Connection on a differential manifold: Newton *versus* Ehresman-Koszul-Nomizu.

What is gravity?

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Connection on a manifold

Historically, connection on a manifold M appeares as a convenient „shortcut” in Riemannian geometry, namely: Christoffel symbols $\Gamma^\lambda_{\mu\nu}$

Nowadays, it is understood as a connection in the tangent bundle TM (or in the principal ``bundle of frames’’).

Three equivalent approaches:
1) Ehresman, 2) Koszul, 3) Nomizu

Paweł Nurowski:
Cartan knew everything!

But such $\Gamma^\lambda_{\mu\nu}$ is not an irreducible object!

It splits authomatically into a pair of different objects:

$$\Gamma^\lambda_{\mu\nu} = \left(\Gamma^\lambda_{(\mu\nu)}, \Gamma^\lambda_{[\mu\nu]}\right)$$

symmetric anti-symmetric

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connection

tensor

Connection on a manifold

$$\Gamma^{\lambda}_{\mu\nu} = \left(\Gamma^{\lambda}_{(\mu\nu)}, \Gamma^{\lambda}_{[\mu\nu]}\right)$$

Symmetric connection **is not** a connection in TM which – it just happened – is symmetric, but is an autonomous geometric object.

$\boxed{\text{Travel = (airplane , orange)}}$

Connection on a manifold

$$\Gamma^{\lambda}_{\mu\nu} = \left(\Gamma^{\lambda}_{(\mu\nu)}, \Gamma^{\lambda}_{[\mu\nu]}\right)$$

Symmetric connection **is not** a connection in TM which – it just happened – is symmetric, but is an autonomous geometric object.

Its definition is based on the fundamental physical idea given by Izaak Newton: the concept of an **inertial frame**.

Newton realized that equations of motion of a freely falling body are of second differential order.

Aristotle: force causes velocity

Newton: force causes acceleration

Second law.

Newton's first law necessary to formulate second law.

What is an inertial frame?

Newton's first law: There is an inertial reference frame!

If no gravity and no other forces, then:

Rectilinear motion with constant velocity.

There are **spacetime** coordinates (y^α) such that:

$$\ddot{y}^\alpha = 0$$

global

Take another coordinate system (x^λ) :

$$\dot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^\nu}\dot{x}^\nu$$

$$\ddot{y}^\alpha = \frac{\partial y^\alpha}{\partial x^\nu}\ddot{x}^\nu + \frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

$$\frac{\partial x^\lambda}{\partial y^\alpha}\ddot{y}^\alpha = \ddot{x}^\lambda + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

$$y^\alpha = y^\alpha\left(x^\lambda\right)$$

Acceleration vanishes!

What is an inertial frame?

$$\frac{\partial x^\lambda}{\partial y^\alpha}\ddot{y}^\alpha = \ddot{x}^\lambda + \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

Coordinate system (x^λ) is inertial iff $\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu} = 0$.

Equivalence relation:

$$\left\{(y^\alpha) \sim \left(x^\lambda\right)\right\} \Leftrightarrow \left\{\frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu} = 0\right\} .$$

Inertial reference frame: an equivalence class of **global** coordinate systems.

Newton's 1st law ⟺ Affine structure of spacetime.

What is an inertial frame?

$$\frac{\partial x^\lambda}{\partial y^\alpha}\ddot{y}^\alpha = \ddot{x}^\lambda + \underbrace{\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}_{\Gamma^\lambda_{\mu\nu}}\dot{x}^\mu\dot{x}^\nu = 0$$

Motion in a non-inertial frame: $\ddot{x}^\lambda = -\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$

What is an inertial frame?

$$\frac{\partial x^\lambda}{\partial y^\alpha}\ddot{y}^\alpha = \ddot{x}^\lambda + \underbrace{\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}_{\Gamma^\lambda_{\mu\nu}}\dot{x}^\mu \dot{x}^\nu = 0$$

Motion in a non-inertial frame: $m\ddot{x}^\lambda = -m\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = F^\lambda$

Ficticious forces (centrifugal, Coriolis etc.) which can be always elliminated **globally** if we use inertial frame.

<u>GRAVITATION:</u> replace Newton 1st law by the Einstein's 1st law!

