Recurrence on homogeneous spaces

Yves Benoist - Jean-François Quint

Paris-Sud - Paris-Nord

Orbit closure on G/Λ Stationary measures on G/Λ Actions on tori Strategy

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1. Dynamics on G/Λ

Yves Benoist – Jean-François Quint Recurrence

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Topological Theorem

Let *G* be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$ and $\Gamma \subset G$ a subgroup. Let $\mathfrak{g} := \operatorname{Lie}(G)$ and $H_{\Gamma} \subset \operatorname{GL}(\mathfrak{g})$ be the Zariski closure of Ad Γ .

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Ex. 2 : $G = SL(2, \mathbb{R}) \ltimes \mathbb{R}^2$, $\Lambda = SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$, $H_{\Gamma} = SL(2, \mathbb{R})$.

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If H_{Γ} is connected semisimple with no compact factor, every Γ -orbit closure $F = \overline{\Gamma x_0}$ in X is homogeneous.

i.e. The stabilizer of *F* in *G* acts transitively on *F*.

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A similar result is true for *p*-adic groups.

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Measure Theorem

Let *G* be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$, $\mu \in \mathcal{P}(G)$ a probability with compact support. Set Γ_{μ} to be the subgroup generated by $\text{Supp}(\mu)$ and $H := H_{\Gamma_{\mu}}$.

If *H* is connected semisimple with no compact factor, every μ -ergodic μ -stationary probability ν on *X* is homogeneous.

i.e. $\mu * \nu = \nu$ where $\mu * \nu = \int_G g_* \nu d\mu(g)$ i.e. The stabilizer of ν in *G* acts transitively on Supp(ν).

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Example Let $X = \mathbb{T}^d$, $\mu \in \mathcal{P}(\mathrm{SL}(d,\mathbb{Z}))$ a probability with finite support generating a subgroup Γ_{μ} whose Zariski closure is connected semisimple with no compact factor.

Corollary Every μ -ergodic μ -stationary probability ν on X is a finite sum of Haar measures on affine subtori.

The above corollary is due to Bourgain, Furman, Lindenstrauss, Mozes when Γ is irreducible and contains proximal elements.

Corollary (Muchnik and Guivarc'h, Starkov) Every Γ-orbit closure in *X* is a finite union of affine subtori.

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Why does Measure Theorem implies Topological Theorem?

Choose $\mu \in \mathcal{P}(G)$ with support *S* generating Γ . Let $x_0 \in X$. Use Kakutani's trick: Any weak sublimit ν_0 of the sequence $\frac{1}{n}(\mu * \delta_{x_0} + \dots + \mu^{*n} * \delta_{x_0})$ is μ -stationary and supported on $\overline{\Gamma x_0}$. One has to check that ν_0 is a probability. For that, one has to use and to extend ideas of Eskin-Margulis that I will explain now.

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Recurrence Foster criterion Furstenberg positivity

2. μ -recurrence

Yves Benoist – Jean-François Quint Recurrence

Recurrence Foster criterion Furstenberg positivity

Recurence Theorem

Let *G* be a real Lie group, $\Lambda \subset G$ a lattice, $X = G/\Lambda$. Let $\mu \in \mathcal{P}(G)$ be a probability with an exponential moment. i.e. $\int_{G} ||Adg||^{\delta} d\mu(g) < \infty$ for some $\delta > 0$. Definition *X* is μ -recurrent if $\forall \varepsilon > 0, \forall x_0 \in X, \exists K \subset X \text{ compact}, \exists n_0 \ge 1, \forall n \ge n_0,$ $\mu^{*n} * \delta_{x_0}(K) \ge 1 - \varepsilon.$

If *H* is semisimple, then *X* is μ -recurrent.

This theorem is due to Eskin-Margulis when Γ is not included in a parabolic subgroup of *G*.

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Recurrence Foster criterion Furstenberg positivity

Recurence Theorem

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Recurrence Foster criterion Furstenberg positivity

Foster recurrence criterion

Assume that, for all $x_0 \in X$, there exists $f : X \to [0, \infty]$ such that (*i*) $f(x_0) < \infty$, (*ii*) *f* is proper, (*iii*) $A_\mu f \le af + b$, for some a < 1, b > 0, then *X* is μ -recurrent.

where $A_{\mu}f(x) = \int_G f(gx) \, \mathrm{d}\mu(g)$

We have now to construct f. We will assume that

 $G = \operatorname{SL}(d, \mathbb{R})$ and $\Lambda = \operatorname{SL}(d, \mathbb{Z})$

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so that X is the set of lattices x of $V:=\mathbb{R}^d$ of covolume 1.

Recurrence Foster criterion Furstenberg positivity

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Assume that, for all $x_0 \in X$, there exists $f : X \to [0, \infty]$ such that (*i*) $f(x_0) < \infty$, (*ii*) *f* is proper, i.e. $f^{-1}([0, L])$ is compact, for all $L < \infty$, (*iii*) $A_{\mu}f \le af + b$, for some a < 1, b > 0, then X is μ -recurrent.

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Furstenberg positivity of the first Lyapounov

Let $\mu \in \mathcal{P}(SL(m, \mathbb{R}))$ be a probability with an exponential moment. Let $\varphi : \mathbb{R}^d \to [0, \infty); v \mapsto \varphi(v) = ||v||.$

If Γ_{μ} is unbounded and acts irreducibly on \mathbb{R}^{m} , then there exist $n_{0} \geq 1$, $\delta > 0$ and a < 1 such that

 $A^{n_0}_{\mu}\varphi^{-\delta} \leq \boldsymbol{a}\varphi^{-\delta}.$

We will assume $n_0 = 1$.

