Journal of Number Theory 261 (2024) 36–54

Contents lists available at ScienceDirect

Journal of Number Theory

journal homepage: www.elsevier.com/locate/jnt

General Section

Relations between values of arithmetic Gevrey series, and applications to values of the Gamma function

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A R T I C L E   I N F O

Article history:
Received 4 September 2023
Received in revised form 24 February 2024
Accepted 26 February 2024
Available online 21 March 2024
Communicated by S.J. Miller

MSC:
primary 11J91
secondary 33B15

Keywords:
E-functions
$\ni$-functions
G-functions
Gamma function
Siegel-Shidlovskii theorem

A B S T R A C T

We investigate the relations between the rings $\mathbf{E}$, $\mathbf{G}$ and $\mathbf{D}$ of values taken at algebraic points by arithmetic Gevrey series of order either -1 (E-functions), 0 (analytic continuations of G-functions) or 1 (renormalization of divergent series solutions at ∞ of E-operators) respectively. We prove in particular that any element of $\mathbf{G}$ can be written as multivariate polynomial with algebraic coefficients in elements of $\mathbf{E}$ and $\mathbf{D}$, and is the limit at infinity of some E-function along some direction. This prompts to defining and studying the notion of mixed functions, which generalizes simultaneously E-functions and arithmetic Gevrey series of order 1. Using natural conjectures for arithmetic Gevrey series of order 1 and mixed functions (which are analogues of a theorem of André and Beukers for E-functions) and the conjecture $\mathbf{D}\cap\mathbf{E}=\overline{\mathbb{Q}}$ (but not necessarily all these conjectures at the same time), we deduce a number of interesting Diophantine results such as an analogue for mixed functions of Beukers' linear independence theorem for values of E-functions, the transcendence of the values of the Gamma function and its derivatives at all non-integral

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https://doi.org/10.1016/j.jnt.2024.02.016

algebraic numbers, the transcendence of Gompertz constant as well as the fact that Euler's constant is not in **E**.

1. Introduction

A power series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be an E-function when it is solution of a linear differential equation over $\overline{\mathbb{Q}}(x)$ (holonomic), and $|\sigma(a_n)|$ (for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) and the least common denominator of $a_0, a_1, \ldots, a_n$ grow at most exponentially in n. They were defined and studied by Siegel in 1929, who also defined the class of G-functions: a power series $\sum_{n=0}^{\infty} a_n x^n \in \overline{\mathbb{Q}}[[x]]$ is said to be a G-function when $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$ is an E-function. In this case, $\sum_{n=0}^{\infty} n! a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is called an $\mathfrak{I}$-function, following the terminology introduced by André in [1]. E-functions are entire, while G-functions have a positive radius of convergence, which is finite except for polynomials. Here and below, we see $\overline{\mathbb{Q}}$ as embedded into $\mathbb{C}$. Following André again, E-functions, G-functions and $\mathfrak{I}$-functions are exactly arithmetic Gevrey series of order $s = -1, 0, 1$ respectively. Actually André defines arithmetic Gevrey series of any order $s \in \mathbb{Q}$, but the set of values at algebraic points is the same for a given $s \neq 0$ as for $s/|s|$ using [1, Corollaire 1.3.2].

$\mathfrak{I}$-functions are divergent series, unless they are polynomials. Given an $\mathfrak{I}$-function $\mathfrak{f}$ and any $\theta \in \mathbb{R}$, except finitely many values mod 2π (namely anti-Stokes directions of $\mathfrak{f}$), one can perform Ramis' 1-summation of $\mathfrak{f}(1/z)$ in the direction θ, which coincides in this setting with Borel-Laplace summation (see [12] or [7]). This provides a function denoted by $\mathfrak{f}_\theta(1/z)$, holomorphic on the open subset of $\mathbb{C}$ consisting in all $z \neq 0$ such that $\theta - \frac{\pi}{2} - \varepsilon < \arg z < \theta + \frac{\pi}{2} + \varepsilon$ for some $\varepsilon > 0$, of which $\mathfrak{f}(1/z)$ is the asymptotic expansion in this sector (called a large sector bisected by θ). Of course $\mathfrak{f}(1/z)$ can be extended further by analytic continuation, but this asymptotic expansion may no longer be valid. When an $\mathfrak{I}$-function is denoted by $\mathfrak{f}_j$, we shall denote by $\mathfrak{f}_{j,\theta}$ or $\mathfrak{f}_{j;\theta}$ its 1-summation and we always assume (implicitly or explicitly) that θ is not an anti-Stokes direction.

In [6], [7] and [8, §4.3], we have studied the sets **G**, **E** and **D** defined respectively as the sets of all the values taken by all (analytic continuations of) G-functions at algebraic points, of all the values taken by all E-functions at algebraic points and of all values $\mathfrak{f}_\theta(1)$ where $\mathfrak{f}$ is an $\mathfrak{I}$-function ($\theta = 0$ if it is not an anti-Stokes direction, and $\theta > 0$ is very small otherwise.) These three sets are countable sub-rings of $\mathbb{C}$ that all contain $\overline{\mathbb{Q}}$; conjecturally, they are related to the set of periods and exponential periods, see §3. (The ring **D** is denoted by $\boldsymbol{\mathfrak{I}}$ in [8].)

We shall prove the following result in §3.

Theorem 1. *Every element of* **G** *can be written as a multivariate polynomial (with coefficients in* $\overline{\mathbb{Q}}$*) in elements of* **E** *and* **D**.

Moreover, **G** *coincides with the set of all convergent integrals* $\int_0^\infty F(x)dx$ *where* F *is an* E*-function, or equivalently with the set of all finite limits of* E*-functions at* ∞ *along some direction.*

Above, a convergent integral $\int_0^\infty F(x)dx$ means a finite limit of the E-function $\int_0^z F(x)dx$ as $z \to \infty$ along some direction; this explains the equivalence of both statements.

We refer to Eq. (3.2) in §3 for an expression of log(2) as a polynomial in elements in **E** and **D**; the number π could be similarly expressed by considering z and iz instead of z and $2z$ there. Examples of the last statement are the identities (see [10] for the second one):

$$\int_0^{+\infty} \frac{\sin(x)}{x}dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^{+\infty} J_0(ix)e^{-3x}dx = \frac{\sqrt{6}}{96\pi^3}\Gamma\Big(\frac{1}{24}\Big)\Gamma\Big(\frac{5}{24}\Big)\Gamma\Big(\frac{7}{24}\Big)\Gamma\Big(\frac{11}{24}\Big).$$

It is notoriously difficult to prove/disprove that a given element of **G** is transcendental; it is known that a Siegel-Shidlovskii type theorem for G-functions can not hold *mutatis mutandis*. Theorem 1 suggests that an alternative approach to the study of the Diophantine properties of elements of **G** can be through a better understanding of joint study of the elements of **E** and **D**, modulo certain conjectures to begin with. Our applications will not be immediately directed to the elements of **G** but rather to the understanding of the (absence of) relations between the elements of **E** and **D**.

It seems natural (see [7, p. 37]) to conjecture that $\mathbf{E} \cap \mathbf{G} = \overline{\mathbb{Q}}$, and also that $\mathbf{G} \cap \mathbf{D} = \overline{\mathbb{Q}}$, though both properties seem currently out of reach. In this paper, we suggest (see §2) a possible approach towards the following analogous conjecture.

Conjecture 1. *We have* $\mathbf{E} \cap \mathbf{D} = \overline{\mathbb{Q}}$.

