Random walk on p-adic flag varieties

Yves Benoist
joint work with J.F. Quint

Paris, March 24^{nd} 2014

Abstract

We will describe equidistribution results for random walks on projective spaces when the linear action is semisimple.

The main question will be:

Which algebraic homogeneous spaces support stationary probability measures?

1. **Stationary measures**
2. **Random trajectories**
3. **Homogeneous spaces**
4. **Flag varieties**

1. Stationary measures

Notation

Let $k = \mathbb{R}$ **or** $\mathbb{Q}_p$**,** $V = k^d$**,** $X = \mathbb{P}(V)$**,**
$G \subset SL(V)$ **connected semisimple algebraic group,**
μ **probability measure on** G**,**
$\Gamma \subset G$ **the semigroup spanned by the support of** μ**.**

We assume that Γ **is Zariski dense in** G**,**
i.e. G **is the smallest algebraic group containing** Γ**.**

Definition

A probability ν on X is

- **μ-stationary if $\nu = \mu * \nu := \int_G g_* \nu \, d\mu(g)$,**
- **ergodic it it is extremal among μ-stationary.**

A compact subset $F \subset X$ is

- **Γ-invariant if, for all g in Γ, $gF \subset F$,**
- **minimal if it contains no Γ-invariant closed subset.**

Theorem 1

A. The maps $\nu \mapsto F := supp\nu$ is a bijection between

$$\left\{ \begin{array}{l} \textbf{ergodic } \mu\textbf{-stationary} \\ \textbf{probability } \nu \textbf{ on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textbf{minimal } \Gamma\textbf{-invariant} \\ \textbf{compact } F \textbf{ in } X \end{array} \right\}.$$

B. When $k = \mathbb{R}$, the map $F \mapsto O := G F$ is a bijection

$$\left\{ \begin{array}{l} \textbf{minimal } \Gamma\textbf{-invariant} \\ \textbf{compact } F \textbf{ in } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textbf{compact } G\textbf{-orbit} \\ O \textbf{ in } X \end{array} \right\}.$$

What is easy?

- *supp*ν **is** Γ**-invariant: by definition.**
- **surjectivity: for** x **in** F**, a weak limit measure of** $\frac{1}{n}\sum_{\ell=1}^{n}\mu^{*\ell}*\delta_x$ **is** μ**-stationary.**

What has to be proven?

- **supp**ν **is minimal + injectivity.**

Remember: there exist homeomorphisms of $\mathbb{T}^2$

- **ergodic and not minimal,**
- **minimal and not uniquely ergodic.**

What is known?

Say Γ is proximal if $\overline{k\Gamma}$ contains rank one matrices.

Fact (Furstenberg) **When V is irreducible and Γ is proximal, then ν is unique.**

Remark **One can find V irreducible with uncountably many ergodic μ-stationary probability ν on $X = \mathbb{P}(V)$.**

2. Motivations: random trajectories

Let $g_1, \ldots, g_\ell, \ldots$ be independant G-valued random variables of law μ on a probability space $(\Omega, \mathcal{B}, \mathbb{P})$.

Theorem 2 **Assume V is irreducible. For all x in X,**

$\nu_x = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \mu^{*\ell} * \delta_x$ **exists and is μ-stationary,**

$\nu_{x,\omega} = \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=1}^{n} \delta_{g_\ell \cdots g_1 x}$ **exists and is ergodic μ-stationary.**

One has $\nu_x = \mathbb{E}(\nu_{x,\omega})$.

Sketch of proof of Theorem 2

Let $P : \mathcal{C}^0(X) \to \mathcal{C}^0(X)$ **given by**

$$P\varphi(x) = \int_G \varphi(gx)\, d\mu(g).$$

Definition P **is equicontinuous if, for all** φ **in** $\mathcal{C}^0(X)$**, the family** $(P^n\varphi)_{n\geq 1}$ **is equicontinuous in** $\mathcal{C}^0(X)$**.**

Fact (Raugi) When P **is equicontinuous, Theorems** $1.A$ **and** 2 **are satisfied.**

Key lemma If V **is irreducible,** P **is equicontinuous.**

Idea of proof: $P^n\varphi(x)$ **is the average of** $\varphi(g_n \cdots g_1 x)$**.**
Check $\forall \varepsilon > 0$**,** $\exists M_\varepsilon > 0$**, for all** x, x' **in** X**,** $n \geq 1$

$$\mathbb{P}(d(g_n \cdots g_1 x, g_n \cdots g_1 x') \leq M_\varepsilon d(x, x')) \geq 1 - \varepsilon.$$

3. Homogeneous spaces

The G-orbits are locally closed. One reformulates

Theorem 1 **Let H be an algebraic subgroup of G. The map $\nu \mapsto F := supp\nu$ is a bijection between**

$$\left\{ \begin{array}{l} \textbf{ergodic } \mu\textbf{-stationary} \\ \textbf{probability } \nu \textbf{ on } G/H \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textbf{minimal } \Gamma\textbf{-invariant} \\ \textbf{compact } F \textbf{ in } G/H \end{array} \right\}.$$

When $k = \mathbb{R}$, these sets
(i) are empty when G/H is not compact,
(ii) have only one element when G/H is compact.

