Legendre Polynomials (1782)

$$L_n(x) \equiv \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= x \\
L_2(x) &= \tfrac{1}{2}(3x^2-1) \\
L_3(x) &= \tfrac{1}{2}(5x^3-3x) \\
L_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3) \\
L_5(x) &= \tfrac{1}{8}(63x^5-70x^3+15x) \\
L_6(x) &= \tfrac{1}{16}(231x^6-315x^4+105x^2-5) \\
\vdots \; &\vdots \; \vdots
\end{aligned}$$

Papoulis Polynomials (1958)

$$P_{2k+1}(w) \equiv \frac{1}{2(k+1)^2} \int_{-1}^{2w-1} [L_0(x) + 3L_1(x) + \cdots + (2k+1)L_k(x)]^2 dx$$

$$\begin{aligned}
P_1(w) &= w \\
P_3(w) &= 3w^3 - 3w^2 + w \\
P_5(w) &= 20w^5 - 40w^4 + 28w^3 - 8w^2 + w \\
P_7(w) &= 175w^7 - 525w^6 + 615w^5 - 355w^4 + 105w^3 \\
&\quad - 15w^2 + w \\
P_9(w) &= 1764w^9 - 7056w^8 + 11704w^7 - 10416w^6 + 5376w^5 \\
&\quad - 1624w^4 + 276w^3 - 24w^2 + w \\
&\vdots \quad \vdots \quad \vdots
\end{aligned}$$

Poincaré series

$$H_n(t) \equiv \sum_{k\geq 0} \dim\{\text{Killing tensors on } \mathbb{CP}_n \text{ of rank } k\}\, t^k.$$

$$H_1(t) = \frac{1}{(1-t)^3} \qquad H_2(t) = \frac{1+t+t^2}{(1-t)^7} \quad \text{(Delong 1982)}$$

$$H_3(t) = \frac{1+4t+10t^2+4t^3+t^4}{(1-t)^{11}} = 1+15t+120t^2+664t^3+\cdots$$

$$H_4(t) = \frac{1+9t+45t^2+65t^3+45t^4+9t^5+t^6}{(1-t)^{15}} \qquad \cdots$$

$$\boxed{H_n(t) = -\frac{1}{t(1-t)^{2n}} P_{2n-1}\left(\frac{t}{t-1}\right)}$$

Fubini-Study pyramid

					1					
				1	1	1				
			1	4	10	4	1			
		1	9	45	65	45	9	1		
	1	16	136	416	626	416	136	16	1	
1	25	325	1700	4550	6202	4550	1700	325	25	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
?	?	?	?	?	?	?	?	?	?	?

Round sphere pyramid (Narayana numbers $\sim$1954)

					1					
				1		1				
			1		3		1			
		1		6		6		1		
	1		10		20		10		1	
1		15		50		50		15		1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