# Finite Fourier transform

**Yves Benoist**
**CNRS and Paris-Saclay University**



**Cetraro July 2025** **Alex Eskin $60^{th}$ birthday**

# Finite Fourier transform

**Yves Benoist**
**CNRS and Paris-Saclay University**

1. **Gaussian functions**
2. **Using Floer homology**
3. **Dirichlet characters**
4. **Jacobi sums**
5. **Legendre character**
6. **Finiteness of $H$-functions**
7. **Open questions**

**Cetraro July 2025** **Alex Eskin $60^{th}$ birthday**

## 1 Gaussian function

**Let** $p \geq 3$ **be prime,** $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ **cyclic group**

**Definition** $f : \mathbb{F}_p \to \mathbb{C}$ **is unimodular if** $|f| = 1_{\mathbb{F}_p}$
$f : \mathbb{F}_p \to \mathbb{C}$ **is biunimodular if** $|f| = |\widehat{f}| = 1_{\mathbb{F}_p}$
**where** $\widehat{f}(x) = \frac{1}{\sqrt{p}} \sum_{y \in \mathbb{F}_p} e^{\frac{2i\pi}{p}xy} f(y)$

**Interpretation:** $f$ **is orthogonal to its translates.**

**Remark** If $f$ is biunimodular, then
the functions $x \mapsto f(x + x_0)$ are biunimodular,
the functions $x \mapsto e^{\frac{2i\pi}{p}x_0 x} f(x)$ are biunimodular,
for $a \neq 0$, the functions $x \mapsto f(ax)$ are biunimodular.

## 1 Gaussian function

**Let $p \geq 3$ be prime, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ cyclic group**

**Definition** $f : \mathbb{F}_p \to \mathbb{C}$ **is unimodular if** $|f| = 1_{\mathbb{F}_p}$
$f : \mathbb{F}_p \to \mathbb{C}$ **is biunimodular if** $|f| = |\widehat{f}| = 1_{\mathbb{F}_p}$
**where** $\widehat{f}(x) = \frac{1}{\sqrt{p}} \sum_{y \in \mathbb{F}_p} e^{\frac{2i\pi}{p}xy} f(y)$

**Interpretation: $f$ is orthogonal to its translates.**

**Remark** **If $f$ is biunimodular, then**
**the functions** $x \mapsto f(x + x_0)$ **are biunimodular,**
**the functions** $x \mapsto e^{\frac{2i\pi}{p}x_0 x} f(x)$ **are biunimodular,**
**for $a \neq 0$, the functions** $x \mapsto f(ax)$ **are biunimodular.**

**Example: Gaussian functions** $f(x) = e^{\frac{2i\pi}{p}x^2}$

$$\widehat{f}(x) = \frac{G(\chi_0)}{\sqrt{p}}\,\overline{f}(x/4)$$

**where** $G(\chi_0) = \sum_{x\in\mathbb{F}_p} \chi_0(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p}}]$

**is the Gauss sum with** $\chi_0$ **Legendre character.**

$\chi_0(x) = -1, 0$ **or** $1$**, for** $x$ **non-square, zero or square.**
$G(\chi_0) = \sqrt{p}$ **or** $i\sqrt{p}$ **for** $p \equiv 1$ **or** $3$ **mod** $4$**.**

**This gives** $p(p-1)$ **biunimodular functions.**

**Question** **(Per Enflo 83)**
**Are there other biunimodular functions?**

**Example: Gaussian functions** $f(x) = e^{\frac{2i\pi}{p}x^2}$

$$\widehat{f}(x) = \frac{G(\chi_0)}{\sqrt{p}}\,\overline{f}(x/4)$$

**where** $G(\chi_0) = \sum_{x \in \mathbb{F}_p} \chi_0(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p}}]$
**is the Gauss sum with** $\chi_0$ **Legendre character.**

$\chi_0(x) = -1, 0$ **or** $1$**, for** $x$ **non-square, zero or square.**
$G(\chi_0) = \sqrt{p}$ **or** $i\sqrt{p}$ **for** $p \equiv 1$ **or** $3$ **mod** $4$**.**

**This gives** $p(p-1)$ **biunimodular functions.**

**Question** **(Per Enflo 83)**
**Are there other biunimodular functions?**

**Example: Gaussian functions** $f(x) = e^{\frac{2i\pi}{p}x^2}$

$$\widehat{f}(x) = \frac{G(\chi_0)}{\sqrt{p}}\,\overline{f}(x/4)$$

**where** $G(\chi_0) = \sum_{x\in\mathbb{F}_p} \chi_0(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p}}]$
**is the Gauss sum with** $\chi_0$ **Legendre character.**

$\chi_0(x) = -1, 0$ **or** $1$, **for** $x$ **non-square, zero or square.**
$G(\chi_0) = \sqrt{p}$ **or** $i\sqrt{p}$ **for** $p \equiv 1$ **or** $3$ **mod** $4$.