Proof of Theorem 1 when $k = \mathbb{R}$

Write the Iwasawa decomposition $G = KAN$.

Fact (Guivarch-Raugi) P is equicontinuous on G/AN.

Key Lemma If $Y = G/H$ supports a μ-stationary probability ν, then H contains a conjugate of AN.

First step H **contains a conjugate** N' **of** N **i.e.** $Y^{N'} \neq \emptyset$.

By Chevalley, assume $G/H \hookrightarrow \mathbb{P}(V)$.

Write a random word $g_n \cdots g_1 = k_{1,n} a_n k_{2,n}$ **in the Cartan decomposition** $G = KA^+K$. **By Goldsheid, Margulis and Furstenberg,** a_n **goes away from the walls of** A^+. **Hence the image of any limit** π **in** $End(V)$ **of** $\frac{g_n \cdots g_1}{\|g_n \cdots g_1\|}$ **is in some** $\mathbb{P}(V)^{N'}$.

The probability $\beta \otimes \nu$ **on** $B \times Y$ **is invariant by** $(b, y) \mapsto (Sb, g_1 y)$, **where** $B = G^{\mathbb{N}}$, $\beta = \mu^{\otimes \mathbb{N}}$, $S =$ **shift.**

By Poincare recurrence, for ν**-almost all** y **in** Y, $g_n \cdots g_1 y$ **has limit points** $y_\infty \in Y$. **Hence** $y_\infty \in Im\pi \cap Y$ **is** N'**-invariant.**

Second step H **contains a conjugate of** A**.**

Let $P = MAN$ **be a minimal parabolic subgroup.**
We can assume $H = MA'N$ **with** $\mathfrak{a}/\mathfrak{a}' \simeq \mathbb{R}$**. Then** $Y = G/H \to Z = G/P$ **is a line bundle.**

The action of G **on** $Y \simeq Z \times \mathbb{R}$ **is given by**

$$g.(z,t) = (gz, t + \overline{\sigma}(g,z))$$

where σ **is the Iwasawa cocycle** $\sigma : G \times Z \to \mathfrak{a}$**.**

Assume there exists an ergodic μ**-stationary** ν **on** Y**.**
We want a contradiction.

Let $\kappa : G \to \mathfrak{a}^+$ be the projection given by the Cartan decomposition $G = KA^+K$. For z in Z, the difference $\kappa(g_n \cdots g_1) - \sigma(g_n \cdots g_1, z)$ is bounded by a constant M_ϵ outside a set of trajectories of mass ϵ.

Fix an interval J_0 with $\nu(Z \times J_0) > \frac{1}{2}$. Apply Birkhoff theorem to the same ergodic dynamical system $(B \times Y, \beta \otimes \nu, T)$: for ν almost all (z, t) in Y, one has

$$\lim_{n\to\infty} \frac{1}{n} |\{\ell \leq n \mid \overline{\sigma}(g_\ell \cdots g_1, z) + t \in J_0\}| = \nu(Z \times J_0).$$

Hence these t's are bounded and ν has compact support. This contradicts the fact:

Fact **Y contains no compact Γ-invariant subsets.**

4. Flag variety **Let $P \subset G$ be a minimal parabolic,**

Theorem 3 **There are only finitely many ergodic μ-stationary probability on the flag variety $Z := G/P$.**

Key Lemma **There are only finitely many minimal Γ-invariant subsets of Z.**

Remark **There exists a G-equivariant embedding $Z \hookrightarrow \mathbb{P}(V)$ with V irreducible and G proximal.**
• When $k = \mathbb{R}$, by Goldsheid–Margulis, Γ is also proximal and, by Furstenberg, ν is unique.
• when $k = \mathbb{Q}_p$, Γ is not always proximal. For instance Γ could be a compact open subgroup of G.

Ref: Random walk on projective spaces, Compositio Mathematica or www.math.u-psud.fr/˜benoist