Geometry of homogeneous spaces

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Abstract

The topic of this minicourse is the geometry of the higher rank real semisimple Lie groups, their subgroups and their homogeneous spaces. We will explain a few classical structure theorems for these groups and their subgroups. For the intuition, we will emphasize concrete examples like the group of real unimodular matrices.

As an application, we will introduce various natural constants, like the critical exponent or the decay exponent, associated to such homogeneous spaces. We will see how these constants are related. This is a joint result with Siwei Liang, that solves questions raised by T. Kobayashi and extends previous results of Patterson, Sullivan, Corlette, Leuzinger, Edwards-Oh and Lutsko-Weich-Wolf.

Preface

This text is the written version of a series of four lectures I gave at the SLMath Institute in January 2026. Videos of these lectures are available on the web here[1]. Most of the people in the audience were PostDoc students. I tried to keep the informal style of the four lectures, giving only complete proof on representative examples, focusing on the main ideas, pointing out those ideas that are often useful in this subject, recalling shortly the proof of preliminary classical results, and leaving the technical issues to my book with Jean-François Quint [5] and to my joint paper [4] with Siwei Liang.

Lecture 1 is called “Symmetric spaces”. It is a tribute to Elie Cartan on the occasion of the 100^{th} birthday of his first paper “Sur une classe remarquable d'espaces de Riemann” published in 1926 in which he introduces he “Riemannian symmetric spaces” and classifies them thanks to the semisimple Lie groups.

We first survey this paper and the context in which it occured.

We then recall the classical decomposition of a semisimple Lie groups called the Cartan decomposition. We also explain the behavior of the Haar measure in this decomposition.

Lecture 2 is called “Subgroups of semisimple Lie groups”. We explain the Iwasawa decomposition, the Bruhat decomposition and we also explain how to express the Haar measure in these coordinates. We then focus on two kinds of subgroups, the connected ones and the discrete ones.

On the one hand we will see that the connected subgroups are “almost equal” to an algebraic subgroups, and that they either are reductive or included in a parabolic subgroup.

On the other hand we will see how the Zariski dense discrete subgroups of G look like in the Cartan decomposition.

[1] https://www.slmath.org/workshops/1125

Lecture 3 is called “Critical exponents”. We consider a closed subgroup H of a semisimple Lie group G and we introduce four quantities that we want to compare. The “normalized” critical exponent δ_H, the decay exponent θ_H, the integrability constant p_H and the local exponent β_H.

And state the main new result (joint with Siwei Liang) of this course: the critical exponent δ_H is equal to the decorellation exponent θ_H, is also equal to $1 - p_H^{-1}$ and is larger than or equal to the local exponent β_H.

When the subgroup H is discrete, this theorem is due to Patterson-Sullivan-Corlette when the real rank of G is 1 and to Lutsko, Weich, Wolf, following works of Leuzinger, and Edwards-Oh, when the real rank of G is at least 2.

We give a sketch of proof of this theorem using the decompositions of G of the previous lectures

Lecture 4 is called ”Unitary representations”. The aim of this lecture is to give an interpretation of Lecture 3 in terms of the harmonic analysis of the homogeneous space G/H. This interpretation allows us to obtain the second main new result (also joint with Siwei Liang) of this course: for a connected subgroup H, when the critical exponent δ_H is larger than $1/2$, it is equal to the local exponent β_H. This equality allows to compute explicitely these exponents when H is a connected subgroup.

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Lecture 1. Symmetric spaces

This lecture is a tribute to the french mathematician Elie Cartan.

One century ago in 1926, Elie Cartan wrote a series of two papers, [6] and [7], that appeared in the Bulletin de la Société Mathématique de France called "Sur une classe remarquable d'espaces de Riemann". At that time the concept of a Riemannian manifold and of their curvatures were already well-known. These Riemann spaces, as Elie Cartan calls them, were far-reaching generalizations of the hyperbolic plane of constant curvature -1 which was the first model of non-euclidean geometry. But, beyond the hyperbolic spaces, there was a lack of other interesting examples.

In these papers, Elie Cartan focuses on the Riemann spaces whose curvature tensor is parallel. This is a very natural geometric condition on the Riemann spaces.

The first surprise found by Elie Cartan is that there are many examples of such "espaces de Riemann remarquables".

The second surprise is that Elie Cartan was able to classify these spaces thanks to the real semisimple Lie groups, a nice family of Lie groups that he had already classified in 1914.

The third surprise is that these "espaces de Riemann remarquables" are still, one hundred year later, very useful spaces known under the name "Riemannian symmetric spaces".

In this lecture I will recall the construction of these spaces and recall a few of their basic properties that will be relevant in the next lectures. I refer to [12] or to [5] for more details.

Before beginning this lecture, I am thrilled to quote a sentence in the introduction of Elie Cartan's papers that shows that he was aware of the importance of his discovery:

> `Les résultats obtenus appellent un grand nombre de recherches nouvelles, ne serait-ce que l'étude individuelle des nouveaux espaces, qui semblent devoir jouer un rôle presque aussi important que celui des espaces à courbure constante, et qui sont du reste susceptible d'une définition géométrique directe`[2].

[2]The results we have obtained are calling for a great number of new researches like the individual study of each one of these new spaces, that seem to play a role almost as

We now know, one century later, that the intuitions of Elie Cartan are exact beyond his own expectations and that indeed the humble “presque aussi important” should be replaced by “bien plus important”.

1.1 Lie algebras

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra.

Definition 1.1. The Lie algebra $\mathfrak{g}$ is simple if $\mathfrak{g}$ has no ideals $0 \subsetneq \mathfrak{i} \subsetneq \mathfrak{g}$ and $\dim\mathfrak{g} \geq 2$. The Lie algebra $\mathfrak{g}$ is semisimple if $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ with $\mathfrak{g}_i$ simple ideals.

Example 1.2. *The following Lie algebras are simple*
$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}) := \{M \in \mathcal{M}(n, \mathbb{C}) \mid \mathrm{tr} M = 0\}$.
$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R}) := \{M \in \mathcal{M}(n, \mathbb{R}) \mid \mathrm{tr} M = 0\}$.
$\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) := \{M \in \mathcal{M}(n, \mathbb{C}) \mid M + {}^tM = 0\}$.
$\mathfrak{g} = \mathfrak{so}(p, q) := \{M \in \mathcal{M}(p+q, \mathbb{R}) \mid MI_{p,q} + I_{p,q}{}^tM = 0\}$ *when* $p+q \geq 5$.
Here $I_{p,q}$ *is a block diagonal matrix* $I_{p,q} = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{pmatrix}$.

In 1890 W. Killing gave a list of all possible simple complex Lie algebras. Those are

$$A_n,\ B_n,\ C_n,\ D_n, E_6,\ E_7,\ E_8,\ F_4,\ G_2.$$

The classical ones are $A_n = \mathfrak{sl}(n, \mathbb{C})$, $B_n = \mathfrak{so}(2n+1, \mathbb{C})$, $C_n = \mathfrak{sp}(2n, \mathbb{C})$ and $D_n = \mathfrak{so}(2n, \mathbb{C})$. The others E_6, E_7, E_8, F_4 and G_2 are called the exceptional ones. He gave a construction of G_2.

In 1894, in his PhD thesis, Elie Cartan gave a rigorous proof of this classification. In particular he gave a proof of the existence of the other four exceptional simple Lie algebras.

In 1914, Elie Cartan classified all the real simple Lie algebras. Those are the complex simple Lie algebras seen as real Lie algebra, together with all the real forms of the complex simple Lie algebras. In particular he proved that all the complex simple Lie algebras $\mathfrak{g}$ have a compact real form. This means that $\mathfrak{g}$ is the complexification $\mathfrak{g} = \mathfrak{u}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{u}$ of a compact simple connected Lie group.

Definition 1.3. The Killing form on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form given by, for X, Y in $\mathfrak{g}$,

$$B(X, Y) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}X\, \mathrm{adY}) \ \text{ where } \ \mathrm{ad}X(Z) = [X, Z] \ \text{ for all } \ Z \in \mathfrak{g}.$$

important as the one of the spaces with constant curvature, and that can by the way be defined in a direct geometric way.

