Recurrence on homogeneous spaces

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1. Dynamics on G/Λ

Topological Theorem

Let G be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$ and $\Gamma \subset G$ a subgroup.
Let $\mathfrak{g} := \mathrm{Lie}(G)$ and $H_\Gamma \subset \mathrm{GL}(\mathfrak{g})$ be the Zariski closure of $\mathrm{Ad}\Gamma$.

If H_Γ is connected semisimple with no compact factor, every Γ-orbit closure $F = \overline{\Gamma x_0}$ in X is homogeneous.

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$\mathrm{vol}(X) < \infty$.

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The smallest algebraic subgroup containing $\mathrm{Ad}\Gamma$.

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Ex. 1 : $G = \mathrm{SL}(2,\mathbb{R})$, $\Lambda = \mathrm{SL}(2,\mathbb{Z})$, Γ non elementary.

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Ex. 2 : $G = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, $\Lambda = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, $H_\Gamma = \mathrm{SL}(2,\mathbb{R})$.

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A similar result is true for p-adic groups.

Measure Theorem

Let G be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\mu \in \mathcal{P}(G)$ a probability with compact support.
Set Γ_μ to be the subgroup generated by $\mathrm{Supp}(\mu)$ and $H := H_{\Gamma_\mu}$.

If H is connected semisimple with no compact factor, every μ-ergodic μ-stationary probability ν on X is homogeneous.

i.e. $\mu * \nu = \nu$ where $\mu * \nu = \int_G g_* \nu \, \mathrm{d}\mu(g)$
i.e. The stabilizer of ν in G acts transitively on $\mathrm{Supp}(\nu)$.

Corollary Every Γ-ergodic Γ-invariant probability ν on X is homogeneous.

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$\boxed{\text{Example}}$ Let $X = \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{Z}))$ a probability with finite support generating a subgroup Γ_μ whose Zariski closure is connected semisimple with no compact factor.

Corollary Every μ-ergodic μ-stationary probability ν on X is a finite sum of Haar measures on affine subtori.

The above corollary is due to Bourgain, Furman, Lindenstrauss, Mozes when Γ is irreducible and contains proximal elements.

Corollary (Muchnik and Guivarc'h, Starkov)
Every Γ-orbit closure in X is a finite union of affine subtori.

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$\boxed{\text{Why does Measure Theorem implies Topological Theorem?}}$

Choose $\mu \in \mathcal{P}(G)$ with support S generating Γ.
Let $x_0 \in X$. Use Kakutani's trick:
Any weak sublimit ν_0 of the sequence $\frac{1}{n}(\mu * \delta_{x_0} + \cdots + \mu^{*n} * \delta_{x_0})$
is μ-stationary and supported on $\overline{\Gamma x_0}$.
One has to check that ν_0 is a probability.
For that, one has to use and to extend ideas of Eskin-Margulis that I will explain now.

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2. μ-recurrence

Recurence Theorem

Let G be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$.
Let $\mu \in \mathcal{P}(G)$ be a probability with an exponential moment.
i.e. $\int_G \|\mathrm{Ad} g\|^\delta \, \mathrm{d}\mu(g) < \infty$ for some $\delta > 0$.

Definition X is μ-recurrent if

$$\forall \varepsilon > 0, \forall x_0 \in X, \exists K \subset X \text{ compact}, \exists n_0 \geq 1, \forall n \geq n_0,$$

$$\mu^{*n} * \delta_{x_0}(K) \geq 1 - \varepsilon.$$

If H is semisimple, then X is μ-recurrent.

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Foster recurrence criterion

Assume that, for all $x_0 \in X$, there exists $f : X \to [0, \infty]$ such that
(i) $f(x_0) < \infty$,
(ii) f is proper,
(iii) $A_\mu f \leq a f + b$, for some $a < 1$, $b > 0$,
then X is μ-recurrent.

where $A_\mu f(x) = \int_G f(gx)\, \mathrm{d}\mu(g)$

We have now to construct f. We will assume that

$$\boxed{G = \mathrm{SL}(d, \mathbb{R}) \text{ and } \Lambda = \mathrm{SL}(d, \mathbb{Z})}$$

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Furstenberg positivity of the first Lyapounov

Let $\mu \in \mathcal{P}(\mathrm{SL}(m, \mathbb{R}))$ be a probability with an exponential moment.
Let $\varphi : \mathbb{R}^d \to [0, \infty); v \mapsto \varphi(v) = \|v\|$.

If Γ_μ is unbounded and acts irreducibly on $\mathbb{R}^m$,
then there exist $n_0 \geq 1$, $\delta > 0$ and $a < 1$ such that

$$A_\mu^{n_0} \varphi^{-\delta} \leq a \varphi^{-\delta}.$$

We will assume $n_0 = 1$.

