Young Geometric Group Theory Meeting

Discrete subgroups of Lie groups
and divisible convex sets

Lecture 1: A survey of results

$\boxed{0}$ $\boxed{\text{Definition}}$ **An open subset $\Omega \subset \mathbb{P}^m(\mathbb{R})$ is**

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$\star$ **<u>strictly convex</u> if moreover $\partial\Omega$ does not contain any segment,**

$\star$ **<u>divisible</u> if there exists a discrete subgroup $\Gamma \subset G := \mathrm{SL}(m+1, \mathbb{R})$ acting properly cocompactly on Ω.**

Example: let $q(x) = x_0^2 - x_1^2 - \cdots - x_m^2$ **and**

$\mathbb{H}^m := \{\ell \in \mathbb{P}^m(\mathbb{R}) \ / \ q > 0$ **on** $\ell\}$**.**

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Siegel's construction is arithmetic:
$\Gamma := O^+(q', \mathbb{Z}[\sqrt{d}])$ **where** d **is not a square and** $q'(x_0, \cdots, x_m) := \sqrt{d}\, x_0^2 - x_1^2 - \cdots - x_m^2$**.**

Here are two other easy examples:

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$\boxed{\text{Theorem}}$ **(Borel 1965) Every symmetric convex open set Ω is divisible.**

$\boxed{\textbf{Question}}$ **: For $m \geq 2$, do there exist discrete subgroups $\Gamma \subset \mathrm{SL}(m+1, \mathbb{R})$ dividing a strictly convex open subset $\Omega \not\simeq \mathbb{H}^m$ of $\mathbb{P}^m(\mathbb{R})$?**

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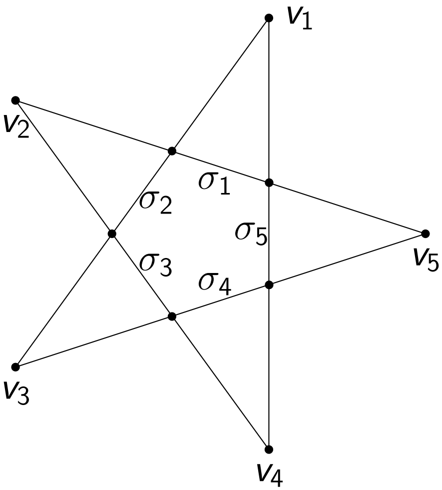
$\boxed{\text{Answer 1}}$ **(Kac-Vinberg, 1970): YES, in small dimension m=2,3,4..., thanks to Coxeter groups.**

$\boxed{\text{Answer 2}}$ **(Koszul, Johnson-Millson, 1970-80) YES, for all m, thanks to a deformation process starting with a tiling of $\mathbb{H}^m$.**

Tits-Vinberg Construction with Coxeter groups



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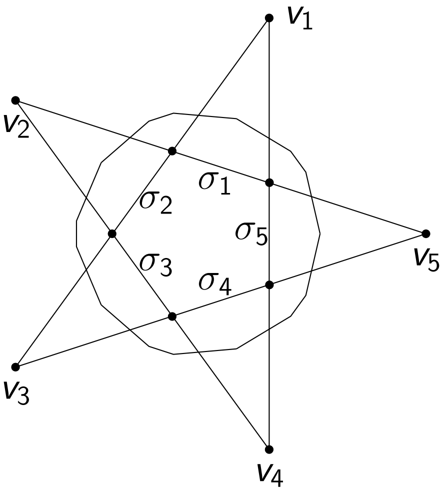

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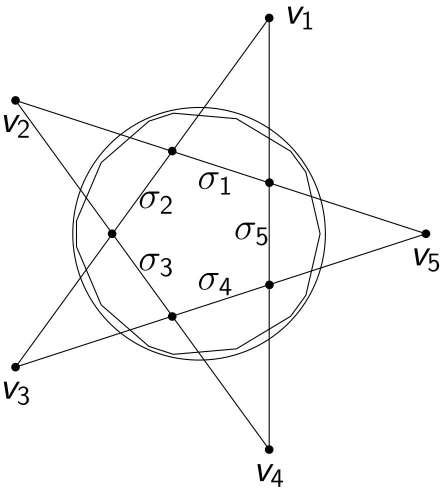

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$\boxed{\text{Corollary}}$ If Ω is strictly convex, then the geodesic flow on $M := \Gamma\backslash\Omega$ is Anosov.
$\boxed{\text{Corollary}}$ If Ω is strictly convex, then $\partial\Omega$ has a $C^{1+\text{Holder}}$ regularity but its curvature is concentrated on a subset of measure zero.

$\boxed{2}$ **Let $\Gamma \subset \mathrm{SL}(m+1, \mathbb{R})$ dividing a properly convex open subset Ω of $\mathbb{P}^m(\mathbb{R})$.**

$\boxed{\text{Theorem}}$ **(Vey 1970) The action of Γ on $\mathbb{R}^{m+1}$ is semisimple.**

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$\boxed{\textbf{Theorem}}$ **(Vey 1970) The action of Γ on $\mathbb{R}^{m+1}$ is semisimple.**

$\boxed{\textbf{Theorem 2}}$ **If Ω is indecomposable and non symmetric, Γ is Zariski dense in $\mathrm{SL}(m+1,\mathbb{R})$.**

$\boxed{3}$ **Let Γ_0 be a group, $G := \mathrm{SL}(m+1, \mathbb{R})$ and let $\mathcal{E}_{\Gamma_0}$ be the space of "discrete faithful morphisms $\rho : \Gamma_0 \longrightarrow G$ dividing some properly convex open subset Ω_ρ in $\mathbb{P}^m(\mathbb{R})$"**

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Hence $\mathcal{E}_{\Gamma_0}$ is a union of components of Hom(Γ_0, G).

Theorem **(Goldman 1990) Let $\Gamma_0 = \pi_1(\Sigma_g)$.**
If $g = 1$, Ω is a triangle.
If $g \geq 2$, $\mathcal{E}_{\Gamma_0}/G \simeq \mathbb{R}^{16(g-1)}$.