There are many equivalent definitions of semisimple Lie algebras:

Fact 1.4. *Let $\mathfrak{g}$ be a Lie algebra. The following are equivalent: $\mathfrak{g}$ is semisimple $\iff B$ is non degenerate $\iff \mathfrak{g}$ contains no abelian ideal $\mathfrak{i} \neq 0$.*

The intimate relationship between semisimple Lie algebras and symmetric spaces comes from the following property discovered by Elie Cartan:

Fact 1.5. *All semisimple Lie algebras admit a Cartan involution θ, that is*

$$\theta \in \mathrm{Aut}(\mathfrak{g}) \ , \ \ \theta^2 = \mathbf{1} \ , \ \ B(\theta X, X) \leq 0 \ \text{ for all } \ X \in \mathfrak{g}.$$

In particular the non-degenerate bilinear symmetric form

$$B_\theta(X, Y) := B(\theta X, Y)$$

is negative definite. One has then the *Cartan decomposition* of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \ \text{ where}$$

$$\mathfrak{k} := \mathfrak{g}^\theta = \{X \in \mathfrak{g} \mid \theta X = X\} \ \text{ and } \ \mathfrak{s} := \{X \in \mathfrak{g} \mid \theta X = -X\}\,.$$

Note that one has the inclusions,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}\ , \ \ [\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}\ , \ \ [\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{k}\,.$$

Example 1.6. *When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the Killing form is given, for $X \in \mathfrak{g}$, by $B(X, X) = 2n\,\mathrm{tr}(X^2)$. The involution $\theta(X) = -{}^tX$ is a Cartan involution and one has $B_\theta(X, X) = -\mathrm{tr}(X^tX)$.*

1.2 Lie groups

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{d}g$ be the left Haar measure on G , let $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation, and let $\Delta_G : G \to \mathbb{R}_{>0}$ be the modulus function $\Delta_G(g) := \mathrm{tr}(\mathrm{Ad}g)$ for all $g \in G$, so that one has the equality, for all function $f \in C_c(G)$ and all $g_0 \in G$,

$$\int_G f(gg_0)\mathrm{d}g \ = \ \Delta_G(g_0) \int_G f(g)\mathrm{d}g.$$

Definition 1.7. A Lie group G is unimodular iff $\mathrm{d}g$ is a right Haar measure iff the modulus function $\Delta_G \equiv 1$.
A Lie group G is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple.

We assume now that G is a connected semisimple Lie group with finite center. In particular G is unimodular.

We write as above $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and we denote by $K \subset G$ the connected Lie subgroup with Lie algebra $\mathfrak{k} = \mathfrak{g}^\theta$.

Fact 1.8. *The group K is a maximal compact subgroup of G. All compact subgroups of G are included in a conjugate of K.*

Elie Cartan noticed that on the homogeneous space $M := G/K$, there is a natural G-invariant Riemannian metric. Indeed on the tangent space $\mathfrak{s} \equiv T_{m_0}M$ at the base point $m_0 := K/K$ of M, the bilinear form B_θ is positive definite and K-invariant.

These Riemannian manifolds discovered by Elie Cartan have very nice geometric properties: they are complete, simply connected, they have non positive sectional curvature and have a parallel curvature tensor.

1.3 Symmetric spaces

Definition 1.9. A Riemanian manifold M is a *symmetric space* if it is complete, simply connected and has parallel tensor curvature $\nabla R = 0$.

Remark 1.10. The condition $\nabla R = 0$ is equivalent to the existence, for every $m_0 \in M$, of a local isometry s_{m_0} of a neighborhood of m_0 such that, for all $X \in T_{m_0}M$ small enough, $s_{m_0}(exp_{m_0}(X)) = exp_{m_0}(-X)$. This equivalent condition, which is also due to Elie Cartan, explains the modern terminology "symmetric spaces" for these spaces.

For instance the flat symmetric spaces are the d-dimensional Euclidean spaces.

Let $\mathcal{E}$ be the set of (isometry equivalence classes of) symmetric spaces.

Let $\mathcal{E}_+$, resp. $\mathcal{E}_-$, the set of symmetric spaces with no Euclidean factors and with non-negative, resp. non-positive, sectional curvature.

Elie Cartan classified all the symmetric spaces. Indeed he proved the following.

Theorem 1.11. (Elie Cartan, 1926) . *a) A Riemannian symmetric space M is a product $M = M_- \times M_0 \times M_+$ with $M_\pm \in \mathcal{E}_\pm$ and M_0 flat.*
b) When M is in $\mathcal{E}_-$, it is isometric to a unique quotient G/K where G is a connected semisimple Lie group with trivial center and no compact factor.

This group G is the connected component of the group $Isom(M)$.
c) There is a bijection $M_- \leftrightarrow M_+$ between $\mathcal{E}_-$ and $\mathcal{E}_+$ given by, when $M_- = G/K$ as above then $M_+ = U/K$ where U is the connected compact Lie group with trivial center whose Lie algebra is $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{s} \subset \mathfrak{g}_{\mathbb{C}}$.

Example 1.12. *The real simple Lie algebra $\mathfrak{g} = \mathfrak{so}(n,1)$ corresponds to the hyperbolic space $\mathbb{H}^n$.*

The real simple Lie algebra $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$ corresponds to the space $M \simeq \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n,\mathbb{R}) \simeq \{\text{Euclidean norms on } \mathbb{R}^n \text{ of volume } 1\}$.

As a consequence of non-positive curvature on the symmetric space $M = G/K$, there is a unique geodesic between two points, hence one gets the decomposition

Fact 1.13. (Cartan decomposition $G = KS$) *Let G be a connected semisimple Lie algebra. Let K and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ as above. For all g in G there is a unique $k \in K$ and $X \in \mathfrak{s}$ such that $g = k\exp(X)$.*

Example 1.14. *When $G = \mathrm{SL}(n,\mathbb{R})$ one gets the familiar fact that any invertible matrix g is the product $g = ks$ of an orthogonal matrix k and a positive definite symmetric matrix s. And this decomposition is unique.*

It might happens that a connected simple Lie group has an infinite center. This is the case with the universal cover of the group $\mathrm{SL}(2,\mathbb{R})$.

The following fact tells us exactly when this happens.

Fact 1.15. *Let G be a connected simple Lie group. The following are equivalent:*
(i) The universal cover $\widetilde{G}$ has infinite center $\Longleftrightarrow$
(ii) The Lie algebra $\mathfrak{k}$ has infinite center $\Longleftrightarrow$
(iii) The symmetric space $M = G/K$ has a G-invariant complex structure.

In this case one says that M is a hermitian symmetric space.

1.4 Cartan decomposition

Definition 1.16. A *Cartan subspace* $\mathfrak{a} \subset \mathfrak{s}$ is a maximal abelian subalgebra of $\mathfrak{g}$ which is included in $\mathfrak{s}$.

Fact 1.17. *All the Cartan subspaces $\mathfrak{a}$ of $\mathfrak{s}$ are $\mathrm{Ad}K$-conjugated. Their dimension r is called the real rank of $\mathfrak{g}$ or the real rank of G.*

The elements of ad$\mathfrak{a}$ are symmetric for the negative definite bilinear form B_θ, therefore one can diagonalize them simultaneously. For $\alpha \in \mathfrak{a}^*$, we introduce the root space

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X \ , \ \forall H \in \mathfrak{a}\}$$

and $m_\alpha := \dim \mathfrak{g}_\alpha$ the multiplicity. We introduce the set of restricted roots

$$\Sigma := \{\alpha \in \mathfrak{a}^* \smallsetminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\,\}$$

so that one has

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \oplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha.$$

Remark that one has, for all α, β in $\mathfrak{a}^*$,

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \quad \text{and} \quad \theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}. \tag{1.1}$$

We denote by $\mathring{\mathfrak{a}}_+$ a connected component of $\mathfrak{a} \smallsetminus \cup_{\alpha\in\Sigma} \operatorname{Ker}\alpha$ and by $\mathfrak{a}_+$ the *Weyl chamber* equal to the closure of $\mathring{\mathfrak{a}}_+$. We also introduce the set of positive roots

$$\Sigma_+ := \{\alpha \in \Sigma \mid \alpha(H) \geq 0 \ \forall H \in \mathfrak{a}_+\}.$$

Fact 1.18. (Cartan decomposition $G = KA_+K$) *Every $g \in G$ can be written as $g = k_1 e^X k_2$ with k_1, k_2 in K and X in $\mathfrak{a}_+$. The element $\kappa(g) := X$ is uniquely defined. The map $\kappa : G \to \mathfrak{a}_+$ is continuous and proper. It is called the Cartan projection.*

The Cartan projection $\kappa(g)$ can be thought of as a sequence of logarithm of various norms of g. This is particularly clear in the following example.

