Invariant subsets and stationary measures on homogeneous spaces

Yves Benoist - Jean-François Quint

Paris-Sud - Paris-Nord

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Stationary measures

Actions on G/Λ Actions on tori

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1. Actions on G/Λ

Yves Benoist – Jean-François Quint Invariant subsets

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Theorem 1

Let *G* be a simple Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\Gamma \subset G$ a Zariski dense subgroup.

a) Every infinite Γ -invariant subset F of X is dense. b) Every atom free Γ -invariant probability ν on X is G-invariant.

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Let *G* be a simple Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\Gamma \subset G$ a Zariski dense subgroup. There are no closed subgroup $\Gamma \subset H \subsetneq G$ with $\#H/H_e < \infty$

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Let *G* be a simple Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\Gamma \subset G$ a Zariski dense subgroup.
$$\begin{split} & G = \mathrm{PSL}(2,\mathbb{R}) \\ & \Lambda = \mathrm{PSL}(2,\mathbb{Z}) \\ & \Gamma \text{ non elementary} \end{split}$$

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Actions on G/Λ Actions on tori

Theorem 2

Let *G* be a simple Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\mu \in \mathcal{P}(G)$ a Zariski dense probability with compact support.

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Choose $\mu \in \mathcal{P}(G)$ with support $S \subset \Gamma$ generating a Zariski dense subgroup.

b) ν is Γ -invariant $\Rightarrow \nu$ is μ -stationary.

a) Let $x_0 \in F$. Use Kakutani's trick: Any weak sublimit ν_0 of the sequence $\frac{1}{n}(\mu * \delta_{x_0} + \cdots + \mu^{*n} * \delta_{x_0})$ is μ -stationary and supported on \overline{F} .

One has to check that ν_0 is an atom free probability. For that, use ideas of Eskin-Margulis.

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Actions on G/Λ Actions on tori

2. Actions on tori

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Actions on G/Λ Actions on tori

Theorem 3

Let $X = \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{Z}))$ a probability with finite support generating a strongly irreducible subgroup Γ .

i.e. no finite union of vector subspaces of \mathbb{R}^d is Γ -invariant

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Every atom free μ -stationary probability ν on X is Haar.

Theorem 3 is due to Bourgain, Furman, Lindenstrauss,Mozes when Γ contains proximal elements.

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Stationary measures Actions on G/ Strategy Actions on tori

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Furstenberg measure Horocyclic flow Non atomic limit measure

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3. Furstenberg measure

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Furstenberg measure Horocyclic flow Non atomic limit measure

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To simplify, let us assume that

 $X = \mathbb{T}^2$, $\mu = \frac{1}{2}(\delta_{g_0} + \delta_{g_1})$ $S = \{g_0, g_1\}$, $\Gamma = <S > \subset SL(2, \mathbb{R})$. Introduce the Bernoulli dynamical system :

 $B = S^{\mathbb{N}}, \mathcal{B}, \beta = \mu^{\otimes \mathbb{N}}, s : (b_0, b_1, ..) \mapsto (b_1, b_2, ..)$ so that $s_* \beta = \beta$.

Proposition (Furstenberg) For β -a.e. *b* in *B*, a) $\nu_b = \lim_{n} (b_0 \cdots b_n)_* \nu$ exists. b) All sublimits π of $\frac{b_0 \cdots b_n}{\|b_0 \cdots b_n\|}$ have same image: a line $V_b \subset \mathbb{R}^2$.

Main step : For β -a.e. *b*, ν_b is V_b -invariant

Remark $\nu = \int_{B} \nu_b \, \mathrm{d}\beta(b)$

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Stationary measures Strategy Furstenberg measure Horocyclic flow Non atomic limit measure

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4. Horocyclic flow

Yves Benoist – Jean-François Quint Invariant subsets

1. Introduce a second dynamical system :

$$B^X = B imes X, \, \mathcal{B}^X, \, \beta^X = \int_B \delta_b \otimes \nu_b \, \mathrm{d}\beta(b), \, T: (b,x) \mapsto (sb, b_0^{-1}x)$$

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 $B^X = B \times X, \ \beta^X, \ \beta^X = \int_B \delta_b \otimes \nu_b \, \mathrm{d}\beta(b), \ T : (b, x) \mapsto (sb, b_0^{-1}x)$ so that $T_*\beta^X = \beta^X$.

