The Uniqueness of Polynomial Crystallographic Actions

Yves Benoist and Karel Dekimpe[1]

Abstract Let Γ be a polycyclic-by-finite group. It is proved in [8] that Γ admits a polynomial action of bounded degree on $\mathbb{R}^n$ which is properly discontinuous and such that the quotient $\Gamma\backslash\mathbb{R}^n$ is compact. We prove here that such an action is unique up to conjugation by a polynomial transformation of $\mathbb{R}^n$.

1 Introduction

Notations Let $\mathrm{P}(\mathbb{R}^n)$ be the group of polynomial bijections of $\mathbb{R}^n$ with polynomial inverse and $\mathrm{P}^d(\mathbb{R}^n)$ be the subset of polynomial bijections p such that the degrees of p and p^{-1} are bounded by d.

Let Γ be a group. A polynomial action of Γ on $\mathbb{R}^n$ is a morphism $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^n)$. The action is said to be of bounded degree if $\rho(\Gamma)$ is included in $\mathrm{P}^d(\mathbb{R}^n)$ for some d. The action is said to be affine if $\rho(\Gamma) \subset \mathrm{P}^1(\mathbb{R}^n)$. The action is said to be crystallographic if the action of Γ on $\mathbb{R}^n$ is properly discontinuous and if the quotient $\Gamma\backslash\mathbb{R}^n$ is compact. If Γ is a subgroup of $\mathrm{P}(\mathbb{R}^n)$, and ρ is the inclusion, we will say in short, that Γ (instead of ρ) is of bounded degree, affine or crystallographic.

Two polynomial actions ρ_1 and ρ_2 of Γ are said to be polynomially conjugated if there exists p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all g in Γ, one has $p \circ \rho_1(g) = \rho_2(g) \circ p$.

The group Γ is said to be polycyclic-by-finite if one can find an increasing finite sequence $\Gamma_0 = \{1\} \subset \cdots \subset \Gamma_i \subset \cdots \subset \Gamma_n = \Gamma$ of normal subgroups such that, for all $1 \leq i \leq n$, Γ_i/Γ_{i-1} is either finite or abelian of finite rank.

Motivations The affine crystallographic actions have been studied for a

[1]Postdoctoral Fellow of the Fund for Scientific Research – Flanders (F.W.O.)

long time. It is widely believed that any group admitting an affine crystallographic action is polycyclic-by-finite.

(This is sometimes refered to as the Auslander conjecture, see [1] for the latest results on this conjecture.) Let us quote a few known results in this subject:
- Not all polycyclic-by-finite group admit an affine crystallographic action on some $\mathbb{R}^n$ ([3],[5]).
- All polycyclic-by-finite groups admit a polynomial crystallographic action of bounded degree on some $\mathbb{R}^n$ ([8]).
- Two affine crystallographic actions on $\mathbb{R}^n$ of a polycyclic-by-finite group are polynomially conjugated ([9]).

Main result The following theorem which generalizes this last statement completes nicely this list.

Theorem 1.1 *Let Γ be a polycyclic-by-finite group. Then any two polynomial crystallographic actions of Γ of bounded degree on some $\mathbb{R}^n$ are polynomially conjugated.*

The main tool in the proof is the notion of algebraic hull of a polynomial action of bounded degree.

Notice that both our main result and the last one in the list above are generalizations of the second Bieberbach theorem stating that any isomorphism between two (Euclidean) crystallographic groups can be obtained as a conjugation with an affine map. See [13, Theorem 3.2.2] or [6, Theorem 4.1] for more details.

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2 Polynomial and linear actions

Let $\mathrm{P}(\mathbb{R}^m,\mathbb{R}^n)$ be the vector space of polynomial maps from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathrm{P}^d(\mathbb{R}^m,\mathbb{R}^n)$ be the subspace of polynomial maps of degree bounded by d. The map $p \mapsto (p, p^{-1})$ is an injection of $\mathrm{P}^d(\mathbb{R}^n)$ into $\mathrm{P}^d(\mathbb{R}^n,\mathbb{R}^n)\times\mathrm{P}^d(\mathbb{R}^n,\mathbb{R}^n)$. Its image $\{(p,q) \mid p\circ q = q\circ p = 1\}$ is Zariski closed. In this way $\mathrm{P}^d(\mathbb{R}^n)$ is a real algebraic variety[2] and the injections $\mathrm{P}^d(\mathbb{R}^n) \to \mathrm{P}^{d+1}(\mathbb{R}^n)$ are closed

[2] We will use abusively the terminology *real algebraic variety* for the set X of real points of an affine algebraic variety defined over $\mathbb{R}$. We will denote by $\mathbb{R}[X]$ the algebra of real

immersions[3].