Einstein's first law: there is an inertial frame $\boxed{\text{but}}$

only local, **at each spacetime point separately!**

Maybe, no global inertal frame!

What is a local inertial frame?

There are local inertial coordinates (y^α) at $m \in M$,
i.e. such that freely falling bodies obey **locally**:

spacetime

$$\ddot{y}^\alpha(m) = 0$$

2-nd order
diff. eqn.

Take another coordinate system (x^λ) :

$$\frac{\partial x^\lambda}{\partial y^\alpha}\ddot{y}^\alpha = \ddot{x}^\lambda + \frac{\partial x^\lambda}{\partial y^\alpha}\boxed{\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}}\dot{x}^\mu \dot{x}^\nu = 0$$

It is also inertial at $m \in M$ **iff** second derivatives vanish at $m \in M$.

Equivalence relation between local coordinate systems at $m \in M$

$$\left\{(y^\alpha) \sim_m \left(x^\lambda\right)\right\} \Leftrightarrow \left\{\frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(m) = 0\right\} .$$

What is an inertial frame?

$$\left\{ (y^\alpha) \sim_m \left(x^\lambda \right) \right\} \Leftrightarrow \left\{ \frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(m) = 0 \right\} .$$

Local reference frame at $m \in M$: an equivalence class.

What is an inertial frame?

$$\left\{(y^\alpha) \sim_m \left(x^\lambda\right)\right\} \Leftrightarrow \left\{\frac{\partial^2 y^\alpha}{\partial x^\mu x^\nu}(m) = 0\right\}.$$

Local inertial reference frame at $m \in M$: an equivalence class.

Collection of all local referece frames: a fiber boundle over spacetime.

Given a coordinate system (x^λ) in a neighbourhood of $m \in M$, any reference frame $[(y^\alpha)]$ at this point can be uniquely parametrized by the following table of numbers:

$$\Gamma^\lambda_{\mu\nu}(m) := \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m)$$

Does not depend upon the choice of a representative (y^α) .

What is gravity

$$[(y^\alpha)] \Longleftrightarrow \Gamma^\lambda_{\mu\nu}(m) := \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m)$$

Hence, $\left(x^\lambda, \Gamma^\lambda_{\mu\nu}\right)$ local coordinates in the **bundle of frames**, compatible with the bundle structure.

Einstein version of the first law: at each spacetime point there is a priviledged reference frame which we call **inertial frame**.

Mathematically: a section of the frame bundle. symmetric connection

Its coordinate description: a function

$$M \ni m \longrightarrow \Gamma^\lambda_{\mu\nu}(m) \; .$$

This is precisely the gravitational field: field of inertial frames!

Motion of free falling bodies

Equations of motion of test bodies in a given gravitational field:

$$\ddot{x}^\lambda = \boxed{-\Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = F^\lambda$$

In particular: $\Gamma^\lambda_{\mu\nu}(m) = 0$ means that our coordinates are inertial **at this particular point**.

Gravitational force can be elliminated if we use inertial frame (free falling elevator or a space ship)
but only locally, at a given point $m \in M$.

Typically, inertial coordinates **at a point**, are not inertial at **neighbouring** points, unless spacetime is **flat** in a neighbourhood.

How to verify that spacetime is or is not flat?

Curvature

If (x^λ) are not inertial at $m = (m^\mu)$,

i.e. if $\Gamma^\lambda_{\mu\nu}(m) \neq 0$,

then we may improve them putting:

$$y^\alpha := x^\alpha + \frac{1}{2}\Gamma^\lambda_{\mu\nu}(m)\,(x^\mu - m^\mu)\,(x^\nu - m^\nu)$$

New coordinates are inertial at $m \in M$, because:

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m) \;=\; \Gamma^\lambda_{\mu\nu}(m)$$

To simplify notation we can use coordinates which are **centered** at our point: $m = (m^\mu) = (0, \ldots, 0)$

$$y^\alpha := x^\alpha + \frac{1}{2}\Gamma^\alpha_{\mu\nu}(m) x^\mu x^\nu$$

Curvature

$$y^\alpha := x^\alpha + \frac{1}{2}\Gamma^\alpha_{\mu\nu}(m)x^\mu x^\nu$$

enables us to „kill” connection coefficients at a single point!

