Many values of the Riemann zeta function at odd integers are irrational

Beaucoup de valeurs aux entiers impairs de la fonction zêta de Riemann sont irrationnelles

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March 29, 2018

Abstract

Building upon ideas of the second and third authors, we prove that at least $2^{(1-\varepsilon)\frac{\log s}{\log \log s}}$ values of the Riemann zeta function at odd integers between 3 and s are irrational, where ε is any positive real number and s is large enough in terms of ε. This lower bound is asymptotically larger than any power of $\log s$; it improves on the bound $\frac{1-\varepsilon}{1+\log 2}\log s$ that follows from the Ball–Rivoal theorem.

The proof is based on an elimination process between several linear forms in odd zeta values with related coefficients. Among other tools, we use Schur polynomials to prove the nonvanishing of an auxiliary determinant.

Résumé

En se fondant sur des idées du deuxième et du troisième auteur, on démontre qu'au moins $2^{(1-\varepsilon)\frac{\log s}{\log \log s}}$ valeurs de la fonction zêta de Riemann aux entiers impairs compris entre 3 and s sont irrationnelles, où ε est un réel strictement positif et s un entier impair suffisamment grand en fonction de ε. Ce minorant est plus grand (asymptotiquement) que n'importe quelle puissance de $\log s$, et il améliore la borne $\frac{1-\varepsilon}{1+\log 2}\log s$ qui découle du théorème de Ball–Rivoal.

La preuve est fondée sur un procédé d'élimination entre des formes linéaires en les valeurs de zêta aux entiers impairs dont les coefficients sont reliés. Parmi d'autres outils, on utilise les polynômes de Schur pour démontrer qu'un déterminant auxiliaire ne s'annule pas.

Version française abrégée

Dans cette note on esquisse la preuve du résultat suivant (voir la version anglaise pour un bref historique, et [6] pour les détails de la preuve).

Théorème 1. *Soient $\varepsilon > 0$, et $s \geq 3$ un entier impair suffisamment grand (en fonction de ε). Alors parmi les nombres*

$$\zeta(3),\, \zeta(5),\, \zeta(7),\, \ldots, \zeta(s),$$

au moins

$$2^{(1-\varepsilon)\frac{\log s}{\log \log s}}$$

sont irrationnels.

Soient $\varepsilon > 0$, et s un entier impair suffisamment grand (en fonction de ε). On note D le produit de tous les nombres premiers inférieurs ou égaux à $(1 - 2\varepsilon)\log s$, de telle sorte qu'on a $D \leq s^{1-\varepsilon}$. En suivant essentiellement [15], on considère pour tout entier n la fraction rationnelle

$$R_n(t) = D^{3Dn}\, n!^{s+1-3D}\, \frac{\prod_{j=0}^{3Dn}(t - n + \frac{j}{D})}{\prod_{j=0}^{n}(t + j)^{s+1}}$$

et pour tout $j \in \{1, \ldots, D\}$ on pose

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\Big(m + \frac{j}{D}\Big).$$

Rappelons que la fonction zêta de Hurwitz est définie par $\zeta(i, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^i}$ pour $\alpha > 0$ et $i \geq 2$. En développant $R_n(t)$ en éléments simples, on démontre comme dans [15] que

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3 \leq i \leq s \\ i \text{ impair}}} \rho_i \, \zeta\Big(i, \frac{j}{D}\Big).$$

En outre, en notant d_n le plus petit commun multiple des n premiers entiers, on montre que d_{n+1}^{s+1} est un multiple commun des dénominateurs des nombres rationnels $\rho_{0,j}$ et ρ_i.