The following lemma asserts that every polynomial action of bounded degree can be seen as part of a linear action.

Lemma 2.1 *Let Γ be a group and $\rho : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ be a polynomial action on $\mathbb{R}^n$ of degree bounded by d. Then*
a) there exist an integer $m \geq 1$, a closed immersion $i \in \mathrm{P}(\mathbb{R}^n, \mathbb{R}^m)$ and a linear action τ of Γ on $\mathbb{R}^m$ such that, for every γ in Γ, $\tau(\gamma) \circ i = i \circ \rho(\gamma)$.
b) The Zariski closure $G := A(\rho(\Gamma))$ of $\rho(\Gamma)$ in $\mathrm{P}^d(\mathbb{R}^n)$ is a subgroup of $\mathrm{P}(\mathbb{R}^n)$ which does not depend on the choice of d.

Definition 2.2 *The real algebraic group $A(\rho(\Gamma))$ is called the algebraic hull of $\rho(\Gamma)$.*

A subgroup of $\mathrm{P}(\mathbb{R}^n)$ is said to be Zariski-closed if it is of bounded degree and equal to its algebraic hull.

Proof: a) One can suppose that Γ is a subgroup of $\mathrm{P}(\mathbb{R}^n)$ and that ρ is just the inclusion. The group Γ acts linearly on the vector space $\mathrm{P}(\mathbb{R}^n, \mathbb{R})$ as follows:

$$\forall \gamma \in \Gamma, \forall F \in \mathrm{P}(\mathbb{R}^n, \mathbb{R}) : \ \gamma \cdot F = F \circ \gamma^{-1}$$

Moreover, if F is of degree $\leq d'$, then $\deg(\gamma \cdot F) \leq dd'$. This shows that the action of Γ on $\mathrm{P}(\mathbb{R}^n, \mathbb{R})$ is locally finite. Let $V \subseteq \mathrm{P}(\mathbb{R}^n, \mathbb{R})$ be a Γ-invariant finite dimensional subspace of $\mathrm{P}(\mathbb{R}^n, \mathbb{R})$ containing $\mathrm{P}^1(\mathbb{R}^n, \mathbb{R})$ and τ be the linear representation of Γ in the dual space V^*:

$$\tau : \Gamma \to \mathrm{GL}(V^*), \text{ given by } (\tau(\gamma)\Phi)(F) = \Phi(F \circ \gamma).$$

Now let $i : \mathbb{R}^n \to V^*$ be the map given by the formula:

$$i(v)(F) = F(v).$$

This map is a closed immersion because, for all f in $(\mathbb{R}^n)^*$, f is in V and $(i(v))(f) = f(v)$. Moreover, by construction i satisfies the required intertwining property: $\tau(\gamma) \circ i = i \circ \gamma$.

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regular (i.e. polynomial) functions on X when no confusion is possible. Idem with *real algebraic group*

[3] A regular map $i : X \to Y$ between two real algebraic varieties is said to be a closed immersion if the image $i(X)$ is Zariski closed and if the map $i^* : \mathbb{R}[Y] \to \mathbb{R}[X] : \phi \mapsto \phi \circ i$ is surjective.

b) It is clear that G does not depend on d. It remains to prove that G is a group. The multiplication $\mu : \mathrm{P}^d(\mathbb{R}^n, \mathbb{R}^n) \times \mathrm{P}^d(\mathbb{R}^n, \mathbb{R}^n) \to \mathrm{P}^{2d}(\mathbb{R}^n, \mathbb{R}^n)$ is a polynomial map. Hence the inclusion $\mu(G \times G) \subset G$ is just a consequence of the corresponding inclusion $\mu(\Gamma \times \Gamma) \subset G$. Similarly, the inclusion $G^{-1} \subset G$ is a consequence of the corresponding inclusion $\Gamma^{-1} \subset G$. □

Remark 2.3 *Let us also denote by $\tau : G \to \mathrm{GL}(V^*)$ the linear action of G on V^* given by the same formula. The image $\tau(G)$ is Zariski closed in $\mathrm{GL}(V)$ and τ is an isomorphism of algebraic groups between G and $\tau(G)$.*

3 Polynomial actions of nilpotent Lie groups

Let N be an n-dimensional real (resp. complex) connected and simply connected nilpotent Lie group. The exponential map $\exp : \mathfrak{n} \to N$ is bijective and so by chosing a basis we can identify $\mathbb{R}^n \overset{\text{basis}}{\equiv} \mathfrak{n} \overset{\exp}{\equiv} N$. Therefore, we can speak about a polynomial map of N, by which we mean a map which is expressed via polynomials after this identification of N with $\mathbb{R}^n$. For instance the multiplication $\mu : N \times N \to N$ is a polynomial map. It is well known that we define this way, the unique structure of unipotent real (resp. complex) algebraic group on the Lie group N (cf [7]).

