Linear independence of odd zeta values using Siegel's lemma

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1. Introduction:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{N}, \quad s \geq 2$$

s even : $\zeta(s) = c_s \pi^s$, $c_s \in \mathbb{Q}^* \rightsquigarrow \zeta(s) \notin \overline{\mathbb{Q}}$

Conj : $1, \zeta(3), \zeta(5), \zeta(7), \ldots$ are linearly independent over $\overline{\mathbb{Q}}$.

Th (Apéry, 1978): $\zeta(3) \notin \mathbb{Q}$.

Th (Ball-Rivoal, 2001):

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \ldots, \zeta(s)) \geq \frac{1+o(1)}{1+\log 2} \log s$$

as $s \to +\infty$, s odd.

The number of irrational numbers among $\zeta(3), \ldots, \zeta(s)$

is $\geq \dfrac{1+o(1)}{1+\log 2} \log s$ (Ball-Rivoal, 2001)

is $\geq 2^{(1+o(1))\frac{\log s}{\log \log s}}$ (F.-Sprang-Zudilin, 2019)

is $\geq 1.19 \sqrt{\dfrac{s}{\log s}}$ (Lai-Yu, 2020)

Th 1: $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \ldots, \zeta(s)) \geq 0.21 \sqrt{\frac{s}{\log s}}$

provided s is a sufficiently large odd integer.

2. Linear independence criterion

Prop 1 (Siegel): Let $\theta_1, \ldots, \theta_p \in \mathbb{R}$, $0 < \alpha < 1$, $\beta > 1$.

Assume that for infinitely many $n \in \mathbb{N}$ there exist $\ell_{ij}^{(n)} \in \mathbb{Z}$, for $1 \leq i, j \leq p$, such that:

(i) the matrix $[\ell_{ij}^{(n)}]_{1 \leq i,j \leq p}$ is invertible

(ii) $\forall i, j$ $|\ell_{ij}^{(n)}| \leq \beta^{n(1+o(1))}$ as $n \longrightarrow \infty$

(iii) $\forall j$ $|\ell_{1j}^{(n)} \theta_1 + \ldots + \ell_{pj}^{(n)} \theta_p| \leq \alpha^{n(1+o(1))}$, $n \to \infty$

Then $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\theta_1, \dots, \theta_p) \geq 1 - \frac{\log \alpha}{\log \beta}$.

<u>Here</u>: $(\theta_1, \dots, \theta_p) = (1, \log 2, \zeta(3), \zeta(5), \dots, \zeta(s))$.

<u>Parameters</u>: $\log \alpha \sim -4.55 \sqrt{s \log s}$, $s \to \infty$

$\log \beta \sim 20.93 \log s$

$\beta^n \approx s^{20.93 n}$

<u>Ball-Rivoal Th</u>: $\log \alpha \sim -s \log s$

$\log \beta \sim (1 + \log 2) s$

$\beta^n \approx (2e)^{sn}$

<u>3. Linear forms in zeta values</u>.

<u>Ball-Rivoal</u>: $F_n^{(BR)}(X) = d_n^a \, n!^{a-2r} \frac{(X - rn)_{rn} (X + n + 1)_{rn}}{(X)_{n+1}^a}$

where $d_n = \operatorname{lcm}(1, 2, \ldots, n)$

$(\alpha)_p = \alpha(\alpha+1)\cdots(\alpha+p-1)$

$r = \left\lfloor \dfrac{a}{(\log a)^2} \right\rfloor$ with $a \geq 3$.

⊗ $\displaystyle\sum_{t=1}^{\infty} F_n^{(BR)}(t)$ is a $\mathbb{Q}$-linear combination of 1

and odd zeta values $\zeta(3), \ldots, \zeta(a)$ if $\begin{cases} a \text{ is odd} \\ n \text{ even} \end{cases}$

$F_n^{BR}(-n-X) = -F_n^{BR}(X)$

⊗ $\left| \displaystyle\sum_{t=1}^{\infty} F_n^{BR}(t) \right|$ is small because:

$\begin{cases} F_n^{BR} \text{ vanishes at } 1, 2, \ldots, rn \text{ so that the sum starts at } rn+1 \\ F_n^{BR}(t) = O\left(\dfrac{1}{t^{(a-2r)n}}\right) \text{ as } t \longrightarrow +\infty \end{cases}$

For Th 1: let $a \geq r \geq 1$, $n \in \mathbb{N}$, consider:

$$F_n(X) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{ij}}{(X+j)^i} \text{ with } c_{ij} \in \mathbb{Z} \text{ to be chosen later}$$

Assume that $F_n(t) = O\left(\frac{1}{t^2}\right)$ as $|t| \longrightarrow +\infty$. For $z \in \mathbb{C}$, $|z| = 1$, let

$$S_n(z) = z^{rn} \underbrace{\sum_{t=rn+1}^{+\infty}}_{\text{the sum starts at } rn+1} \Big(F_n(t) z^{-t} \underbrace{- F_n(-t) z^{t}}_{\text{substitute for the symmetry property}} \Big)$$

If $z \neq 1$:

$$S_n(z) = V(z) + \sum_{i=1}^{a} z^{rn} P_i(z) \left(Li_i\left(\frac{1}{z}\right) - (-1)^i Li_i(z) \right)$$

where $V(z) \in \mathbb{Q}[z]$ has degree $\leq 2rn$

$$P_i(z) = \sum_{j=0}^{n} c_{ij}\, z^j \in \mathbb{Z}[z]$$

$$Li_i(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^i}$$

i^{th} polylogarithm

Convergent when $|x| \leq 1$

except if $x = 1$ and $i = 1$

For $k \geq 1$:

$$S_n^{(k-1)}(z) = \underbrace{V_k(z) + \sum_{i=1}^{a} Q_{k,i}(z)}_{\substack{\text{Rational functions:}\\ \text{only possible poles at}\\ 0 \text{ for } Q_{k,i}(z)\\ 0,1 \text{ for } V_k(z)}} \Big(\underbrace{Li_i\left(\tfrac{1}{z}\right) - (-1)^i Li_i(z)}_{\text{at } z=-1 \text{ this is}} \Big)$$

at $z = -1$ this is

$$\begin{cases} 0 & \text{if } i \text{ is even} \\ 2Li_i(-1) = \begin{cases} -2\log 2 & \text{if } i = 1 \\ 2(2^{1-i}-1)\zeta(i) & \text{if } i \geq 3,\ i \text{ odd} \end{cases} \end{cases}$$

Problem: the coefficients of $V(z)$ have very large denominators : approx. $d_{(r+1)n}^{a} \approx e^{a(r+1)n}$

Consider $k \geq 2rn+2$ so that $k-1 > 2rn \geq \deg V$

Prop 2 : For any $k \geq 2rn+2$ there exists $\delta_k \geq 1$ such that

$$\frac{\delta_k}{(k-1)!} S_n^{(k-1)}(-1) = l_{k,0}^{(n)} + l_{k,1}^{(n)} \log 2 + \sum_{\substack{3 \leq i \leq a \\ i \text{ odd}}} l_{k,i}^{(n)} \zeta(i)$$

with $l_{k,i}^{(n)} \in \mathbb{Z}$ and

$$\left| l_{k,i}^{(n)} \right| \leq \left(16 e^3 (2a+1)\right)^k \max_{i,j} |c_{ij}|$$

4. Small linear forms

$$|S_n(-1)| \leq \sum_{t=2rn+1}^{+\infty} |F_n(t)| + |F_n(-t)|$$

Write $F_n(t) = \sum_{d=1}^{\infty} \frac{a_d}{t^d}$ as $|t| \longrightarrow +\infty$

$$|S_n(-1)| \leq \sum_{t=rn+1}^{+\infty} 2 \sum_{d=1}^{\infty} \frac{|a_d|}{t^d} \approx 2 \sum_{d=1}^{\infty} \left| \frac{a_d}{(rn)^d} \right|$$

<u>Trivial bound</u> : $\left| \frac{a_d}{(rn)^d} \right| \leq r^{-d} d^a \max_{i,j} |c_{ij}|$

