Arithmetic and Algebraic Geometry

in honor of Prof. T. Katsura

on the occasion of his 60th birthday

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On finite group actions, Deligne-Mumford stacks, and traces, after W. Zheng et al.

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SETTING

X/k separated, finite type

G finite group acting on X (on the right)

ℓ prime $\neq$ char(k); $\overline{\mathbb{Q}_\ell}$: alg. closure of $\mathbb{Q}_\ell$

G-$\overline{\mathbb{Q}_\ell}$-sheaf L on X :

$g \in G \mapsto a(g) : L \xrightarrow{\sim} g_* L$, $a(gh) = a(g)a(h)$

$D_c^b(X, G, \overline{\mathbb{Q}_\ell})$: objects of $D_c^b(X, \overline{\mathbb{Q}_\ell})$

with G-action compatible with G-action on X

Grothendieck's operations :

$\bigotimes$, $R\mathcal{H}om$,

$(f, u) : (X, G) \to (Y, H)$,

$R(f, u)_*$, $R(f, u)_!$, $(f, u)^*$, $R(f, u)^!$

$[X/G]$ DM-stack $/k$ associated with (X, G)

(DM = Deligne-Mumford)

$D_c^b(X, G, \overline{\mathbb{Q}_\ell}) = D_c^b([X/G], \overline{\mathbb{Q}_\ell})$ (Laszlo-Olsson)

$(f, u) : (X, G) \to (Y, H) \mapsto [(f, u)] : [X/G] \to [Y/H]$,

$R(f, u)_* = R[(f, u)]_*$, etc.

PLAN

1. Rationality : finite fields

2. Rationality : local fields

3. Free actions, vanishing of Lefschetz numbers

4. Equivariant form of Laumon's theorem on Euler characteristics

1, 2 : Zheng

3, 4 : joint work with Zheng

1. RATIONALITY : FINITE FIELDS

$k = \mathbb{F}_q,\ q = p^f$

$(X/k, G)$ as above ; $|X|$: {closed points of X}

$x \in |X|$; $G_d(x)$ = decomposition gp at x

$\overline{x} \longrightarrow x$ alg. geometric pt

$G_{\overline{x}}$: set of pairs $(g \in G_d(x), \varphi \in \mathrm{Aut}(\overline{x}))$ s. t.

$$\begin{array}{ccc} x & \longleftarrow & \overline{x} \\ \downarrow g & & \downarrow \varphi \\ x & \longleftarrow & \overline{x} \end{array}$$

commutes

Traces

L $\overline{\mathbb{Q}_\ell}$-sheaf on X,

$(g, \varphi) \in G_{\overline{x}} \mapsto \mathsf{Tr}((g, \varphi), L_{\overline{x}}) \in \overline{\mathbb{Q}_\ell}$

$L \in D^b_c(X, G, \overline{\mathbb{Q}_\ell}) \mapsto$ Lefschetz number

$\mathsf{Tr}((g, \varphi), L_{\overline{x}}) = \sum(-1)^i \, \mathsf{Tr}((g, \varphi), H^i(L_{\overline{x}}))$

Compatible systems

given an extension $E/\mathbb{Q}$:

family I of embeddings $\iota : E \to \overline{\mathbb{Q}}_\ell$, $\ell \neq p$; $\ell = \ell_\iota$

family $(t_\iota \in \overline{\mathbb{Q}_{\ell_\iota}})_{\iota \in I}$ E-compatible :

there exists $c \in E$ s. t. $t_\iota = \iota(c)$ for all ι

family $L_\iota \in D_c^b(X, G, \overline{\mathbb{Q}_{\ell_\iota}})$ E-compatible if :

$\forall \overline{x} \to x \in |X|$, $\forall (g, \varphi) \in W(G_{\overline{x}})$,

$(\mathrm{Tr}((g, \varphi), (L_\iota)_{\overline{x}})$ is E-compatible

$W(G_{\overline{x}})$: Weil group : $\{(g, \varphi) \in G_{\overline{x}} | \varphi = F^n\}$, $n \in \mathbb{Z}$,

$F : a \mapsto a^{1/q} \in \mathrm{Aut}(k(\overline{x}))$

Remark : E-compatibility for $\varphi = F^n$, $n \geq N(x)$ ($N(x)$ fixed integer)$\Rightarrow$ E-compatibility

Theorem 1 (Zheng, 2007). E-compatibility stable under Grothendieck's six operations.