Example 1.19. *When $G = \mathrm{SL}(n,\mathbb{R})$, one can choose for Weyl chamber*

$$\mathfrak{a}_+ = \{\operatorname{diag}(\lambda_1, \ldots, \lambda_n) \mid \lambda_1 \geq \cdots \geq \lambda_n \text{ and } \textstyle\sum_k \lambda_k = 0\},$$

and the Cartan projection is given by

$$\kappa(g) = \operatorname{diag}\left(\log\|g\|, \log\frac{\|\Lambda^2 g\|}{\|g\|}, \log\frac{\|\Lambda^3 g\|}{\|\Lambda^2 g\|}, \ldots, \log\frac{1}{\|\Lambda^{n-1} g\|}\right)$$

One has a simple formula for the Haar measure $\mathrm{d}g$ on a connected semisimple Lie group G using the Cartan decomposition. For all $f \in C_c(G)$, one has

$$\int_G f(g)\mathrm{d}g \ = \ \int_{K\times\mathfrak{a}_+\times K} f(k_1 e^X k_2) \prod_{\alpha\in\Sigma_+} \sinh(\alpha(X))^{m_\alpha}\, \mathrm{d}k_1 \mathrm{d}X \mathrm{d}k_2. \tag{1.2}$$

1.5 Cartan projection

The following control on the Cartan projection is very useful. Let $\mathfrak{a}(r)$ denotes the ball $B(0,r)$ of radius r in $\mathfrak{a}$.

Proposition 1.20. *For all compact subset $B \Subset G$, there exists $r > 0$ such that for all g in G, one has*

$$\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r).$$

Proof for $G = \mathrm{SL}(n,\mathbb{R})$. We can assume $B = B^{-1}$. According to Example 1.19, we only need a uniform bound for $|\log \frac{\|\Lambda^k(b_1 g b_2)\|}{\|\Lambda^k(g)\|}|$ when b_1, b_2 are in B and g is in G. Such a bound is given by the constant $\sup\limits_{b\in B} \log \|\Lambda^k b\|^2 < \infty$. □

As a corollary we get an asymptotic control on the volume of these sets BgB.

Corollary 1.21. *For all compact subset $B \Subset G$ of non empty interior, there exists $0 < c < C$ such that for all g in G, one has*

$$c\, e^{2\rho(\kappa(g))} \leq \mathrm{vol}(BgB) \leq C\, e^{2\rho(\kappa(g))}, \tag{1.3}$$

where $\rho := \frac{1}{2} \sum\limits_{\alpha\in\Sigma_+} m_\alpha \alpha$.

Proof of the upper bound. We use Proposition 1.20 together with Equation (1.2) and get, for some $r > 0$,

$$\begin{aligned} \mathrm{vol}(BgB) \;\leq\; \mathrm{vol}(K e^{\kappa(g)+\mathfrak{a}(r)} K) \;&=\; \int_{\kappa(g)+\mathfrak{a}(r)} \prod_{\alpha>0} |\sinh \alpha(X)|^{m_\alpha} \mathrm{d}X \\ \ll \int_{\kappa(g)+\mathfrak{a}(r)} e^{2\rho(X)} \mathrm{d}X \;&\simeq\; C\, e^{2\rho(\kappa(g))}. \end{aligned}$$

This proves the upper bound. □

Proof of the lower bound. The calculation is similar. We can assume that B is a small neighborhood of e in G. We introduce the smaller neighborhood $B' := \cap_{k\in K} kBk^{-1}$. and choose $\varepsilon > 0$ and neighborhood K_ε of e in K such that $B' \supset K_\varepsilon e^{\mathfrak{a}(\varepsilon)} K_\varepsilon$. We notice that for $g = k_1 e^{\kappa(g)} k_2$ one has,

$$\begin{aligned}\mathrm{vol}(BgB) &\geq \mathrm{vol}(B'e^{\kappa(g)}B') \geq \mathrm{vol}(K_\varepsilon e^{\kappa(g)+\mathfrak{a}(\varepsilon)}K_\varepsilon) \\ &\geq \mathrm{vol}_K(K_\varepsilon)^2 \int_{\kappa(g)+\mathfrak{a}(\varepsilon)} \prod_{\alpha>0} \sinh(\alpha(X))^{m_\alpha} \mathrm{d}X \simeq c\, e^{2\rho(\kappa(g))}.\end{aligned}$$

This proves the lower bound. □

Remark 1.22. One may wonder why these sets BgB are important. Here is an answer: when $G = \mathrm{S0}(2,1)$ and when B is assumed to be connected and K-biinvariant, then the images of the sets BgB in the hyperbolic plane $\mathbb{H}^2$ are roughly annuli of bounded width whose radius is roughly equal $\kappa(g)$.

Lecture 2. Subgroups of semisimple groups

In this lecture we focus on the structure of the semisimple Lie groups, we give useful formulas for the Haar measure. We also focus on the structure of the subgroups H of G. First in the case where H is connected and then in the case where H is discrete. I refer to [12] or to [5] for more details.

In this whole lecture G denotes a connected semisimple Lie group. For simplicity we assume that G has trivial center. Thanks to the adjoint representation in its Lie algebra one has an isomorphism

$$\mathrm{Ad} : G \xrightarrow{\sim} \mathrm{Aut}(\mathfrak{g})_e,$$

and this way G is identified with a group of matrices, which is the connected component of the algebraic subgroup $\mathrm{Aut}(\mathfrak{g})$ of $\mathrm{GL}(\mathfrak{g})$.

We keep the notations of the previous lecture and add a few others:
θ is a Cartan involution of $\mathfrak{g}$,
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the associated Cartan decomposition of $\mathfrak{g}$,
$\mathfrak{a} \subset \mathfrak{s}$ is a Cartan subspace and $\mathfrak{m} := \mathfrak{g}_0 \cap \mathfrak{k}$,
$\Sigma \subset \mathfrak{a}^*$ is the restricted root system,
$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \oplus_{\alpha\in\Sigma} \mathfrak{g}_\alpha$ is the root space decomposition,
$\mathfrak{a}_+$ is a Weyl chamber, Σ_+ the corresponding set of positive roots.

2.1 Iwasawa decomposition

Let $\mathfrak{n}$ be the nilpotent subalgebra $\mathfrak{n} := \oplus_{\alpha\in\Sigma_+} \mathfrak{g}_\alpha$, and $\rho = \frac{1}{2} \sum\limits_{\alpha\in\Sigma_+} m_\alpha \alpha$,
so that one has the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. Let K, A and N be the connected subgroups of G with Lie algebra $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$.
Let $\mathfrak{p}$ be the standard minimal parabolic subalgebra $\mathfrak{p} := \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$.
Let P be the minimal parabolic subgroup $P := N_G(\mathfrak{p}) = MAN$, where $M := Z_K(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in K.
Let $G/P = \{\text{minimal parabolic subalgebra } \mathrm{Ad}g(\mathfrak{p})\}$ It is called the flag variety of G or the Furstenberg boundary of G.

Fact 2.1. (Iwasawa decomposition $G = KAN$**)** *Every $g \in G$ can be written in a unique way as a product $g = ke^X n$ with $k \in K$, $X \in \mathfrak{a}$ and $n \in N$.*

The map $(g, \mathfrak{p}) \mapsto \eta(g) = \eta(g, \mathfrak{p}) := X$ is called the Iwasawa cocycle. It is a cocycle on the flag variety: one has $\eta(g_1 g_2, \mathfrak{p}) = \eta(g_1, \mathrm{Ad} g_2(\mathfrak{p}))\eta(g_2, \mathfrak{p})$ for all g_1, g_2 in G.

Definition 2.2. A standard parabolic subgroup Q of G is a subgroup that contains P. A parabolic subgroup of G is a conjugate of a standard parabolic subgroup.

The parabolic subgroups are easy to classify. One can check that the conjugate subgroup Q containing P is unique. Let Π be the set of simple restricted roots, those are the positive roots that are not sums of two positive roots. This set Π is a basis of $\mathfrak{a}$. Its cardinality is the real rank r of G.