Le point crucial, comme dans [15], est l'identité suivante valable pour tout diviseur d de D et pour tout entier $i \geq 2$:

$$\sum_{j=1}^{d} \zeta\Big(i, \frac{j\frac{D}{d}}{D}\Big) = \sum_{j=1}^{d} \zeta\Big(i, \frac{j}{d}\Big) = \sum_{n=0}^{\infty} \sum_{j=1}^{d} \frac{d^i}{(dn+j)^i} = d^i \zeta(i).$$

En effet, cette relation montre qu'en posant $\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,j\frac{D}{d}}$ on a :

$$\widehat{r}_{n,d} = \sum_{j=1}^{d} \rho_{0,j\frac{D}{d}} + \sum_{\substack{3 \leq i \leq s \\ i \text{ impair}}} \rho_i \, d^i \, \zeta(i).$$

Notons $\mathcal{D}$ l'ensemble des diviseurs de D. Pour tout $d \in \mathcal{D}$ et pour tout entier n, on a donc une combinaison linéaire $\widehat{r}_{n,d}$ de 1, $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(s)$. Pour tout entier i impair compris entre 3 et s, le coefficient de $\zeta(i)$ est $\rho_i \, d^i$ (où le nombre rationnel ρ_i dépend implicitement de n, mais pas de d). Si des entiers w_d, $d \in \mathcal{D}$, vérifient $\sum_{d \in \mathcal{D}} w_d \, d^i = 0$ pour un certain i, alors la combinaison linéaire

$$\widetilde{r}_n = \sum_{d \in \mathcal{D}} w_d \, \widehat{r}_{n,d} \tag{0.1}$$

ne fait plus apparaître $\zeta(i)$. Le point crucial de ce procédé d'élimination (mis au point par le troisième auteur [18] pour $D = 2$, et généralisé par le deuxième auteur [15]) est que le coefficient d^i dépend de d mais pas de n : on peut choisir les entiers w_d indépendamment de n.

Pour démontrer le théorème 1, on suppose que parmi $\zeta(3)$, $\zeta(5)$, $\ldots$, $\zeta(s)$ le nombre d'irrationnels est inférieur à $2^{(1-3\varepsilon)\frac{\log s}{\log \log s}}$; il est alors strictement inférieur à $\delta = \operatorname{Card} \mathcal{D}$, qui est le nombre de diviseurs de D. Il existe donc des indices impairs $i_1 < i_2 < \ldots < i_{\delta-1}$ compris entre 3 et s tels que toute valeur irrationnelle $\zeta(i)$, avec i impair compris entre 3 et s, soit l'une des $\zeta(i_j)$. On peut choisir des entiers relatifs w_d non tous nuls (pour $d \in \mathcal{D}$) tels que

$$\sum_{d \in \mathcal{D}} w_d \, d^{i_j} = 0 \quad \text{pour tout } j \in \{1, \ldots, \delta-1\}. \tag{0.2}$$

La relation (0.1) définit alors une suite $(\widetilde{r}_n)_{n \geq 1}$ de nombres réels qui sont des combinaisons linéaires à coefficients rationnels de 1 et des $\zeta(i)$ pour i impair, $3 \leq i \leq s$. La propriété cruciale est que les $\zeta(i_j)$ ne figurent plus dans ces combinaisons linéaires, si bien que tous les $\zeta(i)$ qui y apparaissent avec un coefficient non nul sont par hypothèse des nombres rationnels. En multipliant par un dénominateur commun A de ces nombres rationnels, et aussi par d_{n+1}^{s+1} (qui est un dénominateur commun des coefficients ρ_i et $\rho_{0,j}$), on obtient une suite d'entiers relatifs. Une étude asymptotique

des suites $(r_{n,j})_{n\geq 1}$ montre que cette suite d'entiers relatifs tend vers 0, donc est identiquement nulle à partir d'un certain rang ; précisément on a quand $n \to \infty$:

$$Ad_{n+1}^{s+1}\widetilde{r}_n = \left(\sum_{d\in\mathcal{D}} w_d\, d + o(1)\right) Ad_{n+1}^{s+1} r_{n,1}, \quad \text{avec } 0 < \lim_{n\to\infty} \left(Ad_{n+1}^{s+1} r_{n,1}\right)^{1/n} < 1$$

