Young Geometric Group Theory Meeting

Discrete subgroups of Lie groups

and divisible convex sets

Lecture 4: Dimension 3

0 **Definition** **An open subset $\Omega \subset \mathbb{P}^m(\mathbb{R})$ is**

$\star$ **<u>properly convex</u> if it is convex and bounded in some affine chart,**

$\star$ **<u>strictly convex</u> if moreover $\partial\Omega$ does not contain any segment,**

$\star$ **<u>divisible</u> if there exists a discrete subgroup $\Gamma \subset G := \mathrm{SL}(m+1, \mathbb{R})$ acting properly cocompactly on Ω.**

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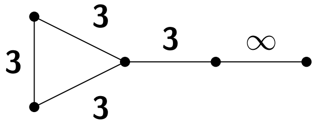

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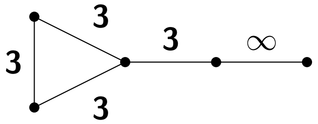

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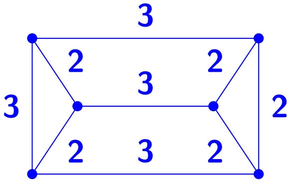

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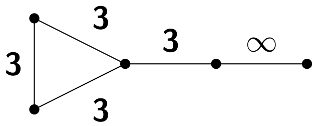

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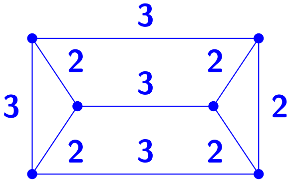

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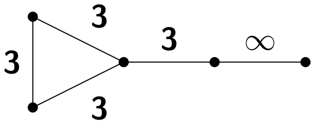

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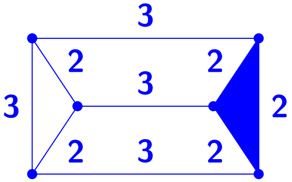

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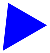

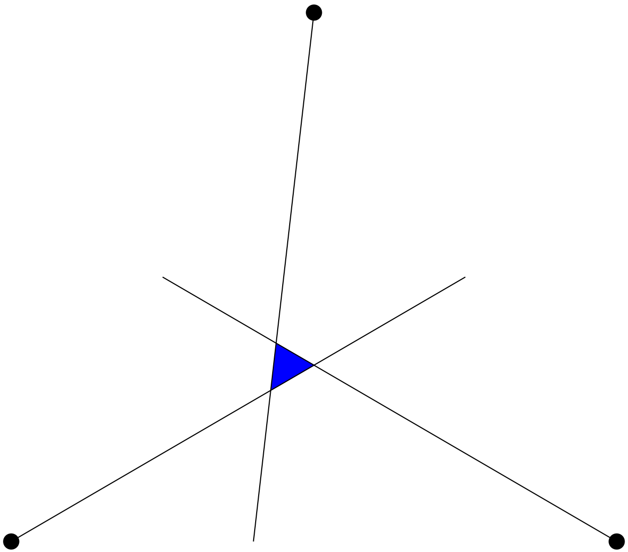

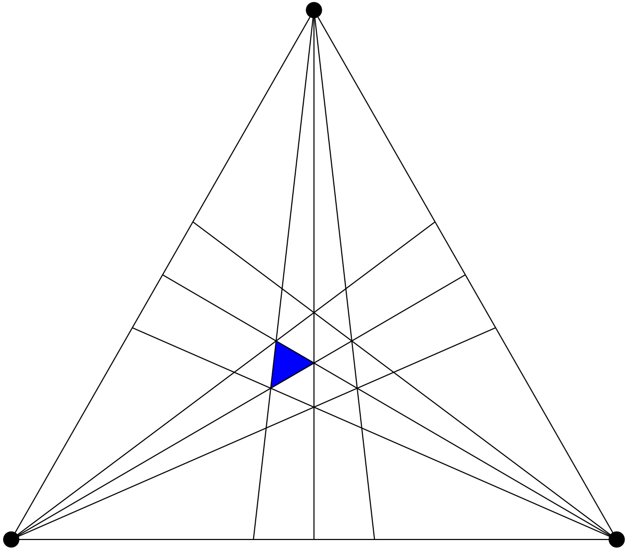

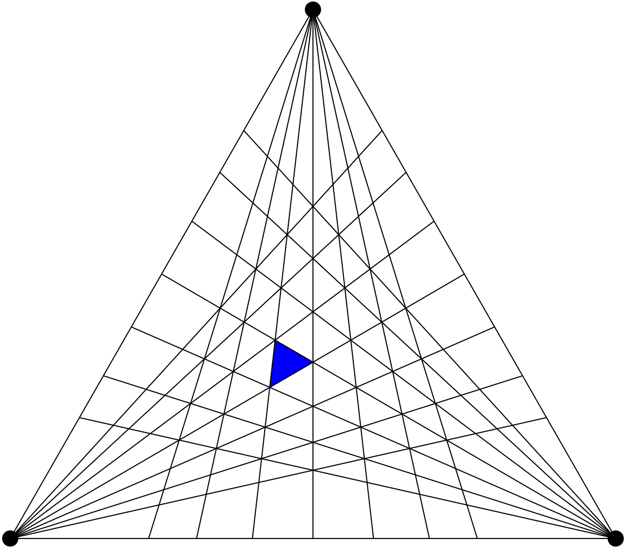

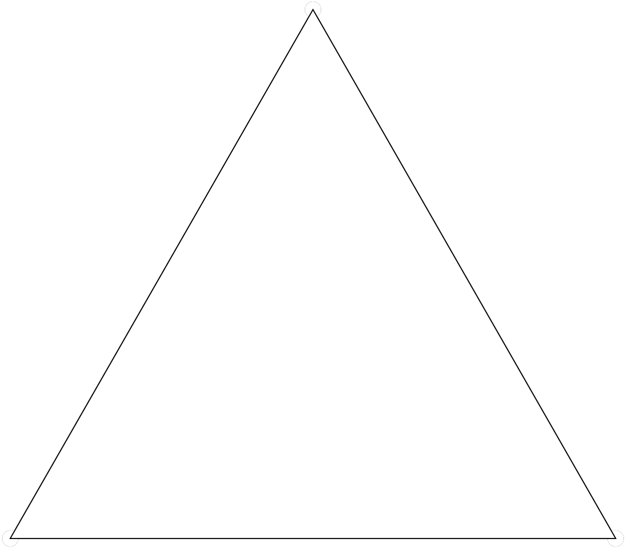

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Theorem 4 **A) The union of PETs projects in $M := \Gamma \backslash \Omega$ onto a finite union of disjoint tori and Klein bottles.**
B) Conversely, every $\mathbb{Z}^2$ subgroup of Γ stabilizes a unique PET.
C) Every segment in $\partial\Omega$ is on the boundary of a unique PET. If Ω is not strictly convex, the vertices of these triangles are dense in $\partial\Omega$.

Proof of Theorem 4

Recall **(Benzecri)**
$X = \{$ **properly convex open set in** $\mathbb{P}^n(\mathbb{R})\}$**,** $G = \mathrm{SL}(m+1, \mathbb{R})$ **and** $\Omega \in X$ **is divisible. Then the** G**-orbit of** Ω **in** X **is closed.**

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Corollary a) Ω non strictly convex $\Longrightarrow$ Ω contains a PET.

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Corollary **a)** Ω **non strictly convex** $\Longrightarrow \Omega$ **contains a PET.**
b) Ω **indecomposable** $\Longrightarrow \partial\Omega$ **does not contain open flat subsets.**

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Step 6 **$\mathcal{L}$ has compact leaves.**