefix of w with length n_i .

7.1. Proof of Theorem 1.3

Let ψ be a reduced function. As in Section 4.2, we denote by $(t_k)_{k \geq 0}$ the family of all indexes n (in increasing order) such that either $1 \leq \psi(n) \leq n - 2$ or $\psi(n) \in \mathcal{A}$.

Assume that $\delta(\psi) < 2$. Then $\psi(i) \in \mathbb{N}^*$ for any sufficiently large integer i , hence:

- Either the family (t_k) is finite, that is $\psi(i) = i - 1$ for any sufficiently large i . This obviously implies $\delta(\psi) = 1$.
- Or the family (t_k) is infinite, and for k sufficiently large we have $\psi(t_k) < t_{k-1}$.

The first condition corresponds to periodic words (see Corollary 4.13). In the second condition, the special case where $\psi(t_k) = t_{k-1} - 1$ for any $k \geq 2$ corresponds to characteristic Sturmian words (see Example 4.6). The following result shows that any reduced function ψ with $\delta(\psi) < \sqrt{3}$ is either “periodic” or “asymptotically Sturmian.”

Lemma 7.1. *Let ψ be a reduced function such that $\delta(\psi) < \sqrt{3}$. Then either the family (t_k) is finite, or for any k sufficiently large we have*

$$\psi(t_k) = t_{k-1} - 1.$$

The value $\sqrt{3}$ in this lemma is optimal (see the end of Section 7.3). We postpone the proof (which is completely elementary, but rather technical) to Appendix A.4.

Now let us prove Theorem 1.3 stated in the Introduction, namely $\mathcal{D}_0 \cap [1, \sqrt{3}] = \mathcal{D}' \cap [1, \sqrt{3}]$. First of all, it is readily seen that $\sqrt{3} = [1, 1, 2, 1] \in \mathcal{D}' \subset \mathcal{D}_0$ (see the beginning of Section 7.4). Let ψ be a reduced function such that $\delta(\psi) < \sqrt{3}$. If w_ψ is periodic then Corollary 4.13 implies $\delta(\psi) = 1 \in \mathcal{D}'$. Otherwise the associated sequence (t_k) is infinite, and satisfies $\psi(t_k) = t_{k-1} - 1$ for any sufficiently large k thanks to Lemma 7.1. Let $s_k = t_k - t_{k-1}$ for $k \geq 1$, and ψ' be the function defined from (s_k) in Example 4.6. Proposition 6.2 shows that $\delta(\psi) = \delta(\psi') = \delta(c_\alpha)$, where c_α is the characteristic Sturmian word with slope $[0, s_1, s_2, \dots]$. Thanks to Theorem 6.3, this concludes the proof of Theorem 1.3. \square

Actually the result we have proved is slightly more precise than Theorem 1.3: for any nonperiodic word w such that $\delta(w) < \sqrt{3}$ we have found a characteristic Sturmian word c_α such that the palindromic prefixes of w and those of c_α satisfy (asymptotically) the same recurrence relation.

In the next section, we show that two additional assumptions (namely abundance of palindromic prefixes and \mathcal{A} -strictness) imply $w = c_\alpha$.

However there is a characteristic Sturmian word w , and a non-episturmian \mathcal{A} -strict word w' with abundant palindromic prefixes, such that $\delta(w) = \delta(w') = \sqrt{3}$ (see the end of Section 7.3). This shows that $\delta(\psi)$ does not characterize a reduced function ψ .

7.2. Initial values and strict words

Let us consider (on the three-letter alphabet $\mathcal{A} = \{a, b, c\}$) the finite word $\pi_4 = abacaba$. It has four palindromic prefixes: $\pi_1 = \varepsilon$, $\pi_2 = a$, $\pi_3 = aba$ and π_4 . It is a palindromic prefix of the word w_ψ constructed (as in Section 4.1) from any function ψ such that $\psi(1) = a$, $\psi(2) = b$ and $\psi(3) = c$. Now if ψ is given by $\psi(i) = i - 2$ for any $i \geq 4$ then w_ψ behaves “asymptotically” like the Fibonacci word (see Example 4.7); for instance, $\delta(w_\psi) = \gamma$.