In §2 we shall make a functional conjecture, namely Conjecture 3, that implies Conjecture 1. We also prove that Conjecture 1 has very important consequences, as the following result shows.

Theorem 2. *Assume that Conjecture 1 holds. Then* $\Gamma^{(s)}(a)$ *is a transcendental number for any rational number* $a > 0$ *and any integer* $s \geq 0$*, except of course if* $s = 0$ *and* $a \in \mathbb{N}$*.*

One of the aims of this paper is to show that combining Ǝ- and E-functions may lead to very important results in transcendental number theory. Let us recall now briefly the main known results on E-functions.

Point (i) in the following result is due to André [2] for E-functions with rational Taylor coefficients, and to Beukers [4] in the general case. André used this property to obtain a new proof of the Siegel-Shidlovskii Theorem, and Beukers to prove an optimal refinement of this theorem (namely, (*ii*) below).

Theorem A.

(*i*) [*André, Beukers*] *If an E-function $F(z)$ is such that $F(1) = 0$, then $\frac{F(z)}{z-1}$ is an E-function.*

(*ii*) [*Beukers*] *Let $\underline{F}(z) := {}^t(f_1(z), \dots, f_n(z))$ be a vector of E-functions solution of a differential system $\underline{F}'(z) = A(z)\underline{F}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$.*
Let $\xi \in \overline{\mathbb{Q}}^*$ which is not a pole of a coefficient of A. Let $P \in \overline{\mathbb{Q}}[X_1, \dots, X_n]$ be a homogeneous polynomial such that

$$P(f_1(\xi), \dots, f_n(\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \dots, X_n]$, homogeneous in the X_i, such that

$$Q(z, f_1(z), \dots, f_n(z)) = 0 \text{ identically and } P(X_1, \dots, X_n) = Q(\xi, X_1, \dots, X_n).$$

In particular, we have

$$\operatorname{trdeg}_{\overline{\mathbb{Q}}}(f_1(\xi), \dots, f_n(\xi)) = \operatorname{trdeg}_{\overline{\mathbb{Q}}(z)}(f_1(z), \dots, f_n(z)).$$

The Siegel-Shidlovskii Theorem itself is the final statement about equality of transcendence degrees.

In this paper we state conjectural analogues of these results, involving Э-functions. The principal difficulty is that these functions are divergent power series, and the exact analogue of Theorem A is meaningless. André discussed the situation in [2] and even though he did not formulate exactly the following conjecture, it seems plausible to us. From it, we will show how to deduce an analogue of the Siegel-Shidlovskii theorem for Э-functions. Ferguson [5, p. 171, Conjecture 1] essentially stated this conjecture when $\mathfrak{f}(z)$ has *rational* coefficients and when $\theta = 0$.

Conjecture 2. *Let $\mathfrak{f}(z)$ be an Э-function and $\theta \in (-\pi/2, \pi/2)$ be such that $\mathfrak{f}_\theta(1) = 0$. Then $\frac{\mathfrak{f}(z)}{z-1}$ is an Э-function.*

In other words, the conclusion of this conjecture asserts that $\frac{z}{1-z}\mathfrak{f}(1/z)$ is an Э-function in $1/z$; this is equivalent to $\frac{\mathfrak{f}(1/z)}{z-1}$ being an Э-function in $1/z$ (since we have $\frac{\mathfrak{f}(1/z)}{z-1} = O(1/z)$ unconditionally as $|z| \to \infty$).

Following Beukers' proof [4] yields the following result (see [3, §4.6] for a related conjecture).

Theorem 3. *Assume that Conjecture 2 holds.*

Let $\underline{\mathfrak{f}}(z) := {}^t(\mathfrak{f}_1(z), \ldots, \mathfrak{f}_n(z))$ *be a vector of* $\mathfrak{I}$*-functions solution of a differential system* $\underline{\mathfrak{f}}'(z) = A(z)\underline{\mathfrak{f}}(z)$ *for some matrix* $A(z) \in M_n(\overline{\mathbb{Q}}(z))$*. Let* $\xi \in \overline{\mathbb{Q}}^*$ *and* $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$*; assume that* ξ *is not a pole of a coefficient of* A*, and that* θ *is anti-Stokes for none of the* $\mathfrak{f}_j$*.*

Let $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ *be a homogeneous polynomial such that*

$$P(\mathfrak{f}_{1,\theta}(1/\xi), \ldots, \mathfrak{f}_{n,\theta}(1/\xi)) = 0.$$

Then there exists $Q \in \overline{\mathbb{Q}}[Z, X_1, \ldots, X_n]$*, homogeneous in the* X_i*, such that*

$$Q(z, \mathfrak{f}_1(z), \ldots, \mathfrak{f}_n(z)) = 0 \text{ identically and } P(X_1, \ldots, X_n) = Q(1/\xi, X_1, \ldots, X_n).$$

In particular, we have

$$\mathrm{trdeg}_{\overline{\mathbb{Q}}}(\mathfrak{f}_{1,\theta}(1/\xi), \ldots, \mathfrak{f}_{n,\theta}(1/\xi)) = \mathrm{trdeg}_{\overline{\mathbb{Q}}(z)}(\mathfrak{f}_1(z), \ldots, \mathfrak{f}_n(z)).$$

As an application of Theorem 3, we shall prove the following corollary. Note that under his weaker version of Conjecture 2, Ferguson [5, p. 171, Theorem 2] proved that Gompertz's constant is an irrational number.

Corollary 1. *Assume that Conjecture 2 holds. Then for any* $\alpha \in \overline{\mathbb{Q}}$*,* $\alpha > 0$*, and any* $s \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$*, the number* $\int_0^\infty (t+\alpha)^s e^{-t} dt$ *is a transcendental number.*

In particular, Gompertz's constant $\delta := \int_0^\infty e^{-t}/(t+1) dt$ *is a transcendental number.*

In this text we suggest an approach towards Conjecture 1, based on the new notion of *mixed functions* which enables one to consider E- and $\mathfrak{I}$-functions at the same time. In particular we shall state a conjecture about such functions, namely Conjecture 3 in §2, which implies both Conjecture 1 and Conjecture 2. The following result is a motivation for this approach.

Proposition 1. *Assume that both Conjectures 1 and 2 hold. Then neither Euler's constant* $\gamma := -\Gamma'(1)$ *nor* $\Gamma(a)$ *(with* $a \in \mathbb{Q}^+ \setminus \mathbb{N}$*) are in* $\boldsymbol{E}$*.*

It is likely that none of these numbers is in **G**, but (as far as we know) there is no "functional" conjecture like Conjecture 3 that implies this. It is also likely that none is in **D** as well, but we don't know if this can be deduced from Conjecture 3.

The structure of this paper is as follows. In §2 we define and study mixed functions, a combination of E- and $\mathfrak{I}$-functions. Then in §3 we express any value of a G-function as a polynomial in values of E- and $\mathfrak{I}$-functions, thereby proving Theorem 1. We study derivatives of the Γ function at rational points in §4, and prove Theorem 2 and Proposition 1. At last, §5 is devoted to adapting Beukers' method to our setting: this approach yields Theorem 3 and Corollary 1.

2. Mixed functions

2.1. Definition and properties

In view of Theorem 1, it is natural to study polynomials in E- and $\ni$-functions. We can prove a Diophantine result that combines both Theorems A(*ii*) and 3 but under a very complicated polynomial generalization of Conjecture 2. We opt here for a different approach to mixing E- and $\ni$-functions for which very interesting Diophantine consequences can be deduced from a very easy to state conjecture which is more in the spirit of Conjecture 2. We refer to §2.3 for proofs of all properties stated in this section (including Lemma 1 and Proposition 2), except Theorem 4.