Fact 2.3. *The map $Q \mapsto \{\alpha \in \Pi \mid \mathfrak{g}_{-\alpha} \subset \mathfrak{q}\}$ is a bijection between the set of standard parabolic subgroups of G and the set of subsets of Π.*

The minimal parabolic P corresponds to the empty set $\emptyset \subset \Pi$.
The group G corresponds to the full set Π.
In particular, there are 2^r conjugacy classes of parabolic subgroups of G.

Example 2.4. *When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the group P is the subgroup of upper triangular matrices in G. The flag variety G/P is also the set of complete flags $\xi = (\xi_k)_{1 \le k \le n}$ of $\mathbb{R}^n$. This means that ξ_k is a k-dimensional vector subspace of $\mathbb{R}^n$ and that $\xi_k \subset \xi_{k+1}$, for all k.*

The Iwasawa decomposition tells us that every flag ξ can be defined thanks to an orthonormal basis $(u_1, \ldots, u_n)$ of $\mathbb{R}^n$ by $\xi_k = \langle u_1, \ldots, u_k \rangle$, for all k. This is the familiar Gram-Schmidt orthonormalization process: any invertible matrix g is the product $g = kt$ of an orthogonal matrix k and an upper triangular matrix t with positive diagonal coefficients. And this decomposition is unique.

In this case the standard parabolic subgroups are parametrized by finite sequences $(n_1, \ldots, n_\ell)$ with $n_j \ge 1$ and $n_1 + \cdots + n_\ell = n$: the corresponding group Q is the group of elements of G which are block-upper-triangular with ℓ blocks of successive sizes $n_1, \ldots, n_\ell$.

One has a simple formula for the Haar measure $\mathrm{d}g$ on a connected semisimple Lie group G using the Iwasawa decomposition: for all $f \in C_c(G)$,

$$\int_G f(g)\mathrm{d}g \;=\; \int_{K \times \mathfrak{a} \times N} f(k e^X n) e^{2\rho(X)}\, \mathrm{d}k \mathrm{d}X \mathrm{d}n. \tag{2.1}$$

Indeed both sides of (2.1) define a left K-invariant and right AN-invariant measure on G.

2.2 Bruhat Decomposition

We need some extra notation. Let $M^* := N_K(\mathfrak{a})$ be the normalizer of $\mathfrak{a}$ in K and let $W := M^*/M$ be the Weyl group

Fact 2.5. *The group $W \hookrightarrow \mathrm{GL}(\mathfrak{a})$ is generated by the root reflections s_α. The group W acts simply transitively on the set of Weyl chambers in $\mathfrak{a}$.*

Let $\mathfrak{n}_- := \theta(\mathfrak{n})$ be the opposite nilpotent subalgebra so that one has the Bruhat decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{p}.$$

Let $P_- := \theta(P)$ be the opposite minimal parabolic subgroup. One has the equality $P_- = MAN_-$ where $N_- := \theta(N)$. The Bruhat decomposition of G describes the N_--orbits in the flag variety G/P of G.

Fact 2.6. (Bruhat decomposition of G)
a) One has the decomposition as a disjoint union $G = \cup_{w\in W} N_- w P$.
b) The cell $\Omega := N_- P$ is open and the others $N_- wP$ have codimension ≥ 1.
c) Every $g \in \Omega$ can be written in a unique way as a product $g = n_- m e^X n$ with $n_- \in N_-$, $m \in M$, $X \in \mathfrak{a}$ and $n \in N$.

Example 2.7. *When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, we have seen that the flag variety G/P is the set of complete flags $\xi = (\xi_k)_{1\leq k\leq n}$ of $\mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. The standard parabolic P is the stabilizer of the flag ξ_0 for which $\xi_{0,k} = \langle e_1, \ldots, e_k\rangle$ while the parabolic P_- is the stabilizer of the flag ξ_0^- with $\xi_{0,k}^- = \langle e_n, \ldots, e_{n-k+1}\rangle$. The group W is isomorphic to the group $\mathcal{S}_n$ of permutations of $\{1, \ldots, n\}$. The open set Ω/P is then the set of flags ξ which are in general position with ξ_0^-, that is, for all k, one has $\xi_k \oplus \xi_{0,n-k+1}^- = \mathbb{R}^n$.*

One has a simple formula for the Haar measure $\mathrm{d}g$ on a connected semisimple Lie group G using the Bruhat decomposition. For all $f \in C_c(G)$, one has

$$\int_G f(g)\mathrm{d}g \;=\; \int_{N_-\times M\times\mathfrak{a}\times N} f(n_- m e^X n) e^{2\rho(X)}\, \mathrm{d}n_- \mathrm{d}m \mathrm{d}X \mathrm{d}n. \qquad (2.2)$$

Indeed both sides of (2.1) define a left N_--invariant and right MAN-invariant measure on G.

As a corollary we get an asymptotic control on the volume of the sets $gBg^{-1} \cap B$.

Corollary 2.8. *For $B \Subset G$ small neighborhood of e in G, there exist constants $0 < c < C$ such that for all g in G, one has*

$$c\, e^{-2\rho(\kappa(g))} \leq \mathrm{vol}(gBg^{-1} \cap B) \leq C\, e^{-2\rho(\kappa(g))}. \tag{2.3}$$

Proof. As in Corollary 1.21, we can assume that $g = e^X$ with X in $\mathfrak{a}_+$, and that B is a product $B = B_{N_-} B_M B_A B_N$ where these sets B_H are suitable small neighborhoods of the identity in the corresponding subgroup H, chosen so that $gB_{N_-}g^{-1} \subset B_{N_-}$ and $B_N \subset gB_N g^{-1}$. Then one has the equality

$$gBg^{-1} \cap B \;\;=\;\; (gB_{N_-}g^{-1})B_M B_A B_N$$

We use Equation (2.2). and get,

$$\mathrm{vol}_G(gBg^{-1} \cap B) \;\simeq\; c'\mathrm{vol}_{N_-}(gB_{N_-}g^{-1}) \;\simeq\; c''\mathrm{det}_{\mathfrak{n}_-}(\mathrm{Ad}g) \;\simeq\; c''\, e^{-2\rho(\kappa(g))}.$$

This proves both inequalities. □

2.3 Connected subgroups

We recall that the semisimple Lie group $G \simeq \mathrm{Aut}_e(\mathfrak{g})$ is a group of matrices. Hence it is endowed with the Zariski topology.

Definition 2.9. A subgroup of G is algebraic in G if it is Zariski closed in G.

For instance, the subgroups K, MA, N and P are algebraic subgroups of G. The Zariski closure in G of a subgroup Γ of G is an algebraic subgroup of G.

The following proposition tells us that the connected subgroups H of G are not "far" from an algebraic group.

Proposition 2.10. *Let $H \subset G$ be a connected subgroup. Then there exists an algebraic subgroup H_0 of G containing H such that H is a normal subgroup of H_0 and the quotient H_0/H is abelian.*

Proof. We choose H_0 to be the Zariski closure of H in G. In each step of the following proof, the key point is the fact that an algebraic property satisfied on H is also satisfied on H_0.

Step 1 H_0 is a subgroup of G. Indeed, since H is included in H_0,

$$\begin{aligned} \text{for all } h_1,\, h_2 \text{ in } H \quad &\text{one has} \quad h_1 h_2^{-1} \in H_0, \\ \text{hence, for all } h_1,\, h_2 \text{ in } H_0 \quad &\text{one has} \quad h_1 h_2^{-1} \in H_0. \end{aligned}$$

This tells us that H_0 is a subgroup of G.

Step 2 H is a normal subgroup of H_0. Indeed, since H normalizes $\mathfrak{h}$,

$$\begin{aligned} \text{for all } h \text{ in } H \quad &\text{one has} \quad (\mathrm{Ad}h - \mathbf{1})\mathfrak{h} \subset \mathfrak{h}, \\ \text{hence, for all } h \text{ in } H_0 \quad &\text{one has} \quad (\mathrm{Ad}h - \mathbf{1})\mathfrak{h} \subset \mathfrak{h}. \end{aligned}$$

This implies that $[\mathfrak{h}_0, \mathfrak{h}] \subset \mathfrak{h}$ and H is a normal subgroup of H_0.

