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Grothendieck and the six operations

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Plan

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1. Serre's duality theorem

Theorem 1 (ICM Amsterdam, 1954)

k algebraically closed,
X/k smooth, projective, irreducible, of dimension m,
V a vector bundle on X, $V^\vee = \mathcal{H}om(V, \mathcal{O}_X)$,
$\Omega^i := \Lambda^i \Omega^1_{X/k}$. Then:

(a) $\dim_k H^m(X, \Omega^m) = 1$;

(b) For all $q \in \mathbf{Z}$, the pairing

$$H^q(X, V) \otimes H^{m-q}(X, V^\vee \otimes \Omega^m) \to H^m(X, \Omega^m)(\stackrel{\sim}{\to} k)$$

is perfect.

Remarks

(a) Serre had previously proved (FAC) that, for any coherent sheaf $\mathcal{F}$ on X, and all q, $\dim_k H^q(X, \mathcal{F}) < \infty$ and $H^q(X, \mathcal{F}) = 0$ for $q > m$.

(b) Serre doesn't exhibit a distinguished basis of $H^m(X, \Omega^m)$.
Proof by induction on m, his vanishing theorems on $H^q(X, \mathcal{F}(n))$ for $q > 0$ and n large play a key role.
Construction of a distinguished basis crucial in further work by Grothendieck et al.

(c) Serre proved analogue for $X/\mathbf{C}$ smooth, compact analytic, V a vector bundle on X (Comm. Helv., 1955). Quite different techniques.

Th. 1 revisited by Grothendieck:

1955-56, Sém. Bourbaki 149, May 1957

X/k smooth, projective, irreducible, dimension m as above.

Theorem 2 (Grothendieck, loc. cit., Th. 2, 3).

(1) A canonical basis $\varepsilon_X \in H^m(X, \Omega^m)$ (called fundamental class) is constructed.

(2) For all $q \in \mathbf{Z}$, the canonical pairing

$$H^q(X, \mathcal{F}) \otimes \mathrm{Ext}^{m-q}(\mathcal{F}, \Omega^m) \to H^m(X, \Omega^m)(\xrightarrow{\sim} k)$$

is perfect.

Remarks

(a) Pairing in (2): usual pairing on Ext groups, using $H^i(X, \mathcal{G}) = \mathrm{Ext}^i(\mathcal{O}_X, \mathcal{G})$ (Grothendieck, Tôhoku).

(b) (1) combines 2 ingredients:

- the fundamental local isomorphism for $x \in X(k)$

$$\mathrm{Ext}^m(k(x), \Omega^m) \xrightarrow{\sim} k(x)$$

- the trace map (an isomorphism as X is irreducible)

$$\mathrm{Tr} : H^m(X, \Omega^m) \xrightarrow{\sim} k.$$

At the core of all further work on coherent duality.

2. Derived categories: Grothendieck's revolution

How to generalize Th. 2 to:

- singular varieties X/k?
- morphisms $f : X \to Y$?

Grothendieck ICM Edimburgh (1958): Ω^n should be replaced by complex K_X (later called residual or dualizing),

and isomorphism deduced from perfect pairing (2) in Th. 2

$$H^q(X, \mathcal{F})^\vee \xrightarrow{\sim} \mathrm{Ext}^{m-q}(\mathcal{F}, \Omega^m)$$

by spectral sequences converging to H^* of a common complex E, to be constructed.

To solve the question, builds new foundations to homological algebra: derived categories:

Main ideas in Résidus et dualité, pré-notes pour un séminaire Hartshorne (summer of 1963). Formal construction and development of the theory left to, and worked out by, Verdier.

$\mathcal{A}$: an abelian category, $C(\mathcal{A})$: complexes of $\mathcal{A}$

To $L = (\cdots \to L^n \xrightarrow{d} L^{n+1} \to \cdots)$ in $C(\mathcal{A})$, wants to associate an object finer than H^*L, but coarser than L itself, that could remember classical derived functors (Cartan-Eilenberg, Tôhôku) in a functorial way.