This word w_ψ is a standard episturmian word, which is not Sturmian (it cannot be written on a two-letter alphabet), but which behaves like a Sturmian word. To avoid this kind of examples, one usually restricts to \mathcal{A} -strict standard episturmian words [8, §4.2], also known as characteristic Arnoux–Rauzy words.

Now let us turn to the (more general) case of words with abundant palindromic prefixes. There are words which have abundant palindromic prefixes, behave like a Sturmian word (as above), but are not Sturmian—and not episturmian either. We introduce the following definition (recall that with any word w with abundant palindromic prefixes we associate in Definition 4.17 its word of first letters $\Delta = \delta_1 \delta_2 \dots$).

Definition 7.2. A word w on an alphabet \mathcal{A} , with abundant palindromic prefixes, is said to be \mathcal{A} -strict if every letter in \mathcal{A} occurs infinitely many times in the word of first letters of w .

This definition extends that of \mathcal{A} -strict standard episturmian words [8, §4.2]. It allows us to state the following result (recall from [8, Theorem 4] that \mathcal{A} -strict standard episturmian words on a two-letter alphabet \mathcal{A} are exactly characteristic Sturmian words):

Theorem 7.3. Let w be any nonperiodic word with abundant palindromic prefixes. We assume w to be \mathcal{A} -strict, and $\delta(w) < \sqrt{3}$. Then \mathcal{A} contains exactly two letters, and w is characteristic Sturmian.

As a special case, we get the following result.

Corollary 7.4. Let w be any nonperiodic \mathcal{A} -strict standard episturmian word, with $\delta(w) < \sqrt{3}$. Then \mathcal{A} contains exactly two letters, and w is characteristic Sturmian.

Proof of Theorem 7.3. Denote by (n_i) the increasing sequence of all lengths of palindromic prefixes of w , and by ψ be the directive function of w . Lemma 7.1 shows that $\psi(t_k) = t_{k-1} - 1$ for any sufficiently large k . Denote by Δ the word of first letters associated with ψ (i.e., with w). Then $\delta_n = \delta_{n-1}$ if n is not among the t_k 's, and $\delta_{t_k} = \delta_{t_{k-1}-1} = \delta_{t_{k-2}}$ for any sufficiently large k . Therefore δ_n takes infinitely many times at most two values. Since w is \mathcal{A} -strict, \mathcal{A} contains at most two letters. If \mathcal{A} is reduced to a single letter then w is nothing but the periodic repetition of this letter. Otherwise w is a standard \mathcal{A} -strict episturmian word on a two-letter alphabet, hence is characteristic Sturmian [8, Theorem 4]. \square

7.3. Non-episturmian examples near $\sqrt{3}$

The following proposition, together with Proposition 7.7 proved in Section 7.4, shows that the value $\sqrt{3}$ is optimal in Theorem 1.3.

Proposition 7.5. *There exists a decreasing sequence in \mathcal{D}_0 with limit $\sqrt{3}$.*

To prove this, consider for any integer $n \geq 2$ the function ψ_n defined as follows (with the two-letter alphabet $\mathcal{A} = \{a, b\}$).

Let $\phi_n : \{0, \dots, 4n\} \rightarrow \{1, 2, 3\}$ be defined by:

- $\phi_n(0) = 3$.
- For $i \in \{1, \dots, n\}$, $\phi_n(i) = 2$.
- For $i \in \{n + 1, \dots, 4n\}$ with $i \equiv n + 1 \pmod{3}$, $\phi_n(i) = 2$.
- For $i \in \{n + 1, \dots, 4n\}$ with $i \equiv n + 2 \pmod{3}$, $\phi_n(i) = 1$.
- For $i \in \{n + 1, \dots, 4n\}$ with $i \equiv n \pmod{3}$, $\phi_n(i) = 3$.

Then we let $\psi_n(1) = a$, $\psi_n(2) = b$ and $\psi_n(i) = i - \phi_n(i')$ for any $i \geq 3$, where i' is the integer between 0 and $4n$ which is congruent to i modulo $4n + 1$.