Let ω, Ω be such that $1 \leq \omega < \Omega < a$. Assume that:

$$\begin{cases} a_d = 0 \text{ for any } d < \omega n \\ \left| \frac{a_d}{(rn)^d} \right| \leq \underline{r^{-\Omega n}} d^a \max_{i,j} |c_{ij}| \text{ for } \omega n \leq d < \Omega n \end{cases}$$

Then $|S_n(-1)| \leq r^{-\Omega n(1+o(1))} \max_{i,j} |c_{ij}|$ as $n \to \infty$

Parameters: $r = 3.9$, $\omega = 11.58$

$\Omega \simeq 3.9 \sqrt{a \log a} \rightsquigarrow$ α is very close to $r^{-\Omega}$

$$\log \alpha \sim -\Omega \log r$$
$$\sim -5.31 \sqrt{a \log a}.$$

5. Application of Siegel's lemma

$$F_n(X) = \sum_{i=1}^{a} \sum_{j=0}^{n} \frac{c_{ij}}{(X+j)^i} = \sum_{d=1}^{\infty} \frac{a_d}{X^d}, \quad |X| \to \infty$$

Prop 3: Let $a \geq 3$, $r \geq 1$, $1 \leq \omega < \Omega < a$, $n \geq 1$ be such that $rn, \omega n, \Omega n \in \mathbb{N}$. Then there exist $c_{ij} \in \mathbb{Z}$, not all zero, such that:

(i) $|c_{ij}| \leq X^{n(1+o(1))}$ as $n \to \infty$, with

$$X = \exp\left(\frac{4\omega^2 \log(a+1) + \frac{1}{2}\Omega^2 \log r}{a - \omega} \right)$$

(ii) $a_d = 0$ for any $1 \leq d < \omega n$

(iii) $\left| \dfrac{a_d}{(rn)^d} \right| \leq r^{-\Omega n} d^{a} X^{n(1+o(1))}$ for any $\omega n \leq d < \Omega n$

<u>Parameters</u>: $\log X \sim \frac{1}{2} \frac{\Omega^2 \log r}{a} \sim 10.35 \log a$

<u>Siegel's lemma</u>: consider a linear system

$$(S) \quad \sum_{i=1}^{N} a_{ik} x_i = 0 \quad \text{for any } k \in [\![1, K]\!]$$

with coefficients $a_{ik} \in \mathbb{Z}$ such that $K < N$ and

$$\sqrt{\sum_{i=1}^{N} a_{ik}^2} \leq H \quad \text{for any } k \in [\![1, K]\!].$$

Let $G_1, \dots, G_D \geq 1$ be real numbers, and let

$$X = \left(H^K G_1 \dots G_D\right)^{\frac{1}{N-K}}.$$

Let $\lambda_{i,d} \in \mathbb{Z}$ for $1 \leq i \leq N$ and $1 \leq d \leq D$. Then $\exists (x_1, \dots, x_N) \in \mathbb{Z}^N \setminus \{(0, \dots, 0)\}$ such that (S) holds,

$$|x_i| \leq X \quad \text{for any } i \in [\![1, N]\!]$$

and

$$\left|\sum_{i=1}^{N} \lambda_{i,d}\, x_i\right| \leq \frac{X \sum_{i=1}^{N} |\lambda_{id}|}{G_d} \quad \text{for any } d \in [\![1, D]\!]$$

Problem: a_d is a linear form in the c_{ij} with huge coefficients (when d is large)

$$P_i(z) = \sum_{j=0}^{n} c_{ij} z^j$$

$$R_n(z) = \sum_{i=1}^{a} P_i(z) (-1)^{i-1} \frac{(\log z)^{i-1}}{(i-1)!}, \quad z \in \mathbb{C}, \ |z-1|<1$$

Then $$R_n(z) = \sum_{d=1}^{\infty} a_d (-1)^{d-1} \frac{(\log z)^{d-1}}{(d-1)!} \qquad \text{(F.-Rivoal, 2003)}$$

$(\forall d \in [\![1,K]\!] \ a_d = 0) \iff R_n$ vanishes at 1 with order $\geq K$

$K = \omega n - 1$ $\qquad \iff R_n^{(k-1)}(1) = 0$ for any $k \leq K$.

$$R_n^{(k-1)}(z) = \sum_{i=1}^{a} \underline{\underline{P_{k,i}(z)}} (-1)^{i-1} \frac{(\log z)^{i-1}}{(i-1)!}$$

$$R_n^{(k-1)}(1) = P_{k,1}(1)$$

6. Extrapolation.

Problem: for infinitely many n we have to produce $\frac{a+3}{2}$ linearly independent linear forms in $1, \log 2, \zeta(3), \dots, \zeta(a)$.