**This gives** $p(p-1)$ **biunimodular functions.**

**Question** **(Per Enflo 83)**
**Are there other biunimodular functions?**

**Answer 1** (Björck 85) YES. There exists a $(\mathbb{F}_p^*)^2$-invariant biunimodular function.
given by elementary formulas

$$f = \delta_0 + \frac{1}{1+i\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p}}{1+i\sqrt{p}}\chi_0, \quad \text{for } p \equiv 3 \bmod 4.$$

$$f = \delta_0 + \frac{1}{1+\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p+2\sqrt{p}}}{1+\sqrt{p}}\chi_0, \text{ for } p \equiv 1 \bmod 4.$$

This gives $4p^2$ functions.

**Answer 2** (Haagerup 08) There exist only finitely many biunimodular functions $f$ with $f(0) = 1$.

**Answer 3** (Björck, Haagerup, Gabidulin, Shorin) YES. There are more functions when $p > 7$, $p \equiv 1 \bmod 3$. One can choose them $(\mathbb{F}_p^*)^3$-invariant, given by explicit formulas.

**Answer 1** (Björck 85) YES. There exists a $(\mathbb{F}_p^*)^2$-invariant biunimodular function.

given by elementary formulas

$$f = \delta_0 + \frac{1}{1+i\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p}}{1+i\sqrt{p}}\,\chi_0, \quad \text{for } p \equiv 3 \bmod 4.$$

$$f = \delta_0 + \frac{1}{1+\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p+2\sqrt{p}}}{1+\sqrt{p}}\,\chi_0, \text{ for } p \equiv 1 \bmod 4.$$

This gives $4p^2$ functions.

**Answer 2** (Haagerup 08) There exist only finitely many biunimodular functions $f$ with $f(0) = 1$.

**Answer 3** (Björck, Haagerup, Gabidulin, Shorin) YES. There are more functions when $p > 7$, $p \equiv 1 \bmod 3$. One can choose them $(\mathbb{F}_p^*)^3$-invariant, given by explicit formulas.

Answer 1 (Björck 85) YES. There exists a $(\mathbb{F}_p^*)^2$-invariant biunimodular function.

given by elementary formulas

$$f = \delta_0 + \frac{1}{1+i\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p}}{1+i\sqrt{p}}\chi_0, \quad \text{for } p \equiv 3 \bmod 4.$$

$$f = \delta_0 + \frac{1}{1+\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p+2\sqrt{p}}}{1+\sqrt{p}}\chi_0, \quad \text{for } p \equiv 1 \bmod 4.$$

This gives $4p^2$ functions.

Answer 2 (Haagerup 08) There exist only finitely many biunimodular functions $f$ with $f(0) = 1$.

Answer 3 (Björck, Haagerup, Gabidulin, Shorin) YES. There are more functions when $p > 7$, $p \equiv 1 \bmod 3$. One can choose them $(\mathbb{F}_p^*)^3$-invariant, given by explicit formulas.

**Answer 1** **(Björck 85) YES. There exists a $(\mathbb{F}_p^*)^2$-invariant biunimodular function.**

**given by elementary formulas**

$$f = \delta_0 + \frac{1}{1+i\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p}}{1+i\sqrt{p}}\,\chi_0, \quad \text{for } p \equiv 3 \bmod 4.$$

$$f = \delta_0 + \frac{1}{1+\sqrt{p}} 1_{\mathbb{F}_p^*} + i\frac{\sqrt{p+2\sqrt{p}}}{1+\sqrt{p}}\,\chi_0, \text{ for } p \equiv 1 \bmod 4.$$

**This gives $4p^2$ functions.**

**Answer 2** **(Haagerup 08) There exist only finitely many biunimodular functions $f$ with $f(0) = 1$.**

**Answer 3** **(Björck, Haagerup, Gabidulin, Shorin) YES. There are more functions when $p > 7$, $p \equiv 1 \bmod 3$. One can choose them $(\mathbb{F}_p^*)^3$-invariant, given by explicit formulas.**

## 2. Using Floer homology

**Theorem 1** **YES. When $p \geq 11$, there exists a biunimodular function $f$ neither gaussian, nor björckian.**

No formulas are expected

$X := \mathbb{P}(\mathbb{C}^n)$ = projective space
$T := \{[z_1, \ldots, z_n] \in X \mid |z_1| = \cdots = |z_n|\}$ = Clifford torus
$u \in U := U(n)$ = unitary transformation.

**Fact** (Biran, Entov, Polterovich + Cho 04)
a) $T \cap uT \neq \emptyset$. Interpretation: $U$ = DVD.
b) If $T \cap uT$ is transversal, then $\# T \cap uT \geq 2^{n-1}$.