The word of first letters associated with ψ_n is (in the case where $n \geq 3$ is odd)

$$\Delta_n = (ab)^n (bba)^n b^\omega.$$

By definition, for any $i \geq 3$ we have $\delta_{\psi_n(i)} = \delta_i$, with $\Delta_n = \delta_1 \delta_2 \dots$. Moreover, if $i \geq 3$ is not a multiple of $4n + 1$ then $\psi_n(i)$ is the largest index i' such that $\delta_{i'} = \delta_i$. But when i is a multiple of $4n + 1$, we have $\psi_n(i) = i - 3$ with $\delta_{i-3} = \delta_{i-2} = \delta_i = b$. Moreover, it is easily checked that ψ_n is reduced. Therefore Theorem 4.12 shows that w_{ψ_n} is not a standard episturmian word.

It is possible to prove that $\delta(\psi_n)$ is greater than $\sqrt{3}$, and tends to $\sqrt{3}$ as n tends to infinity.

Using the same ideas, it is possible to construct a word w_ψ (which is not standard episturmian) such that $\delta(w_\psi) = \sqrt{3}$, by choosing an increasing sequence (n_k) , with sufficiently fast growth, and building a function ϕ by concatenating the functions ϕ_{n_k} . This proves that the conclusion of Lemma 7.1 does not hold for any functions ψ such that $\delta(\psi) = \sqrt{3}$.

Remark 7.6. The word w_ψ constructed in this way is not episturmian; however, the value of $\delta(w_\psi)$, namely $\sqrt{3}$, is equal to $\delta(w')$ for some characteristic Sturmian word w' (see the beginning of Section 7.4). This proves that knowing $\delta(w)$ does not provide information on the structure of w .

7.4. The Sturmian spectrum near $\sqrt{3}$

To prove that

$$\sqrt{3} = [1, 1, \overline{2, 1}] = 1.7320\dots$$

belongs to \mathcal{D}' , it is enough to apply Eq. (1) with $s_k = 1$ for k even and $s_k = 2$ for k odd. Now taking $s_k = 3$ for any k yields another element of \mathcal{D}' :

$$\frac{7 + \sqrt{13}}{6} = [1, 1, \overline{3}] = 1.7675\dots$$

The following proposition proves that there is nothing inbetween.

Proposition 7.7. *There is no element of \mathcal{D}' between $\sqrt{3}$ and $\frac{7+\sqrt{13}}{6}$.*

Proof. For a sequence $b = (b_1, b_2, \dots)$ of positive integers, and a nonnegative integer k , we let $T^k b = (b_{k+1}, b_{k+2}, \dots)$. Cassaigne proved in [3] that \mathcal{D}' is the set of numbers $[1, 1, b_1, b_2, \dots]$ where the sequence b satisfies $[b] \geq [T^k b]$ for any $k \geq 0$ (where $[b]$ is the continued fraction $[b_1, b_2, \dots]$). Let b be such a sequence, with $\sqrt{3} < [1, 1, b_1, b_2, \dots] < \frac{7+\sqrt{13}}{6}$.

First of all, let us prove that $b_i \in \{1, 2\}$ for any $i \geq 1$. Indeed, the assumption $[1, 1, b_1, b_2, \dots] < [1, 1, \bar{3}]$ means $[b] < [\bar{3}]$ hence $b_1 \leq 3$. If $b_1 \leq 2$ then the assertion is proved (since $[b] \geq [T^k b]$ for any k). Otherwise $b_1 = 3$ and $b_i \in \{1, 2, 3\}$ for any i . But $[b_2, b_3, \dots] > [\bar{3}]$ yields $b_2 \geq 3$ hence $b_2 = 3$. Now $[b] \geq [Tb]$ gives $[3, b_3, \dots] < [b_3, \dots]$ hence $b_3 = 3$. Repeating these arguments gives $b_i = 3$ for any i , in contradiction with the assumption.

Now we have $[b] > [2, \bar{1}]$ with $b_i \in \{1, 2\}$ for all i . This gives $b_1 = 2$ and $[b_2, b_3, \dots] < [1, \bar{2}]$ hence $b_2 = 1$. Repeating this process yields $[b] = [2, \bar{1}]$, that is a contradiction.

This concludes the proof. \square

8. Open questions

This section is devoted to open questions about words w with abundant palindromic prefixes. Some of them were asked by various specialists I would like to thank.

We let w be a word with abundant palindromic prefixes.

Do letters (and more generally factors) have frequencies in w ? Is it possible to compute these frequencies in terms of the directive function of w (see Definition 4.15)?

What is the complexity of w ?

What are the recurrence quotient, and the critical exponent of w (in terms of ψ)?