Lemma 3.1 *Let N be a connected and simply connected nilpotent Lie group and assume that $\theta : N \times \mathbb{R}^n \to \mathbb{R}^n$ is an action which is polynomial in both variables. Let v_0 be a point in $\mathbb{R}^n$. Then*
a) the isotropy group of v_0 is a closed connected subgroup of N and therefore N_{v_0} is diffeomorphic to a vector space.
b) The orbit $N.v_0$ is closed.
c) If the action of N on $\mathbb{R}^n$ is simply transitive, then the map $\theta_{v_0} : N \to \mathbb{R}^n : x \mapsto x \cdot v_0$ has a polynomial inverse.

Proof: Points a) and b) are well known when the action of N on $\mathbb{R}^n$ is linear (cf [7] §3.1). The general case is then a consequence of the Lemma 2.1.a. with $\Gamma = N$, if one notices that the linear action τ of N constructed in this lemma is still polynomial in both variables. Note that the statement that N_{v_0} is diffeomorphic to a vector space follows from the fact that any connected subgroup of a simply connected nilpotent Lie group, is itself simply connected.

c) Let $N_{\mathbb{C}}$ be the complexification of N. Then $N_{\mathbb{C}}$ acts on $\mathbb{C}^n$, say via the map $\theta^{\mathbb{C}}$, which is given by the same expression as θ, but now seen as a map from $N_{\mathbb{C}} \times \mathbb{C}^n$ to $\mathbb{C}^n$. In order to prove the lemma it suffices to show that the map $\theta_{v_0}^{\mathbb{C}} : N_{\mathbb{C}} \to \mathbb{C}^n : x \mapsto x \cdot v_0$ has a polynomial inverse.

Since the orbit $N.v_0$ is open in $\mathbb{R}^n$, the orbit $N_{\mathbb{C}}.v_0$ is open in $\mathbb{C}^n$ (note that the Jacobian of $\theta_{v_0}^{\mathbb{C}}$ at the neutral element of $N_{\mathbb{C}}$ is given by the same matrix as the Jacobian of θ_{v_0}). We conclude that the isotropy group of v_0 in $N_{\mathbb{C}}$ is discrete, hence trivial by a). This proves that $\theta_{v_0}^{\mathbb{C}}$ is injective. We also know by b), that the orbit $N_{\mathbb{C}}.v_0$ is closed in $\mathbb{C}^n$. This proves that $\theta_{v_0}^{\mathbb{C}}$ is surjective. Since $\theta_{v_0}^{\mathbb{C}}$ is bijective it has a polynomial inverse ([11]). $\square$

4 The algebraic hull of a crystallographic group

For our purposes we need to understand the action of $A(\Gamma)$, in the case where $\Gamma \subseteq \mathrm{P}(\mathbb{R}^n)$ is a polycyclic-by-finite crystallographic subgroup of bounded degree and in the situation where Γ is a solvable Lie group acting continuously and simply transitively on $\mathbb{R}^n$. A crucial observation in this context is the following lemma.

Lemma 4.1 *Let T be a real algebraic torus acting algebraically on $\mathbb{R}^n$, then the set of fixed points $(\mathbb{R}^n)^T$ is non-empty.*

Proof: Let $T_{\mathbb{C}}$ be the complexified torus acting algebraically on $\mathbb{C}^n$. Moreover, we let $\mathbb{Z}_2$ act on $\mathbb{C}^n$ by complex conjugation. In order to prove the proposition, it suffices to show that $(\mathbb{C}^n)^{\mathbb{Z}_2 \times T_{\mathbb{C}}}$ is non-empty. The elements of $T_{\mathbb{C}}$ of order dividing 2^ℓ form a finite 2-group T_ℓ. By noetherianity of the Zariski topology, the descending sequence of Zariski closed subsets $((\mathbb{C}^n)^{T_\ell})_{\ell \geq 1}$ stabilizes. Moreover, as the union T_∞ of the subgroups T_ℓ is Zariski dense in $T_{\mathbb{C}}$, there exists an integer ℓ_0 for which $(\mathbb{C}^n)^{T_{\mathbb{C}}} = (\mathbb{C}^n)^{T_{\ell_0}}$. Hence, we find that $(\mathbb{R}^n)^T = (\mathbb{C}^n)^{\mathbb{Z}_2 \times T_{\ell_0}}$. As $\mathbb{Z}_2 \times T_{\ell_0}$ is a 2-group, we can use [4, Theorem 7.11] to conclude that the set $(\mathbb{R}^n)^T$ is non-empty. $\square$

The following proposition generalizes a theorem of L.Auslander ([2]).