## 2. Using Floer homology

**Theorem 1** **YES. When $p \geq 11$, there exists a biunimodular function $f$ neither gaussian, nor björckian.**

No formulas are expected

$X := \mathbb{P}(\mathbb{C}^n)$ **= projective space**
$T := \{[z_1, \ldots, z_n] \in X \mid |z_1| = \cdots = |z_n|\}$ **= Clifford torus**
$u \in U := U(n)$ **= unitary transformation.**

**Fact** **(Biran, Entov, Polterovich + Cho 04)**
$a)$ $T \cap uT \neq \emptyset$. Interpretation: $U = DVD$.
$b)$ **If** $T \cap uT$ **is transversal, then** $\# T \cap uT \geq 2^{n-1}$.

## 2. Using Floer homology

**Theorem 1** **YES. When $p \geq 11$, there exists a biunimodular function $f$ neither gaussian, nor björckian.**
**No formulas are expected**

$X := \mathbb{P}(\mathbb{C}^n)$ **= projective space**
$T := \{[z_1, \ldots, z_n] \in X \mid \ |z_1| = \cdots = |z_n|\}$ **= Clifford torus**
$u \in U := U(n)$ **= unitary transformation.**

**Fact** **(Biran, Entov, Polterovich + Cho 04)**
$a)$ $T \cap uT \neq \emptyset$. **Interpretation: $U = DVD$.**
$b)$ **If** $T \cap uT$ **is transversal, then** $\# T \cap uT \geq 2^{n-1}$.

**Proof of Theorem 1** **Choose**
$n = p$, $\mathbb{C}^n = \{f : \mathbb{F}_p \to \mathbb{C}\}$, $X = \mathbb{P}(\mathbb{C}^n)$
$T = \{$**unimodular functions**$\} \subset X$,
$u =$ **Fourier transform.**

**Check that** $T \cap uT$ **is transversal**
**at gaussian and björckian functions.**
**but** $2^{p-1} > p(p-1) + 4p^2$.
**There must exist another biunimodular function.** $\square$

**Proof of Theorem 1** **Choose**
$n = p$, $\mathbb{C}^n = \{f : \mathbb{F}_p \to \mathbb{C}\}$, $X = \mathbb{P}(\mathbb{C}^n)$
$T = \{\text{unimodular functions}\} \subset X$,
$u =$ **Fourier transform.**

**Check that** $T \cap uT$ **is transversal**
**at gaussian and björckian functions.**
**but** $2^{p-1} > p(p-1) + 4p^2$.
**There must exist another biunimodular function.** □

## 3. Dirichlet character

**Definition** $f : \mathbb{F}_p \to \mathbb{C}$ **is unimodular on** $\mathbb{F}_p^*$ **if** $|f| = 1_{\mathbb{F}_p^*}$

$f : \mathbb{F}_p \to \mathbb{C}$ **is biunimodular on** $\mathbb{F}_p^*$ **if** $|f| = |\widehat{f}| = 1_{\mathbb{F}_p^*}$

**Example:** $f = \chi$ **non trivial multiplicative character**
$\chi(1) = 1$, $\chi(xy) = \chi(x)\chi(y)$, **for all** $x, y \in \mathbb{F}_p$.

$$\widehat{\chi} = \frac{G(\chi)}{\sqrt{p}}\,\overline{\chi}$$

**where** $G(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}, e^{\frac{2i\pi}{p}}]$

**One has** $|G(\chi)| = \sqrt{p}$.

## 3. Dirichlet character

**Definition** $f : \mathbb{F}_p \to \mathbb{C}$ **is unimodular on** $\mathbb{F}_p^*$ **if** $|f| = 1_{\mathbb{F}_p^*}$

$f : \mathbb{F}_p \to \mathbb{C}$ **is biunimodular on** $\mathbb{F}_p^*$ **if** $|f| = |\widehat{f}| = 1_{\mathbb{F}_p^*}$

**Example:** $f = \chi$ **non trivial multiplicative character**
$\chi(1) = 1$, $\chi(xy) = \chi(x)\chi(y)$, **for all** $x, y \in \mathbb{F}_p$.

$$\widehat{\chi} = \frac{G(\chi)}{\sqrt{p}}\,\overline{\chi}$$

**where** $G(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}, e^{\frac{2i\pi}{p}}]$

**One has** $|G(\chi)| = \sqrt{p}$.

## 3. Dirichlet character

**Definition** $f : \mathbb{F}_p \to \mathbb{C}$ **is unimodular on** $\mathbb{F}_p^*$ **if** $|f| = 1_{\mathbb{F}_p^*}$
$f : \mathbb{F}_p \to \mathbb{C}$ **is biunimodular on** $\mathbb{F}_p^*$ **if** $|f| = |\widehat{f}| = 1_{\mathbb{F}_p^*}$