Key new notion: $f : K \to L$ a quasi-isomorphism if $H^i f : H^i K \to H^i L$ is an isomorphism for all i

Derived category $D(\mathcal{A}) := C(\mathcal{A})[\mathrm{Quasi-isomorphisms}]^{-1}$

Left and right calculus of fractions from $K(A)$, where $\mathrm{Hom}_{K(A)}(K, L) = \mathrm{Hom}_{C(A)}(K, L)/(\{f = dh + hd)\}$

Triangulated structure on $K(\mathcal{A})$, $D(\mathcal{A})$ (given by distinguished triangles coming from mapping cones of morphisms, or short exact sequences of complexes): axiomatized by Verdier. Key axiom: octahedral axiom.

Derived functors

For $\mathcal{A}$ having enough injectives, or projectives, or suitable substitutes,

$$RF,\ LF : D(\mathcal{A}) \to D(\mathcal{B})$$

of additive $F : \mathcal{A} \to \mathcal{B}$ replace old R^iF, L^iF of Cartan-Eilenberg and Tôhôku: $H^iRF(L) = R^iF(L)$, etc. They are triangulated, i.e., send distinguished triangles to distinguished triangles.

Generalized to triangulated multi-functors, such as

$$K(\mathcal{A})^0 \times K(\mathcal{A}) \to K(\mathrm{Ab}),\ (K, L) \mapsto \mathrm{Hom}^\bullet(K, L),$$

giving

$$R\mathrm{Hom} : D(\mathcal{A})^0 \times D(\mathcal{A}) \to D(\mathrm{Ab}).$$

Under mild assumptions, spectral sequences of composite functors replaced by transitivity isomorphisms

$$RG \circ RF \xrightarrow{\sim} R(G \circ F).$$

Main (rough) idea: though $F : K(\mathcal{A}) \to K(\mathcal{B})$ doesn't extend to $D(\mathcal{A}) \to D(\mathcal{B})$ (as F(quasi-iso) is not a quasi-iso in general), for suitable resolutions, i.e., certain quasi-iso $K \to K'$, $F(K')$ in $D(\mathcal{B})$ "doesn't depend" on the "suitable" K', and $K \mapsto F(K')$ is the "closest" functor from $D(\mathcal{A})$ to $D(\mathcal{B})$ to a (in general, nonexistent) functor extending F.

Changed the face of homological algebra. All that had been done before became suddenly obsolete, except spectral sequences, still giving deeper insight into derived categories.

The four basic functors

(called operations by Grothendieck in *Récoltes et Semailles*)

- internal

$$\otimes^L, \; R\mathcal{H}om,$$

on a topos X with commutative ring $\mathcal{O}_X$,

- external

$$Lf^*, \; Rf_*$$

for a morphism of ringed topoi $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, and

$$R\Gamma(X, -) := Rf_*$$

for $Y = \mathrm{pt}$, satisfying

Adjunction isomorphisms

$$\mathrm{Hom}(K \otimes^L L, M) \xrightarrow{\sim} \mathrm{Hom}(K, R\mathcal{H}om(L, M)),$$

souped up to

$$R\mathcal{H}om(K \otimes^L L, M) \xrightarrow{\sim} R\mathcal{H}om(K, R\mathcal{H}om(L, M)),$$

$$\mathrm{Hom}(K, Rf_*L) \xrightarrow{\sim} \mathrm{Hom}(Lf^*K, L)$$

souped up to

$$R\mathcal{H}om(K, Rf_*L) \xrightarrow{\sim} Rf_*R\mathcal{H}om(Lf^*K, L)$$

(trivial global duality),

and various Canonical isomorphisms, such as

$$Lf^*(K \otimes^L L) \xrightarrow{\sim} Lf^*K \otimes^L Lf^*L,$$

and

$$Lf^*Lg^* \xrightarrow{\sim} L(gf)^*, \; Rg_*Rf_* \xrightarrow{\sim} R(gf)_*$$

for a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$, etc.