If spacetime is (locally) flat then we may kill them not only at a point, but also in its neighbourhood.

As a first step in this procedure, try to kill also their derivatives at this point:

$$\partial_\kappa \Gamma^\lambda_{\mu\nu}(m) =: \Gamma^\lambda_{\mu\nu\kappa}(m)$$

$$\frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m) \;=\; \Gamma^\lambda_{\mu\nu}(m)$$

$$y^\alpha := x^\alpha + \frac{1}{2}\Gamma^\lambda_{\mu\nu}(m)x^\mu x^\nu + \frac{1}{6}W^\lambda_{\mu\nu\kappa}x^\mu x^\nu x^\kappa$$

We see that 0-order and 1-order corrections are irrelevant, 2-order correction is already fixed, 4-order and higher corrections are irrelevant, so only 3-order corrections may do the job!

Curvature

$$y^\alpha := x^\alpha + \frac{1}{2}\Gamma^\lambda_{\mu\nu}(m)x^\mu x^\nu + \frac{1}{6}W^\lambda_{\mu\nu\kappa}x^\mu x^\nu x^\kappa$$

This correction of coordinates implies the following change:

$$\Gamma^\lambda_{\mu\nu}(m) \Longrightarrow 0 \qquad \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial^2 y^\alpha}{\partial x^\mu \partial x^\nu}(m) = \Gamma^\lambda_{\mu\nu}(m)$$

$$\Gamma^\lambda_{\mu\nu\kappa}(m) \Longrightarrow \Gamma^\lambda_{\mu\nu\kappa}(m) + W^\lambda_{\mu\nu\kappa}$$

But $W^\lambda_{\mu\nu\kappa}$ is totally symmetric. Hence, we are able to kill the totally symmertic part of derivatives $\Gamma^\lambda_{\mu\nu\kappa}$.

What remains is an invariant **curvature tensor**:

$$K^\lambda_{\mu\nu\kappa} = \Gamma^\lambda_{\mu\nu\kappa} - \Gamma^\lambda_{(\mu\nu\kappa)} + \Gamma^\lambda_{\gamma\kappa}\Gamma^\gamma_{\mu\nu} - \Gamma^\lambda_{\gamma(\kappa}\Gamma^\gamma_{\mu\nu)}$$

If non-zero, there is no chance to kill derivatives $\Gamma^\lambda_{\mu\nu\kappa}$, i.e. spacetime is not flat!

Curvature

$$K^{\lambda}_{\mu\nu\kappa} = \Gamma^{\lambda}_{\mu\nu\kappa} - \Gamma^{\lambda}_{(\mu\nu\kappa)} + \Gamma^{\lambda}_{\gamma\kappa}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\gamma(\kappa}\Gamma^{\gamma}_{\mu\nu)}$$

Symmetries: $K^{\lambda}_{\mu\nu\kappa} = K^{\lambda}_{\nu\mu\kappa}$ $\quad K^{\lambda}_{(\mu\nu\kappa)} = 0$

Bianchi
I-st type

Riemann tensor: $R^{\lambda}_{\mu\nu\kappa} = -R^{\lambda}_{\mu\kappa\nu}$ $\quad R^{\lambda}_{[\mu\nu\kappa]} = 0$

Both carry the same information:

$$K^{\lambda}_{\mu\nu\kappa} = -\frac{2}{3}R^{\lambda}_{(\mu\nu)\kappa} \quad ; \quad R^{\lambda}_{\mu\nu\kappa} = -2K^{\lambda}_{\mu[\nu\kappa]}.$$

No metric tensor so far!

$$\ddot{x}^{\lambda} = -\Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$$

Active role of gravity described **completely**. What about passive…

General Relativity: needs generalization?

Everything we know about gravity comes from observations of the solar system, $\boxed{\text{but}}$

People desperately look for generalizations, because:
dark matter, dark energy, exotic matter…

Extrapolation of a succesfull theory by 20 orders of magnitude: (from the solar system scale, to the cosmic scale) was never sucesfull in the history of humanity!