Step 3 The quotient H_0/H is abelian. Indeed, since H is normal in H_0,

$$\begin{aligned} \text{for all } h \text{ in } H \quad &\text{one has} \quad (\mathrm{Ad}h - \mathbf{1})\mathfrak{h}_0 \subset \mathfrak{h}, \\ \text{hence, for all } h \text{ in } H_0 \quad &\text{one has} \quad (\mathrm{Ad}h - \mathbf{1})\mathfrak{h}_0 \subset \mathfrak{h}. \end{aligned}$$

This implies that $[\mathfrak{h}_0, \mathfrak{h}_0] \subset \mathfrak{h}$ and H_0/H is abelian. □

Definition 2.11. A connected algebraic subgroup H of G is reductive if H is a product $H = SZ$ with S semisimple, with Z central in H and such that, for all z in Z, the matrix $\mathrm{Ad}(z) \in \mathrm{GL}(\mathfrak{g})$ is semisimple.

Fact 2.12. *Let $H \subset G$ be a reductive subgroup. Then there exists a Cartan decomposition $G = KA_+K$ such that $H = (K \cap H)(A \cap H)(K \cap H)$.*

Interpretation: this means that such a subgroup H corresponds to a totally geodesic symmetric subspace of G/K.

Note that in this fact one can not replace $A \cap H$ by $A_+ \cap H$. Think of the case $H = A$.

Fact 2.13. *A connected subgroup H of G is either reductive or is included in a proper parabolic subgroup $H \subset Q \subsetneq G$.*

2.4 Zariski dense subgroups

We have seen that a connected proper subgroup of G is never Zariski dense in G. More generally, one has

Proposition 2.14. *Let $H \subsetneq G$ be a closed Zariski dense subgroup of G. Assume that G is simple. Then the group H is discrete.*

Proof. We argue as in the proof of Proposition 2.10. The Lie algebra $\mathfrak{h}$ is normalized by H, hence it is also normalized by G, hence it is an ideal of $\mathfrak{g}$, hence it is trivial and H is discrete. □

Now I want to discuss how one can understand the behavior of H at infinity thanks to the Cartan projection $\kappa : G \to \mathfrak{a}^+$.

Let L_H be the limit cone of $\kappa(H)$,

$$L_H := \{v \in \mathfrak{a}^+ \mid \exists h_n \in H \ , \, t_n \to 0 \, \text{such that} \, v = \lim_{n\to\infty} t_n \kappa(h_n)\}$$

Proposition 2.15. *If the subgroup H of G is Zariski dense, then the limit cone L_H is a convex cone of $\mathfrak{a}_+$ of non empty interior*

Sketch of proof of convexity of L_H. One would like an equality of the form $\kappa(h_1h_2) \overset{?}{\simeq} \kappa(h_1) + \kappa(h_2)$. Such a statement may be completely wrong, for instance when $h_2 = h_1^{-1}$. But we can use □

Lemma 2.16. *Let $H \subset G$ be a Zariski dense subgroup. Then there exists a finite subset $F \subset H$ and a constant $r > 0$ such that, for all h_1, h_2 in H, there exists f in F such that*

$$\kappa(h_1 f h_2) \in \kappa(h_1) + \kappa(h_2) + \mathfrak{a}(r).$$

Lecture 3. Critical exponents

This lecture is based on my joint paper [4] with Siwei Liang.

Let G be a connected semisimple Lie group and H be a closed subgroup of G. Our aim is to introduce a few natural exponents associated to H, to explain their meanings and to compare them.

To simplify the notation, we will assume that the group G has trivial center, and that the subgroup H is unimodular and non-compact. In this case the homogeneous space G/H admits a G-invariant measure $\mathrm{d}(gH)$, and, for all $\varphi \in \mathcal{C}_c(G)$, one has

$$\int_G \varphi(g)\mathrm{d}g \;=\; \int_{G/H} \left(\int_H \varphi(gh)\mathrm{d}h \right) \mathrm{d}(gH). \tag{3.1}$$

We recall the notation from the previous lecture: $G = K\exp\mathfrak{a}_+ K$ is a Cartan decomposition, $\kappa : G \to \mathfrak{a}_+$ is the corresponding Cartan projection, and $\rho \in \mathfrak{a}^*$ is $\rho := \frac{1}{2} \sum\limits_{\alpha>0} m_\alpha \alpha$.

3.1 A few exponents

3.1.1 The critical exponent δ_H

We fix a compact subset B in G of non-empty interior. The critical exponent δ_H is the relative growth speed in H of $BgB \cap H$ compared to the growth speed of BgB in G. More precisely

$$\begin{aligned} \delta_H \;&:=\; \limsup_{g\to\infty} \frac{\log \mathrm{vol}_H(BgB\cap H)}{\log \mathrm{vol}_G(BgB)} \\ &:=\; \inf\{t>0 \mid \int_H e^{-t\rho(\kappa(h))}\mathrm{d}h \;<\; \infty\} \end{aligned} \tag{3.2}$$

In our definition of the critical exponent there is a renormalization so that

$$\delta_H \in [0,1] \;\text{ and }\; \delta_G = 1.$$

For instance when the real rank of G is 1, the relation with the classical critical exponent introduced by Patterson and Sullivan is $\delta_H = \delta_H^{clas}/\delta_G^{class}$

3.1.2 The decay exponent θ_H

The decay speed σ_H is the worst exponential decay speed of the correlation function of two compact sets D_1 and D_2, which is the volume of the intersection of gD_1 with D_2 where g is a large element of G.

The decay exponent is $\theta_H := 1 - \sigma_H$. More precisely,

$$\begin{aligned}\theta_H \;=\; & \sup\{\theta \in [0,1] \mid \text{for all } D_1, D_2 \Subset G/H \text{ there exists } C > 1 \\ & \text{such that, for all } g \in G,\ \mathrm{vol}_{G/H}(gD_1 \cap D_2) \leq C\, e^{-(1-\theta)2\rho(\kappa(g))}\,\}.\end{aligned} \tag{3.3}$$

3.1.3 The integrability constant

The integrability constant controls the integrability properties of the powers of the correlation functions. More precisely,

$$\begin{aligned}p_H \;=\; & \inf\{p \geq 1 \mid \text{for all } D_1, D_2 \Subset G/H \text{ the function} \\ & g \mapsto \mathrm{vol}_{G/H}(gD_1 \cap D_2) \text{ belongs to } L^p(G)\}.\end{aligned} \tag{3.4}$$

3.1.4 The local exponent

The local exponent β_H is a purely algebraic quantity that controls the exponential decay of the function $h \mapsto \mathrm{vol}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{Ad}hD_0 \cap D_0)$ where D_0 is a compact neighborhood of 0 in $\mathfrak{g}/\mathfrak{h}$. More precisely, denoting $t_+ := \max(t, 0)$, one sets

$$\beta_H \;:=\; \sup_{\substack{X \in \mathfrak{h} \\ \text{not nilpotent}}} \frac{\rho_{\mathfrak{h}}(X)}{\rho_{\mathfrak{g}}(X)} \quad \text{where} \quad \rho_{\mathfrak{h}}(X) := \frac{1}{2} \sum_{\lambda \in \mathrm{Sp}(\mathrm{ad}_{\mathfrak{h}}(X))} Re(\lambda)_+ \tag{3.5}$$

When all the elements X in $\mathfrak{h}$ are nilpotent, we set $\beta_H = 0$.

3.2 Comparing the exponents

The main new result in this lecture series is the following comparison theorem for all these exponents.

Theorem 3.1. (Y. B., Siwei Liang) *Let G be a connected semisimple Lie group with trivial center and H be a unimodular subgroup.*
a) One has $1 - \frac{1}{p_H} = \theta_H = \delta_H \geq \beta_H$.
b) If H is reductive, one also has $\delta_H = \beta_H$.
c) If H is connected and if $\delta_H > \frac{1}{2}$, one also has $\delta_H = \beta_H$.

Previously known results ⋆ When H is connected, and p_H is an even integer, this theorem is due to Y. B. and T. Kobayashi in [2] and [3].
⋆ When H is discrete, this theorem is due to Lutsko, Weich and Wolf in [15].

Example 3.2. *The statements in b) and c) are optimal. Indeed, when H is connected, one does not always have the equality $\delta_H = \beta_H$. For instance when $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ and $H = N \times A$ where N is the upper unipotent triangular subgroup and A the diagonal subgroup. One can compute $\delta_H = \frac{1}{2}$ and $\beta_H = 0$.*

The aim of this lecture is not to give a full proof of this theorem. Such a proof can be found in our paper [4]. Instead we will only sketch the proof and explain how to use what we know about the geometry of H and G to prove these equalities and inequalities.