Remark

Classically those four functors were not defined on the whole derived categories, but on certain full subcategories defined by degree restrictions. Recall

$$D^+(\mathcal{A}),\ D^-(\mathcal{A}),\ D^b(\mathcal{A}) \subset D(\mathcal{A})$$

consisting of complexes cohomologically bounded below, above, bounded. For example, $Lf^* : D^-(Y) \to D^-(X)$, $Rf_* : D^+(X) \to D^+(Y)$, $\otimes^L : D(X) \times D^-(X) \to D(X)$, etc. And some canonical isomorphisms needed further restrictive hyotheses.

Grothendieck (1965) asked: could one get rid of those degree restrictions?

Question solved by N. Spaltenstein (1988), using the notion of homotopically injective (resp. projective) resolution. Generalized by Kashiwara-Schapira (2006).

4. The $f^!$ functor: duality in the coherent setting

For $f : X \to Y$ of finite type between Noetherian schemes, and smoothable, i.e., of the form gi, for $g : X' \to Y$ smooth and $i : X \hookrightarrow X'$ a closed immersion, Grothendieck defines a new functor

$$f^! : D^+_{\mathrm{qcoh}}(Y) \to D^+_{\mathrm{qcoh}}(X)$$

(where $(-)_{\mathrm{qcoh}}$ means quasi-coherent H^*: image of $D^+(\mathrm{Qcoh}(-))$ by fully faithful functor to $D^+(-)$) by the curious formula

$$f^!(M) := R\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{O}_X, g^*(M) \otimes \Omega^d_{X'/Y})|X,$$

where d is the relative dimension of g. Indeed, a miracle: RHS doesn't depend on factorization $f = gi$, up to transitive system of isomorphisms. A corollary of the so-called fundamental local isomorphism

$$R\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_X, E) \xrightarrow{\sim} i^*E \otimes^L_{\mathcal{O}_X} \Lambda^r \mathcal{N}^\vee_{X/Z}[-r],$$

for a regular immersion $i : X \hookrightarrow Z$ of codimension r, ideal $\mathcal{I}$, $E \in D^+_{\mathrm{qcoh}}(Z)$, and $\mathcal{N}_{X/Z} := \mathcal{I}/\mathcal{I}^2$ the conormal bundle.

Transitivity isomorphism $(f_2 f_1) \xrightarrow{\sim} f_1^! f_2^!$ for a composition, etc.

Second miracle: Grothendieck shows that for X, Y of finite Krull dimension, and f projective, $f^!$ turns out to be a partial right adjoint to $Rf_* : D_{\mathrm{qcoh}}(X) \to D_{\mathrm{qcoh}}(Y)$. Indeed, he constructs a so-called trace map for $M \in D_{\mathrm{qcoh}}(Y)$,

$$\mathrm{Tr}_f : Rf_* f^! M \to M,$$

giving rise to an isomorphism, the duality isomorphism,

$$(*) \qquad Rf_* R\mathcal{H}om(L, f^! M) \xrightarrow{\sim} R\mathcal{H}om(Rf_* L, M)$$

for $L \in D_{\mathrm{qcoh}}(X)$, $M \in D^+_{\mathrm{qcoh}}(Y)$, and, in particular, to the adjunction isomorphism

$$\mathrm{Hom}(L, f^! M) \to \mathrm{Hom}(Rf_* L, M),$$

deduced from (*) by applying $H^0 R\Gamma$.

Applying $H^{-q} R\Gamma$, we recover, for $Y = \mathrm{Spec}(k)$, X/k projective and smooth of dimension m, Grothendieck-Serre's isomorphism

$$(H^q(X, \mathcal{F}))^\vee \xrightarrow{\sim} \mathrm{Ext}^{m-q}(\mathcal{F}, \Omega^m_{X/k}).$$

Duality isomorphism

$$(*) \qquad Rf_* R\mathcal{H}om(L, f^! M) \xrightarrow{\sim} R\mathcal{H}om(Rf_* L, M)$$

generalized by Grothendieck to f proper, X, Y satisfying mild additional assumptions, by means of a theory of residual complexes, fitting with a theory of generalized residues (Hartshorne, R. and D.).