Special cases

- $G = \{1\}$: Gabber's th. (around 1980's)

(cf. Fujiwara, Azumino 2000)

- $(L_\iota) = (\overline{\mathbb{Q}_\ell})_{\ell \neq p}$, $f : X \to \operatorname{Spec} k$,

$\overline{k}$ = alg. closure of k, $f_!$, f_* :

$\forall g \in G$, $\forall n \in \mathbb{Z}$,

$\operatorname{Tr}(gF^n, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell))$ and

$\operatorname{Tr}(gF^n, H^*(X_{\overline{k}}, \mathbb{Q}_\ell)) \in \mathbb{Q}$, independent of ℓ

Remarks

- H_c^* : Deligne-Lusztig (1976)

(gF^n, $n > 0$, *is a Frobenius*)

- $n \geq 0$: Tr $\in \mathbb{Z}$ (Deligne's integrality th.)

Zheng's proof : independent of Gabber's

ingredients :

- de Jong's equivariant alterations :

reduce to lisse, tame family (L_ι) on $U = X - D$, X/k smooth, D G-strict dnc and E-compatibility of $Rj_* L_\iota$, $j : U \to X$

- Deligne's generic constructibility th. :

reduce to $\dim X = 1$

- Deligne-Lusztig's trick (1976) :

get rid of G-action

- Deligne's th. (Antwerp 1972) for curves

$\Rightarrow$ compatibility of Rj_*L_ι

Generalization to algebraic stacks

$\mathcal{X}/k$: alg. stack of finite type $/k$

(i. e. : separated, finite type diagonal + lisse cover by k-scheme of finite type)

family $(L_\iota \in D^b_c(\mathcal{X}, \overline{\mathbb{Q}_{\ell_\iota}}))_{\iota \in I}$ E-compatible :

$\forall i : x \to \mathcal{X}$, $x = \operatorname{Spec} k'$, $[k' : k] < \infty$,

$(i^*(L_\iota) \in D^b_c(x, \overline{\mathbb{Q}_{\ell_\iota}}))$ E-compatible

($\Leftrightarrow \forall$ smooth $f : X \to \mathcal{X}$, X/k affine,

$(f^*(L_\iota) \in D_c^b(X, \overline{\mathbb{Q}_{\ell_\iota}}))$ E-compatible)

Theorem 1' (Zheng, 2007). E-compatibility on alg. stacks stable under

- $\otimes$, $R\mathcal{H}om$, f^*, $Rf^!$,
- $Rf_!$, Rf_* if f relatively Deligne-Mumford

Proof :

for $Rf_!$, reduce to $f : [X/G] \to \operatorname{Spec} k'$, then to th. 1

2. RATIONALITY : LOCAL FIELDS

K : local field, i. e. $K = k(\eta)$,

$\eta =$ gen. pt of S

$S = \mathrm{Spec}\, A$, A : excellent henselian dvr

closed pt s , $k = k(s)$ finite, $|k| = q = p^f$

X/K separated, finite type

G finite group acting on X

$|X|$: $\{$closed points of $X\}$

family I of embeddings $\iota : E \to \overline{\mathbb{Q}}_\ell$, $\ell \neq p$; $\ell = \ell_\iota$

family $L_\iota \in D^b_c(X, G, \overline{\mathbb{Q}_{\ell_\iota}})$ E-compatible if :

$\forall \overline{x} \to x \in |X|$, $\forall (g, \varphi) \in W(G_{\overline{x}})$,

$(\mathrm{Tr}((g, \varphi), (L_\iota)_{\overline{x}})$ is E-compatible

$W(G_{\overline{x}})$: Weil group : $\{(g, \varphi) \in G_{\overline{x}} | \rho(\varphi) = F^n\}$, $n \in \mathbb{Z}$,