Proposition 4.2 *Let $G \subseteq \mathrm{P}(\mathbb{R}^n)$ be a solvable Lie group acting continuously and simply transitively on $\mathbb{R}^n$, then, the unipotent radical $U(G)$ of $A(G)$ also acts simply transitively on $\mathbb{R}^n$.*

Notice that a connected Lie group included in $\mathrm{P}(\mathbb{R}^n)$ whose action on $\mathbb{R}^n$ is continuous, is automatically of bounded degree ([8]).

Proof: The group $A(G)$ is a Zariski connected algebraic group, and therefore it splits as a semidirect product $A(G) = U(G) \rtimes T$ where T is a real algebraic torus. As $G \subseteq A(G)$ already acts transitively on $\mathbb{R}^n$, the group $A(G)$ acts transitively on $\mathbb{R}^n$. Moreover, by Lemma 4.1, the group $U(G)$ acts transitively on $\mathbb{R}^n$. By [10, Lemma 4.36], we know that $\dim U(G) \leq \dim G$, and as $U(G)$ acts transitively on $\mathbb{R}^n$ we have that $\dim U(G) = \dim G = n$. It follows that $U(G)$ acts with discrete isotropy groups, and so by Lemma 3.1.a, with trivial isotropy groups. This allows us to conclude that $U(G)$ acts simply transitively on $\mathbb{R}^n$. □

Let us now give the corresponding statement for discrete groups.

Proposition 4.3 *Let $\Gamma \subseteq \mathrm{P}(\mathbb{R}^n)$ be a polycyclic-by-finite crystallographic subgroup of bounded degree. Then the unipotent radical $U(\Gamma)$ of $A(\Gamma)$ acts simply transitively on $\mathbb{R}^n$.*

Proof: Replacing Γ by a subgroup of finite index, we can assume that $A(\Gamma)$ is a Zariski connected solvable algebraic group. In this case we have again a semidirect product decomposition $A(\Gamma) = U(\Gamma) \rtimes T$ for some real algebraic torus T. Let $v_0 \in \mathbb{R}^n$ be a fixed point for the action of T. By Lemma 3.1, the orbit $U(\Gamma) \cdot v_0$ is closed and is diffeomorphic to some vector space $\mathbb{R}^p$ with $p \leq d$ where d is the dimension of $U(\Gamma)$. As $U(\Gamma) \cdot v_0$ is $A(\Gamma)$–invariant, it is also Γ–invariant. Moreover, by [10, Lemma 4.36] we have the inequality $d \leq h(\Gamma)$ where $h(\Gamma)$ is the Hirsch length of Γ. Notice that, since Γ is crystallographic, one has $h(\Gamma) = n$. The action of Γ on $U(\Gamma) \cdot v_0$ is also properly discontinuous. This implies that $h(\Gamma) \leq p$. This discussion proves that $d = p = n$ and that $U(\Gamma) \cdot v_0 = \mathbb{R}^n$. We deduce from Lemma 3.1.a that the action of $U(\Gamma)$ on $\mathbb{R}^n$ is simply transitive as claimed. □

As a conclusion of this section, we will prove one more property concerning the algebraic hull of a group of polynomial diffeomorphisms, which will be needed in the sequel of this paper.

Lemma 4.4 *Let $G \subseteq \mathrm{P}(\mathbb{R}^n)$ be a solvable Lie group acting simply transitively on $\mathbb{R}^n$ or a polycyclic-by-finite crystallographic subgroup of bounded degree and let $U(G)_{\mathbb{C}}$ be the unipotent radical of $A(G)_{\mathbb{C}}$. Then, the centralizer of $U(G)_{\mathbb{C}}$ in $A(G)_{\mathbb{C}}$ coincides with the center of $U(G)_{\mathbb{C}}$.*

Proof: ¿From Propositions 4.2, 4.3 and the proof of Lemma 3.1.c, we know that $U(G)_{\mathbb{C}}$ acts simply transitively on $\mathbb{C}^n$. The centralizer C of

