Invariant subsets and stationary measures on homogeneous spaces

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1. Actions on G/Λ

Theorem 1

Let G be a simple Lie group,
$\Lambda \subset G$ a lattice, $X = G/\Lambda$,
$\Gamma \subset G$ a Zariski dense subgroup.

a) Every infinite Γ-invariant subset F of X is dense.
b) Every atom free Γ-invariant probability ν on X is G-invariant.

i.e. $\nu(\{x\}) = 0, \ \forall x \in X$

Corollary (Eskin, Margulis)
Every discrete Γ-orbit in X is finite.

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$\mathrm{vol}(X) < \infty$

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There are no closed subgroup $\Gamma \subset H \subsetneq G$ with $\#H/H_e < \infty$

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Why does Theorem 2 implies Theorem 1?

Choose $\mu \in \mathcal{P}(G)$ with support $S \subset \Gamma$ generating a Zariski dense subgroup.

b) ν is Γ-invariant $\Rightarrow$ ν is μ-stationary.

a) Let $x_0 \in F$. Use Kakutani's trick: Any weak sublimit ν_0 of the sequence $\frac{1}{n}(\mu * \delta_{x_0} + \cdots + \mu^{*n} * \delta_{x_0})$ is μ-stationary and supported on $\overline{F}$.

One has to check that ν_0 is an atom free probability. For that, use ideas of Eskin-Margulis.

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2. Actions on tori

Theorem 3

Let $X = \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathrm{SL}(d, \mathbb{Z}))$ a probability with finite support generating a strongly irreducible subgroup Γ.

i.e. no finite union of vector subspaces of $\mathbb{R}^d$ is Γ-invariant

Every atom free μ-stationary probability ν on X is Haar.

Theorem 3 is due to Bourgain, Furman, Lindenstrauss,Mozes when Γ contains proximal elements.

Corollary (Muchnik and Guivarc'h, Starkov)
Every infinite Γ-invariant subset F of X is dense.

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3. Furstenberg measure

To simplify, let us assume that

$X = \mathbb{T}^2$, $\mu = \frac{1}{2}(\delta_{g_0} + \delta_{g_1})$ $S = \{g_0, g_1\}$, $\Gamma = < S > \subset \mathrm{SL}(2, \mathbb{R})$.

Introduce the Bernoulli dynamical system :

$B = S^{\mathbb{N}}$, $\mathcal{B}$, $\beta = \mu^{\otimes \mathbb{N}}$, $s : (b_0, b_1, ..) \mapsto (b_1, b_2, ..)$
so that $s_* \beta = \beta$.

Proposition (Furstenberg) For β-a.e. b in B,
a) $\nu_b = \lim_n (b_0 \cdots b_n)_* \nu$ exists.
b) All sublimits π of $\frac{b_0 \cdots b_n}{\|b_0 \cdots b_n\|}$ have same image: a line $V_b \subset \mathbb{R}^2$.

Main step : For β-a.e. b, ν_b is V_b-invariant

Remark $\nu = \int_B \nu_b \, \mathrm{d}\beta(b)$

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4. Horocyclic flow

1. Introduce a second dynamical system :

$$B^X = B \times X,\ \mathcal{B}^X,\ \beta^X = \int_B \delta_b \otimes \nu_b \,\mathrm{d}\beta(b),\ T : (b, x) \mapsto (sb, b_0^{-1} x)$$

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$\int_B \log |C_b| \, \mathrm{d}\beta(b) > 0$

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Proof Since
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6. One wants to find $t = t_{(b,x)}$ small such that a.e.

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5. Non atomic limit measure

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