$F : a \mapsto a^{1/q} \in \mathrm{Aut}(k(\overline{x_0}))$

x_0 : closed pt of $\mathrm{Spec}\, R_x$, $R_x = \mathcal{O}_{k(x)}$

$$\begin{array}{ccccc} \overline{x_0} & \longrightarrow & \mathrm{Spec}\, R_{\overline{x}} & \longleftarrow & \overline{x} \\ \downarrow & & \downarrow & & \downarrow \\ x_0 & \longrightarrow & \mathrm{Spec}\, R_x & \longleftarrow & x \end{array}$$

$\rho : \mathrm{Gal}(k(\overline{x})/k(x)^{G_d(x)}) \to \mathrm{Gal}(k(\overline{x}_0)/k(x_0)^{G_d(x)})$

Remark : E-compatibility for $\rho(\varphi) = F^n$, $n \geq N(x)$

($N(x)$ fixed integer)$\Rightarrow$ E-compatibility

Theorem 2 (W. Zheng, 2007). E-compatibility stable under Grothendieck's six operations.

In particular : $\forall g \in G$, $\forall \sigma \in W(\overline{K}/K)$,

Tr$(g\sigma, H_c^*(X_{\overline{K}}, \mathbb{Q}_\ell))$ and Tr$(g\sigma, H^*(X_{\overline{K}}, \mathbb{Q}_\ell)) \in \mathbb{Q}$, independent of ℓ

($\overline{K}$ = alg. closure of K, $W(\overline{K}/K)$ = Weil group)

Remarks

- H_c^* : Ochiai, Vidal
- $\mathrm{Tr}(g\sigma) \in \mathbb{Z}$ if $\sigma \mapsto F^n \in \mathrm{Gal}(\overline{k}/k)$, $n \geq 0$,

F = geometric Frobenius

($\Leftarrow$ generalization (Zheng) of Deligne-Esnault's integrality theorem)

Proof :

same ingredients as for the finite field case, plus :

Theorem 3 (W. Zheng, 2007). E-compatibility stable under $R\Psi$.

Th. 3 used to treat curve case over K

(no analogue of Deligne's Antwerp th. available)

Generalizations to algebraic stacks :

similar to finite field case

3. FREE ACTIONS : VANISHING OF LEFSCHETZ NUMBERS

k alg. closed of char. p, ℓ prime $\neq p$,

X/k separated, finite type

$G =$ finite group acting admissibly on X/k ($\Rightarrow X/G =$ scheme).

Theorem 4. $\forall g \in G$,

Tr$(g, H^*_c(X, \mathbb{Q}_\ell))$ and Tr$(g, H^*(X, \mathbb{Q}_\ell))$

are in $\mathbb{Z}$ and inpendent of ℓ.

Remark : H_c^* : Deligne-Lusztig (1976).

Proof :

spreading out $\Rightarrow$ reduce to $k =$ alg. closure of $k_0 = \mathbb{F}_q$,

$X = X_0 \otimes k$, X_0/k_0 separated, f. t., G acting on X_0

and showing :

$\forall n > 0$, $\forall g \in G$,

$Tr(gF^n, H_c^*(X, \mathbb{Q}_\ell))$ and $\mathrm{Tr}(gF^n, H^*(X, \mathbb{Q}_\ell))$ are in $\mathbb{Q}$

and independent of ℓ

follows from th. 1

(equivalently : Deligne-Lusztig trick $\Rightarrow$: replace gF^n by F^n

H_c^* : Grothendieck trace formula

H^* : Gabber)

Theorem 5.

Assume moreover G acts freely on X/k. Then :

(1) $R\Gamma_c(X, \mathbb{Z}_\ell)$ and $R\Gamma(X, \mathbb{Z}_\ell)$

are perfect complexes of $\mathbb{Z}_\ell[G]$-modules

(2) $\forall g \in G$, order(g) not a power of p,

$\mathrm{Tr}(g, H_c^*(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(g, H^*(X, \mathbb{Q}_\ell)) = 0$.