2. Choose $v_b \in V_b$, with $||v_b|| = 1$. Let $C_b \in \mathbb{R}^*$, so that $b_0 v_{sb} = C_b v_b$.

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Proposition (Furstenberg-Kesten) $\int_{B} \log |C_b| \, d\beta(b) > 0$

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3. Introduce the horocyclic flow Φ_t on B^X : $\Phi_t(b, x) = (b, x + t v_b)$.

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5. Introduce the tail σ -algebra $\mathcal{Q}_{\infty} := \bigcap_{n \geq 1} T^{-n} \mathcal{B}^{X}$.

Let us go on as if C_b were a constant $C_b = C_0 > 1$.

3. Introduce the horocyclic flow Φ_t on B^X : $\Phi_t(b, x) = (b, x + t v_b).$

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$$\begin{split} \frac{1}{2^n} \sum_{\mathrm{I}} \psi(b_{\mathrm{I}}, x_{\mathrm{I}}) &= \mathbb{E}(\psi \mid T^{-n}(\mathcal{B}^X))(b, x) \to \mathbb{E}(\psi \mid \mathcal{Q}_{\infty})(b, x) \\ \text{where} \begin{cases} \mathrm{I} &= (i_0, \dots, i_{n-1}) \in \{0, 1\}^n, \\ b_{\mathrm{I}} &= (g_{i_0}, \dots, g_{i_{n-1}}, b_n, \dots), \\ x_{\mathrm{I}} &= g_{i_0} .. g_{i_{n-1}} b_{n-1}^{-1} .. b_0^{-1} x. \end{cases} \end{split}$$

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\mathbf{I} = (i_0, \dots, i_{n-1}) \in \{0, 1\}^n, \\
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- Write $y_{I} = x_{I} + v_{I}$ with

$$v_{I} = g_{i_{0}}..g_{i_{n-1}}b_{n-1}^{-1}..b_{0}^{-1}(y-x) \simeq t v_{b_{I}}.$$

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5. Non atomic limit measure

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To begin the drift argument one needs:

Lemma For β^X -a.e. (b, x), one has $\nu_b(x + V_b) = 0$.

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Lemma For β^X -a.e. (b, x), one has $\nu_b(x + V_b) = 0$.

Main step : There does not exist $\kappa : B \to X \mid \nu_b = \delta_{\kappa(b)}$.

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$$\check{\mu}^{*n} * v \leq a^{n}v + (1 + \dots + a^{n-1})C.$$

$$\frac{1}{\rho} \sum_{0 \leq n < \rho} (\check{\mu}^{*n} * v) \leq \frac{1}{\rho(1-a)}v + \frac{1}{1-a}C.$$

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$$\frac{1}{p} \sum_{0 \leq n < p} (\check{\mu}^{*n} * v) \leq \frac{1}{p(1-a)}v + \frac{1}{1-a}C.$$

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2. $\forall \varepsilon > 0, \exists K_0 \subset B \beta(K_0) = 1 - \varepsilon, \kappa|_{K_0}$ is continuous.

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$$\forall \varepsilon > 0$$
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If ν not Dirac, choose b_0 and b'_0 with $\kappa(b_0) \neq \kappa(b'_0)$. One gets

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When $p \to \infty$, one gets $(1 - 4\varepsilon)M \le C/(1 - a)$. Contradiction.

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Apply Chacon, Ornstein: for β -a.e. b, $\lim_{p\to\infty} \frac{1}{p} \sum_{0 \le n < p} (L^n_{\mu} \mathbf{1}_{K_0})(b) = \beta(K_0) = 1 - \varepsilon.$

If ν not Dirac, choose b_0 and b'_0 with $\kappa(b_0) \neq \kappa(b'_0)$. One gets $\forall M, \exists p_0, \frac{1}{p} \sum_{0 \le n < p} (\check{\mu}^{*n} * \nu)(\kappa(b_0), \kappa(b'_0)) \ge (1 - 4\varepsilon)M - \frac{p_0}{p}$

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$$\forall M, \exists p_0, \quad \frac{1}{p} \sum_{0 \le n < p} (\check{\mu}^{*n} * v)(\kappa(b_0), \kappa(b'_0)) \ge (1 - 4\varepsilon)M - \frac{p_0}{p}$$

When $p \to \infty$, one gets $(1 - 4\varepsilon)M \le C/(1 - a)$. Contradiction.

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Stationary measures Strategy Non atomic limit measure

When $p \to \infty$, one gets $(1 - 4\varepsilon)M \le C/(1 - a)$. Contradiction.

Yves Benoist – Jean-François Quint Invariant subsets