**Example:** $f = \chi$ **non trivial multiplicative character**
$\chi(1) = 1$, $\chi(xy) = \chi(x)\chi(y)$, **for all** $x, y \in \mathbb{F}_p$.

$$\widehat{\chi} = \frac{G(\chi)}{\sqrt{p}}\,\overline{\chi}$$

**where** $G(\chi) := \sum_{x \in \mathbb{F}_p} \chi(x) e^{\frac{2i\pi}{p}x} \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}, e^{\frac{2i\pi}{p}}]$

**One has** $|G(\chi)| = \sqrt{p}$.

Question (Harvey Cohn 94)
**Are there other biunimodular functions on $\mathbb{F}_p^*$?**

Answer 1 (Biro 99) There exist only finitely many biunimodular functions on $\mathbb{F}_p^*$ with $f(1) = 1$.

Answer 2 (Kurlberg 02) All biunimodular functions $f$ on $\mathbb{F}_p^*$ for which $f^N = 1_{\mathbb{F}_p^*}$ for some $N \geq 1$ and $f(1) = 1$ are Dirichlet characters.

Difficulty: the condition $f(0) = 0$ is not Fourier invariant.

Trick : look for an odd biunimodular function on $\mathbb{F}_p^*$.
There are only $\frac{p-1}{2}$ odd Dirichlet characters.

Question (Harvey Cohn 94)
**Are there other biunimodular functions on $\mathbb{F}_p^*$?**

**Answer 1 (Biro 99) There exist only finitely many biunimodular functions on $\mathbb{F}_p^*$ with $f(1) = 1$.**

**Answer 2 (Kurlberg 02) All biunimodular functions $f$ on $\mathbb{F}_p^*$ for which $f^N = 1_{\mathbb{F}_p^*}$ for some $N \geq 1$ and $f(1) = 1$ are Dirichlet characters.**

Difficulty: the condition $f(0) = 0$ is not Fourier invariant.

Trick : look for an odd biunimodular function on $\mathbb{F}_p^*$.
There are only $\frac{p-1}{2}$ odd Dirichlet characters.

**Question** (Harvey Cohn 94)
Are there other biunimodular functions on $\mathbb{F}_p^*$?

**Answer 1** (Biro 99) There exist only finitely many biunimodular functions on $\mathbb{F}_p^*$ with $f(1) = 1$.

**Answer 2** (Kurlberg 02) All biunimodular functions $f$ on $\mathbb{F}_p^*$ for which $f^N = 1_{\mathbb{F}_p^*}$ for some $N \geq 1$ and $f(1) = 1$ are Dirichlet characters.

**Difficulty:** the condition $f(0) = 0$ is not Fourier invariant.

Trick : look for an odd biunimodular function on $\mathbb{F}_p^*$.
There are only $\frac{p-1}{2}$ odd Dirichlet characters.

**Question** (Harvey Cohn 94)
Are there other biunimodular functions on $\mathbb{F}_p^*$?

**Answer 1** (Biro 99) There exist only finitely many biunimodular functions on $\mathbb{F}_p^*$ with $f(1) = 1$.

**Answer 2** (Kurlberg 02) All biunimodular functions $f$ on $\mathbb{F}_p^*$ for which $f^N = 1_{\mathbb{F}_p^*}$ for some $N \geq 1$ and $f(1) = 1$ are Dirichlet characters.

Difficulty: the condition $f(0) = 0$ is not Fourier invariant.

Trick : look for an odd biunimodular function on $\mathbb{F}_p^*$.
There are only $\frac{p-1}{2}$ odd Dirichlet characters.

**Theorem 2** **YES. When $p \geq 11$, there exists an odd biunimodular function $f$ on $\mathbb{F}_p^*$ which is not a Dirichlet character.**

No formulas are expected

**Proof of Theorem 2** Choose $n = \frac{p-1}{2}$, $\mathbb{C}^n = \{f : \mathbb{F}_p \to \mathbb{C} \mid \text{odd}\}$, $X = \mathbb{P}(\mathbb{C}^n)$, $T = \{\text{odd unimodular functions on } \mathbb{F}_p^*\} \subset X$, $u =$ Fourier transform.

One has $2^{n-1} > n$. This looks good!

Difficulty $T \cap uT$ is not always transversal.

**Theorem 2** **YES. When $p \geq 11$, there exists an odd biunimodular function $f$ on $\mathbb{F}_p^*$ which is not a Dirichlet character.**

No formulas are expected

**Proof of Theorem 2** Choose $n = \frac{p-1}{2}$, $\mathbb{C}^n = \{f : \mathbb{F}_p \to \mathbb{C} \mid \text{odd}\}$, $X = \mathbb{P}(\mathbb{C}^n)$, $T = \{\text{odd unimodular functions on } \mathbb{F}_p^*\} \subset X$, $u =$ Fourier transform.