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3. Construction of f

$\boxed{d = 2 \text{ and } H = \mathrm{SL}(2, \mathbb{R})}$

The elements $x \in X$ are lattices of covolume 1 in $\mathbb{R}^2$.

$\alpha(x) = \inf\{\|v\| \mid v \in x \smallsetminus \{0\}\}$
$f(x) = \alpha(x)^{-\delta}$

Key fact The lattices x never contain two non-colinear vectors of norm at most 1.

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$d \geq 2$ and $H = \mathrm{SL}(d, \mathbb{R})$

$\alpha_i(x) = \inf\{\|v\| \mid v \in \Lambda^i x \smallsetminus \{0\}$ pure tensor $\}$
$f(x) = \sum_i \varepsilon_0^{(d-i)i} \alpha_i(x)^{-\delta}$, for δ and ε_0 small.

i.e. $v = v_1 \wedge \cdots \wedge v_i$, with $v_j \in x$.

Key fact: if x contains two non-colinear vectors of small norm, $\Lambda^2 x$ will contain a pure tensor of much smaller norm.

Key inequality: $\forall u, v, w \in \Lambda^*(\mathbb{R}^d)$, pure tensors,

$$\|u\| \, \|u \wedge v \wedge w\| \leq \|u \wedge v\| \, \|u \wedge w\|.$$

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$\boxed{d = 3 \text{ and } H = \text{upper left } \mathrm{SL}(2, \mathbb{R})}$

One has $V = V_+ \oplus V_0 = \mathbb{R}^2 \oplus \mathbb{R}$,
$\Lambda^2 V = V^* = V_+^* \oplus V_0^* = \mathbb{R}^2 \oplus \mathbb{R}$,
write $v = v_+ + v_0$, for $v \in V$ or $\Lambda^2 V$.

$\alpha(x) = \inf\{\|v_+\| \mid v \in x \text{ or } \Lambda^2 x\ ,\ v \neq 0,\ \|v_0\| < \varepsilon_0\}$
$f(x) = \alpha(x)^{-\delta}$

$\boxed{\text{Key inequality}}$: For all v, w in V,
$\|(v \wedge w)_+\| \leq \|v_0\|\,\|w_+\| + \|v_+\|\,\|w_0\|$

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write $v = v_+ + v_0$, for $v \in V$ or $\Lambda^2 V$.

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$\boxed{\text{Key inequality}}$: For all v, w in V,
$\|(v \wedge w)_+\| \leq \|v_0\| \, \|w_+\| + \|v_+\| \, \|w_0\|$

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$d \geq 2$ and H semisimple

Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{h}$, and H_0 be an element in the interior of a Weyl chamber $\mathfrak{a}^+$.
For $\lambda \in \mathfrak{a}^*$, let q_λ be the projector on the sum of the irreducible representation of H of highest weight λ.

$$\varphi(v) = \max_{\lambda \neq 0} \varepsilon_0^{(d-i)i/\lambda(H_0)} \|q_\lambda(v)\|^{1/\lambda(H_0)}$$

$\alpha(x) = \inf\{\varphi(v) \mid v \in \Lambda^* x \smallsetminus \{0\}$ pure tensor with $\|q_0(v)\| < \varepsilon_0\}$
$f(x) = \alpha(x)^{-\delta}$, for δ and ε_0 small.

Main inequality: $\forall \lambda, \mu \in \mathfrak{a}^*$, for $u, v, w \in \Lambda^*(R^d)$ pure tensors,

$$\|q_\lambda(u)\| \, \|q_\mu(u \wedge v \wedge w)\| = O\Big(\sum_{\nu+\rho \geq \lambda+\mu} \|q_\nu(u \wedge v)\| \, \|q_\rho(u \wedge w)\|\Big).$$

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$$\psi : \Lambda^{r+s} V \otimes \Lambda^{r+t} V \to \Lambda^r V \otimes \Lambda^{r+s+t} V$$

such that for all pure tensors $u \in \Lambda^r V$, $v \in \Lambda^s V$, $w \in \Lambda^t V$,

$$\psi((u \wedge v) \otimes (u \wedge w)) = u \otimes (u \wedge v \wedge w).$$

Step 2 Let E be a representation of H. One has, for $x, y \in E$,

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Exercise: a special case of Main Inequality

If $V = V_0 \oplus V_+$, then for all vectors u, v, w in V,

$$\|u_0\| \, \|u_0 \wedge v_+ \wedge w_+ + u_+ \wedge v_0 \wedge w_+ + u_+ \wedge v_+ \wedge w_0\|$$

$$= O(\|u_0 \wedge v_+ + u_+ \wedge v_0\| \, \|u_0 \wedge w_+ + u_+ \wedge w_0\|)$$