Recurrence on the space of lattices

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Abstract

We will describe the orbit closures $F = \overline{\Gamma x_0}$ in the space X of lattices of $\mathbb{R}^d$ when the Zariski closure of the group Γ is semisimple.

We will need to know that a random walk on X is recurrent as soon as the support of its transition law μ is compact and spans Γ.

A key point will be the construction of a proper function f on X which is contracted by the law μ.

1. Dynamics of Γ on G/Λ.

Let G be a real Lie group,
$\Lambda \subset G$ a lattice, $X = G/\Lambda$,
$\Gamma \subset G$ a subgroup such that the Zariski closure of $\mathrm{Ad}(\Gamma)$ is semisimple with no compact factor.

i.e. $vol(G/\Lambda) < \infty$.
i.e. the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ containing $\mathrm{Ad}(\Gamma)$.

Example: $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$,
i.e. $X = \{$ lattices x in $\mathbb{R}^d$ of covolume 1 $\}$,
$\Gamma =$ Zariski dense subgroup in
$H = \mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$, $d = d_1 + d_2$.

Orbit closure Theorem 1 For all x_0 in X, the orbit closure $F := \overline{\Gamma x_0}$ in X is homogeneous of finite volume. (Margulis-Shah Conjecture)

i.e. $G_F := \{g \in G \mid gF = F\}$ is transitive on F.

i.e. F supports a G_F-invariant probability ν_F.

Let $\mu \in \mathcal{P}(G)$ be a probability measure on G whose support is compact and spans Γ.

Stationary Measure Theorem 2 **Every μ-ergodic μ-stationary probability measure ν on X is Γ-invariant and homogeneous.** **(Furstenberg Conjecture)**

i.e. ν is extremal among the μ-stationary ones.
i.e. $\mu * \nu = \nu$ where $\mu * \nu = \int_G g_* \nu \, \mathrm{d}\mu(g)$.
i.e. $G_\nu := \{g \in G \mid g_* \nu = \nu\}$ is transitive on $\mathrm{supp}(\nu)$.

Remark. The set of Γ-invariant, ergodic and homogeneous probability measures on X is a countable union of L-orbits where L is the centralizer of Γ.

Equidistribution Theorem 3 For all x_0 in X, the sequence of probability $\nu_n := \frac{1}{n}\sum_{k=1}^{n} \mu^{*k} * \delta_{x_0}$ converges to ν_F with $F = \overline{\Gamma x_0}$.

Why does Theorem 2 implies Theorems 1 and 3?

Any weak sublimit ν_∞ of the sequence ν_n is μ-stationary and supported on F.

One has to check that

(i) ν_∞ is a probability measure,

(ii) $\nu_\infty(LF') = 0$ for any Γ-invariant finite volume homogeneous set F' that does not meet Lx.

The end of this talk is devoted to (i) no mass escape.

2. Random walk on $X = G/\Lambda$

Recurrence Theorem 4 **For all x_0 in X, $\varepsilon > 0$, there exists a compact set $K \subset X$ such that, for all $n \geq 1$,** $\mu^{*n} * \delta_{x_0}(K^c) \leq \varepsilon$. **(Eskin-Margulis Conjecture)**

It is enough to check the

Existence of f **There exists a proper function $f : X \to [0, \infty]$, finite at x_0, and contracted by μ.**

i.e. for all $T > 0$, $f^{-1}([0, T])$ is relatively compact.
i.e. there exist $a < 1$, $b > 0$ such that $P_\mu f \leq a f + b$,
where $P_\mu f(x) = \int_G f(gx)\, \mathrm{d}\mu(g)$.

When the centralizer L is trivial, Theorem 4 is due to Eskin-Margulis and the function f is finite everywhere.

A. $X = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, $H = \mathrm{SL}(2,\mathbb{R})$

Choose $\alpha_1(x) := \min\{\|v\| \mid v \in x \setminus 0\}$ **and** $f = \alpha_1^{-\delta}$.

$\star$ f **is proper, by Mahler compactness criterion.**

$\star$ f **is contracted by** μ**, as soon as** δ **is given by the**

General Fact **(Furstenberg, Eskin, Margulis)**
When V **is an irreducible non-trivial representation of** H**, there exist** $a < 1$, $n_0 \geq 1$, $\delta > 0$ **such that** $\int_G \|gv\|^{-\delta} d\mu^{*n_0}(g) \leq a\|v\|^{-\delta}$ **, for all** v **in** V.

Indeed, when the min is almost achieved at two non-colinear vectors v **and** w**, the value** $f(x)$ **is uniformly bounded, because**

$$1 \leq \|v \wedge w\| \leq \|v\|\,\|w\|.$$

Hence $P_\mu f \leq a f + b$.