Point *a*) and *b*) will be discussed in this lecture. Point *c*) will be postponed to the next lecture since it relies on unitary representation.

3.3 Critical exponent and decay exponent

3.3.1 Sketch of the inequality $1 - \frac{1}{p_H} \leq \theta_H$

We use the formula (1.2) for the Haar measure and compute, for all $\theta > \theta_H$,

$$\int_G \mathrm{vol}_{G/H}(gD_1 \cap D_2)^p \mathrm{d}g \;\leq\; C^p \int_{\mathfrak{a}_+} e^{-2(1-\theta)p\,\rho(X)} e^{2\rho(X)} \mathrm{d}X.$$

This quantity is finite as soon as $1 - \frac{1}{p} > \theta$. This proves that $1 - \frac{1}{p_H} \leq \theta$ and hence $1 - \frac{1}{p_H} \leq \theta_H$.

3.3.2 Sketch of the inequality $\theta_H \leq \delta_H$

We fix two positive functions f_k in $\mathcal{C}_c(G/H)$, with $k = 1, 2$, and we want to bound the integral

$$I_g \;:=\; \int_{G/H} f_1(xH)\, f_2(gxH) \mathrm{d}(xH).$$

The first idea is to notice that, one can find positive functions $\varphi_k \in \mathcal{C}_c(G)$ such that these functions f_k can be written as

$$f_k(xH) = \int_H \varphi_k(xh) \mathrm{d}h.$$

After splitting these functions φ_k in small pieces, one can assume that the support $B_k := \mathrm{supp}\varphi_k$ is such that the compact set $\widetilde{B_k} := B_k B_k^{-1}$ is a small neighborhood of e in G. We will use the following lemma:

Lemma 3.3. *a) One has*

$$I_g = \int_H \Phi(g,h)\mathrm{d}h \ \text{ where } \ \Phi(g,h) := \int_G \varphi_1(x)\varphi_2(gxh^{-1})\mathrm{d}x.$$

b) One has the following bound where the constant C is uniform in g and h,

$$\Phi(g,h) \ \leq \ C\,\mathrm{vol}_G(\widetilde{B_1}\cap g^{-1}\widetilde{B_2}g)\ \mathbf{1}_{B_2^{-1}gB_1}(h). \tag{3.6}$$

We first take for granted this lemma and prove our inequality. For all $\delta > \delta_H$, we compute, using both the controls (1.3) and (2.3), and setting $B = B_1 \cup B_2^{-1}$,

$$\begin{aligned} I_g &\leq C\,\mathrm{vol}_G(\widetilde{B_1}\cap g^{-1}\widetilde{B_2}g)\mathrm{vol}_H(B_2^{-1}gB_1\cap H) \\ &\leq C'\,\frac{\mathrm{vol}_H(BgB\cap H)}{\mathrm{vol}_G(BgB)} \ \leq \ C''\,e^{-2(1-\delta)\rho(\kappa(g))}. \end{aligned}$$

This proves that $\theta_H \leq \delta$ and hence $\theta_H \leq \delta_H$.

Sketch for Lemma 3.3. *a*) We compute using (3.1)

$$\begin{aligned} I_g &= \int_{G/H}\int_H \varphi_1(xh_1)\mathrm{d}h_1 \int_H \varphi_2(gxh_1h_2)\mathrm{d}h_2\,\mathrm{d}(xH) \\ &= \int_G\int_H \varphi_1(x)\varphi_2(gxh_2)\,\mathrm{d}h_2 \ = \ \int_H \Phi(g,h)\,\mathrm{d}h. \end{aligned}$$

b) If $\Phi(g,h)$ is non zero, there exists x_0 in $B_1\cap g^{-1}B_2h$. In particular h belongs to $B_2^{-1}gB_1$. Moreover, when $\varphi_1(x)\varphi_2(gxh^{-1})$ is non zero, then the element x is also in $B_1\cap g^{-1}B_2h$, and hence it belongs to $\widetilde{B_1}x_0\cap g^{-1}\widetilde{B_2}gx_0$ This proves the inequality (3.6) with the constant $C = \|\varphi_1\|_\infty\|\varphi_2\|_\infty$. □

3.3.3 Sketch of the inequality $\delta_H \leq 1 - \frac{1}{p_H}$

We skip this part which relies on similar computations. See [4].

3.4 Critical exponent and local exponent

3.4.1 Sketch of the inequality $\beta_H \leq \theta_H$

To avoid a few technicalities, we will assume that H is algebraic. We denote by $\mathfrak{f}$ a Levi factor of $\mathfrak{h}$. This means that $\mathfrak{f}$ is a maximal reductive subalgebra of $\mathfrak{h}$. We denote by $\mathfrak{a}_\mathfrak{h}$ a Cartan subspace of $\mathfrak{f}$. We can assume that the Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$ contains $\mathfrak{a}_h$. In the definition (3.5) of β_H we can restrict to those X belonging to $\mathfrak{a}_\mathfrak{h}$. In the definition (3.3), we focus on elements $g^t = e^{-tX}$ with X in $\mathfrak{a}_\mathfrak{h}$ and $t > 0$ large. We fix an $\mathfrak{a}_\mathfrak{h}$ invariant complementary subspace $\mathfrak{c}$ of $\mathfrak{h}$ in $\mathfrak{g}$. We choose D to be $D = \exp(D_0)$ where D_0 is a small neighborhood D_0 of 0 in $\mathfrak{c}$. The local action of g^t on D is the same as the local action of $\mathrm{Ad} g^t = e^{-t\mathrm{ad}X}$ on D_0. One computes

$$\mathrm{vol}_{G/H}(e^{-t\mathrm{ad}X} D \cap D) \;\geq\; C\, \mathrm{vol}_{\mathfrak{g}/\mathfrak{h}}(e^{-t\mathrm{ad}X} D_0 \cap D_0) \;\simeq\; C'\, e^{-2t\rho_{\mathfrak{g}/\mathfrak{h}}(X)}.$$

On the other hand, for all $\theta > \theta_H$,

$$\mathrm{vol}_{G/H}(e^{-t\mathrm{ad}X} D \cap D) \;\leq\; C''\, e^{-2(1-\theta)\rho(\kappa(g^t))} \;=\; C''\, e^{-2t(1-\theta)\rho_\mathfrak{g}(X)}.$$

Since this is true for all $t > 0$, one gets

$$\rho_{\mathfrak{g}/\mathfrak{h}}(X) \geq (1-\theta)\rho_\mathfrak{g}(X), \quad \text{that is,} \quad \rho_\mathfrak{h}(X) \leq \theta \rho_\mathfrak{g}(X).$$

This proves that $\beta_H \leq \theta$, and hence $\beta_H \leq \theta_H$.

3.4.2 Sketch of the inequality $\theta_H \leq \beta_H$ for H reductive

One key point is to use compatible Cartan decompositions

$$G = K \exp(\mathfrak{a}) K \quad \text{and} \quad H = (K \cap H) \exp(\mathfrak{a}_\mathfrak{h})\, (K \cap H), \tag{3.7}$$

where $\mathfrak{a}_\mathfrak{h} := \mathfrak{a} \cap \mathfrak{h}$. We fix a compact set $B \Subset G$ and we are looking for an upper bound of the volume $\mathrm{vol}_H(BgB \cap H)$. We can assume that g is in H and, more precisely that $g = e^X$ with X in $\mathfrak{a}_\mathfrak{h}$. In this case one has $\kappa(g) = wX$ for some w in the Weyl group W of G. Therefore, there exists $r > 0$ such that, one has, for all X in $\mathfrak{a}_\mathfrak{h}$,

$$BgB \cap H \subset \cup_{w \in W} (K \cap H) \exp((wX + \mathfrak{a}(r)) \cap \mathfrak{a}_\mathfrak{h})\, (K \cap H).$$

For all w, we choose an element Y_w in $(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_{\mathfrak{h}}$ when it exists, and set $Y_w = 0$ otherwise. One has then an upper bound for the volume,

$$\begin{aligned} \mathrm{vol}_H(BgB \cap H) &\leq C \sum_w e^{2\rho_{\mathfrak{h}}(Y_w)} \leq C \sum_w e^{2\beta_H \rho_{\mathfrak{g}}(Y_w)} \\ &\leq C' \sum_w e^{2\beta_H \rho_{\mathfrak{g}}(wX)} \leq C'' e^{2\beta_H \rho(\kappa(g))}. \end{aligned}$$

This proves that $\theta_H \leq \beta_H$.