Global duality (*) tightly linked with:

• Local duality, dualizing complexes (R. and D., SGA 2)

• Hodge cohomology classes (Grothendieck's Prenotes for R. and D.; Angeniol, El Zein, 1978)

Dualizing complexes

On a Noetherian scheme S, a dualizing complex is an object K of $D^b_{\mathrm{coh}}(S)$ which is of finite injective dimension, and such that, if $D : R\mathcal{H}om(-, K)$, then for any $L \in D^b_{\mathrm{coh}}(S)$, the canonical map

$$L \to DDL$$

is an isomorphism ($\Leftrightarrow \mathcal{O}_S \xrightarrow{\sim} DD\mathcal{O}_S$).

A dualizing complex is unique up to shift, and twist by an invertible sheaf).

It exists if S is the spectrum of a complete local ring, more generally, if and only if S is a closed subscheme of a finite dimensional Gorenstein scheme (Sharp's conjecture, proved by Kawasaki, 2002).

Exchange formulas

If K is dualizing on S, for $a : X \to S$ of finite type, $K_X := Ra^! K$ is dualizing. for $f : X \to Y$ projective, the global duality theorem implies (with $D_X = R\mathcal{H}om(-, K_X)$, $D_Y = R\mathcal{H}om(-, K_Y)$)

$$Rf_* D_X L \stackrel{\sim}{\to} D_Y Rf_* L.$$

For f of finite type, D exchanges Lf^* and $f^!$.

Link with local duality

For S local, of closed point $i : \{s\} \to S$, if K is dualizing on S, then $i^! K = k(s)[d]$ for some $d \in \mathbf{Z}$. If $d = 0$, then $R\Gamma_s(K)$ is an injective envelope of $k(s)$, and, for any $M \in D^b_{coh}(S)$, the natural map

$$R\Gamma_s(M) \to \mathrm{RHom}(DM, R\Gamma_s(K)),$$

where $DM := R\mathcal{H}om(M, K)$, and
$\Gamma_s(-) := \mathrm{Ker}(\Gamma(S, -) \to \Gamma(S - s, -))$, is an isomorphism (local duality theorem).

Verdier's categorical approach to $f^!$

Meanwhile, other similar duality theories had emerged:

• Duality for the cohomology of profinite groups (Verdier, 1963)

• Duality in étale cohomology (SGA 4, Artin-Grothendieck-Verdier, 1964)

• Duality in the cohomology of locally compact spaces (Verdier, 1965)

This last work was proposed to Verdier by Grothendieck, who thought the theory would be analogous to (and easier than) duality in the étale setting. But in it Verdier introduced a new idea (already apparent in his work on profinite groups), namely:

• Prove *a priori* the existence of $f^!$ as a right adjoint to $Rf_!$, calculate it afterwards for nice morphisms f.

Let $f : X \to Y$, continuous, between locally compact spaces, and let k be a Noetherian ring.
Then the direct image with proper support functor $f_! : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_Y)$ has a derived functor

$$Rf_! : D^+(X) \to D^+(Y),$$

where $D(X) := D(X, k_X)$, $D(Y) := D(Y, k_Y)$.

Assume $f_!$ has finite cohomological dimension. Then, using a calculation process for $Rf_!$ on the level of complexes (based on soft resolutions), Verdier constructs a right adjoint $f^!$ to $Rf_!$, giving rise to an isomorphism

$$(**) \qquad Rf_* R\mathcal{H}om(L, f^! M) \xrightarrow{\sim} R\mathcal{H}om(Rf_! L, M)$$

similar to (*), for $L \in D^-(X)$, $M \in D^+(Y)$.

From the knowledge of $H^*_c(\mathbf{R}^n, k)$ Verdier deduces from (**) that, for an n-dimensional manifold X, and $f : X \to Y = \mathrm{pt}$, one has

$$f^! k = \omega_X[n]$$

where ω_X, the orientation sheaf, is the (invertible) sheaf associated to $U \mapsto H^n_c(U, k)^\vee$. Then the adjunction map $Rf_! f^! \to \mathrm{Id}$ gives a trace map $\mathrm{Tr}_X : H^n(X, \omega_X) \to k$, and, for k a field, (**) yields a duality isomorphism similar to the Grothendieck-Serre one, namely,

$$(H^q(X, \mathcal{F}))^\vee \xrightarrow{\sim} \mathrm{Ext}^{n-q}(\mathcal{F}, \omega^n_X).$$

for any k-sheaf F.