Proof :

(for $R\Gamma_c$; $R\Gamma$: similar)

(1) standard : Grothendieck (1966) ;

$R\Gamma_c(X, \mathbb{Z}_\ell) = R\Gamma_c(X/G, f_*\mathbb{Z}_\ell)$, $f : X \to X/G$,

and $f_*\mathbb{Z}_\ell$ loc. free rk 1 / $\mathbb{Z}_\ell[G]$

(2) Brauer theory :

P projective, finite type / $\mathbb{Z}_\ell[G] \Rightarrow$

$\mathrm{Tr}(g, P \otimes \mathbb{Q}_\ell) = 0$ for g ℓ-singular

(i. e. $\ell|$order(g)) ;

then apply independence of ℓ (th. 4)

Corollary 1. If moreover $(p, |G|) = 1$, then :

$\chi_c(X, G, \mathbb{Q}_\ell) = \chi_c(X/G, \mathbb{Q}_\ell)\,\mathrm{Reg}_{\mathbb{Q}_\ell}(G)$

(resp. $\chi(X, G, \mathbb{Q}_\ell) = \chi(X/G, \mathbb{Q}_\ell)\,\mathrm{Reg}_{\mathbb{Q}_\ell}(G)$)

in $R_{\mathbb{Q}_\ell}(G)$,

$R_{\mathbb{Q}_\ell}(G)$: Grothendieck gp of finite dim. $\mathbb{Q}_\ell[G]$-modules

$\mathrm{Reg}_{\mathbb{Q}_\ell}(G)$: regular representation

$\chi_c(X, G, \mathbb{Q}_\ell) = \sum(-1)^i[H^i_c(X, \mathbb{Q}_\ell)]$ (resp. ...)

([] = class in Grothendieck gp)

Remarks

- char$(k) = 0$, H_c^* : Verdier (1973)

topological variants (Verdier, K. Brown)

- assumption $(p, |G|) = 1$ can be replaced by

tameness assumption on Galois cover $X \to X/G$

Definition.

$f : X \to Y$ étale Galois cover of Y of group G,

with X, Y normal connected

called tame (relative to k) if :

there exists $\overline{Y}$ = normal compactification of Y/k

s. t. if $\overline{X}$ = normalization of $\overline{Y}$ in X,

$\forall$ p-Sylow P of G,

P acts freely on $\overline{X}$

Corollary 2. With X, $Y = X/G$ as above,
same conclusion as in Cor. 1
assuming only $X \to Y$ tame.

Proof : H_c^* : Deligne (1977) ;
H^* : similar, using independence of ℓ (th. 4)

Remarks :

• $X \to Y$ tame $\Leftrightarrow \mathrm{Im}(\pi_1(Y)_{\mathrm{w}} \to G) = \{1\}$,

$\pi_1(Y)_{\mathrm{w}}$ = Vidal's local (at ∞)) wild part of $\pi_1(Y)$

• $\Leftrightarrow$ $f_*\mathbb{F}_\ell$ tame in Vidal's sense

(virtual local wild ramification of $f_*\mathbb{F}_\ell - |G|$ vanishes)

(Gabber-Vidal)

($\Rightarrow$ definition and Cor. 2 generalize to Y separated, finite type $/k$)

• Kato-Saito (2007) : finer results, involving

Swan class

Serre's congruences

k : field of char. p, $\overline{k}$ = alg. closure

X/k separated, finite type

G = ℓ-group ($\ell \neq p$) acting admissibly, freely on X

Then :

• $\forall \sigma \in \mathrm{Gal}(\overline{k}/k)$: $\mathrm{Tr}(\sigma, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) \equiv 0 \mod |G|$

(Serre, 2005)

• $\forall g \in G$, $g \neq 1$: $\mathrm{Tr}(g, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) = 0$ (th. 5).

Theorem 6.

$\forall g \in G, \forall \sigma \in$ Gal$(\overline{k}/k)$:

Tr$(g\sigma, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) \equiv 0 \mod |Z_G(g)|$,

$Z_G(g)$ = centralizer of g in G.

Proof :

similar to Serre's :

spreading out + (generalized) Chebotarev $\Rightarrow$

reduced to showing : if $k = \mathbb{F}_q$, F = geom. Frobenius,

Tr$(gF, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) \equiv 0 \mod |Z_G(g)|$

follows from

Deligne-Lusztig + Grothendieck trace formula :

$\mathrm{Tr}(gF, H_c^*(X_{\overline{k}}, \mathbb{Q}_\ell)) = |X(\overline{k})^{gF}|$

Question : how about H^* (instead of H_c^*) ?