Lecture 4. Unitary representations

Let G be a connected semisimple Lie group with finite center. The aim of this lecture is to reinterpret the main theorem of the previous lecture as a relation between the critical exponent δ_H of a closed subgroup H and the harmonic analysis on G/H. In particular when $H = \Gamma$ is discrete and the real rank of G is one we will recover the relation between the critical exponent and the bottom spectrum of the Laplacian in the quotient $\Gamma\backslash G/K$

This reinterpretation is one of the key ingredient for the equality $\beta_H = \theta_H$ when H is connected and $\delta_H \geq 1/2$.

For more on this chapter, one can read [18], [13], [1], [14] and [4].

4.1 Coefficients decay

Definition 4.1. A unitary representation $(\mathcal{H}, \pi)$ of G is a morphism π from G to the unitary group of a Hilbert space $\mathcal{H}$ such that for all v in $\mathcal{H}$ the map $g \mapsto \pi(g)v$ is continuous from G to $\mathcal{H}$.

For v_1, v_2 in $\mathcal{H}$, their *coefficient* is the function $c_{v_1,v_2} : G \to \mathbb{C}$, given by

$$c_{v_1,v_2}(g) := \langle \pi(g)v_1, v_2 \rangle$$

Example 4.2. *Let $H \subset G$ be a unimodular subgroup. The regular representation $\pi = \lambda_{G/H}$ of G in the Hilbert space of square integrable functions $\mathcal{H} = L^2(G/H)$ is given by, for all g in G, f in $L^2(G/H)$ and x in G/H*

$$\lambda_{G/H}(g)f\,(x) \;=\; f(g^{-1}x).$$

For $v_1 = \mathbf{1}_{D_1}$ and $v_2 = \mathbf{1}_{D_2}$ with D_1, $D_2 \Subset G/H$, the coefficient is nothing but the correlation:

$$c_{v_1,v_2}(g) = \mathrm{vol}_{G/H}(gD_1 \cap D_2).$$

Example 4.3. *Let $P = MAN$ be a minimal parabolic subgroup of G. This group is not unimodular, so that one has to modify the previous construction. We denote by $\mathrm{d}\xi$ the K-invariant probability on G/P and one sets*

$$\begin{aligned} L^2(G/P) &:= \{\text{square integrable half densities on } G/P\} \\ &= \{\mu = f(\xi)d\xi^{\frac{1}{2}} \mid \int_{G/P} |f(\xi)|^2 \mathrm{d}\xi \;<\; \infty\,\}. \end{aligned}$$

The unitary action of G on $L^2(G/P)$ is twisted by the square root of the Radon-Nikodym cocycle

$$\lambda_{G/P}(g)\mu \;=\; f(g^{-1}\xi)\left(\frac{\mathrm{d}g^{-1}\xi}{\mathrm{d}\xi}\right)^{\frac{1}{2}}\,\mathrm{d}\xi^{1/2}$$

Note that when $\xi = kP$ with k in K, this cocycle is given by the Iwasawa cocycle η

$$\left(\frac{\mathrm{d}g^{-1}\xi}{\mathrm{d}\xi}\right)^{\frac{1}{2}} \;=\; e^{-\rho(\eta(g^{-1}k))}$$

For v_0 the K-invariant vector $v_0 = \mathrm{d}\xi^{\frac{1}{2}} \in L^2(G/P)$ the coefficient is the Harish-Chandra spherical function:

$$\Xi_0(g) := c_{v_0,v_0}(g) = \int_K e^{-\rho(\eta(g^{-1}k))}\,\mathrm{d}k.$$

The following fact is a very general and useful "mixing property" or "decay property".

Fact 4.4. (Howe, Moore) *Let $(\mathcal{H},\pi)$ be a unitary representation of a simple Lie group G which has no non zero G-invariant vector. Then, for all v_1, v_2 in $\mathcal{H}$, one has*

$$\lim_{g\to\infty}\langle\pi(g)v_1, v_2\rangle = 0.$$

Sketch of proof. The key point is to use the Cartan decomposition $G = KA_+K$ and the Iwasawa decomposition KAN together with the weak compacity of the unit ball in $\mathcal{H}$. Indeed, if there exists a non-zero weak limit v_∞ of a sequence $\pi(g_n)v_1$ with g_n going to infinity.

By using the Cartan decomposition, we can assume that g_n is in A^+ and by using the Iwasawa decomposition we can check that the weak limit is indeed N invariant.

Using then a trick called Mautner lemma, we deduce that v_∞ is A-invariant. Using again both parabolic subgroups $P = MAN$ and $P_- = MAN_-$ we deduce that this vector v_∞ is N_--invariant. □

4.2 Weak containment

Let σ and π be two unitary representations of G.

Definition 4.5. We say that σ is weakly contained in π and we write $\sigma \prec \pi$ if all the coefficients of σ are limits, uniformly on compact sets, of linear combinations of coefficients of π.

When f is in $L^1(G)$ we set $\pi(f) := \int_G f(g)\pi(g)\,\mathrm{d}g$. Here is a useful equivalent definition of weak containement, see [1] or [11].

Fact 4.6. *One has the equivalence:*
$\sigma \prec \pi \iff$ *For all f in $L^1(G)$, one has $\|\sigma(f)\|_{\mathrm{op}} \leq \|\pi(f)\|_{\mathrm{op}}$.*

We recall that λ_G is the regular representation of G in $L^2(G)$.

Definition 4.7. π is tempered if $\pi \prec \lambda_G$

When $(\mathcal{H}, \pi)$ is a unitary representation of G we set

$$\mathcal{H}^{(K)} := \{v \in \mathcal{H} \mid \dim\langle Kv\rangle < \infty\} \tag{4.1}$$

There are three theorems that characterize the unitary representations that are tempered in terms of the coefficients. We set $L^{2+\varepsilon}(G) := \cap_{p>2} L^p(G)$.

Fact 4.8. (Harish-Chandra) *The representation $(\mathcal{H}, \pi)$ is tempered if and only if there exists a dense set of v in $\mathcal{H}$ such that $c_{v,v}$ is in $L^{2+\varepsilon}(G)$.*

Fact 4.9. (Kunze, Stein, Cowling [8]) *The representation $(\mathcal{H}, \pi)$ is tempered if and only if, for all v, w in $\mathcal{H}$, the coefficient $c_{v,w}$ is in $L^{2+\varepsilon}(G)$.*

Fact 4.10. (Cowling, Haagerup, Howe [9]) *The representation $(\mathcal{H}, \pi)$ is tempered if and only if, for all v, w in $\mathcal{H}^{(K)}$, and all g in G, one has*

$$|\langle \pi(g)v, w\rangle| \;\;\leq\;\; \dim\langle Kv\rangle^{\frac{1}{2}} \dim\langle Kw\rangle^{\frac{1}{2}}\, \Xi_0(g).$$

As a consequence of these facts we get the following corollary of our main theorem 3.1

Corollary 4.11. (Y. B., Siwei Liang) *Let G be a connected semisimple Lie group with finite center and $H \subset G$ be a closed subgroup. One has the equivalence:*

$$\lambda_{G/H} \text{ is tempered} \iff \delta_H \leq 1/2.$$

This corollary is due to Lutsko, Weich, Wolf in [15] when H is discrete.

Corollary 4.12. (Y. B., T. Kobayashi) *Let G be a connected semisimple Lie group with finite center and $H \subset G$ be a connected closed subgroup. One has the equivalence:*

$$\lambda_{G/H} \text{ is tempered } \iff \beta_H \leq 1/2.$$

Example 4.13. *When $G = \mathrm{SL}(n, \mathbb{R})$ and $H = SL(m, \mathbb{R})$ as a upper-left block, one has*

$$\lambda_{G/H} \text{ is tempered } \iff 2m \leq n + 1.$$

4.3 Integrability constant

We can now explain the meaning of the integrability constant p_H that we defined in (3.4). For that we introduce the integrability constant of a unitary representation.