Formula for $f^! k$ generalized by Verdier to $f^! M = f^* M \otimes \omega_{X/Y}[n]$ for maps $f : X \to Y$ satisfying suitable condition of smoothness of relative dimension n.

Deligne's adaptation to the coherent setting

Let $f : X \to Y$ proper, X, Y Noetherian. Deligne observes that, for formal reasons, as in the topological case, $Rf_* : D^+_{\mathrm{qcoh}}(X) \to D^+_{\mathrm{qcoh}}(Y)$ admits a right adjoint $f^! : D^+_{\mathrm{qcoh}}(Y) \to D^+_{\mathrm{qcoh}}(Y)$:

$$\mathrm{Hom}(Rf_* K, L) \xrightarrow{\sim} \mathrm{Hom}(K, Rf^! L)$$

(finite type, instead of proper, would even suffice). Why is that?

- Can calculate $Rf_* K$ by functorial process on level of complexes:

$$Rf_* K := f_* \mathcal{C}^\bullet(K),$$

$\mathcal{C}^\bullet(K)$ finite acyclic resolution, compatible with filtering inductive limits, with each $\mathcal{C}^q(K)$ exact in K (Cech, or modified Godement)

• Lemma. If $\mathcal{A}$ is Grothendieck abelian category, i.e., an abelian category admitting a generator and exact filtering inductive limits, any contravariant functor F on $\mathcal{A}$ with values in abelian groups, transforming arbitrary (small) inductive limits into projective ones is representable.

Apply it to $F = \mathrm{Hom}(f_* \mathcal{C}^q(K), I)$ for fixed q and injective quasi-coherent I.

Precursors of Lemma in Gabriel's thesis (II 4). See [Kashiwara-Schapira, Categories and Sheaves, 5.2.6] for a generalization.

Homotopical variants in triangulated categories

Inspired by Neeman's form of Brown's representability theorem, see Kashiwara-Schapira (*loc. cit.*, 10.5.3, 14.2.3):

Theorem

Let D, D' be triangulated categories, with D admitting small direct sums and a system of t-generators, i.e. a (small) system of generators C such that $\mathrm{Hom}(C, -)$ detects the vanishing of any countable sum $\oplus u_i : X_i \to Y_i$ by the vanishing of $\mathrm{Hom}(C, u_i)$ (a weaker condition than the notion of compactly generated introduced by Neeman), e.g., (*loc. cit.*, 14.2.1) $D = D(\mathcal{A})$, $\mathcal{A}$ a Grothendieck abelian category.

Then any triangulated functor $F : D \to D'$ commuting with direct sums admits a right adjoint (which is triangulated).

Calculation of $f^!$

Definition of $f^!$ extended to f compactifiable by

$$f^! = j^* g^!$$

for $f = gj$, $j : X \hookrightarrow Z$ open, $g : Z \to Y$ proper.

Independence of compactification proved independently by

• Deligne (App. to R. and D.), using a $j_! : \mathrm{Coh}(X) \to \mathrm{ProCoh}(Z)$ functor

• Verdier (1968), using a base change formula for $f^!$ (f proper, flat base change).

Comparison with Grothendieck's $f^!$

Delicate issues: Lipman, Hashimoto, Neeman, Conrad, Iyengar, Yekutieli, Nayak-Sastry (2019)

4. Duality in étale cohomology and the six operations

Coefficient ring $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$, ℓ invertible on base (much later (end of 1970's), $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, $\overline{\mathbf{Q}}_\ell$)

$D(-) := D(-, \Lambda)$, D_c : constructible $\mathcal{H}^i$.

The $Rf_!$ functor

Artin-Grothendieck: can't imitate the topological case: for $k = \overline{k}$, X/k an affine curve, $\mathcal{F}$ on X, sections of $\mathcal{F}$ on X with proper support $= \oplus_{x \in X(k)} \Gamma_x(\mathcal{F})$, with bad derived functors $\oplus H^i_x(\mathcal{F})$ (want: $H^2_c(\mathbf{A}^1_k, \Lambda) = \Lambda(1)$).