où $o(1)$ est une suite qui tend vers 0 quand n tend vers l'infini. Cela impose $\sum_{d\in\mathcal{D}} w_d\, d = 0$. Autrement dit, tout famille $(w_d)_{d\in\mathcal{D}}$ d'entiers relatifs vérifiant (0.2) devrait vérifier aussi $\sum_{d\in\mathcal{D}} w_d\, d = 0$. Ce n'est pas le cas, car la matrice $[d^{i_j}]_{d\in\mathcal{D},\, 0\leq j\leq \delta-1}$ (dans laquelle on pose $i_0 = 1$) est inversible : son déterminant est le produit d'un déterminant de Vandermonde et d'un polynôme de Schur, qui est un polynôme à coefficients entiers naturels évalué en la famille des diviseurs de D.

1 Introduction

When $s \geq 2$ is an even integer, the value $\zeta(s)$ of the Riemann zeta function is a non-zero rational multiple of π^s and, therefore, a transcendental number. On the other hand, very few results are known on the zeta values $\zeta(s)$ when $s \geq 3$ is odd, though we expect them all to be transcendental.

It was only in 1978 when Apéry astonished the mathematics community by his proof [1] of the irrationality of $\zeta(3)$, with the next breakthrough in this direction taken in 2000 by Ball and Rivoal [2, 12], who proved the following:

Theorem 1 (Ball–Rivoal). *Let $\varepsilon > 0$, and $s \geq 3$ be an odd integer sufficiently large with respect to ε. Then*
$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1,\, \zeta(3),\, \zeta(5),\, \zeta(7),\, \ldots, \zeta(s)) \geq \frac{1-\varepsilon}{1+\log 2} \log s.$$

In spite of several refinements [2, 17, 5] for small s, the lower bound in Theorem 1 has never been improved for large values of s. The proof of Theorem 1 applies Nesterenko's linear independence criterion [11] to certain linear combinations of odd zeta values. To improve on this bound using the same strategy, one has to find linear combinations that are considerably smaller, with not too large coefficients, — this comes out to be a rather difficult task.

The situation has drastically changed when the third author introduced [18] a new method, which has been generalized by the second author in [15]. For a given integer $D > 1$ and a certain rational function $R(t)$ the series

$$\sum_{j=1}^{d} \sum_{t=1}^{\infty} R\left(t + \frac{j}{d}\right), \quad \text{where } d \mid D,$$

give $\mathbb{Q}$-linear combinations of $1, d^3\zeta(3), d^5\zeta(5), \ldots, d^s\zeta(s)$ with coefficients independent of d (except that of 1). This makes it possible to eliminate from the entire collection of these linear combinations essentially as many odd zeta values as the number of divisors of D. For applications of this idea, see [18, 9, 15, 14]. Building upon the approach in [18] and [15], we sketch in this note the proof of the following result (see [6] for details).

Theorem 2. *Let $\varepsilon > 0$, and $s \geq 3$ be an odd integer sufficiently large with respect to ε. Then among the numbers*
$$\zeta(3),\, \zeta(5),\, \zeta(7),\, \ldots, \zeta(s),$$
at least
$$2^{(1-\varepsilon)\frac{\log s}{\log\log s}}$$
are irrational.

In this result, the lower bound is asymptotically greater than $\exp(\sqrt{\log s})$, and than any power of $\log s$; "to put it roughly, [it is] much more like a power of s than a power of $\log s$" [7, Chapter XVIII, §1].

In comparison, Theorem 1 gives only $\frac{1-\varepsilon}{1+\log 2} \log s$ irrational odd zeta values, but they are linearly independent over the rationals, whereas Theorem 2 ends up only with their irrationality.