Definition 1. We call *mixed (arithmetic Gevrey) function* any formal power series

$$\sum_{n\in\mathbb{Z}} a_n z^n$$

such that $\sum_{n\geq 0} a_n z^n$ is an E-function in z, and $\sum_{n\geq 1} a_{-n} z^{-n}$ is an $\ni$-function in $1/z$.

In other words, a mixed function is defined as a formal sum $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ where F is an E-function and $\mathfrak{f}$ is an $\ni$-function. In particular, such a function is zero if, and only if, both F and $\mathfrak{f}$ are constants such that $F + \mathfrak{f} = 0$; obviously, F and $\mathfrak{f}$ are uniquely determined by Ψ upon assuming (for instance) that $\mathfrak{f}(0) = 0$. The set of mixed functions is a $\overline{\mathbb{Q}}$-vector space stable under multiplication by z^n for any $n \in \mathbb{Z}$. Unless $\mathfrak{f}(z)$ is a polynomial, such a function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is purely formal: there is no $z \in \mathbb{C}$ such that $\mathfrak{f}(1/z)$ is a convergent series. However, choosing a direction θ which is not anti-Stokes for $\mathfrak{f}$ allows one to evaluate $\Psi_\theta(z) = F(z) + \mathfrak{f}_\theta(1/z)$ at any z in a large sector bisected by θ. Here and below, such a direction will be said *not anti-Stokes for* Ψ and whenever we write $\mathfrak{f}_\theta$ or Ψ_θ we shall assume implicitly that θ is not anti-Stokes.

Definition 1 is a formal definition, but one may identify a mixed function with the holomorphic function it defines on a given large sector by means of the following lemma.

Lemma 1. *Let Ψ be a mixed function, and $\theta \in \mathbb{R}$ be a non-anti-Stokes direction for Ψ. Then Ψ_θ is identically zero (as a holomorphic function on a large sector bisected by θ) if, and only if, Ψ is equal to zero (as a formal power series in z and $1/z$).*

Any mixed function $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ is solution of an E-operator. Indeed, this follows from applying [1, Theorem 6.1] twice: there exist an E-operator L such that $L(\mathfrak{f}(1/z)) = 0$, and an E-operator M such that $M(L(F(z))) = 0$ (because $L(F(z))$ is an E-function). Hence $ML(F(z)+\mathfrak{f}(1/z)) = 0$ and by [1, p. 720, §4.1], ML is an E-operator.

We formulate the following conjecture, which implies both Conjecture 1 and Conjecture 2.

Conjecture 3. *Let $\Psi(z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Then $\frac{\Psi(z)}{z-1}$ is an mixed function.*

The conclusion of this conjecture is that $\Psi(z) = (z-1)\Psi_1(z)$ for some mixed function Ψ_1. This conclusion can be made more precise as follows; see §2.3 for the proof.

Proposition 2. *Let $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ be an mixed function, and $\theta \in (-\pi/2, \pi/2)$ be such that $\Psi_\theta(1) = 0$. Assume that Conjecture 3 holds for Ψ and θ.*

Then both $F(1)$ and $\mathfrak{f}_\theta(1)$ are algebraic, and $\frac{\mathfrak{f}(1/z)-\mathfrak{f}_\theta(1)}{z-1}$ is an Э-function.

Of course, in the conclusion of this proposition, one may assert also that $\frac{F(z)-F(1)}{z-1}$ is an E-function using Theorem A(i).

Conjecture 3 already has far reaching Diophantine consequences: Conjecture 2 and Theorem 2 stated in the introduction, and also the following result that encompasses Theorem 3 in the linear case.

Theorem 4. *Assume that Conjecture 3 holds.*

Let $\boldsymbol{\Psi}(z) := {}^t(\Psi_1(z), \dots, \Psi_n(z))$ be a vector of mixed functions solution of a differential system $\boldsymbol{\Psi}'(z) = A(z)\boldsymbol{\Psi}(z)$ for some matrix $A(z) \in M_n(\overline{\mathbb{Q}}(z))$. Let $\xi \in \overline{\mathbb{Q}}^*$ and $\theta \in (\arg(\xi) - \pi/2, \arg(\xi) + \pi/2)$; assume that ξ is not a pole of a coefficient of A, and that θ is anti-Stokes for none of the Ψ_j.

Let $\lambda_1, \dots, \lambda_n \in \overline{\mathbb{Q}}$ be such that

$$\sum_{i=1}^{n} \lambda_i \Psi_{i,\theta}(\xi) = 0.$$

Then there exist $L_1, \dots, L_n \in \overline{\mathbb{Q}}[z]$ such that

$$\sum_{i=1}^{n} L_i(z)\Psi_i(z) = 0 \textit{ identically and } L_i(\xi) = \lambda_i \textit{ for any } i.$$

In particular, we have

$$\mathrm{rk}_{\overline{\mathbb{Q}}}(\Psi_{1,\theta}(\xi), \dots, \Psi_{n,\theta}(\xi)) = \mathrm{rk}_{\overline{\mathbb{Q}}(z)}(\Psi_1(z), \dots, \Psi_n(z)).$$

The proof of Theorem 4 follows exactly the linear part of the proof of Theorem 3 (see §5.1), which is based on [4, §3]. The only difference is that Э-functions have to be replaced with mixed functions, and Conjecture 2 with Conjecture 3.

However a product of mixed functions is not, in general, a mixed function. Therefore the end of [4, §3] does not adapt to mixed functions, and there is no hope to obtain in this way a result on the transcendence degree of a field generated by values of mixed functions.

As an application of Theorem 4, we can consider the mixed functions $1, e^{\beta z}$ and $\mathfrak{f}(1/z) := \sum_{n=0}^{\infty}(-1)^n n! z^{-n}$, where β is a fixed non-zero algebraic number. These three functions are linearly independent over $\mathbb{C}(z)$ and form a solution of a differential system with only 0 for singularity (because $(\mathfrak{f}(1/z))' = (1+1/z)f(1/z)-1)$, hence for any $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$ and any $\varrho \in \overline{\mathbb{Q}}^*$, the numbers $1, e^{\varrho}, \mathfrak{f}_0(1/\alpha) := \int_0^{\infty} e^{-t}/(1+\alpha t)dt$ are $\overline{\mathbb{Q}}$-linearly independent (for a fixed α, take $\beta = \varrho/\alpha$).

2.2. Values of mixed functions

We denote by $\mathbf{M}_G$ the set of values $\Psi_\theta(1)$, where Ψ is a mixed function and $\theta = 0$ if it is not anti-Stokes, $\theta > 0$ is sufficiently small otherwise. This set is obviously equal to $\mathbf{E} + \mathbf{D}$.

Proposition 3. *For every integer $s \geq 0$ and every $a \in \mathbb{Q}^+$, $a \neq 0$, we have $\Gamma^{(s)}(a) \in e^{-1}\mathbf{M}_G$.*

This result follows immediately from Eq. (4.4) below (see §4.2), written in the form

$$\Gamma^{(s)}(a) = e^{-1}\big((-1)^s e s! E_{a,s+1}(-1) + \mathfrak{f}_{a,s+1;0}(1)\big),$$

because $e^z E_{a,s+1}(-z)$ is an E-function and $\mathfrak{f}_{a,s+1;0}(1)$ is the 1-summation in the direction 0 of an $\mathfrak{I}$-function.