General Relativity Theory was derived from the Hilbert (metric) variational principle (second order):

$$L_{Hilbert} = \frac{1}{16\pi}\sqrt{|\det g|}\ R$$

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General Relativity Theory was derived from the Hilbert (metric) variational principle (second order):

$$L_{Sacharov} = \sqrt{|\det g|}\left(\frac{1}{16\pi}R + \Lambda + cR^2\right)$$

$$L = \sqrt{|\det g|}\, f(R) \qquad\qquad L = L\left(g_{\mu\nu}, {R^\kappa}_{\lambda\mu\nu}\right)$$

Affine variational principle

$$K^{\lambda}_{\mu\nu\kappa} = \Gamma^{\lambda}_{\mu\nu\kappa} - \Gamma^{\lambda}_{(\mu\nu\kappa)} + \Gamma^{\lambda}_{\gamma\kappa}\Gamma^{\gamma}_{\mu\nu} - \Gamma^{\lambda}_{\gamma(\kappa}\Gamma^{\gamma}_{\mu\nu)}$$

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No metric tensor so far! $\quad \ddot{x}^{\lambda} = -\Gamma^{\lambda}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}$

Dynamics of the gravitational field $\Gamma^{\lambda}_{\mu\nu}$ should be derived from a variational principle

$$L = L\left(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu\kappa}\right) = L\left(K^{\lambda}_{\mu\nu\kappa}\right)$$

Gravitational Field dynamics

$$L = L\left(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu\kappa}\right) = L\left(K^{\lambda}_{\mu\nu\kappa}\right)$$

Must be a scalar density!

No metric tensor to rise or lower indices!

Canonical decomposition into traces and the tracelss part:

$$K^{\lambda}_{\mu\nu\kappa} = -\frac{1}{9}\left(\delta^{\lambda}_{\mu}K_{\nu\kappa} + \delta^{\lambda}_{\nu}K_{\mu\kappa} - 2\delta^{\lambda}_{\kappa}K_{\mu\nu}\right) - \frac{1}{5}\left(\delta^{\lambda}_{\mu}F_{\nu\kappa} + \delta^{\lambda}_{\nu}F_{\nu\kappa}\right) + U^{\lambda}_{\mu\nu\kappa}$$

Weyl tensor

$$R_{\mu\nu} := R^{\lambda}_{\mu\lambda\nu} = K_{\mu\nu} + F_{\mu\nu}$$

First guess: only $K_{\mu\nu}$ enter into this game?

$$L = C \cdot \sqrt{|\det K_{\mu\nu}|}$$

Gravitational Field dynamics

$$L = C \cdot \sqrt{|\det K_{\mu\nu}|}$$

$$\partial_\kappa \frac{\partial L}{\partial \Gamma^\lambda_{\mu\nu\kappa}} = \frac{\partial L}{\partial \Gamma^\lambda_{\mu\nu}}$$

$$\partial \frac{\partial L}{\partial K_{\mu\nu}} \qquad \frac{\partial L}{\partial K_{\mu\nu}} \cdot \Gamma$$

Denote momentum canonically conjugate with $K_{\mu\nu}$:

$$\frac{\partial L}{\partial K_{\mu\nu}} =: \pi^{\mu\nu} \qquad \frac{\partial L}{\partial \Gamma^\lambda_{\mu\nu\kappa}} = \delta^\kappa_\lambda \pi^{\mu\nu} - \delta^{(\mu}_\lambda \pi^{\nu)\kappa}$$

Hence, Euler-Lagrange equations:

$$\partial_\kappa \pi^{\mu\nu} = \pi \cdot \Gamma \quad \Leftrightarrow \quad \nabla_\kappa \pi^{\mu\nu} = 0$$

Gravitational Field dynamics

Euler-Lagrange equations: $\nabla_\kappa \pi^{\mu\nu} = 0$

Hence, $\pi^{\mu\nu}$ is nothing but the spacetime metric tensor.
More precisely:

$$\pi^{\mu\nu} = \frac{1}{16\pi} \underbrace{\sqrt{|\det g|}\, g^{\mu\nu}}_{\text{V. A. Fock}}$$

Conclusion: $\Gamma^\lambda_{\mu\nu}$ must be the metric connection

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right)$$

which otherwise is assumed *a priori* .