Recurrence Foster criterion Furstenberg positivity

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 $\begin{array}{l} \sigma = 2, H = SL(2)\\ \text{Dynamics on } G/\Lambda \\ \text{Recurrence} \\ \textbf{Construction of } f. \\ \end{array} \begin{array}{l} d \geq 2, H = SL(2)\\ d \geq 3, H = SL(2)\\ d \geq 2, H \text{ semisimple}\\ \text{Main Inequality} \end{array}$

3. Construction of *f*

Yves Benoist – Jean-François Quint Recurrence

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Dynamics on G/Λ d = 2, H = SL(2)Dynamics on G/Λ $d \ge 2, H = SL(d)$ Recurrenced = 3, H = SL(2)Construction of f. $d \ge 2, H$ semisimple
Main Inequality

$$d = 2$$
 and $H = SL(2, \mathbb{R})$

The elements $x \in X$ are lattices of covolume 1 in \mathbb{R}^2 .

$$\alpha(x) = \inf\{ \|v\| \mid v \in x \setminus \{0\} \}$$

$$f(x) = \alpha(x)^{-\delta}$$

Key fact The lattices *x* never contain two non-colinear vectors of norm at most 1.

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The elements $x \in X$ are lattices of covolume 1 in \mathbb{R}^2 .

$$\alpha(\mathbf{x}) = \inf\{\|\mathbf{v}\| \mid \mathbf{v} \in \mathbf{x} \setminus \{\mathbf{0}\}\}\$$

$$f(\mathbf{x}) = \alpha(\mathbf{x})^{-\delta}$$

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Dynamics on G/Λ Recurrence $d \ge$ Construction of f. $d \ge$ Mair

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$$\begin{split} \alpha_i(x) &= \inf\{ \|v\| \mid v \in \Lambda^i x \smallsetminus \{0\} \text{ pure tensor } \}\\ f(x) &= \sum_i \varepsilon_0^{(d-i)^i} \alpha_i(x)^{-\delta}, \quad \text{ for } \delta \text{ and } \varepsilon_0 \text{ small.} \end{split}$$

i.e. $v = v_1 \land \cdots \land v_i$, with $v_j \in x$.

Key fact : if x contains two non-colinear vectors of small norm, $\Lambda^2 x$ will contain a pure tensor of much smaller norm.

Key inequality : $\forall u, v, w \in \Lambda^*(\mathbb{R}^d)$, pure tensors,

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$$d = 3$$
 and $H =$ upper left SL(2, \mathbb{R})

One has
$$V = V_+ \oplus V_0 = \mathbb{R}^2 \oplus \mathbb{R}$$
,
 $\Lambda^2 V = V^* = V^*_+ \oplus V^*_0 = \mathbb{R}^2 \oplus \mathbb{R}$,
write $v = v_+ + v_0$, for $v \in V$ or $\Lambda^2 V$.

 $\begin{aligned} \alpha(x) &= \inf\{\|v_+\| \mid v \in x \text{ or } \Lambda^2 x, v \neq 0, \|v_0\| < \varepsilon_0\} \\ f(x) &= \alpha(x)^{-\delta} \end{aligned}$

Key inequality: For all v, w in V, $\|(v \land w)_+\| \le \|v_0\| \|w_+\| + \|v_+\| \|w_0\|$

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$d \ge 2$ and H semisimple

Let \mathfrak{a} be a Cartan subspace of \mathfrak{h} , and H_0 be an element in the interior of a Weyl chamber \mathfrak{a}^+ .

For $\lambda \in \mathfrak{a}^*$, let q_{λ} be the projector on the sum of the irreducible representation of *H* of highest weight λ .

$$\begin{split} \varphi(v) &= \max_{\lambda \neq 0} \varepsilon_0^{(d-i)i/\lambda(H_0)} \| q_\lambda(v) \|^{1/\lambda(H_0)} \\ \alpha(x) &= \inf\{\varphi(v) \mid v \in \Lambda^* x \smallsetminus \{0\} \text{ pure tensor with } \| q_0(v) \| < \varepsilon_0\} \\ f(x) &= \alpha(x)^{-\delta}, \qquad \text{for } \delta \text{ and } \varepsilon_0 \text{ small.} \end{split}$$

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Proof of Main Inequality : For $u, v, w \in \Lambda^*(\mathbb{R}^d)$ pure tensors,

$$\|q_\lambda(u)\|\,\|q_\mu(u\wedge v\wedge w)\|=O(\sum_{
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Step 1 There exists a GL(V)-equivariant linear map

 $\psi: \Lambda^{r+s} V \otimes \Lambda^{r+t} V \to \Lambda^r V \otimes \Lambda^{r+s+t} V$

such that for all pure tensors $u \in \Lambda^r V$, $v \in \Lambda^s V$, $w \in \Lambda^t V$,

 $\psi((u \wedge v) \otimes (u \wedge w)) = u \otimes (u \wedge v \wedge w).$

Step 2 Let *E* be a representation of *H*. One has, for $x, y \in E$,

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Exercise: a special case of Main Inequality

If $V = V_0 \oplus V_+$, then for all vectors u, v, w in V,

 $\|u_0\| \|u_0 \wedge v_+ \wedge w_+ + u_+ \wedge v_0 \wedge w_+ + u_+ \wedge v_+ \wedge w_0\|$

 $= O(\|u_0 \wedge v_+ + u_+ \wedge v_0\| \|u_0 \wedge w_+ + u_+ \wedge w_0\|)$

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