It would be interesting to know if $\Gamma^{(s)}(a)$ belongs to $\mathbf{M}_G$. We did not succeed in proving it does, and we believe it does not. Indeed, for instance if we want to prove that $\gamma \in \mathbf{M}_G$, a natural strategy would be to construct an E-function $F(z)$ with asymptotic expansion of the form $\gamma + \log(z) + \mathfrak{f}(1/z)$ in a large sector, and then to evaluate at $z = 1$. However this strategy cannot work since there is no such E-function (see the footnote in the proof of Lemma 1 in §2.3).

2.3. Proofs concerning mixed functions

To begin with, let us take Proposition 2 for granted and prove that Conjecture 3 implies both Conjecture 1 and Conjecture 2. Concerning Conjecture 2 it is clear. To prove that it implies Conjecture 1, let $\xi \in \mathbf{D}$, i.e. $\xi = \mathfrak{f}_\theta(1)$ is the 1-summation of an $\mathfrak{I}$-function $\mathfrak{f}(z)$ in the direction $\theta = 0$ if it is not anti-Stokes, and $\theta > 0$ close to 0 otherwise. Assume that ξ is also in $\mathbf{E}$: we have $\xi = F(1)$ for some E-function $F(z)$. Therefore, $\Psi(z) = F(z) - \mathfrak{f}(1/z)$ is a mixed function such that $\Psi_\theta(1) = 0$. By Conjecture 3 and Proposition 2, we have $\xi = \mathfrak{f}_\theta(1) \in \overline{\mathbb{Q}}$. This concludes the proof that Conjecture 3 implies Conjecture 1.

Let us prove Proposition 2 now. Assuming that Conjecture 3 holds for Ψ and θ, there exists a mixed function $\Psi_1(z) = F_1(z) + \mathfrak{f}_1(1/z)$ such that $\Psi(z) = (z-1)\Psi_1(z)$. We have

$$F(z) - (z-1)F_1(z) + \mathfrak{f}(1/z) - (z-1)\mathfrak{f}_1(1/z) = 0 \tag{2.1}$$

as a formal power series in z and $1/z$. Now notice that $z-1=z(1-\frac{1}{z})$, and that we may assume $\mathfrak{f}$ and $\mathfrak{f}_1$ to have zero constant terms. Denote by α the constant term of $\mathfrak{f}(1/z)-z(1-\frac{1}{z})\mathfrak{f}_1(1/z)$. Then we have

$$F(z)-(z-1)F_1(z)+\alpha+\mathfrak{f}_2(1/z)=0$$

for some $\Im$-function $\mathfrak{f}_2$ without constant term, so that $\mathfrak{f}_2=0$, $F(z)=(z-1)F_1(z)-\alpha$ and $F(1)=-\alpha\in\overline{\mathbb{Q}}$. This implies $\mathfrak{f}_\theta(1)=\alpha$, and $\frac{\mathfrak{f}(1/z)-\mathfrak{f}_\theta(1)}{z-1}=\mathfrak{f}_1(1/z)$ is an $\Im$-function since $\mathfrak{f}_2=0$. This concludes the proof of Proposition 2.

At last, let us prove Lemma 1. We write $\Psi(z)=F(z)+\mathfrak{f}(1/z)$ and assume that Ψ_θ is identically zero. Modifying θ slightly if necessary, we may assume that the asymptotic expansion $-\mathfrak{f}(1/z)$ of $F(z)$ in a large sector bisected by θ is given explicitly by [7, Theorem 5] applied to $F(z)-F(0)$; recall that such an asymptotic expansion is unique (see [7]). As in [7] we let $g(z)=\sum_{n=1}^{\infty}a_n z^{-n-1}$ where the coefficients a_n are given by $F(z)-F(0)=\sum_{n=1}^{\infty}\frac{a_n}{n!}z^n$. For any $\sigma\in\mathbb{C}\setminus\{0\}$ there is no contribution in $e^{\sigma z}$ in the asymptotic expansion of $F(z)$, so that $g(z)$ is holomorphic at σ. At $\sigma=0$, the local expansion of g is of the form $g(z)=h_1(z)+h_2(z)\log(z)$ with G-functions h_1 and h_2, and the coefficients of h_2 are related to those of $\mathfrak{f}$; however we shall not use this special form ([1]). Now recall that $g(z)=G(1/z)/z$ where G is a G-function; then G is entire and has moderate growth at infinity (because ∞ is a regular singularity of G), so it is a polynomial due to Liouville's theorem. This means that $F(z)$ is a polynomial in z. Recall that asymptotic expansions in large sectors are unique. Therefore both F and $\mathfrak{f}$ are constant functions, and $F+\mathfrak{f}=0$. This concludes the proof of Lemma 1.

3. Proof of Theorem 1: values of *G*-functions

In this section we prove Theorem 1. Let us begin with an example, starting with the relation proved in [13, Proposition 1] for $z\in\mathbb{C}\setminus(-\infty,0]$:

$$\gamma+\log(z)=zE_{1,2}(-z)-e^{-z}\mathfrak{f}_{1,2;0}(1/z) \tag{3.1}$$

where $E_{1,2}$ is an E-function, and $\mathfrak{f}_{1,2}$ is an $\Im$-function, both defined below in §4.2.

Apply Eq. (3.1) at both z and $2z$, and then substract one equation from the other. This provides a relation of the form

$$\log(2)=F(z)+e^{-z}\mathfrak{f}_{1;0}(1/z)+e^{-2z}\mathfrak{f}_{2;0}(1/z) \tag{3.2}$$

valid in a large sector bisected by 0, with an E-function F and $\Im$-functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$. Choosing arbitrarily a positive real algebraic value of z yields an explicit expression of

[1] Actually we are proving that the asymptotic expansion of a non-polynomial E-function is never a $\mathbb{C}$-linear combination of functions $z^\alpha\log^k(z)\mathfrak{f}(1/z)$ with $\alpha\in\mathbb{Q}$, $k\in\mathbb{N}$ and $\Im$-functions $\mathfrak{f}$: some exponentials have to appear.

$\log(2) \in \mathbf{G}$ as a multivariate polynomial in elements of $\mathbf{E}$ and $\mathbf{D}$. But this example shows also that a polynomial in E- and $\ni$-functions may be constant even though there does not seem to be any obvious reason. In particular, the functions 1, $F(z)$, $e^{-z}\mathfrak{f}_{1;0}(1/z)$, and $e^{-2z}\mathfrak{f}_{2;0}(1/z)$ are linearly dependent over $\mathbb{C}$. However we see no reason why they would be linearly dependent over $\overline{\mathbb{Q}}$. This could be a major drawback to combine in E- and $\ni$-functions, since functions that are linearly dependent over $\mathbb{C}$ but not over $\overline{\mathbb{Q}}$ can not belong to any Picard-Vessiot extension over $\overline{\mathbb{Q}}$.

Let us come now to the proof of Theorem 1. We first prove the second part, which runs as follows (it is reproduced from the unpublished note [14]).