One has $2^{n-1} > n$. This looks good!

Difficulty $T \cap uT$ is not always transversal.

**Theorem 2** **YES.** When $p \geq 11$,
there exists an odd biunimodular function $f$ on $\mathbb{F}_p^*$ which is not a Dirichlet character.

No formulas are expected

**Proof of Theorem 2** Choose
$n = \frac{p-1}{2}$, $\mathbb{C}^n = \{f : \mathbb{F}_p \to \mathbb{C} \mid \text{odd }\}$, $X = \mathbb{P}(\mathbb{C}^n)$,
$T = \{$odd unimodular functions on $\mathbb{F}_p^*\} \subset X$,
$u =$ Fourier transform.

One has $2^{n-1} > n$. This looks good!

Difficulty $T \cap uT$ is not always transversal.

There exists f biunimodular on $\mathbb{F}_p^*$ not a Dirichlet character.

## 4. Jacobi sums

**Proposition** **Let** $\chi : \mathbb{F}_p \to \mathbb{C}$ **odd Dirichlet character.**
**One has** $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.
$(i)$ $T \cap uT$ **is not transversal at** $[\chi]$.
$(ii)$ **There exists an even Dirichlet character** $\psi$
**such that** $J(\chi, \psi) = J(\overline{\chi}, \psi)$.
$(iii)$ $\chi$ **has order** 2 **or** 4, **this means** $\chi^4 = 1_{\mathbb{F}_p^*}$.

**Here** $J(\chi, \psi) := \sum_{x \in \mathbb{F}_p} \chi(x)\psi(1-x) \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}]$.

**The character of order** 2 **is the Legendre character** $\chi_0$.
**When** $p \equiv 3 \bmod 4$, $T$ **and** $uT$ **are tangent at** $[\chi_0]$.
**When** $p \equiv 1 \bmod 8$, $T \cap uT$ **is transversal at all** $[\chi]$.

There exists f biunimodular on $\mathbb{F}_p^*$ not a Dirichlet character.

## 4. Jacobi sums

**Proposition** **Let $\chi : \mathbb{F}_p \to \mathbb{C}$ odd Dirichlet character.**
**One has** $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.
$(i)$ $T \cap uT$ **is not transversal at** $[\chi]$.
$(ii)$ **There exists an even Dirichlet character** $\psi$
**such that** $J(\chi, \psi) = J(\overline{\chi}, \psi)$.
$(iii)$ $\chi$ **has order** 2 **or** 4, **this means** $\chi^4 = 1_{\mathbb{F}_p^*}$.

**Here** $J(\chi, \psi) := \sum_{x \in \mathbb{F}_p} \chi(x)\psi(1-x) \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}]$.

The character of order 2 is the Legendre character $\chi_0$.
When $p \equiv 3 \bmod 4$, $T$ and $uT$ are tangent at $[\chi_0]$.
When $p \equiv 1 \bmod 8$, $T \cap uT$ is transversal at all $[\chi]$.

There exists f biunimodular on $\mathbb{F}_p^*$ not a Dirichlet character.

## 4. Jacobi sums

**Proposition** **Let $\chi : \mathbb{F}_p \to \mathbb{C}$ odd Dirichlet character.**
**One has** $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.
$(i)$ $T \cap uT$ **is not transversal at** $[\chi]$.
$(ii)$ **There exists an even Dirichlet character** $\psi$
**such that** $J(\chi, \psi) = J(\overline{\chi}, \psi)$.
$(iii)$ $\chi$ **has order** 2 **or** 4, **this means** $\chi^4 = 1_{\mathbb{F}_p^*}$.

**Here** $J(\chi, \psi) := \sum_{x \in \mathbb{F}_p} \chi(x)\psi(1-x) \in \mathbb{Z}[e^{\frac{2i\pi}{p-1}}]$.

**The character of order** 2 **is the Legendre character** $\chi_0$.
**When** $p \equiv 3 \bmod 4$, $T$ **and** $uT$ **are tangent at** $[\chi_0]$.
**When** $p \equiv 1 \bmod 8$, $T \cap uT$ **is transversal at all** $[\chi]$.

Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Remark** *a*) $\chi * \psi = J(\chi,\psi)\,\chi\psi$.
*b*) **For** $\chi$, $\psi$, $\chi\psi \neq 1_{\mathbb{F}_p^*}$ **, one has** $J(\chi,\psi) = \frac{G(\chi)G(\psi)}{G(\chi\psi)}$,
*c*) **In that case** $|J(\chi,\psi)| = \sqrt{p}$.