What is the bound one should put instead of $\sqrt{3}$ in Theorem 7.3 to ensure that \mathcal{A} contains at most 3 letters? More generally, for any integer p one could study the set of values taken by $\delta(w)$, for words w written on an alphabet of at most p letters.

Are there other ways to construct the set of all words with abundant palindromic prefixes (as for standard episturmian words)?

Is it possible to write words with abundant palindromic prefixes as fixed points of morphisms?

Is there a way to define “nonstandard” words with abundant palindromic prefixes? This would be a class of words that behaves with respect to words with abundant palindromic prefixes in the same way as Sturmian words with respect to characteristic Sturmian words, and in the same way as episturmian words with respect to standard episturmian words.

What does \mathcal{D}' look like between the least accumulation point σ_∞ and $\sqrt{3}$ (see [2, §8])? What does \mathcal{D}_0 look like above $\sqrt{3}$? In which intervals is it dense?

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Appendix A. Study of some recurrence relations

In this appendix, we study the linear recurrence relation satisfied by the sequence (n_i) of all lengths of palindromic prefixes of a word w with abundant palindromic prefixes. We focus on asymptotic properties of this sequence (and especially on $\limsup n_{i+1}/n_i$). The point of view is to forget everything about words: all statements and proofs are completely elementary, and rely only on the recurrence relation associated with a reduced function ψ .

The point is to prove two technical statements used in the text: Proposition 6.2 and Lemma 7.1. Moreover, the tools introduced here are used in [9].

A.1. Definitions

Recall from Section 4.1 that we consider functions $\psi : \mathbb{N}^* \rightarrow \mathbb{N}^* \sqcup \mathcal{A}$ such that, for each $n \geq 1$:
 either $\psi(n) \in \mathcal{A}$ or $1 \leq \psi(n) \leq n - 1$.

With such a function we associate in Section 4.2 a (finite or infinite) family $(t_k)_{k \geq 0}$, namely the family of all indexes $n \geq 1$ (in increasing order) such that either $1 \leq \psi(n) \leq n - 2$ or $\psi(n) \in \mathcal{A}$.

We recall that ψ is *reduced* (see Definition 4.9) if, for any $k \geq 1$, the following two conditions are satisfied:

- $\psi(t_k) \neq \psi(t_{k-1})$.
- Either $\psi(t_k) \in \mathcal{A}$ or $\psi(t_k) < t_{k-1}$.

In the special case where the family (t_k) is finite (i.e., $\psi(n) = n - 1$ for n sufficiently large), we assume in this definition that both properties hold for any k such that t_k exists.

Moreover, with any reduced function ψ we associate (as in Section 4.3) the increasing sequence of nonnegative integers $(n_i)_{i \geq 1}$ defined by $n_1 = 0$ and, for all $i \geq 1$:

$$n_{i+1} = 2n_i - n_{\psi(i)} \quad \text{if } \psi(i) \in \mathbb{N}^*$$

and

$$n_{i+1} = 2n_i + 1 \quad \text{if } \psi(i) \in \mathcal{A},$$

and we let

$$\delta(\psi) = \limsup \frac{n_{i+1}}{n_i}.$$

A.2. First lemmas

Lemma A.1. *Let ψ be a reduced function, and $(n_i)_{i \geq 1}$ be the associated sequence. Then for any $k \geq 1$ we have, with $i = t_k$:*

$$n_{i+1} > n_i + n_{i-1}.$$

Proof. We proceed by induction on k . We have $t_0 = 1$ hence $\psi(t_1) \in \mathcal{A}$ since ψ is reduced. Therefore $n_{t_1+1} = 2n_{t_1} + 1 > n_{t_1-1}$, which proves the result for $k = 1$. Assume it holds for k . If $\psi(t_{k+1}) \in \mathcal{A}$ then it clearly holds for $k + 1$; otherwise we have:

$$n_{t_{k+1}+1} \geq 2n_{t_{k+1}} - n_{t_{k-1}} = n_{t_{k+1}} + n_{t_{k+1}-1} + n_{t_{k+1}} - n_{t_k} - n_{t_{k-1}}.$$

This concludes the proof. \square

Lemma A.2. *Let ψ be a reduced function. Then $\delta(\psi) < 2$ if, and only if, there is an integer B such that, for any sufficiently large i :*

$$\psi(i) \in \mathbb{N}^* \quad \text{and} \quad \psi(i) \geq i - B.$$

Proof. Denote by (n_i) the sequence associated with ψ .