Shidlovsky's lemma $\rightsquigarrow$ if we produce "reasonably" $a(n+1) - \omega n + c$ linear forms (for some c

independent from n) then $\frac{a+3}{2}$ among them will be linearly independent.

* Make k vary : $2rn+2 \leq k \leq \kappa n$
where $r = 3.9$, $\kappa = 10.58$

* Extrapolate: consider $F_n^{(p)}(t)$ for $0 \leq p \leq h$
where $h = 0.36\, a$

$$(h+1)(\kappa - 2r)n > a(n+1)$$

$\rightsquigarrow$ linear combinations of $1, \log 2, \zeta(3), \zeta(5), \ldots, \zeta(a+h)$

$s = a+h$

Shidlovsky's lemma (1955):

Let $N \geq 1$, $A \in M_N(\mathbb{C}(z))$, $S_0, \ldots, S_{N-1} \in \mathbb{C}[z]$ of degree $\leq m$

Let $y_0, \ldots, y_{N-1}$ be holomorphic at 0 such that $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \end{pmatrix}$

is a solution of $Y' = AY$.

Let μ be the order of a linear diff. equation (with polynomial coefficients) satisfied by

$$R(Y)(z) = \sum_{i=0}^{N-1} S_i(z)\, y_i(z) \qquad \text{"remainder"}$$

<u>Theorem</u>: $\mathrm{ord}_0 R(Y) \leq m\mu + c$ (if $R(Y) \neq 0$)

where c depends only on Y.

Generalizations:

Bertrand–Beukers 1985 using differential Galois theory

several points instead of just 0

Bertrand 2012: at each point, several solutions Y

F. 2018: ∞ can be one of these points

Solutions don't have to be holomorphic

(moderate growth is enough)

Application of Shidlovsky's lemma:

We have linear forms

$$\ell^{(n)}_{k,p,0}\, 1 + \sum_{i=1}^{a+h} \ell^{(n)}_{k,p,i} \left(1-(-1)^i\right) \mathrm{Li}_i(-1)$$

If these linear forms are linearly dependent then $\exists\, x_0, \dots, x_{a+h} \in \mathbb{Q}$, not all zero, such that

$$\forall k\ \forall p \qquad \sum_{i=0}^{a+h} l_{k,p,i}^{(n)}\, x_i = 0.$$

Since -1 is not a singularity of the differential system satisfied by 1 and $Li_i\left(\frac{1}{z}\right) - (-1)^i Li_i(z)$: there exists a solution $(g_0, \dots, g_{a+h})$ of this system such that $\forall i \quad g_i(-1) = x_i$.

Then $\forall k\ \forall p \qquad \sum_{i=0}^{a+h} l_{k,p,i}^{(n)}\, g_i(-1) = 0.$

Let $f_p(z) = \underbrace{T_p(z)}_{\substack{\text{polynomial} \\ \text{of degree} \\ \leq 2rn \\ \text{chosen so that}}} + \sum_{i=0}^{a+h} Q_i^{[p]}(z)\, g_i(z)$

$$\boxed{Q_i^{[0]}(z) = z^{rn} P_i(z)}$$

$$f_p(z) = O\left((z+1)^{2rn+1}\right)$$

Point: $\forall k \in [\![2rn+2, K^r n]\!]$ $f_p^{(k-1)}(-1) = \sum_{i=0}^{a+h} l_{k,p,i}^{(n)} g_i(-1) = 0.$

$$f_p(z) = O\left((z+1)^{K^r n}\right) \text{ as } z \to -1.$$

$$\Sigma = \{0, 1, -1, \infty\}$$

Gon equations $Q_d = 0$