Good hybrid definition for f compactifiable, $f = gj$, $j : X \hookrightarrow Z$ open, $g : Z \to Y$ proper:

$$Rf_! := Rg_* \circ j_!$$

but $Rf_!$ not derived functor of $H^0 Rf_!$
Proper base change th. $\Rightarrow$ independence of compactification, $Rf_!$ commutes with base change, compatible with composition.

The $f^!$ functor

(1) Grothendieck's definition for f smoothable, i.e., of the form hi, for $h : X' \to Y$ smooth and $i : X \hookrightarrow X'$ a closed immersion:

$$f^! L := Ri^! h^* L(d)[2d],$$

where d = relative dimension of h, and $Ri^!$ = derived functor of $M \mapsto \mathcal{H}^0_X(X', M)|X$.

Relative purity for smooth pairs $i : Y \hookrightarrow Z$, Y/S, Z/S smooth, i.e. $Ri^! \Lambda = \Lambda(-r)[-2r]$ (r = codimension of i) (SGA 4 XVI) implies independence of factorization $f = hi$, compatibility with composition.

For $f : X \to Y$ quasi-projective, using structure of $H^*(\mathbf{P}^r_S)$ ($r = 1$, in fact, suffices), Grothendieck defines a trace map

$$\mathrm{Tr}_f : Rf_! f^! \to \mathrm{Id}$$

making $f^!$ a right adjoint to $Rf_!$, and giving rise to a global duality isomorphism

$$Rf_* R\mathcal{H}om(L, f^! M) \to R\mathcal{H}om(Rf_! L, M)$$

for $L \in D^-(X)$, $M \in D^+(Y)$. (Proof by reduction to relative curves. Written up by Verdier in (Driebergen, Local Fields, 1967).)

(2) Deligne's method

Similar to the coherent setting, and written up later. For f compactifiable, $f = gj$, $j : X \hookrightarrow Z$ open, $g : Z \to Y$ proper, can calculate $Rf_! K$ by functorial process

$$Rf_! K = g_*(\mathcal{C}(j_! K))$$

where $\mathcal{C}$ is a modified Godement resolution, with each component $g_*\mathcal{C}^q(j_! K)$ exact and commuting with filtering inductive limits, hence (by the representability lemma) admitting a right adjoint, giving rise to $f^! : D^+(Y) \to D^+(X)$, right adjoint to $Rf_!$, yielding global duality isomorphism similar to the above:

$$Rf_* R\mathcal{H}om(L, f^! M) \to R\mathcal{H}om(Rf_! L, M).$$

Difficulty shifted to comparing abstractly defined $f^!$ with Grothendieck's $f^!$.

Done in [SGA 4 XVIII], using theory of fundamental class, or trace map to construct, for $f : X \to Y$ flat of relative dimension d, and compactifiable, a canonical map

$$t_f : f^* M(d)[2d] \to f^! M$$

(for $M \in D^+(Y)$), which is an isomorphism for f smooth.

As in (1), proof of t_f being an isomorphism eventually reduced to relative curve case.

Constructibility and the six operations

$D(-)$ too big, want to stay within $D_{\mathrm{c}}(-)$ (possibly $D^b_{\mathrm{c}}(-)$ or $D_{\mathrm{ctf}}(-)$, "ctf" for "constructible" and "tor-dimension finie").

At the time of Grothendieck, known that $Rf_!$ sends D^b_c to D^b_c, but not (except under assumptions of existence of resolution of singularities and validity of absolute purity conjecture) for Rf_* (even for $f : X \to \mathrm{Spec}(k)$, k algebraically closed).

Stability of D_{ctf} under f^* and $\otimes^L$ trivial, but Deligne (1973, SGA 4 1/2, Th. finitude) proved stability of D_{ctf} under the remaining four operations

$$R\mathcal{H}om,\ Rf_*,\ Rf_!,\ f^!,$$

for schemes separated and of finite type over a regular base of dimension ≤ 1.

Method of proof (global to local) often imitated afterwards.

Two additional important results by Deligne, that are not corollaries of the above, but are proved by the same method:

• The nearby cycle functor $R\Psi$ send D_{ctf} to D_{ctf}, and commutes with surjective base change of traits.