From the stability of the class of E-functions by $\frac{d}{dz}$ and $\int_0^z$, we deduce that the set of convergent integrals $\int_0^\infty F(x)dx$ of E-functions and the set of finite limits of E-functions along some direction as $z \to \infty$ are the same. Theorem 2(iii) in [7] implies that if an E-function has a finite limit as $z \to \infty$ along some direction, then this limit must be in $\mathbf{G}$. Conversely, let $\beta \in \mathbf{G}$. By Theorem 1 in [6], there exists a G-function $G(z) = \sum_{n=0}^\infty a_n z^n$ of radius of convergence ≥ 2 (say) such that $G(1) = \beta$. Let $F(z) = \sum_{n=0}^\infty \frac{a_n}{n!} z^n$ be the associated E-function. Then for any z such that $\mathrm{Re}(z) > \frac{1}{2}$, we have

$$\frac{1}{z}G\Big(\frac{1}{z}\Big) = \int_0^{+\infty} e^{-xz}F(x)dx.$$

Hence, $\beta = \int_0^{+\infty} e^{-x}F(x)dx$ where $e^{-z}F(z)$ is an E-function.

We shall now prove the first part of Theorem 1. In fact, we shall prove a slightly more general result, namely Theorem 5 below. We first recall a few notations. Denote by $\mathbf{S}$ the $\mathbf{G}$-module generated by all derivatives $\Gamma^{(s)}(a)$ (with $s \in \mathbb{N}$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$), and by $\mathbf{V}$ the $\mathbf{S}$-module generated by $\mathbf{E}$. Recall that $\mathbf{G}$, $\mathbf{S}$ and $\mathbf{V}$ are rings. Conjecturally, $\mathbf{G} = \mathcal{P}[1/\pi]$ and $\mathbf{V} = \mathcal{P}_e[1/\pi]$ where $\mathcal{P}$ and $\mathcal{P}_e$ are the ring of periods and the ring of exponential periods over $\overline{\mathbb{Q}}$ respectively (see [6, §2.2] and [8, §4.3]). We have proved in [8, Theorem 3] that $\mathbf{V}$ is the $\mathbf{S}$-module generated by the numbers $e^\rho\chi$, with $\rho \in \overline{\mathbb{Q}}$ and $\chi \in \mathbf{D}$.

Theorem 5. *The ring* $\mathbf{V}$ *is the ring generated by* $\mathbf{E}$ *and* $\mathbf{D}$*. In particular, all values of* G*-functions belong to the ring generated by* $\mathbf{E}$ *and* $\mathbf{D}$*.*

In other words, the elements of $\mathbf{V}$ are exactly the sums of products ab with $a \in \mathbf{E}$ and $b \in \mathbf{D}$.

Proof of Theorem 5. We already know that $\mathbf{V}$ is a ring, and that it contains $\mathbf{E}$ and $\mathbf{D}$. To prove the other inclusion, denote by U the ring generated by $\mathbf{E}$ and $\mathbf{D}$. Using Proposition 3 proved in §2.2 and the functional equation of Γ, we have $\Gamma^{(s)}(a) \in U$ for any $s \in \mathbb{N}$ and any $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Therefore for proving that $\mathbf{V} \subset U$, it is enough to prove that $\mathbf{G} \subset U$.

Let $\xi \in \mathbf{G}$. Using [9, Theorem 3] there exists an E-function $F(z)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $F(z)$ in a large sector bisected by θ. As the proof of [9, Theorem 3] shows, we can assume that ξ is the coefficient of e^z in this expansion.

Denote by L an E-operator of which F is a solution, and by μ its order. André has proved [1] that there exists a basis $(H_1(z), \ldots, H_\mu(z))$ of formal solutions of L at infinity such that for any j, $e^{-\rho_j z} H_j(z) \in \mathrm{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ for some algebraic number ρ_j. We recall that elements of $\mathrm{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ are arithmetic Nilsson-Gevrey series of order 1 with algebraic coefficients, i.e. $\overline{\mathbb{Q}}$-linear combinations of functions $z^k (\log z)^\ell \mathfrak{f}(1/z)$ with $k \in \mathbb{Q}$, $\ell \in \mathbb{N}$ and $\mathfrak{I}$-functions $\mathfrak{f}$. Expanding in this basis the asymptotic expansion of $F(z)$ in a large sector bisected by θ (denoted by $\widetilde{F}$), there exist complex numbers $\kappa_1, \ldots, \kappa_d$ such that $\widetilde{F}(z) = \kappa_1 H_1(z) + \ldots + \kappa_\mu H_\mu(z)$. Then we have $\xi = \kappa_1 c_1 + \ldots + \kappa_\mu c_\mu$, where c_j is the coefficient of e^z in $H_j(z) \in e^{\rho_j z} \mathrm{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$. We have $c_j = 0$ if $\rho_j \neq 1$, and otherwise c_j is the constant coefficient of $e^{-z} H_j(z)$: in both cases c_j is an algebraic number. Therefore to conclude the proof that $\xi \in U$, it is enough to prove that $\kappa_1, \ldots, \kappa_\mu \in U$.

For simplicity let us prove that $\kappa_1 \in U$. Given solutions $F_1, \ldots, F_\mu$ of L, we denote by $W(F_1, \ldots, F_\mu)$ the corresponding wronskian matrix. Then for any z in a large sector bisected by θ we have

$$\kappa_1 = \frac{\det W(F(z), H_{2,\theta}(z), \ldots, H_{\mu,\theta}(z))}{\det W(H_{1,\theta}(z), \ldots, H_{\mu,\theta}(z))}$$

where $H_{j,\theta}(z)$ is the 1-summation of $H_j(z)$ in this sector. The determinant in the denominator belongs to $e^{az} \mathrm{NGA}\{1/z\}_1^{\overline{\mathbb{Q}}}$ with $a = \rho_1 + \ldots + \rho_\mu \in \overline{\mathbb{Q}}$. As the proof of [8, Theorem 6] shows, there exist $b, c \in \overline{\mathbb{Q}}$, with $c \neq 0$, such that

$$\det W(H_{1,\theta}(z), \ldots, H_{\mu,\theta}(z)) = c z^b e^{az}.$$

We take $z = 1$, and choose $\theta = 0$ if it is not anti-Stokes for L (and $\theta > 0$ sufficiently small otherwise). Then we have

$$\kappa_1 = c^{-1} e^{-a} \Big(\det W(F(z), H_{2,\theta}(z), \ldots, H_{\mu,\theta}(z)) \Big)_{|z=1} \in U.$$

This concludes the proof. $\square$

Remark 1. The second part of Theorem 1 suggests the following comments. It would be interesting to have a better understanding (in terms of $\mathbf{E}$, $\mathbf{G}$ and $\mathbf{D}$) of the set of convergent integrals $\int_0^\infty R(x)F(x)dx$ where R is a rational function in $\overline{\mathbb{Q}}(x)$ and F is an E-function, which are thus in $\mathbf{G}$ when $R = 1$ (see [14] for related considerations). Indeed, classical examples of such integrals are $\int_0^{+\infty} \frac{\cos(x)}{1+x^2} dx = \pi/(2e) \in \pi\mathbf{E}$, Euler's constant $\int_0^{+\infty} \frac{1-(1+x)e^{-x}}{x(1+x)} dx = \gamma \in \mathbf{E} + e^{-1}\mathbf{D}$ (using Eq. (3.1) and [15, p. 248, Example 2]) and

Gompertz constant $\delta := \int_0^{+\infty} \frac{e^{-x}}{1+x} dx \in \mathbf{D}$. A large variety of behaviors can thus be expected here.