Fact 4.14. *Let $(\mathcal{H}, \pi)$ be a unitary representation of G and $p, p' > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. One has the equivalence:*
for all $v, w \in \mathcal{H}$, $c_{v,w} \in L^p(G) \iff$
for all $f \in L^1(G) \cap L^\infty(G)$, there exists $C > 1$ such that $\|\pi(f)\|_{\mathrm{op}} \leq C \|f\|_{L^{p'}}$

This fact follows directly from the closed graph theorem and the duality L^p, $L^{p'}$.

Definition 4.15. For $(\mathcal{H}, \pi)$ a unitary representation of G, we set

$$p(\pi) \ := \ \inf\{p > 1 \mid \text{for all } v, w \in \mathcal{H}, \ \ c_{v,w} \in L^p(G)\}$$

A first remark tells us that the integrability constant is compatible with the weak containment.

Remark 4.16. If $\sigma \prec \pi$, then one has $p(\sigma) \leq p(\pi)$.

This remark follows directly from Facts 4.6 and 4.14.

A second remark gives a lower bound for $p(\pi)$.

Remark 4.17. When G is non compact one always has $p(\pi) \geq 2$

This remark follows from from Fact 4.6 and the fact that on G, there does not exist $\varepsilon > 0$ and a non zero integrable function f on G such that $f * L^2(G) \subset L^{2-\varepsilon}(G)$, or, equivalently, such that $f * L^{2+\varepsilon}(G) \subset L^2(G)$

The following facts give clean generalisations of the characterisation of temperedness.

Fact 4.18. (Samei, Wiersma [16]) *Let $(\mathcal{H}, \pi)$ be a unitary representation of G. Set*

$$\widetilde{p}(\pi) \;\; := \;\; \inf\{p > 1 \mid \text{the set } \{v \in \mathcal{H},\;\; c_{v,v} \in L^p(G)\} \text{ is dense inSameiWiersma } \mathcal{H}\}.$$

Then one has the equality $p(\pi) = \max(\widetilde{p}(\pi), 2)$.

Fact 4.19. (Cowling [10]) *Let $(\mathcal{H}, \pi)$ be a unitary representation of G and $p := p(\pi)$. Then, for all v, w in $\mathcal{H}^{(K)}$, and all g in G, one has*

$$|\langle \pi(g)v, w\rangle| \;\; \leq \;\; \dim\langle Kv\rangle^{\frac{1}{2}} \dim\langle Kw\rangle^{\frac{1}{2}}\, \Xi_{\chi_p}(g),$$

where Ξ_{χ_p} is the spherical function given by

$$\Xi_{\chi_p}(g) = \int_K e^{-\frac{2}{p}\rho(\eta(g^{-1}k))}\, \mathrm{d}k.$$

Example 4.20. *One can compute $p(\lambda_G) = 2$ and $\widetilde{p}(\lambda_G) = 1$.*

Let $\widehat{G}$ be the set of (equivalence classes) of irreducible unitary representations of G. For a unitary representation π of G, we define its support

$$S_\pi \;\; := \;\; \{\sigma \in \widehat{G} \mid \sigma \prec \pi\}.$$

It is known that G is type I. This is a fact due to Harish-Chandra which says that π can be decomposed in a "unique" way as an integral

$$\pi \;\; = \;\; \int_{S_\pi}^{\oplus} \sigma^{m_\sigma(\pi)}\, \mathrm{d}\mu(\sigma),$$

where μ is a measure on $\widetilde{G}$ and where $m_\sigma(\pi)$ is a multiplicity.

The integrability constant $p(\pi)$ is useful since it give some information on the support of π. Indeed one has the following corollary

Corollary 4.21. *Let $(\mathcal{H}, \pi)$ be a unitary representation of G. Then one has*

$$p(\pi) = \sup\{p(\sigma) \mid \sigma \in S_\pi\}$$

As a consequence of these facts we get the following relation between the integrability constant of $\lambda_{G/H}$ and the integrability constant of H.

Corollary 4.22. *Let G be a connected semisimple Lie group with finite center and $H \subset G$ be a closed subgroup. Then one has the equality*

$$p(\lambda_{G/H}) = \max(p_H, 2) \quad \text{where} \quad p_H = \frac{1}{1-\delta_H}.$$

When the real rank of G is 1, the desintegration of $L^2(G/H)$ as irreducible unitary representations of G gives precise informations on the spectral decomposition of the Laplacian on $\Gamma\backslash G/K$.

For instance one gets the following corollary.

Corollary 4.23. (Elstrodt, Patterson Sullivan,[17]) *Let $G = SO(n+1,1)$, let $\Gamma \subset G$ be a discrete subgroup and $M := \Gamma \subset \mathbb{H}^n$. We normalize the metric on $\mathbb{H}^n$ so that it has constant curvature $-1/n$. We denote by*

$$\lambda_0(M) := \inf_{\psi \in \mathcal{C}^2_c(M)} \frac{\|\mathrm{d}\psi\|^2_{L^2}}{\|\psi\|^2_{L^2}}. \quad \text{Then one has}$$

$$\begin{aligned} \lambda_0(M) &= 1/4 && \text{when} \quad \delta_\Gamma \le 1/2 \\ \lambda_0(M) &= \delta_\Gamma(1-\delta_\Gamma) && \text{when} \quad \delta_\Gamma \ge 1/2 \end{aligned}$$

We recall that here $\delta_\Gamma \in [0,1]$ is the critical value of Γ normalized so that $\delta_G = 1$.

4.4 Critical exponent and coamenability

We have not proven yet Theorem 3.1.*c*, i.e. the equality $\beta_H = \delta_H$, when the closed subgroup $H \subset G$ is connected and $\delta_H > \frac{1}{2}$.

The first step is to reduce to the case where the group H is algebraic in G. For that we replace H by its Zariski closure H_0. Since the quotient H_0/H is an abelian group, we can apply:

Proposition 4.24. (Y. B., Siwei Liang) *a) Let $H_1 \subset H_2 \subset G$ be two closed subgroups then one has $\delta_{H_1} \le \delta_{H_2}$.*
b) If H_1 is coamenable in H_2, i.e. if $\mathbf{1}_{H_2} \prec \lambda_{H_2/H_1}$ as representations of H_2 and if $\delta_{H_2} > 1/2$, then one has $\delta_{H_1} = \delta_{H_2}$.

Sketch of proof. *a*) We recall that for simplicity all our groups are assumed to be unimodular. The following calculation is an instance of the so-called

"Hertz majoration principle". For all $f \in \mathcal{C}_c(G/H_1)$ we introduce the function $F \in \mathcal{C}_c(G/H_2)$ given by, for all g in G,

$$F(gH_2) \;=\; \left(\int_{H_2/H_1} |f(ghH_1)|^2 \,\mathrm{d}(hH_1)\right)^{\frac{1}{2}}.$$

We compute

$$\begin{aligned}\langle \lambda_{G/H_1}(g)f, f\rangle \;&=\; \int_{G/H_2}\left(\int_{H_2/H_1} f(g^{-1}xh_2H_1)f(xh_2H_1)\,\mathrm{d}(h_2H_1))\right)\mathrm{d}(xH_2)\\ &\leq\; \int_{G/H_2} F(g^{-1}xH_2)F(xH_2)\mathrm{d}(xH_2) \;=\; \langle \lambda_{G/H_1}(g)F, F\rangle.\end{aligned}$$

These coefficients of λ_{G/H_1} are bounded by coefficients of λ_{G/H_2}. Therefore one has $\theta_{H_1} \leq \theta_{H_2}$ and hence $\delta_{H_1} \leq \delta_{H_2}$.

b) Since the group H_1 is coamenable in H_2 the representation λ_{G/H_2} of G is weakly contained in λ_{G/H_1}. Therefore by the remark above, one has $p(\lambda_{G/H_2}) \leq p(\lambda_{G/H_1})$. Since, by assumption, $p_{H_2} > 2$, by Corollary 4.22, one also has $p_{H_2} \leq p_{H_1}$ and therefore $\delta_{H_2} \leq \delta_{H_1}$. □

The second step in the proof of Theorem 3.1.*c* we assume that H is algebraic. We can then introduce a parabolic subgroup Q containing H of minimal dimension. In this case, one can write $Q = LU$ in such a way that $H = (L \cap H)(U \cap H)$ and that $L \cap H$ is a reductive subgroup of L. The idea then is to prove our statement for the pair (G, H) by using the fact that we know it for the pair $(L, L \cap H)$.

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