Our proof of Theorem 2 follows the above-mentioned strategy from [18] and [15]. The main new ingredient, compared to the proof in [15], is taking D large (about $s^{1-2\varepsilon}$) and equal to the product of the first prime numbers (the so-called primorial) — such a number has asymptotically the largest possible number of divisors with respect to its size (see [7, Chapter XVIII, §1]).

2   Small linear forms in values of the Hurwitz zeta function

This section essentially follows [15], except for Lemma 3. Let s and D be positive integers such that s is odd and $s \geq 3D$. Let n be a positive integer, such that Dn is even. We consider the following rational function:

$$R_n(t) = D^{3Dn} \, n!^{s+1-3D} \, \frac{\prod_{j=0}^{3Dn}(t - n + \frac{j}{D})}{\prod_{j=0}^{n}(t + j)^{s+1}}$$

which, of course, depends also on s and D, and for $j \in \{1, \ldots, D\}$ we let

$$r_{n,j} = \sum_{m=1}^{\infty} R_n\Big(m + \frac{j}{D}\Big).$$

We write d_n for $\mathrm{lcm}(1, 2, \ldots, n)$. Expanding $R_n(t)$ into partial fractions yields the following $\mathbb{Q}$-linear combinations of values of the Hurwitz zeta function.

Lemma 1. *For each $j \in \{1, \ldots, D\}$, we have*

$$r_{n,j} = \rho_{0,j} + \sum_{\substack{3 \leq i \leq s \\ i \text{ odd}}} \rho_i \, \zeta\Big(i, \frac{j}{D}\Big)$$

with

$$d_n^{s+1-i}\rho_i \in \mathbb{Z} \quad \text{for } i \in \{3, 5, \ldots, s\} \quad \text{and} \quad d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z} \quad \text{for any } j \in \{1, \ldots, D\}.$$

We point out that the coefficient ρ_i of $\zeta(i, \frac{j}{D})$ in the linear form $r_{n,j}$ does not depend on j. The expansion of $r_{n,j}$ is very classical; only odd zeta values appear because of the symmetry phenomenon from [2]. The proof that $d_n^{s+1-i}\rho_i \in \mathbb{Z}$ for odd i follows that of [4, Lemma 4.5]. The last assertion, namely $d_{n+1}^{s+1}\rho_{0,j} \in \mathbb{Z}$, is proved as in [15, Lemma 1.4].

An important feature of the linear forms $r_{n,j}$ is that they are simultaneously very small: even when multiplied by a common denominator of the rational coefficients, they still tend to 0.

Lemma 2. *Assume that $\frac{s}{D \log D}$ is sufficiently large. Then we have*

$$\lim_{n \to \infty} r_{n,j}^{1/n} = g(x_0) < 3^{-(s+1)} \quad \text{and} \quad \lim_{n \to \infty} \frac{r_{n,j'}}{r_{n,j}} = 1 \quad \text{for any } j, j' \in \{1, \ldots, D\},$$

where

$$g(x) = D^{3D} \, \frac{(x+3)^{3D}(x+1)^{s+1}}{(x+2)^{2(s+1)}}$$

and x_0 is the unique positive root of the polynomial $(X+3)^D(X+1)^{s+1} - X^D(X+2)^{s+1}$.

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The following lemma is used to eliminate unwanted irrational odd zeta values (in §3 below).

Lemma 3. *For $t \geq 1$, let $x_1, \ldots, x_t$ be pairwise distinct positive real numbers and $\alpha_1, \ldots, \alpha_t$ pairwise distinct non-negative integers. Then the matrix $[x_j^{\alpha_i}]_{1 \leq i,j \leq t}$ has non-zero determinant.*

As pointed out in [8, §2.1], the generalized Vandermonde determinant in question is closely related to Schur polynomials. Indeed, the determinant of the matrix $[x_j^{\alpha_i}]$ is a product of a (non-zero) Vandermonde determinant and a Schur polynomial; the latter is a polynomial in $x_1, \ldots, x_t$ with non-negative integral coefficients (see for instance [10]) and, therefore, is positive.