For instance, using various explicit formulas in [11, Chapters 6.5–6.7], it can be proved that

$$\int_0^{+\infty} R(x) J_0(x) dx \in \mathbf{G} + \mathbf{E} + \gamma \mathbf{E} + \log(\overline{\mathbb{Q}}^\star) \mathbf{E}$$

for any $R(x) \in \overline{\mathbb{Q}}(x)$ without poles on $[0, +\infty)$, where $J_0(x) = \sum_{n=0}^{\infty} (ix/2)^{2n}/n!^2$ is a Bessel function.

A second class of examples is when $R(x)F(x)$ is an even function without poles on $[0, +\infty)$ and such that $\lim_{|x|\to\infty, \mathrm{Im}(x)\geq 0} x^2 R(x)F(x) = 0$. Then by the residue theorem,

$$\int_0^{+\infty} R(x)F(x)dx = i\pi \sum_{\rho,\, \mathrm{Im}(\rho)>0} \mathrm{Res}_{x=\rho}\big(R(x)F(x)\big) \in \pi \mathbf{E}$$

where the summation is over the poles of R in the upper half plane.

4. Derivatives of the Γ function at rational points

In this section we prove Theorem 2 and Proposition 1 stated in the introduction, dealing with $\Gamma^{(s)}(a)$. To begin with, we define E-functions $E_{a,s}(z)$ in §4.1 and prove a linear independence result concerning these functions. Then we prove in §4.2 a formula for $\Gamma^{(s)}(a)$, namely Eq. (4.4), involving $E_{a,s+1}(-1)$ and the 1-summation of an Ϡ-function. This enables us to prove Theorem 2 in §4.3 and Proposition 1 in §4.4.

4.1. Linear independence of a family of E-functions

To study derivatives of the Γ function at rational points, we need the following lemma. For $s \geq 1$ and $a \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, we consider the E-function $E_{a,s}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!(n+a)^s}$.

Lemma 2.

(*i*) *For any $a \in \mathbb{Q} \setminus \mathbb{Z}$ and any $s \geq 1$, the functions*

$$1, e^z, e^z E_{a,1}(-z), e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

(*ii*) *For any $a \in \mathbb{N}^*$ and any $s \geq 2$, the functions*

$$1, e^z, e^z E_{a,2}(-z), \dots, e^z E_{a,s}(-z)$$

are linearly independent over $\mathbb{C}(z)$.

Remark 2. Part (i) of the lemma is false if $a \in \mathbb{N}^*$ because $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{Q}(z)$-linearly dependent in this case (see the proof of Part (ii) below).

Proof. (i) For simplicity, we set $\psi_s(z) := e^z E_{a,s}(-z)$. We proceed by induction on $s \geq 1$. Let us first prove the case $s = 1$. The derivative of $\psi_1(z)$ is $(1 + (z - a)\psi_1(z))/z$. Let us assume the existence of a relation $\psi_1(z) = u(z)e^z + v(z)$ with $u, v \in \mathbb{C}(z)$ (a putative relation $U(z) + V(z)e^z + W(z)\psi_1(z) = 0$ forces $W \neq 0$ because $e^z \notin \mathbb{C}(z)$). Then after differentiation of both sides, we end up with

$$\frac{1 + (z - a)\psi_1(z)}{z} = \big(u(z) + u'(z)\big)e^z + v'(z).$$

Hence,

$$\frac{1 + (z - a)\big(u(z)e^z + v(z)\big)}{z} = \big(u(z) + u'(z)\big)e^z + v'(z).$$

Since $e^z \notin \mathbb{C}(z)$, the function $v(z)$ is a rational solution of the differential equation $zv'(z) = (z-a)v(z)+1$: $v(z)$ cannot be identically 0, and it cannot be a polynomial (the degrees do not match on both sides). It must then have a pole at some point ω, of order $d \geq 1$ say. We must have $\omega = 0$ because otherwise the order of the pole at ω of $zv'(z)$ is $d+1$ while the order of the pole of $(z-a)v(z)+1$ is at most d. Writing $v(z) = \sum_{n\geq -d} v_n z^n$ with $v_{-d} \neq 0$ and comparing the term in z^{-d} of $zv'(z)$ and $(z-a)v(z)+1$, we obtain that $d = a$. This forces a to be an integer ≥ 1, which is excluded. Hence, $1, e^z, e^z E_{a,1}(-z)$ are $\mathbb{C}(z)$-linearly independent.

Let us now assume that the case $s - 1 \geq 1$ holds. Let us assume the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=1}^{s-1} u_j(z)\psi_j(z). \tag{4.1}$$

(A putative relation $V(z)+U_0(z)e^z+\sum_{j=1}^{s} U_j(z)\psi_j(z) = 0$ forces $U_s \neq 0$ by the induction hypothesis.) Differentiating (4.1) and because $\psi_j'(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 1$ (where we have let $\psi_0(z) = 1$), we have

$$A(z)\psi_s(z) + \frac{1}{z}\psi_{s-1}(z) = v'(z) + \big(u_0(z) + u_0'(z)\big)e^z + \sum_{j=1}^{s-1} u_j'(z)\psi_j(z) + \sum_{j=1}^{s-1} u_j(z)\Big(A(z)\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)\Big), \tag{4.2}$$

where $A(z) := 1-a/z$. Substituting the right-hand side of (4.1) for $\psi_s(z)$ on the left-hand side of (4.2), we then deduce that

$$v'(z) - A(z)v(z) + \big(u_0'(z) + (1 - A(z))u_0(z)\big)e^z$$
$$+ \frac{1}{z}(z-a)u_1(z)\psi_1(z) + \sum_{j=1}^{s-1} u_j'(z)\psi_j(z) + \frac{1}{z}\sum_{j=1}^{s-1} u_j(z)\psi_{j-1}(z) - \frac{1}{z}\psi_{s-1}(z) = 0.$$

This is a non-trivial $\mathbb{C}(z)$-linear relation between $1, e^z, \psi_1(z), \psi_2(z), \dots, \psi_{s-1}(z)$ because the coefficient of $\psi_{s-1}(z)$ is $u_{s-1}'(z) - 1/z$ and it is not identically 0 because $u_{s-1}'(z)$ cannot have a pole of order 1. But by the induction hypothesis, we know that such a relation is impossible.

(*ii*) The proof can be done by induction on $s \geq 2$ similarly. In the case $s = 2$, assume the existence of a relation $\psi_2(z) = u(z)e^z + v(z)$ with $u(z), v(z) \in \mathbb{C}(z)$. By differentiation, we obtain

$$\Big(1 - \frac{a}{z}\Big)\psi_2(z) = -\frac{1}{z}\psi_1(z) + \big(u(z) + u'(z)\big)e^z + v'(z).$$

By induction on $a \geq 1$, we have $\psi_1(z) = (a-1)!e^z/z^a + w(z)$ for some $w(z) \in \mathbb{Q}(z)$. Hence, we have

$$\Big(1 - \frac{a}{z}\Big)u(z) = -\Big(\frac{(a-1)!}{z^{a+1}} + 1\Big)u(z) + u'(z)$$

which is not possible. Let us now assume that the case $s - 1 \geq 2$ holds, as well as the existence of a relation over $\mathbb{C}(z)$

$$\psi_s(z) = v(z) + u_0(z)e^z + \sum_{j=2}^{s-1} u_j(z)\psi_j(z). \tag{4.3}$$

We proceed exactly as above by differentiation of both sides of (4.3). Using the relation $\psi_j'(z) = (1 - \frac{a}{z})\psi_j(z) + \frac{1}{z}\psi_{j-1}(z)$ for all $j \geq 2$ and the fact that $\psi_1(z) = (a-1)!e^z/z^a + w(z)$, we obtain a relation $\widetilde{v}(z) + \widetilde{u}_0(z)e^z + \sum_{j=2}^{s-1}\widetilde{u}_j(z)\psi_j(z) = 0$ where $\widetilde{u}_{s-1}(z) = u_{s-1}'(z) - 1/z$ cannot be identically 0. The induction hypothesis rules out the existence of such a relation. □