The p-adic side

k alg. closed field of char. $p > 0$,

K = fraction field of $W(k)$

X/k separated, finite type,

acted on by finite group G

$H^*_{c,rig}(X/K)$ = rigid cohomology with compact supports (Berthelot)

$H^*_{c,rig}(X/K) = H^* R\Gamma_{rig}(X/K)$

$R\Gamma_{c,rig}(X/K) \in D(K[G])$

$R\Gamma_{c,rig}(X/K) \in D^b_c$ (Berthelot's finiteness th.)

If X/k proper, smooth,

$R\Gamma_{c,rig}(X/K) = R\Gamma(X/W) \otimes K$,

$R\Gamma(X/W) \in D^b_c(W[G])$ = crystalline cohomogy complex

Theorem 7.

(1) $\forall g \in G$, $\mathrm{Tr}(g, H^*_{c,rig}(X/K)) \in \mathbb{Z}$

and $\mathrm{Tr}(g, H^*_{c,rig}(X/K)) = \mathrm{Tr}(g, H^*_c(X, \mathbb{Q}_\ell))$ $(\ell \neq p)$.

(2) If X/k proper, smooth, and G acts freely, then

$R\Gamma(X/W)$ = perfect complex of $W[G]$-modules and

$\forall g \in G$, $g \neq 1$, $\mathrm{Tr}(g, H^*_{rig}(X/K)) = 0$

Proof :

(1) use de Jong's equivariant alterations

(as in proof of Zheng's ths 1, 2)

to reduce to (well known)

Lemma. X/k projective, smooth,

$s =$ endomorphism of X, $\ell \neq p$

Then :

$\mathrm{Tr}(s, H^*(X/W) \otimes K)) = \mathrm{Tr}(s, H^*(X, \mathbb{Q}_\ell)) = (\Gamma_s . \Delta)$

($\Gamma_s =$ graph of s, $\Delta =$ diagonal (in $X \times X$))

(2) similar to ℓ-adic case (th. 5), using

$R\Gamma(X/W) \otimes^L k = R\Gamma_{\mathsf{dR}}(X/k)$,

$\mathsf{gr}\, R\Gamma_{\mathsf{dR}}(X/k) = R\Gamma_{\mathsf{Hdg}}(X/k)$

(gr for Hodge filtration)

Remarks

• If G acts freely, but X/k not proper,

in general, there exists no perfect complex P $/W[G]$

s. t. $P \otimes K = R\Gamma_{c,rig}(X/K)$

(e. g. : $s : x \to x + 1$ on $\mathbb{A}^1_k$:

$s^p = 1$ but $\mathrm{Tr}(s) = 1$)

In the proper case : ?

- in th. 7 (1), $\mathrm{Tr}(g, H^*_{c,rig}) = \mathrm{Tr}(g, H^*_{rig})$?

(open question)

- ℓ-adic analogue : OK : next section

4. EQUIVARIANT FORM OF LAUMON'S THEOREM ON EULER CHARACTERISTICS

Setting

as in the beginning : X/k, G, $D_c^b(X, G, \overline{\mathbb{Q}_\ell})$

$K(X, G, \overline{\mathbb{Q}_\ell})$ = Grothendieck gp of $D_c^b(X, G, \overline{\mathbb{Q}_\ell})$

$f : (X, G) \to (Y, H)$: Rf_*, $Rf_!$ induce

$f_*, f_! : K(X, G, \overline{\mathbb{Q}_\ell}) \to K(Y, H, \overline{\mathbb{Q}_\ell})$

(similarly with f^*, $Rf^!$, ...)

Define :

$\tilde{K}(X, G, \overline{\mathbb{Q}_\ell}) = K(X, G, \overline{\mathbb{Q}_\ell}) / < [\overline{\mathbb{Q}_\ell}(1)] - 1 >$

($<, >$ = ideal generated by)

f_*, $f_!$ induce

$f_*, f_! : \tilde{K}(X, G, \overline{\mathbb{Q}_\ell}) \to \tilde{K}(Y, H, \overline{\mathbb{Q}_\ell})$

Theorem 8.