3   Elimination of odd zeta values

Take $0 < \varepsilon < \frac{1}{3}$, and let s be odd and sufficiently large with respect to ε. We let D be the product of all primes less than or equal to $(1 - 2\varepsilon) \log s$, so that $D \leq s^{1-\varepsilon}$. Notice that D has $\delta = 2^{\pi((1-2\varepsilon)\log s)}$ divisors, with $\log \delta \geq (1 - 3\varepsilon)(\log 2) \frac{\log s}{\log \log s}$. Assume that the number of irrational odd zeta values between $\zeta(3)$ and $\zeta(s)$ is less than δ. Let $3 = i_1 < i_2 < \ldots < i_{\delta-1} \leq s$ be odd integers such that if $\zeta(i) \notin \mathbb{Q}$ and i is odd, $3 \leq i \leq s$, then $i = i_j$ for some j. Moreover, we let $i_0 = 1$, and consider the set $\mathcal{D}$ of all divisors of D, so that $\operatorname{Card} \mathcal{D} = \delta$. Lemma 3 shows that the matrix $[d^{i_j}]_{d \in \mathcal{D}, 0 \leq j \leq \delta-1}$ is invertible. Therefore, there exist integers $w_d \in \mathbb{Z}$, where $d \in \mathcal{D}$, such that

$$\sum_{d \in \mathcal{D}} w_d \, d^{i_j} = 0 \quad \text{for any } j \in \{1, \ldots, \delta-1\} \quad \text{and} \quad \sum_{d \in \mathcal{D}} w_d \, d \neq 0. \tag{3.1}$$

The crucial point (as in [15, §3]) is that for any $d \in \mathcal{D}$ and any $i \geq 2$,

$$\sum_{j=1}^{d} \zeta\left(i, \frac{j \frac{D}{d}}{D}\right) = \sum_{j=1}^{d} \zeta\left(i, \frac{j}{d}\right) = \sum_{n=0}^{\infty} \sum_{j=1}^{d} \frac{d^i}{(dn + j)^i} = d^i \zeta(i).$$

Lemma 1 implies that the quantities $\widehat{r}_{n,d} = \sum_{j=1}^{d} r_{n,j\frac{D}{d}}$ are linear forms in the odd zeta values:

$$\widehat{r}_{n,d} = \sum_{j=1}^{d} \rho_{0,j\frac{D}{d}} + \sum_{\substack{3 \leq i \leq s \\ i \text{ odd}}} \rho_i \, d^i \, \zeta(i), \tag{3.2}$$

while Lemma 2 leads to the following asymptotics:

$$\widehat{r}_{n,d} = (d + o(1)) r_{n,1} \quad \text{with} \quad \lim_{n \to \infty} r_{n,1}^{1/n} = g(x_0). \tag{3.3}$$

We shall use now the integers w_d to eliminate $\delta - 1$ odd zeta values, including all irrational ones, as in [18] and [15]. For that, introduce $\widetilde{r}_n = \sum_{d \in \mathcal{D}} w_d \widehat{r}_{n,d}$. Eqns. (3.1)–(3.3) imply that

$$\widetilde{r}_n = \left(\sum_{d \in \mathcal{D}} w_d \sum_{j=1}^{d} \rho_{0,j\frac{D}{d}} \right) + \sum_{i \in I} \rho_i \left(\sum_{d \in \mathcal{D}} w_d \, d^i \right) \zeta(i) = \left(\sum_{d \in \mathcal{D}} w_d \, d + o(1) \right) r_{n,1},$$