4.2. A formula for $\Gamma^{(s)}(a)$

Let $z > 0$ and $a \in \mathbb{Q}^+$, $a \neq 0$. We have

$$\Gamma^{(s)}(a) = \int_0^\infty t^{a-1}\log(t)^s e^{-t}dt = \int_0^z t^{a-1}\log(t)^s e^{-t}dt + \int_z^\infty t^{a-1}\log(t)^s e^{-t}dt.$$

On the one hand,

$$\begin{aligned}\int_0^z t^{a-1}\log(t)^s e^{-t}dt &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^z t^{a+n-1}\log(t)^s dt\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\sum_{k=0}^{s}(-1)^k\frac{s!}{(s-k)!}\frac{z^{n+a}\log(z)^{s-k}}{(n+a)^{k+1}}\\ &= \sum_{k=0}^{s}\frac{(-1)^k s!}{(s-k)!}z^a\log(z)^{s-k}E_{a,k+1}(-z);\end{aligned}$$

recall that $E_{a,j}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!(n+a)^j}$. On the other hand,

$$\begin{aligned}\int_z^{\infty} t^{a-1}\log(t)^s e^{-t}dt &= e^{-z}\int_0^{\infty}(t+z)^{a-1}\log(t+z)^s e^{-t}dt\\ &= z^{a-1}e^{-z}\sum_{k=0}^{s}\binom{s}{k}\log(z)^{s-k}\int_0^{\infty}(1+t/z)^{a-1}\log(1+t/z)^k e^{-t}dt.\end{aligned}$$

Now $z>0$ so that

$$\mathfrak{f}_{a,k+1;0}(z) := \int_0^{\infty}(1+tz)^{a-1}\log(1+tz)^k e^{-t}dt = \frac{1}{z}\int_0^{\infty}(1+x)^{a-1}\log(1+x)^k e^{-x/z}dx$$

is the 1-summation at the origin in the direction 0 of the $\mathfrak{I}$-function

$$\sum_{n=0}^{\infty} n!u_{a,k,n}z^n,$$

where the sequence $(u_{a,k,n})_{n\geq 0}\in\mathbb{Q}^{\mathbb{N}}$ is defined by the expansion of the G-function:

$$(1+x)^{a-1}\log(1+x)^k = \sum_{n=0}^{\infty}u_{a,k,n}x^n.$$

Note that if $k=0$ and $a\in\mathbb{N}^*$, then $u_{a,k,n}=0$ for any $n\geq a$, and $\mathfrak{f}_{a,k+1;0}(1/z)$ is a polynomial in $1/z$. Therefore, we have for any $z>0$:

$$\Gamma^{(s)}(a)=\sum_{k=0}^{s}\frac{(-1)^k s!}{(s-k)!}z^a\log(z)^{s-k}E_{a,k+1}(-z)+z^{a-1}e^{-z}\sum_{k=0}^{s}\binom{s}{k}\log(z)^{s-k}\mathfrak{f}_{a,k+1;0}(1/z).$$

In particular, for $z=1$, this relation reads

$$\Gamma^{(s)}(a)=(-1)^s s!E_{a,s+1}(-1)+e^{-1}\mathfrak{f}_{a,s+1;0}(1). \tag{4.4}$$

Since $\gamma=-\Gamma'(1)$ we obtain as a special case the formula

$$\gamma = E_{1,2}(-1) - e^{-1}\mathfrak{f}_{1,2;0}(1), \tag{4.5}$$

which is also a special case of Eq. (3.1) proved in [13].

4.3. Proof of Theorem 2

Let us assume that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{Q}^+ \setminus \mathbb{N}$ and $s \geq 0$. Then $e^z\Gamma^{(s)}(a) + (-1)^{s+1}s!e^zE_{a,s+1}(-z)$ is an E-function. The relation $e\Gamma^{(s)}(a)+(-1)^{s+1}s!eE_{a,s+1}(-1) = \mathfrak{f}_{a,s+1;0}(1)$ proved at the end of §4.2 shows that $\alpha := e\Gamma^{(s)}(a) + (-1)^{s+1}s!eE_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$-linear relation between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector

$$Y(z) := {}^t(1, e^z, e^zE_{a,1}(-z), \dots, e^zE_{a,s+1}(-z)).$$

It is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+3}(\overline{\mathbb{Q}}(z))$ (see the computations in the proof of Lemma 2 above). Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$-linearly independent by Lemma 2(i), we deduce from Beukers' [4, Corollary 1.4] that

$$1,\ e,\ eE_{a,1}(-1),\ \dots,\ eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$-linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$-linearly independent. This concludes the proof if $a \in \mathbb{Q}^+ \setminus \mathbb{N}$.

Let us assume now that $\Gamma^{(s)}(a) \in \overline{\mathbb{Q}}$ for some $a \in \mathbb{N}^*$ and $s \geq 1$. Then $e^z\Gamma^{(s)}(a) + (-1)^{s+1}s!e^zE_{a,s+1}(-z)$ is an E-function. The relation $\Gamma^{(s)}(a) + (-1)^{s+1}s!E_{a,s+1}(-1) = e^{-1}\mathfrak{f}_{a,s+1;0}(1)$ shows that $\alpha := e\Gamma^{(s)}(a) + (-1)^{s+1}s!eE_{a,s+1}(-1) \in \mathbf{E} \cap \mathbf{D}$. Hence α is in $\overline{\mathbb{Q}}$ by Conjecture 1 and we have a non-trivial $\overline{\mathbb{Q}}$-linear relation between $1, e$ and $eE_{a,s+1}(-1)$: we claim that this is not possible. Indeed, consider the vector $Y(z) := {}^t(1, e^z, e^zE_{a,2}(-z), \dots,\ e^zE_{a,s+1}(-z))$: it is solution of a differential system $Y'(z) = M(z)Y(z)$ where 0 is the only pole of $M(z) \in M_{s+2}(\overline{\mathbb{Q}}(z))$. Since the components of $Y(z)$ are $\overline{\mathbb{Q}}(z)$-linearly independent by Lemma 2(ii), we deduce again from Beukers' theorem that

$$1,\ e,\ eE_{a,2}(-1),\ \dots,\ eE_{a,s+1}(-1)$$

are $\overline{\mathbb{Q}}$-linearly independent, and in particular that $1, e$ and $eE_{a,s+1}(-1)$ are $\overline{\mathbb{Q}}$-linearly independent. This concludes the proof of Theorem 2.

4.4. Proof of Proposition 1

Recall that Eq. (4.5) proved in §4.2 reads $eE_{1,2}(-1) - e\gamma = \mathfrak{f}_{1,2;0}(1)$. Assuming that $\gamma \in \mathbf{E}$, the left-hand side is in $\mathbf{E}$ while the right-hand side is in $\mathbf{D}$. Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. Note that, by integration by parts,

$$\mathfrak{f}_{1,2;0}(1) = \int_0^\infty \log(1+t)e^{-t}dt = \int_0^\infty \frac{e^{-t}}{1+t}dt$$

is Gompertz's constant. Hence, by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{1,2;0}(1)$ is not in $\overline{\mathbb{Q}}$. Consequently, $\gamma \notin \mathbf{E}$.

