Automorphisms of Cartan geometries

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7 March 2024

Cartan geometries: notation

$$\begin{tikzcd}
\mathscr{G} \arrow[d] & H \arrow[l] \\
M &
\end{tikzcd}$$

Cartan geometries: terminology

$$\begin{tikzcd} \mathscr{G}_{\text{total space}} \arrow[d] & H_{\text{structure group}} \arrow[l] \\ M_{\text{base}} \end{tikzcd}$$

Cartan geometries: terminology

Constant vector field: on $\mathscr{G}$, Cartan connection $=$ constant

The main theorem

Theorem

$$\begin{tikzcd} \mathscr{G} \arrow[d] & \mathrm{Aut} \arrow[l] \\ \mathscr{G}/\mathrm{Aut} & \end{tikzcd}$$

is a principal bundle for a unique smooth structure on Aut*.*

Riemannian approach

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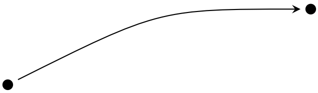
Freedom

Theorem (Cartan)

Each automorphism is determined by where it maps a point of $\mathscr{G}$.

Proof.



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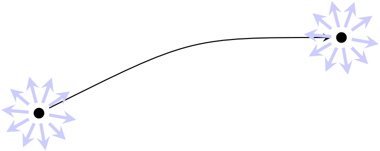

Constant vector fields commute. $\square$

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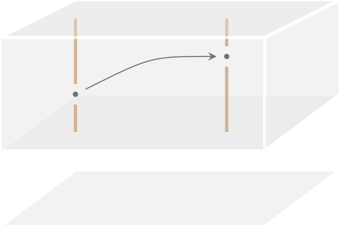

Structure group commutes. $\square$

Freedom

Corollary

Aut *acts freely on* $\mathscr{G}$.

Corollary

Each orbit is an injection of Aut $\to \mathscr{G}$.

Freedom

1. Defines the *orbit topology* on Aut.

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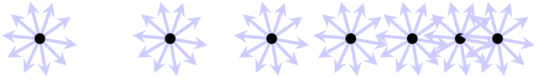
2. Independent of choice of orbit: flows of constant vector fields and structure group.

Closed orbits

Theorem (Kobayashi)

Orbits are closed.

Proof.



Commuting with constant vector fields, structure group: a closed condition. □

Palais's theorem: the Lie algebra

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5. Define $\Gamma^0 \subseteq \Gamma$: the subgroup generated by those flows.

Palais's theorem

Theorem (Palais)

If $\operatorname{Lie}\Gamma$ *is finite dimensional then* Γ^0 *is a connected Lie group acting smoothly.*

Palais's theorem: translate

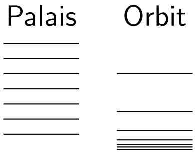
Left translate Γ^0:

$\cdots$  $\cdots$

Smooth structure on Γ.

Oops!

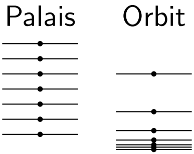
Back to Cartan geometries.
Kobayashi does not distinguish the two topologies.



Why can't Palais components squeeze in orbit?

Oops!

Kobayashi assumes the two topologies agree.



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3. Look out at your orbit:

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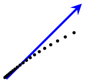
Seeing stars

1. Suppose they converge to you.

2. The celestial sphere is compact: a subsequence converges in direction.
3. You see stars line up, in your telescope.

What is a telescope?

1. A unique constant vector field points in each direction



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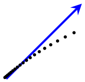
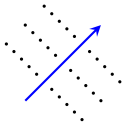
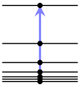


2. Suitable powers converge to any point along the "telescope direction"



What is a telescope?

The telescope direction moves along the orbit:



Palais is orbit

Theorem

The Palais and orbit topologies coincide in any Cartan geometry.

Review of complex theory

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1. Holomorphic Cartan geometry $\mathscr{G} \to M$
2. M smooth projective variety
3. Exists: $M \to \bar{M}$ hol fiber bundle.
4. Every fiber is (i) a flag variety or (ii) a point: $M = \bar{M}$.
5. Cartan geometry on M arises, trivially, from $\bar{M}$.
6. $\bar{M}$ is *minimal model* (Mori), no rational curves

Automorphisms

Theorem

Automorphism group of $\mathscr{G} \to \bar{M}$ is discrete extension of abelian variety.

Open problems

1. Smooth ok: how rough can you get?

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1. Smooth ok: how rough can you get?
2. Gromov: main theorem is true for rigid geometric structures. No proof.