Assume there is an integer B be such that $\psi(i) \geq i - B$ for all i sufficiently large. Then for i sufficiently large we have $\psi(i) \leq 3n_i$ hence $n_{\psi(i)} \geq 3^{-B}n_i$, and therefore $\limsup n_{i+1}/n_i \leq 2 - 3^{-B}$.

Now, assume $\limsup \frac{n_{i+1}}{n_i} < 2 - \varepsilon$ with $\varepsilon > 0$. Then obviously we have $\psi(i) \in \mathbb{N}^*$ for i sufficiently large. Moreover, Lemma A.1 yields, for k large enough:

$$n_{t_k-1} < (1 - \varepsilon)n_{t_k}. \tag{A.1}$$

Let $s_k = t_k - t_{k-1}$; the previous inequality yields

$$n_{t_k} = n_{t_{k-1}} + s_k(n_{t_k} - n_{t_{k-1}}) > n_{t_{k-1}} + s_k \varepsilon n_{t_k},$$

therefore $s_k \leq 1/\varepsilon$: the sequence (s_k) is bounded. Now for i large enough, and λ such that $\psi(i) < t_{k-\lambda} < t_k \leq i$, Eq. (A.1) gives

$$\varepsilon n_i \leq n_{i+1} - 2n_i = n_{\psi(i)} \leq (1 - \varepsilon)^{\lambda+1} n_i$$

that is, an upper bound on λ . This concludes the proof of the lemma. \square

A.3. An independence property

This section is devoted to a proof of Proposition 6.2, which means that $\delta(\psi)$ can be defined using the asymptotic behavior of any solution (n'_i) of the associated recurrence relation: the initial values of the sequence (n_i) do not matter.

Let (n'_i) be as in the statement of Proposition 6.2 (see Section 6.2). Lemma A.2 provides an integer B such that $\psi(i) \in \mathbb{N}^*$ and $\psi(i) \geq i - B$ for all sufficiently large i . Let $\varepsilon'_i = n'_{i+1} - n'_i$. The recurrence relation satisfied by (n'_i) yields (for i sufficiently large)

$$\varepsilon'_{i+1} = \sum_{j=\psi(i+1)}^i \varepsilon'_j.$$

This implies that (ε'_i) is nondecreasing for i sufficiently large, and tends to infinity as i tends to infinity (except in the special case where $\psi(i) = i - 1$ for any sufficiently large i , which is easily dealt with). Moreover, the same properties hold for the sequence $\varepsilon_i = n_{i+1} - n_i$. Now, let

$$\alpha_i = \varepsilon_i / \varepsilon'_i.$$

The relation above yields, for i sufficiently large:

$$\sum_{j=\psi(i+1)}^i \alpha_j \varepsilon'_j = \sum_{j=\psi(i+1)}^i \varepsilon_j = \varepsilon_{i+1} = \alpha_{i+1} \varepsilon'_{i+1} = \alpha_{i+1} \sum_{j=\psi(i+1)}^i \varepsilon'_j,$$

hence

$$\alpha_{i+1} = \sum_{j=\psi(i+1)}^i \frac{\varepsilon'_j}{\varepsilon'_{\psi(i+1)} + \dots + \varepsilon'_i} \alpha_j. \tag{A.2}$$

Now let i_0 be sufficiently large, and for $i \geq i_0$ let I_i be the convex hull of $\alpha_{i+1-B}, \dots, \alpha_i$ (that is, the smallest segment in \mathbb{R}_+^* that contains these points). We shall deduce from (A.2) the following claim (where $|I|$ denotes the length of a segment I):

$$I_{i+B} \subset I_i \quad \text{and} \quad |I_{i+B}| \leq (1 - B^{-B})|I_i| \quad \text{for any } i \geq i_0.$$

If the claim holds then the intersection of all I_i 's is reduced to a positive real number, which is the limit of the sequence (α_i) . As ε'_i tends to infinity with i , it is a classical consequence that

$$\frac{n_i}{n'_i} = \frac{n_0 + \varepsilon_0 + \varepsilon_1 + \dots + \varepsilon_{i-1}}{n'_0 + \varepsilon'_0 + \varepsilon'_1 + \dots + \varepsilon'_{i-1}}$$

converges to the same limit. This concludes the proof of the proposition—if the claim holds.