Enstein equations contained in the definition of field momenta:

$$\frac{\partial L}{\partial K_{\mu\nu}} =: \pi^{\mu\nu}$$

Gravitational Field dynamics

$$\frac{\partial L}{\partial K_{\mu\nu}} =: \pi^{\mu\nu} \quad \Longleftrightarrow \quad \delta L =: \pi^{\mu\nu} \delta K_{\mu\nu}$$

$$\delta L = C \cdot \delta \sqrt{|\det K_{\mu\nu}|} = \frac{C}{2} \cdot \sqrt{|\det K_{\mu\nu}|} \left(K^{-1}\right)^{\mu\nu} \delta K_{\mu\nu}$$

$$\pi^{\mu\nu} = \frac{1}{16\pi} \sqrt{|\det g|} g^{\mu\nu} = \frac{C}{2} \cdot \sqrt{|\det K_{\mu\nu}|} \left(K^{-1}\right)^{\mu\nu}$$

Ricci is proportional to the metric. Can be solved:

$$K_{\mu\nu} = \frac{1}{8\pi C} \; g_{\mu\nu} \qquad\qquad G_{\mu\nu} := K_{\mu\nu} - \frac{1}{2} K_{\alpha\beta} g^{\alpha\beta} g_{\mu\nu}$$

$$G_{\mu\nu} = \frac{1}{4\pi C} \; g_{\mu\nu} = \Lambda g_{\mu\nu}$$

No freedom in the choice of the dynamics!

Electrodynamics

$$K^{\lambda}_{\mu\nu\kappa} = -\frac{1}{9}\left(\delta^{\lambda}_{\mu} K_{\nu\kappa} + \delta^{\lambda}_{\nu} K_{\mu\kappa} - 2\delta^{\lambda}_{\kappa} K_{\mu\nu}\right) - \frac{1}{5}\left(\delta^{\lambda}_{\mu} F_{\nu\kappa} + \delta^{\lambda}_{\nu} F_{\nu\kappa}\right) + U^{\lambda}_{\mu\nu\kappa}$$

Antisymmetric part of the Ricci:

$$F_{\mu\nu} = \partial_{\mu}\Gamma^{\lambda}_{\lambda\nu} - \partial_{\nu}\Gamma^{\lambda}_{\lambda\nu}$$

Hence, 1-st pair of Maxwell eqs. satisfied *a priori*:

$$\mathrm{d}F \equiv 0$$

Hermann Weyl 1918

Cannot be directly identified with electromagnetic tensor because physical units do not agree.

But, what about

$$L = \frac{1}{8\pi\Lambda} \cdot \sqrt{|\det R_{\mu\nu}|} = \frac{1}{8\pi\Lambda} \cdot \sqrt{|\det\left(K_{\mu\nu} + F_{\mu\nu}\right)|}$$

Electrodynamics

$$L = \frac{1}{4\pi\Lambda} \cdot \sqrt{|\det R_{\mu\nu}|} = \frac{1}{4\pi\Lambda} \cdot \sqrt{|\det(K_{\mu\nu} + F_{\mu\nu})|}$$

For weak field: $\|F\| \ll \Lambda$ $\quad K_{\mu\nu} \approx \frac{\Lambda}{2}\, g_{\mu\nu}$

$$L \approx \frac{1}{4\pi\Lambda} \cdot \sqrt{\frac{\Lambda^4}{16} \det\left(g_{\mu\nu} + \frac{2}{\Lambda} F_{\mu\nu}\right)} = \frac{\Lambda}{32\pi} \sqrt{|\det g|} \sqrt{\det\left(\mathbb{I} + \frac{2}{\Lambda} g^{-1} F\right)}$$

$$\det\left(\mathbb{I} + \frac{2}{\Lambda} g^{-1} F\right) \approx 1 - \frac{2}{\Lambda^2} F_{\mu\nu} F^{\mu\nu}$$

Hence, for weak field: $\quad L \approx \sqrt{|\det g|} \left(\frac{\Lambda}{32\pi} + \frac{1}{32\pi\Lambda} F_{\mu\nu} F^{\mu\nu}\right)$

If cosmological constant is negative, we put $\quad f_{\mu\nu} := \frac{1}{\sqrt{8\pi|\Lambda|}} F_{\mu\nu}$

$$L \approx \sqrt{|\det g|} \left(\frac{\Lambda}{32\pi} - \frac{1}{4} f_{\mu\nu} f^{\mu\nu}\right)$$