where $I = \{3, 5, 7, \ldots, s\} \setminus \{i_1, \ldots, i_{\delta-1}\}$. In particular, no irrational zeta value $\zeta(i)$, where $3 \leq i \leq s$, appears in this linear combination, and $\lim_{n \to \infty} |\widetilde{r}_n|^{1/n} = g(x_0) < 3^{-(s+1)}$ since $\sum_{d \in \mathcal{D}} w_d \, d \neq 0$. Denoting by A a common denominator of the (rational) numbers $\zeta(i)$, $i \in I$, we deduce from Lemma 1 that $A d_{n+1}^{s+1} \widetilde{r}_n$ is an integer. Now the prime number theorem implies that $d_{n+1}^{1/n} \to e$ as $n \to \infty$, hence from Lemma 2 we conclude that the sequence of integers satisfies

$$0 < \lim_{n \to \infty} |A d_{n+1}^{s+1} \widetilde{r}_n|^{1/n} = e^{s+1} g(x_0) < \left(\frac{e}{3} \right)^{s+1} < 1.$$

This contradiction implies the truth of Theorem 2.

Acknowledgements

References

[1] R. APÉRY – "Irrationalité de $\zeta(2)$ et $\zeta(3)$", in *Journées Arithmétiques (Luminy, 1978)*, Astérisque, no. 61, 1979, p. 11–13.

[2] K. BALL & T. RIVOAL – "Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs", *Invent. Math.* **146** (2001), no. 1, p. 193–207.

[3] N. G. DE BRUIJN – *Asymptotic methods in analysis*, Dover Publications, 1981.

[4] — , "Shidlovsky's multiplicity estimate and irrationality of zeta values", preprint arXiv 1609.09770 [math.NT], J. Austral. Math. Soc., to appear.

[5] S. FISCHLER & W. ZUDILIN – "A refinement of Nesterenko's linear independence criterion with applications to zeta values", *Math. Ann.* **347** (2010), p. 739–763.

[6] S. FISCHLER, J. SPRANG & W. ZUDILIN – "Many odd zeta values are irrational", preprint arXiv:1803.08905 [math.NT], 2018.

[7] G. HARDY & E. WRIGHT – *An introduction to the theory of numbers*, fifth ed., Oxford Science Publications, 1979.

[8] C. KRATTENTHALER – "Advanced determinant calculus", *Sém. Lotharingien Combin.* **42** (1999), 67 pp.

[9] C. KRATTENTHALER & W. ZUDILIN – "Hypergeometry inspired by irrationality questions", preprint arXiv:1802.08856 [math.NT], 2018.

[10] G. MACDONALD – *Symmetric functions and Hall polynomials*, Oxford Univ. Press, London, 1979.

[11] Y. NESTERENKO – "On the linear independence of numbers", *Vestnik Moskov. Univ. Ser. I Mat. Mekh. [Moscow Univ. Math. Bull.]* **40** (1985), no. 1, p. 46–49 [69–74].

[12] T. RIVOAL – "La fonction zêta de Riemann prend une infinité de valeurs irrationnelles aux entiers impairs", *C. R. Acad. Sci. Paris, Ser. I* **331** (2000), no. 4, p. 267–270.

[13] — , "Irrationalité d'au moins un des neuf nombres $\zeta(5)$, $\zeta(7)$, ..., $\zeta(21)$", *Acta Arith.* **103** (2002), no. 2, p. 157–167.

[14] T. RIVOAL & W. ZUDILIN – "A note on odd zeta values", preprint arXiv:1803.03160 [math.NT], 2018.

[15] J. SPRANG – "Infinitely many odd zeta values are irrational. By elementary means", preprint arXiv:1802.09410 [math.NT], 2018.

[16] W. ZUDILIN – "One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational", *Uspekhi Mat. Nauk [Russian Math. Surveys]* **56** (2001), no. 4, p. 149–150 [774–776].

[17] — , "Irrationality of values of the Riemann zeta function", *Izvestiya Ross. Akad. Nauk Ser. Mat. [Izv. Math.]* **66** (2002), no. 3, p. 49–102 [489–542].

[18] — , "One of the odd zeta values from $\zeta(5)$ to $\zeta(25)$ is irrational. By elementary means", *SIGMA* **14** (2018), 028, 8 pages.

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