To prove the claim, write $I_i = [\alpha_i - \delta_i^-, \alpha_i + \delta_i^+]$ and notice that the right-hand side of (A.2) is a linear combination of $\alpha_{i+1-B}, \dots, \alpha_i$ with nonnegative coefficients. Moreover, the coefficient of α_i is at least $1/B$. Therefore bounding α_j in (A.2) from below by $\alpha_i - \delta_i^-$ (respectively from above by $\alpha_i + \delta_i^+$) for $j \neq i$ shows that

$$\alpha_{i+1} \in [\alpha_i - (1 - 1/B)\delta_i^-, \alpha_i + (1 - 1/B)\delta_i^+].$$

Applying this result inductively yields, for any $\ell \geq 0$:

$$\alpha_{i+\ell} \in [\alpha_i - (1 - B^{-\ell})\delta_i^-, \alpha_i + (1 - B^{-\ell})\delta_i^+] \subset I.$$

This proves the claim, thereby concluding the proof of Proposition 6.2. \square

A.4. A special property of $\sqrt{3}$

In this section, we prove Lemma 7.1 stated in Section 7.1.

Let ψ be a reduced function such that $\delta(\psi) < \sqrt{3}$, and (n_i) be the associated sequence, as in Appendix A.1. For any $k \geq 1$ and any $i \in \{t_k, \dots, t_{k+1} - 2\}$, we have $\psi(i + 1) = i$ hence $n_{i+2} - n_{i+1} = n_{i+1} - n_i$. Let us denote by α_k the common value of $n_{i+1} - n_i$ for $i \in \{t_k, \dots, t_{k+1} - 1\}$. Excluding the case where the family (t_k) is finite, it is clear that the sequence (α_k) is increasing. Let $\delta < \sqrt{3}$ and $K \geq 2$ be such that $n_{i+1} \leq \delta n_i$ (hence $\psi(i) \in \mathbb{N}^*$) for all $i \geq t_K$.

Assume there is an index $k > K$ such that $\psi(t_k) < t_{k-1} - 1$.

First of all, let us write (using Lemma A.1 with $i = t_{k-1}$)

$$n_{\psi(t_k)} = 2n_{t_k} - n_{t_{k+1}} \geq (2 - \delta)n_{t_k} \geq (2 - \delta)(n_{t_{k-1}} + n_{t_{k-1}-1}). \tag{A.3}$$

Subtracting $\alpha_{k-2} = n_{t_{k-1}} - n_{t_{k-1}-1}$ from this inequality yields (since $\delta \geq 1$):

$$n_{\psi(t_k)} - \alpha_{k-2} \geq (1 - \delta)n_{t_{k-1}} + (3 - \delta)n_{t_{k-1}-1} \geq (3 - \delta^2)n_{t_{k-1}-1}. \tag{A.4}$$

Since $\delta < \sqrt{3}$, the right-hand side of (A.4) is positive. Therefore Lemma A.1 yields $n_{\psi(t_k)} > \alpha_{k-2} > n_{t_{k-2}-1}$ hence $\psi(t_k) \geq t_{k-2}$. This inequality, together with the assumption $\psi(t_k) < t_{k-1} - 1$, yields $n_{t_{k-1}-1} - \alpha_{k-2} = n_{t_{k-1}-2} \geq n_{\psi(t_k)}$. Combining this with (A.3) gives

$$(2\delta - 3)n_{t_{k-1}-1} \geq (3 - \delta)\alpha_{k-2}, \tag{A.5}$$

which implies, in particular, $\delta \geq 3/2$. Now we also have

$$\delta n_{t_{k-1}} \geq n_{t_{k-1}+1} \geq n_{t_{k-1}} + n_{t_{k-1}-1} = 2n_{t_{k-1}} - \alpha_{k-2},$$

hence

$$\alpha_{k-2} \geq (2 - \delta)n_{t_{k-1}} \geq (2 - \delta)(\alpha_{k-2} + n_{t_{k-1}-1}).$$

Combining this relation with (A.5) yields $\delta \geq \sqrt{3}$, that is a contradiction.

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