$f_* = f_! : \tilde{K}(X, G, \overline{\mathbb{Q}_\ell}) \to \tilde{K}(Y, H, \overline{\mathbb{Q}_\ell})$

Remark :

$G = H = \{1\}$: Laumon's th.

(char. 0 : Grothendieck,

general case : Gabber : unpublished)

Corollary 1

Assume k alg. closed. Then

$\chi_c(X, G, \mathbb{Q}_\ell) = \chi(X, G, \mathbb{Q}_\ell) \in R_{\mathbb{Q}_\ell}(G)$

i. e. $\forall g \in G, \mathrm{Tr}(g, H_c^*(X, \mathbb{Q}_\ell)) = \mathrm{Tr}(g, H^*(X, \mathbb{Q}_\ell))$.

Remark

multiplicativity of χ by tame covers (Cor. 2 to Th. 5) $\Rightarrow$ for $X \to X/G$ tame,

$\chi(X, G, \mathbb{Q}_\ell) = \sum_{S \in \mathcal{S}} \chi(X_S/G, \mathbb{Q}_\ell) I_S$,

$\mathcal{S}$ = set of conjugacy classes of subgroups of G

$X_H = X^H - \cup_{H' \supset H, H' \neq H} X^{H'}$,

$S \in \mathcal{S}$, $X_S = \cup_{H \in S} X_H$,

I_S= class of $\mathbb{Q}_\ell[G/H]$, $H \in S$

(char$(k) = 0$: Verdier (1973))

Corollary 2.

$j : U \to X$ G-equivariant open immersion,

$i : Y = X - U \to X$

Then : $\forall x \in \tilde{K}(U, G, \overline{\mathbb{Q}_\ell})$,

$i^* j_* x = 0$.

Remark

Assume k alg. closed, X/k proper,

G acts freely on U

Then, for $L \in D^b_c(U, G, \overline{\mathbb{Q}_\ell})$, $g \in G$,

Lefschetz-Verdier trace formula $\Rightarrow$

$\mathrm{Tr}(g, H^*_c(U, L)) = \sum_{Z \in \pi_0(Y^g)} a_Z$,

$\mathrm{Tr}(g, H^*(U, L)) = \sum_{Z \in \pi_0(Y^g)} b_Z$,

a_Z, b_Z : local terms at infinity

Cor. 2 $\Rightarrow$: $a_Z = b_Z \ \forall Z$

Proof of th. 8 :

Imitate Laumon's proof

• reduce to $G = H$, then (equivariant compactification)

reduce to Cor. 2

• reduce to Y = divisor,

then to $Y = V(F)$, F = G-invariant equation

• reduce to equivariant form of

Laumon's lemma :

f : $X \to S$ G-equivariant, S = henselian trait, with trivial action of G

s = closed pt, η = generic pt

$i : X_s \to X$, $j : X_\eta \to X$. Then : $\forall x \in K(X_\eta, G, \overline{\mathbb{Q}_\ell})$, image of $i^* j_* x$ in $\tilde{K}(Y, G, \overline{\mathbb{Q}_\ell}) = 0$

- use nearby cycles :

for $K \in D^b_c(X_\eta, \overline{\mathbb{Q}_\ell})$,

$i^* Rj_* K = R\Gamma(I, R\Psi K) = R\Gamma(\mathbb{Z}_\ell(1), (R\Psi K)^{P_\ell})$,

$I \subset \mathrm{Gal}(\overline{\eta}/\eta) =$ inertia

$0 \to P_\ell \to I \to \mathbb{Z}_\ell(1) \to 0.$

$\Rightarrow$ enough to show :

$\forall L \in D^b_c(X_{\overline{s}}, \mathbb{Z}_\ell(1), G, \overline{\mathbb{Q}_\ell})$,

$[R\Gamma(\mathbb{Z}_\ell(1), L)] = 0$ in $\tilde{K}(X_s, G, \overline{\mathbb{Q}_\ell})$

• reduce to L unipotent,

use monodromy operator

$N : L \to L(-1)$,

$N^i : \mathrm{gr}^M_i L \xrightarrow{\sim} \mathrm{gr}^M_{-i} L(-i)$

(M = monodromy filtration)