**How to distinguish Jacobi sums?** By reducing modulo an ideal $\mathfrak{p}$ over $p$. Fix $x_0$ a generator of $\mathbb{F}_p^*$. Choose $\mathfrak{p} = (p, x_0 - \zeta) \subset \mathbb{Z}[\zeta]$ where $\zeta = e^{\frac{2i\pi}{p-1}}$.

**Definition** The Teichmüller character is the Dirichlet character $\omega : \mathbb{F}_p \to \mathbb{C}$ such that $\omega(x) = x \bmod \mathfrak{p}$.

One has $\chi = \omega^j$, $\overline{\chi} = \omega^{-j}$ and $\psi = \omega^k$ where $j, k \in \mathbb{Z}/(p-1)\mathbb{Z}$.

Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Remark** a) $\chi * \psi = J(\chi,\psi)\,\chi\psi$.
b) **For** $\chi$, $\psi$, $\chi\psi \neq 1_{\mathbb{F}_p^*}$ **, one has** $J(\chi,\psi) = \frac{G(\chi)G(\psi)}{G(\chi\psi)}$,
c) **In that case** $|J(\chi,\psi)| = \sqrt{p}$.

**How to distinguish Jacobi sums? By reducing modulo an ideal** $\mathfrak{p}$ **over** $p$. **Fix** $x_0$ **a generator of** $\mathbb{F}_p^*$. **Choose** $\mathfrak{p} = (p, x_0 - \zeta) \subset \mathbb{Z}[\zeta]$ **where** $\zeta = e^{\frac{2i\pi}{p-1}}$.

**Definition** **The Teichmüller character is the Dirichlet character** $\omega : \mathbb{F}_p \to \mathbb{C}$ **such that** $\omega(x) = x$ **mod** $\mathfrak{p}$.

**One has** $\chi = \omega^j$, $\overline{\chi} = \omega^{-j}$ **and** $\psi = \omega^k$ **where** $j, k \in \mathbb{Z}/(p-1)\mathbb{Z}$.

## Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Remark** *a)* $\chi * \psi = J(\chi,\psi)\,\chi\psi$.
*b)* **For** $\chi$, $\psi$, $\chi\psi \neq 1_{\mathbb{F}_p^*}$ **, one has** $J(\chi,\psi) = \frac{G(\chi)G(\psi)}{G(\chi\psi)}$,
*c)* **In that case** $|J(\chi,\psi)| = \sqrt{p}$.

**How to distinguish Jacobi sums? By reducing modulo an ideal $\mathfrak{p}$ over $p$. Fix $x_0$ a generator of $\mathbb{F}_p^*$. Choose $\mathfrak{p} = (p, x_0 - \zeta) \subset \mathbb{Z}[\zeta]$ where $\zeta = e^{\frac{2i\pi}{p-1}}$.**

**Definition** **The Teichmüller character is the Dirichlet character $\omega : \mathbb{F}_p \to \mathbb{C}$ such that $\omega(x) = x$ mod $\mathfrak{p}$.**

**One has $\chi = \omega^j$, $\overline{\chi} = \omega^{-j}$ and $\psi = \omega^k$ where $j, k \in \mathbb{Z}/(p-1)\mathbb{Z}$.**

## Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Lemma** **(Stickelberger 1890) For $0 < j, k < p-1$, let** $J_{-j,-k} := \sum_{x \in \mathbb{F}_p^* \smallsetminus \{1\}} x^{-j}(1-x)^{-k} \in \mathbb{F}_p$.

$a)$ **One has** $J_{-j,-k} = -\frac{(j+k)!}{j!\,k!}$.
$b)$ **Hence** $J_{-j,-k} = 0 \iff j + k \geq p$.

**Proof** **Expand and use** $\sum_{x \in \mathbb{F}_p^*} x^m = 0$ **when** $m \neq 0$:

$$J_{-j,-k} = \sum_{x \in \mathbb{F}_p^*} x^{-j}(1-x)^{p-1-k} = -(-1)^j \binom{p-1-k}{j} = -\binom{j+k}{j}.$$

**Need to check** $J_{j,k} \neq J_{-j,k}$. **Use** $b)$.
**If it does not work... apply Galois. This replaces** $j$ **by** $aj$ **and** $k$ **by** $ak$ **mod** $p-1$, **where** $a \in (\mathbb{Z}/(p-1)\mathbb{Z})^*$.

## Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Lemma** **(Stickelberger 1890) For** $0 < j, k < p-1$**,**
**let** $J_{-j,-k} := \sum_{x \in \mathbb{F}_p^* \setminus \{1\}} x^{-j}(1-x)^{-k} \in \mathbb{F}_p$.