• For $a : X \to S$ separated and of finite type, with S regular, Noetherian, of dimension ≤ 1,

$$K_X := a^! \Lambda_S$$

is a dualizing complex on X, i.e., is of finite injective dimension, and if $D_X := R\mathcal{H}om(-, K_X)$, then, for any $L \in D_c^b(X)$, the canonical map

$$L \to D_X D_X L$$

is an isomorphism. Observed by Grothendieck that such a dualizing complex is unique up to shift, and twist by an invertible Λ-module. He had proved existence only under the assumptions mentioned above (resolution, etc.).

For $f : X \to Y$ an S-morphism, with X/S, Y/S, K_X, K_Y as before, the global duality isomorphism yields, for $L \in D^b_c(X)$

$$Rf_* D_X L \xrightarrow{\sim} D_Y Rf_! L,$$

and,

$$D_X f^* M \xrightarrow{\sim} f^! D_Y M$$

for $M \in D^b_c(Y)$: D exchanges Rf_* and $Rf_!$, resp. f^* and $f^!$.

The Lefschetz-Verdier formula

This is the most famous and useful application of the duality formalism. The results discussed above made it possible to state and prove the LV formula unconditionally (SGA 5 III, 1977). Classically, one works with schemes separated and of finite type over an algebraically closed field k, and the formula consists of two parts:

- Definition of local terms near the fixed points of a cohomological correspondence
- A theorem expressing the compatibility of these local terms with proper push-forward.

Local terms

Write D for D_{ctf}. A cohomological correspondence with support in a correspondence $c = (c_1, c_2) : C \to X \times X$ is the data of $M \in D(X)$ and a morphism

$$u : c_1^* M \to c_2^! M.$$

Example: A pair $(f : X \to X, u : f^* M \to M)$, with $c = {}^{\mathrm{t}}\Gamma_f = (f, Id)$ the (transposed) graph.

In particular, we have the tautological correspondence Id_M with support in the diagonal $\Delta : X \to X \times X$.

One has

$$R\mathcal{H}om(c_1^* M, c_2^! M) = c^!(DM \boxtimes^L M),$$

hence

$$\mathrm{Hom}(c_1^* M, c_2^! M) = H^0(C, c^!(DM \boxtimes^L M))$$

The natural pairing

$$(DM \boxtimes^L M) \otimes^L (M \boxtimes DM) \to K_X \otimes^L K_X \xrightarrow{\sim} K_{X \times X}$$

send the pair $((c, u), (\Delta, \mathrm{Id}_M))$ to the so-called Verdier local term

$$\mathrm{Tr}_{C \cap \Delta}(c, u) \in H^0(C \cap \Delta, K_{C \cap \Delta}),$$

where $C \cap \Delta := C \times_{X \times X} (X, \Delta)$ is the fixed point scheme of c.

The LV formula

This formula expresses compatibility of formation of Verdier local term

$$\mathrm{Tr}_{C \cap \Delta}(c, u) \in H^0(C \cap \Delta, K_{C \cap \Delta}),$$

with proper push-forward of correspondences (SGA 5 III). In particular, for X, C proper over k, the cohomological correspondence (c, u) has a push-forward to $\mathrm{Spec}(k)$, which is a homomorphism

$$(c, u)^* : R\Gamma(X, M) \to R\Gamma(X, M),$$

whose trace is given by the formula (LV formula)

$$\mathrm{Tr}((c, u)^*) = \mathrm{Tr}_{C \cap \Delta/k} \mathrm{Tr}_{C \cap \Delta}(c, u),$$

where

$$\mathrm{Tr}_{C \cap \Delta/k} : H^0(C \cap \Delta, K_{C \cap \Delta}) \to \Lambda$$

is the trace map of the global duality isomorphism defined by the adjunction $Ra_! a^! \to \mathrm{Id}$, $a : C \cap \Delta \to \mathrm{Spec}(k)$.