B. $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$, $H = \mathrm{SL}(3,\mathbb{R})$

Choose $\alpha_1(x) := \min\{\|v\| \mid v \in x \smallsetminus 0\}$**,**

$\alpha_2(x) := \alpha_1(x^*)$ **where** x^* **is the dual lattice,**

$f_1 = \alpha_1^{-\delta}$**,** $f_2 = \alpha_2^{-\delta}$**, and** $f = f_1 + f_2$**.**

One has $P_\mu f_1 \leq a\, f_1 + M\sqrt{f_2}$**, for some** $M > 0$**,**

and also $P_\mu f_2 \leq a\, f_2 + M\sqrt{f_1}$**.**

Indeed, when the min is almost achieved at two non-colinear vectors v **and** w**, the value** $f_1(x)$ **is bounded by a multiple of** $\sqrt{f_2(x)}$**, because**

$$\|v \wedge w\| \leq \|v\|\,\|w\| \text{ , for all } v, w \text{ in } \mathbb{R}^3.$$

Hence $P_\mu f \leq a\, f + b$**.**

C. $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$, $H = \mathrm{SL}(d,\mathbb{R})$

Choose $\alpha_i(x) := \min\{\|u\| \mid u \in \Lambda^i x \smallsetminus 0$ **pure tensor**$\}$,

$f_i = \alpha_i^{-\delta}$, $f_0 = f_d = 1$ **and** $f = \sum_{i=1}^{d-1} \varepsilon_0^{(d-i)i} f_i$.

One has $P_\mu f_i \leq a\, f_i + M \sum_{0<j\leq min(i,d-i)} \sqrt{f_{i+j} f_{i-j}}$.

Indeed, when the min is almost achieved at two non-colinear vectors, one uses the inequality

$$\|u\|\,\|u \wedge v \wedge w\| \leq \|u \wedge v\|\,\|u \wedge w\|,$$

for all $u \in \Lambda^r\mathbb{R}^d$, $v \in \Lambda^s\mathbb{R}^d$, $w \in \Lambda^t\mathbb{R}^d$ **pure tensors.**

Hence $P_\mu f \leq a\, f + b$, **for** ε_0 **small.**

D. $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$, $H = \mathrm{SL}(2,\mathbb{R})$

Write $v = v_+ + v_0 \in \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$ **and fix** $\varepsilon_0 > 0$.

Choose $\alpha_1(x) := \min\{\|v_+\| \mid v \in x \smallsetminus 0$ **with** $\|v_0\| < \varepsilon_0\}$,

$\alpha_2(x) := \alpha_1(x^*)$, $f_1 = \alpha_1^{-\delta}$, $f_2 = \alpha_2^{-\delta}$, **and** $f = f_1 + f_2$.

One has $P_\mu f_1 \leq a\, f_1 + M\varepsilon_0^\delta\, f_2 + b$, **for some** $M > 0$,

and also $P_\mu f_2 \leq a\, f_2 + M\varepsilon_0^\delta f_1 + b$.

Indeed, when the min is almost achieved at two non-colinear vectors v **and** w**, one uses the inequality**

$$\|(v \wedge w)_+\| \leq \|v_0\|\,\|w_+\| + \|v_+\|\,\|w_0\|.$$

Hence $P_\mu f \leq a\, f + b$, **for** ε_0 **small.**

E. $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$, $H = \mathrm{SL}(d_1,\mathbb{R}) \times \mathrm{SL}(d_2,\mathbb{R})$

For $\lambda = (\lambda_1, \lambda_2)$, **write** $|\lambda| = (d_1 - \lambda_1)\lambda_1 + (d_2 - \lambda_2)\lambda_2$.

For v **in** $\Lambda^r\mathbb{R}^d$ **set** $v = \sum v_\lambda$, **with** $v_\lambda \in \Lambda^{\lambda_1}\mathbb{R}^{d_1} \otimes \Lambda^{\lambda_2}\mathbb{R}^{d_2}$,

and set $\varphi(v) = \max_{|\lambda|\neq 0} \varepsilon_0^{\frac{(d-r)r}{|\lambda|}} \|v_\lambda\|^{\frac{1}{|\lambda|}}$. **Choose**

$\alpha(x) := \min\{\varphi(v) \mid v \in x \setminus 0$ **with** $\|v_0\| < \varepsilon_0\}$, $f = \alpha^{-\delta}$.

One has $P_\mu f \leq a f + b$, **for** ε_0 **small. Indeed, when the min is almost achieved at two non-colinear vectors , one uses the** **Mother inequality for** $SL_{d_1} \times SL_{d_2}$

$$\|u_\lambda\| \, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu+\rho=\lambda+\mu \\ \min(\nu_i,\rho_i)\geq\min(\lambda_i,\mu_i)}} \|(u \wedge v)_\nu\| \, \|(u \wedge w)_\rho\|,$$

for all u, v, w **in** $\Lambda^\bullet\mathbb{R}^d$ **pure tensors and all** λ, μ.

This inequality is proven using representation theory. Indeed, its statement is simpler for a general semisimple Lie group $H \subset \mathrm{SL}(d, \mathbb{R})$. Let P^+ be the set of dominant weights of H.

Mother inequality

$$\|u_\lambda\| \, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu,\rho \in P^+ \\ \nu+\rho \geq \lambda+\mu}} \|(u \wedge v)_\nu\| \, \|(u \wedge w)_\rho\|,$$

for all u, v, w pure tensors and all λ, μ in P^+.

Here u_λ means the projection of u on the sum of irreducible subrepresentations with highest weight λ,

and $\geq$ means that the difference is a sum of positive root.

Thank you!