$a)$ **One has** $J_{-j,-k} = -\frac{(j+k)!}{j!\,k!}$.
$b)$ **Hence** $J_{-j,-k} = 0 \iff j + k \geq p$.

**Proof** **Expand and use** $\sum_{x \in \mathbb{F}_p^*} x^m = 0$ **when** $m \neq 0$**:**

$$J_{-j,-k} = \sum_{x \in \mathbb{F}_p^*} x^{-j}(1-x)^{p-1-k} = -(-1)^j \binom{p-1-k}{j} = -\binom{j+k}{j}.$$

**Need to check** $J_{j,k} \neq J_{-j,k}$. **Use** $b)$.
**If it does not work... apply Galois. This replaces** $j$ **by** $aj$ **and** $k$ **by** $ak$ **mod** $p-1$, **where** $a \in (\mathbb{Z}/(p-1)\mathbb{Z})^*$.

## Why does $\chi^4 \neq 1_{\mathbb{F}_p^*} \implies J(\chi,\psi) \neq J(\overline{\chi},\psi)$ ?

**Lemma** **(Stickelberger 1890) For** $0 < j, k < p-1$**,**
**let** $J_{-j,-k} := \sum_{x \in \mathbb{F}_p^* \smallsetminus \{1\}} x^{-j}(1-x)^{-k} \in \mathbb{F}_p$**.**
$a)$ **One has** $J_{-j,-k} = -\frac{(j+k)!}{j!\,k!}$**.**
$b)$ **Hence** $J_{-j,-k} = 0 \iff j + k \geq p$**.**

**Proof** **Expand and use** $\sum_{x \in \mathbb{F}_p^*} x^m = 0$ **when** $m \neq 0$**:**

$$J_{-j,-k} \;=\; \sum_{x \in \mathbb{F}_p^*} x^{-j}(1-x)^{p-1-k} \;=\; -(-1)^j \binom{p-1-k}{j} \;=\; -\binom{j+k}{j}.$$

**Need to check** $J_{j,k} \neq J_{-j,k}$**. Use** $b)$**.**
**If it does not work... apply Galois. This replaces** $j$ **by** $aj$ **and** $k$ **by** $ak$ **mod** $p-1$**, where** $a \in (\mathbb{Z}/(p-1)\mathbb{Z})^*$**.**

There exists f biunimodular on $\mathbb{F}_p^*$ not a Dirichlet character.

## 5. Legendre character

**Let $\chi_0$ be the Legendre character and $p \equiv 3 \bmod 4$.**

**Lemma** *a*) **The tori $T$ and $uT$ are tangent at $[\chi_0]$.**
*b*) **The algebraic multiplicity at $[\chi_0]$ of $T \cap uT$ is $2^{n-1}$.**

*c*) **There exists a continuous family $u_t \in U(n)$ with $u_0 = u$ so that, for $t \neq 0$, near $[\chi_0]$, $T \cap u_t T$ is transversal and contains at most $2^{n-2}$ points.**

## End of proof of Theorem 2

**One still has $2^{n-1} > 2^{n-2} + n-1$. There must exist another odd biunimodular function on $\mathbb{F}_p^*$.** □

## 5. Legendre character

**Let $\chi_0$ be the Legendre character and $p \equiv 3 \bmod 4$.**

**Lemma** *a*) **The tori $T$ and $uT$ are tangent at $[\chi_0]$.**
*b*) **The algebraic multiplicity at $[\chi_0]$ of $T \cap uT$ is $2^{n-1}$.**

*c*) **There exists a continuous family $u_t \in U(n)$ with $u_0 = u$ so that, for $t \neq 0$, near $[\chi_0]$, $T \cap u_t T$ is transversal and contains at most $2^{n-2}$ points.**

**End of proof of Theorem 2**

**One still has $2^{n-1} > 2^{n-2} + n - 1$. There must exist another odd biunimodular function on $\mathbb{F}_p^*$.** □

## 5. Legendre character

**Let $\chi_0$ be the Legendre character and $p \equiv 3 \bmod 4$.**

**Lemma** $a)$ **The tori $T$ and $uT$ are tangent at $[\chi_0]$.**
$b)$ **The algebraic multiplicity at $[\chi_0]$ of $T \cap uT$ is $2^{n-1}$.**

$c)$ **There exists a continuous family $u_t \in U(n)$ with $u_0 = u$ so that, for $t \neq 0$, near $[\chi_0]$, $T \cap u_t T$ is transversal and contains at most $2^{n-2}$ points.**

**End of proof of Theorem 2**

**One still has $2^{n-1} > 2^{n-2} + n-1$. There must exist another odd biunimodular function on $\mathbb{F}_p^*$.** □

## 6. Finiteness of biunimodular functions

**A pair of functions** $f, g : \mathbb{F}_p \to \mathbb{C}$ **is a** $H$**-pair if** $f(x)\,g(x) = \widehat{f}(x)\,\widehat{g}(-x) = 1$ **for all** $x \in \mathbb{F}_p$**.**