Similarly, Eq. (4.4) with $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $s = 0$ reads $e\Gamma(a) - eE_{a,1}(-1) = \mathfrak{f}_{a,1;0}(1)$. Assuming that $\Gamma(a) \in \mathbf{E}$, the left-hand side is in $\mathbf{E}$ while the right-hand side is in $\mathbf{D}$. Hence both sides are in $\overline{\mathbb{Q}}$ by Conjecture 1. But by Corollary 1 (which holds under Conjecture 2), the number $\mathfrak{f}_{a,1;0}(1) = \int_0^\infty (1+t)^{a-1}e^{-t}dt$ is not in $\overline{\mathbb{Q}}$. Hence, $\Gamma(a) \notin \mathbf{E}$.

5. Application of Beukers' method and consequence

In this section we prove Theorems 3 and 4, and Corollary 1 stated in the introduction.

5.1. Proofs of Theorems 3 and 4

The proof of Theorem 3 (resp. Theorem 4) is based on the arguments given in [4], except that E-functions have to be replaced with $\mathfrak{I}$-functions (resp. mixed functions), and 1-summation in non-anti-Stokes directions is used for evaluations. Conjecture 2 (resp. Conjecture 3) is used as a substitute for Theorem A(i).

The main step is the following result.

Proposition 4. *Assume that Conjecture 2 (resp. Conjecture 3) holds.*

Let $\mathfrak{f}$ be an $\mathfrak{I}$-function (resp. a mixed function), $\xi \in \overline{\mathbb{Q}}^*$ and $\theta \in (\arg(\xi)-\pi/2, \arg(\xi)+\pi/2)$. Assume that θ is not anti-Stokes for $\mathfrak{f}$, and that $\mathfrak{f}_\theta(1/\xi) = 0$ (resp. $\mathfrak{f}_\theta(\xi) = 0$). Denote by $Ly = 0$ a differential equation, of minimal order, satisfied by $\mathfrak{f}(1/z)$ (resp. by $\mathfrak{f}(z)$).

Then all solutions of $Ly = 0$ are holomorphic and vanish at ξ; the differential operator L has an apparent singularity at ξ.

We recall that mixed functions (usually denoted by Ψ in this paper) are given by $\Psi(z) = F(z) + \mathfrak{f}(1/z)$ where F is an E-function, and $\mathfrak{f}$ an $\mathfrak{I}$-function; both $\Psi(z)$ and $\mathfrak{f}(1/z)$ are annihilated by E-operators (but neither $\Psi(1/z)$ nor $\mathfrak{f}(z)$ in general).

Proof of Proposition 4. We follow the end of the proof of [4, Corollary 2.2]. Upon replacing $\mathfrak{f}(z)$ with $\mathfrak{f}(z/\xi)$ we may assume that $\xi = 1$. Then we apply Conjecture 2 (resp. Conjecture 3) to $\mathfrak{f}$, since $\mathfrak{f}_\theta(1) = 0$. Accordingly, $g(z) = \frac{-z\mathfrak{f}(z)}{z-1} = \frac{\mathfrak{f}(z)}{\frac{1}{z}-1}$ (resp. $g(z) = \frac{\mathfrak{f}(z)}{z-1}$) is an $\mathfrak{I}$-function (resp. a mixed function). Now $L \circ (z-1)$ is a differential operator, of minimal order, that annihilates $g(1/z)$ (resp. $g(z)$). Since this function is annihilated by an E-operator Φ, there exists $Q \in \overline{\mathbb{Q}}[z] \setminus \{0\}$ such that $Q(z)\Phi$ is a left multiple of $L \circ (z-1)$ in $\overline{\mathbb{Q}}[z, d/dz]$. Now André proved [1, Theorem 4.3] that 1 is not a singularity of Φ, so that all solutions of $L \circ (z-1)$ are holomorphic at 1. This provides a basis of solutions of L, all of which vanish at 1, and concludes the proof of Proposition 4. $\square$

Let us deduce now the linear case of Theorem 3 (namely when $\deg P = 1$) from Proposition 4, by following [4, §3]. The arguments for proving Theorem 4 are exactly the same.

Again we may assume that $\xi = 1$. Letting m denote the rank of $\mathfrak{f}_1, \dots, \mathfrak{f}_n$ over $\overline{\mathbb{Q}}(z)$, [4, Lemma 3.1] yields polynomials $C_{i,j} \in \overline{\mathbb{Q}}[z]$, $1 \le i \le n-m$, $1 \le j \le n$, such that

$$\sum_{j=1}^{n} C_{i,j}(1/z)\mathfrak{f}_j(1/z) = 0 \text{ for any } z \text{ and any } i,$$

and the matrix $[C_{i,j}(1)]$ has rank $n-m$. Assume now that a $\overline{\mathbb{Q}}$-linear relation $\sum_{j=1}^{n} \alpha_j \mathfrak{f}_{j,\theta}(1) = 0$ does not come from specialization at $z=1$ of a $\overline{\mathbb{Q}}(z)$-linear relation between the functions $\mathfrak{f}_j$. Then it is possible (as in [4, proof of Theorem 3.2]) to construct polynomials $A_j \in \overline{\mathbb{Q}}[z]$, $1 \le j \le n$, such that $A_j(1) = \alpha_j$, L has order m and 1 is a regular point of L, where L is a differential operator of minimal order that annihilates $\mathfrak{f}(1/z) = \sum_{j=1}^{n} A_j(1/z)\mathfrak{f}_j(1/z)$. But $\mathfrak{f}$ is an $\mathfrak{I}$-function such that $\mathfrak{f}_\theta(1) = 0$: this contradicts Proposition 4, and concludes the proof of the linear case of Theorem 3.

The general case of Theorem 3 follows by applying the linear case to the family of monomials $\mathfrak{f}_1^{i_1} \dots \mathfrak{f}_n^{i_n}$ where $i_1 + \ldots + i_n = \deg P$, since any product of $\mathfrak{I}$-functions is again an $\mathfrak{I}$-function. But the corresponding property with mixed functions does not hold, so that Theorem 4 is restricted to the linear case.

5.2. *Proof of Corollary 1*

Let $s \in \mathbb{Q} \setminus \mathbb{Z}_{\ge 0}$. The $\mathfrak{I}$-function $\mathfrak{f}(z) := \sum_{n=0}^{\infty} s(s-1)\dots(s-n+1)z^n$ is solution of the inhomogeneous differential equation $z^2\mathfrak{f}'(z) + (1-sz)\mathfrak{f}(z) - 1 = 0$, which can be immediately transformed into a differential system satisfied by the vector of $\mathfrak{I}$-functions ${}^t(1, \mathfrak{f}(z))$. The coefficients of the matrix have only 0 as pole. Moreover, $\mathfrak{f}(z)$ is a transcendental function because $s \notin \mathbb{Z}_{\ge 0}$. Hence, by Theorem 3, $\mathfrak{f}_0(1/\alpha) \notin \overline{\mathbb{Q}}$ when $\alpha \in \overline{\mathbb{Q}}$, $\alpha > 0$, because 0 is not an anti-Stokes direction of $\mathfrak{f}(z)$. It remains to observe that this 1-summation is

$$\int_0^\infty (1+tz)^s e^{-t} dt.$$

Data availability

No data was used for the research described in the article.

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