Born-Infeld

$$L = \frac{1}{4\pi\Lambda} \cdot \sqrt{|\det R_{\mu\nu}|} = \frac{1}{4\pi\Lambda} \cdot \sqrt{|\det (K_{\mu\nu} + F_{\mu\nu})\,|}$$

For weak field: $\|F\| \ll \Lambda$ $\qquad K_{\mu\nu} \approx \frac{\Lambda}{2}\, g_{\mu\nu}$

$$L \approx \frac{\Lambda}{32\pi}\sqrt{|\det g|}\sqrt{\det\left(\mathbb{I} + \frac{2}{\Lambda} g^{-1} F\right)} \qquad f_{\mu\nu} := \frac{1}{\sqrt{8\pi|\Lambda|}} F_{\mu\nu}$$

$$\approx \frac{\Lambda}{32\pi}\sqrt{|\det g|}\sqrt{\det\left(\delta^{\lambda}_{\nu} + \sqrt{\frac{32\pi}{\Lambda}} f^{\lambda}{}_{\nu}\right)}$$

$$\approx \frac{\Lambda}{32\pi}\sqrt{\det\left(g_{\mu\nu} + \sqrt{\frac{32\pi}{\Lambda}} f_{\mu\nu}\right)}$$

The Born-Infeld electrodynamics with cosmological constant playing role of the Born-Infeld constant.

Generic Lagrangian

Canonical decomposition of te curvature tensor:

$$K^{\lambda}_{\mu\nu\kappa} = -\frac{1}{9}\left(\delta^{\lambda}_{\mu} K_{\nu\kappa} + \delta^{\lambda}_{\nu} K_{\mu\kappa} - 2\delta^{\lambda}_{\kappa} K_{\mu\nu}\right) - \frac{1}{5}\left(\delta^{\lambda}_{\mu} F_{\nu\kappa} + \delta^{\lambda}_{\nu} F_{\nu\kappa}\right) + U^{\lambda}_{\mu\nu\kappa}$$

Canonical Levi-Civita tensor-density at our disposal:

$$\epsilon^{\kappa\lambda\mu\nu} = \{-1, 0, +1\}$$

$$L = R^{\kappa}_{\lambda\mu\nu}\ R^{\lambda}_{\kappa\alpha\beta}\ \epsilon^{\mu\nu\alpha\beta}$$

Unfortunately, trivial (complete divergence).

At least two epsilons have to be used:

$$L^2 = R^{\star}_{\star\bullet\bullet}\ R^{\star}_{\star\bullet\bullet}\ R^{\star}_{\star\circ\circ}\ R^{\star}_{\star\circ\circ}\ \epsilon^{\bullet\bullet\bullet\bullet}\ \epsilon^{\circ\circ\circ\circ}$$

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$$L = L\left(K^{\lambda}_{\mu\nu\kappa}\right)$$

Momentum canically conjugate:

$$P^{\mu\nu\kappa}_{\lambda} := \frac{\partial L}{\partial \Gamma^{\lambda}_{\mu\nu\kappa}} = \frac{\partial L}{\partial K^{\lambda}_{\mu\nu\kappa}}$$

Canonical decomposition:

$$P^{\mu\nu\kappa}_{\lambda}\delta K^{\lambda}_{\mu\nu\kappa} \;=\; \pi^{\mu\nu}\delta K_{\mu\nu} + \mathcal{F}^{\mu\nu}\delta F_{\mu\nu} + p^{\mu\nu\kappa}_{\lambda}\delta U^{\lambda}_{\mu\nu\kappa}$$

$$P^{\mu\nu\kappa}_{\lambda} = \left(\delta^{\kappa}_{\lambda}\pi^{\mu\nu} - \delta^{(\mu}_{\lambda}\pi^{\nu)\kappa}\right) - \frac{1}{2}\left(\delta^{\mu}_{\lambda}\mathcal{F}^{\nu\kappa} + \delta^{\nu}_{\lambda}\mathcal{F}^{\mu\kappa}\right) + p^{\mu\nu\kappa}_{\lambda}\,.$$