**Proposition** **(Haagerup 08) There are finitely many** $H$**-pairs with** $f(0) = g(0) = 1$**.**

**Lemma** **(Cebotarev 1920) Let** $f : \mathbb{F}_p \to \mathbb{C} \quad f \neq 0$ **one has**

$$\#\mathrm{supp}(f) + \#\mathrm{supp}(\widehat{f}) \geq p + 1.$$

**Proof of Proposition** **Choose** $(f_n, g_n)$ **going to** $\infty$**. Set** $u_n = \frac{f_n}{\|f_n\|_\infty}$**,** $v_n = \frac{g_n}{\|g_n\|_\infty}$**,** $u_\infty = \lim_{n\infty} u_n$**,** $v_\infty = \lim_{n\infty} v_n$**. They satisfy** $u_\infty v_\infty = \widehat{u_\infty}\widehat{v_\infty} = 0$**. Contradiction with Lemma.**$\square$

## 6. Finiteness of biunimodular functions

**A pair of functions** $f, g : \mathbb{F}_p \to \mathbb{C}$ **is a** $H$**-pair if** $f(x)\, g(x) = \widehat{f}(x)\, \widehat{g}(-x) = 1$ **for all** $x \in \mathbb{F}_p$.

**Proposition (Haagerup 08) There are finitely many** $H$**-pairs with** $f(0) = g(0) = 1$.

**Lemma (Cebotarev 1920) Let** $f : \mathbb{F}_p \to \mathbb{C}$ $\;f \neq 0$ **one has** $\#\mathrm{supp}(f) + \#\mathrm{supp}(\widehat{f}) \geq p + 1$.

**Proof of Proposition** Choose $(f_n, g_n)$ going to $\infty$. Set $u_n = \frac{f_n}{\|f_n\|_\infty}$, $v_n = \frac{g_n}{\|g_n\|_\infty}$, $u_\infty = \lim_{n\infty} u_n$, $v_\infty = \lim_{n\infty} v_n$. They satisfy $u_\infty v_\infty = \widehat{u_\infty} \widehat{v_\infty} = 0$. Contradiction with Lemma.□

## 6. Finiteness of biunimodular functions

**A pair of functions** $f, g : \mathbb{F}_p \to \mathbb{C}$ **is a** $H$**-pair if** $f(x)\,g(x) = \widehat{f}(x)\,\widehat{g}(-x) = 1$ **for all** $x \in \mathbb{F}_p$.

**Proposition** **(Haagerup 08) There are finitely many** $H$**-pairs with** $f(0) = g(0) = 1$.

**Lemma** **(Cebotarev 1920) Let** $f : \mathbb{F}_p \to \mathbb{C} \quad f \neq 0$ **one has** $$\#\mathrm{supp}(f) + \#\mathrm{supp}(\widehat{f}) \geq p + 1.$$

**Proof of Proposition** **Choose** $(f_n, g_n)$ **going to** $\infty$**. Set** $u_n = \frac{f_n}{\|f_n\|_\infty}$, $v_n = \frac{g_n}{\|g_n\|_\infty}$, $u_\infty = \lim_{n\infty} u_n$, $v_\infty = \lim_{n\infty} v_n$. **They satisfy** $u_\infty v_\infty = \widehat{u_\infty}\widehat{v_\infty} = 0$. **Contradiction with Lemma.**□

## 7. Open questions

**Question A** Is there a non gaussian even biunimodular function $f$ on $\mathbb{F}_p$?

**Question B** Is there a biunimodular function $f$ on $\mathbb{F}_p$ such that $x \mapsto f(x)^p$ is one-to-one?

**Question C** For $n$ square-free, are there finitely many biunimodular functions on $\mathbb{Z}/n\mathbb{Z}$ with $f(0) = 1$?

## 7. Open questions

**Question A** Is there a non gaussian even biunimodular function $f$ on $\mathbb{F}_p$?

**Question B** Is there a biunimodular function $f$ on $\mathbb{F}_p$ such that $x \mapsto f(x)^p$ is one-to-one?

**Question C** For $n$ square-free, are there finitely many biunimodular functions on $\mathbb{Z}/n\mathbb{Z}$ with $f(0) = 1$?

## 7. Open questions

**Question A** Is there a non gaussian even biunimodular function $f$ on $\mathbb{F}_p$?

**Question B** Is there a biunimodular function $f$ on $\mathbb{F}_p$ such that $x \mapsto f(x)^p$ is one-to-one?

**Question C** For $n$ square-free, are there finitely many biunimodular functions on $\mathbb{Z}/n\mathbb{Z}$ with $f(0) = 1$?