Remarks on the formalism of six operations

1. Gabber proved stability theorems for D_c^b on quasi-excellent schemes similar to those of Deligne in SGA 4 1/2 (Travaux de Gabber, Astérisque 363-364) (and implying them). In particular, he proved Grothendieck's duality conjecture, to the effect that on a regular, excellent scheme X, the constant sheaf Λ_X is a dualizing complex ($\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$, ℓ invertible on X).

2. Extension of six operations from torsion coefficients to ℓ-adic coefficients, such as $\mathbf{Z}_\ell$, $\mathbf{Q}_\ell$, or $\overline{\mathbf{Q}}_\ell$ was a nontrivial task: Jouanolou (SGA 5), Deligne (1979), Ekedahl (1990), Bhatt-Scholze (2012).

3. Extension of the formalism (both for torsion coefficients and ℓ-adic ones) to algebraic stacks was an equally nontrivial task: Laszlo-Olsson (2008), Liu-Zheng (2012).

Remarks on the LV formula

1. An analogue of LV in the coherent setting was proved in (SGA 5, III). Implies the so-called (algebraic) Woods-Hole fixed point formula (a conjecture of Shimura) (Hartshorne-Mumford-Verdier, 1964).

2. The LV formula implies Grothendieck's trace formula (GTF) for the Frobenius correspondence over finite fields. However, in SGA 5, Grothendieck had proved GTF independently of LV, by the so-called Nielsen-Wecken method of non-commutative traces (report by Deligne in SGA 4 1/2).

3. But LV turned out to be crucial in the proof of Deligne's conjecture on Frobenius twisted correspondences (Fujiwara (1977), generalization by Varshavsky (2007)). Used by L. Lafforgue in his proof of the Langlands correspondence for GL_n over function fields.

4. The Verdier local term decomposes into

$$\mathrm{Tr}_{C\cap\Delta}(c,u) = \sum_{Z} \mathrm{Tr}_Z(c,u)$$

where Z runs through the connected components of $C \cap \Delta$. Each term is of étale local nature around Z. Even for an isolated fixed point $Z = \{x\}$, and a correspondence of the form $({}^t\Gamma_f, u)$, $f : X \to X$, $u : f^*M \to M$, the calculation of $\mathrm{Tr}_x(f,u)$ is difficult. In this case, (f,u) induces an endomorphism $(f,u)_x$ of M_x, and one can ask whether

$$(*) \qquad \mathrm{Tr}_x(f,u) = \mathrm{Tr}((f,u)_x).$$

Formula

$$(*) \qquad \mathrm{Tr}_x(f,u) = \mathrm{Tr}((f,u)_x).$$

was proved by Verdier (Driebergen (1967), SGA 5 III B) for X/k a smooth curve and the graph of f assumed to be transversal at x to the diagonal. Deligne conjectured that (*) holds in any dimension, even for X singular at x, provided that f^* has no fixed vector on the Zariski cotangent space $T_x^*(X)$. Conjecture recently proved by Varshavski in a stronger form ($T_x^*(X)$ replaced by the conormal cone).

5. Shown by Liu-Zheng (2019) that cohomological correspondences form a symmetric monodoidal 2-category in a natural way, and that the LV formula is a formal consequence of pairings in such categories. They even prove a relative LV formula under local acyclicity hypotheses. Such categorical trace arguments, that can be traced back to Dold-Puppe (1978), were used by Gaitsgory et al. in the past few years.

5. Further developments

On derived and triangulated categories

- Filtered derived categories
- t-structures on triangulated categories, perverse sheaves
- ∞-enhancements of $D(\mathcal{A})$.

Duality and six operations in other contexts

• Coherent sheaves on complex analytic spaces:
Ramis-Ruget-Verdier (1971)

• $\mathcal{D}$-modules: Bernstein, Kashiwara-Schapira, Mebkhout, Malgrange (1970 - 1980)

• Mixed Hodge theory: Deligne, M. Saito (1990 - ...)

• Crystalline cohomology, rigid cohomology, arithmetic $\mathcal{D}$-modules: Berthelot, Le Stum, Ekedahl, Kedlaya, Caro, Caruso, ... (1970 - ...).

• Logarithmic geometry: C. Nakayama, Tsuji (1997 - ...)

• p-adic Hodge theory: still to come

Thank You!