Generalization to DM stacks

$\mathcal{X}$: DM stack of finite type $/k$

$D_c^b(\mathcal{X}, \overline{\mathbb{Q}_\ell})$

$K(\mathcal{X}, \overline{\mathbb{Q}_\ell})$, $\tilde{K}(\mathcal{X}, \overline{\mathbb{Q}_\ell})$

$f : \mathcal{X} \to \mathcal{Y}$ gives

Rf_*, $Rf_!$: $D_c^b(\mathcal{X}, \overline{\mathbb{Q}_\ell}) \to D_c^b(\mathcal{Y}, \overline{\mathbb{Q}_\ell})$,

f_*, $f_!$: $K(\mathcal{X}, \overline{\mathbb{Q}_\ell}) \to K(\mathcal{Y}, \overline{\mathbb{Q}_\ell})$

(f_*, $f_!$: $\tilde{K} \to \tilde{K}$)

Theorem 9.

$$f_* = f_! : \tilde{K}(\mathcal{X}, \overline{\mathbb{Q}_\ell}) \to \tilde{K}(\mathcal{Y}, \overline{\mathbb{Q}_\ell})$$

$\mathcal{X} = [X/G]$, $\mathcal{Y} = [Y/H]$, f associated with equivariant $(f, u) : (X, G) \to (Y, H)$: th. 8

Proof of th. 9 :

• for $a \in \tilde{K}(\mathcal{X}, \overline{\mathbb{Q}_\ell})$,

enough to check $f_* a = f_! a$ on stalks

i. e. $i_y^* f_* a = i_y^* f_! a \ \forall y \in \mathcal{Y}$,

$i_y : \mathcal{G}_y \to \mathcal{Y}$: residue gerbe

($\mathcal{G}_y = [\operatorname{Spec} K/G]$ for some finite type extension K/k,

finite group G acting on $\operatorname{Spec} K$)

follows from injectivity of

$\tilde{K}(\mathcal{Y}, \overline{\mathbb{Q}_\ell}) \to \prod_{y \in \mathcal{Y}} \tilde{K}(\mathcal{G}_y, \overline{\mathbb{Q}_\ell})$

• $\mathcal{G}_y \to \mathcal{Y}$ factors through smooth map

$[Y/H] \to \mathcal{Y}$,

H finite gp acting on Y/k affine, finite type

smooth base change $\Rightarrow$

reduce to $\mathcal{Y} = [Y/H]$

• induction on $\dim \mathcal{X} \Rightarrow$ reduce to th. 8

Orbifold Euler characteristics

Deligne-Rapoport (1973) :

k = alg. closed field of char. 0,

$\mathcal{X}/k$ DM-stack of finite type $\mapsto$

$\chi(\mathcal{X})_{orb} \in \mathbb{Q}$,

(orbifold) Euler char. of $\mathcal{X}$

satisfies :

- $\chi(\mathcal{X})_{orb} = \chi(\mathcal{X})$ if $\mathcal{X}$ = scheme

- $\chi(\mathcal{X})_{orb} = \chi(\mathcal{Y})_{orb} + \chi(\mathcal{U})_{orb}$
($\mathcal{Y} \subset \mathcal{X}$ closed, $\mathcal{U} = \mathcal{X} - \mathcal{Y}$)

- $\chi(\mathcal{X})_{orb} = d\ \chi(\mathcal{Y})_{orb}$
for $\mathcal{X}/\mathcal{Y}$ finite étale of degree d.

In particular :

$\chi(BG/k)_{orb} = 1/|G|$,

$\chi([X/G])_{orb} = \chi(X)/|G|$

Example (Harer-Zagier, 1986) :

$$\chi(M_g)_{orb} = \zeta(1-2g)$$

($g = 1$: Deligne-Rapoport)

Question

k alg. closed of char. p,

$\mathcal{X}/k$ DM-stack of finite type

can one define tameness of $\mathcal{X}/k$

and, for ℓ prime $\neq p$, $\mathcal{X}/k$ tame

$\chi(\mathcal{X}, \mathbb{Q}_\ell)_{orb} \in \mathbb{Q}$

(independent of ℓ),

with similar properties ?