$U(G)_{\mathbb{C}}$ in $A(G)_{\mathbb{C}}$ is an algebraic group and therefore, for any element c in C, the unipotent part c_u and the semisimple part c_s of c are also in C. Assume that there is an element c of C which does not belong to $U(G)_{\mathbb{C}}$. Then, $c_s \neq 1$ and c_s also belongs to C. Now, there are two possibilities:

- c_s is of finite order. In this case, we can assume that the order of c_s is a prime number p. It then follows, again from [4, Theorem 7.11], that $(\mathbb{C}^n)^{c_s}$ is a non-empty proper Zariski closed subset of $\mathbb{C}^n$.
- c_s is of infinite order. In this case, we can assume that c_s belongs to an algebraic torus of $A(G)_{\mathbb{C}}$. As in Lemma 4.1, we can conclude also in this case that $(\mathbb{C}^n)^{c_s}$ is a non-empty proper Zariski closed subset of $\mathbb{C}^n$.

However, as c_s centralizes $U(G)_{\mathbb{C}}$, the set $(\mathbb{C}^n)^{c_s}$ is $U(G)_{\mathbb{C}}$-invariant. This contradicts the transitivity of $U(G)_{\mathbb{C}}$ on $\mathbb{C}^n$. It follows that C is included in $U(G)_{\mathbb{C}}$. □

Remark 4.5 *The properties above show that $A(G)$, for G a simply transitive subgroup of $\mathrm{P}(\mathbb{R}^n)$ or a crystallographic subgroup of bounded degree, is an $\mathbb{R}$–algebraic hull in the language of [10].*

5 Uniqueness of simply transitive polynomial actions of nilpotent groups

It was already known for a long time ([12]) that any simply transitive nilpotent subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ is unipotent. Let us remark that the analogous statement for polynomial actions is also correct even though we will not use it in this article:

Lemma 5.1 *Let $N \subseteq \mathrm{P}(\mathbb{R}^n)$ be a nilpotent Lie group acting simply transitively on $\mathbb{R}^n$. Then $N = A(N) = U(N)$.*

Proof: As N is nilpotent, its algebraic closure $A(N)$ is also nilpotent. It follows that $A(N) = U(N) \times S(N)$, where $S(N)$ denotes the set of semi-simple elements of $A(N)$. So $S(N)$ centralizes $U(N)$. However, by Lemma 4.4, we know that the centralizer of $U(N)$ equals the center of $U(N)$, showing that $S(N)$ has to be trivial. □

Remark 5.2 *For a nilpotent crystallographic subgroup $N \subset \mathrm{P}(\mathbb{R}^n)$ of bounded degree one analogously shows that $A(N) = U(N)$ and thus $N \subseteq U(N)$.*

Corollary 5.3 *Let $\rho : N \times \mathbb{R}^n \to \mathbb{R}^n$ be a simply transitive action of a nilpotent Lie group N by means of polynomial diffeomorphisms, then ρ is polynomial in both variables*

Proof: By Lemma 5.1, we know that the action of N is the same as the algebraic action of $U(N)$. □

The following proposition is the starting point for our uniqueness statements.

Proposition 5.4 *Let N be a connected and simply connected nilpotent Lie group and $\rho_1, \rho_2 : N \to \mathrm{P}(\mathbb{R}^n)$ be two continuous and simply transitive polynomial actions of N on $\mathbb{R}^n$. Then for every v, w in $\mathbb{R}^n$ there exists a unique element p in $\mathrm{P}(\mathbb{R}^n)$ conjugating ρ_1 and ρ_2 and such that $p(v) = w$.*

Proof: One must have $p(\rho_1(n)v) = \rho_2(n)w$. Hence p is unique. By the Lemma 3.1.c, the bijective map $\theta_{1,v} : N \to \mathbb{R}^n : n \mapsto \rho_1(n)v$ is polynomial with polynomial inverse. Same for the map $\theta_{2,w} : N \to \mathbb{R}^n : n \mapsto \rho_2(n)w$. Hence $p := \theta_{2,w} \circ \theta_{1,v}^{-1}$ is in $\mathrm{P}(\mathbb{R}^n)$ and conjugates ρ_1 and ρ_2. □

6 Polynomial conjugacy of polynomial actions

Proposition 6.1 *For $i = 1, 2$, let G_i be a Zariski closed subgroup of $\mathrm{P}(\mathbb{R}^n)$, suppose that the Zariski connected component of G_i is solvable and that the unipotent radical U_i of G_i acts simply transitively on $\mathbb{R}^n$.*

Then, for any isomorphism of algebraic groups $F : G_1 \to G_2$, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all g_1 in G_1, $p \circ g_1 = F(g_1) \circ p$.

Proof: a) Suppose first that G_1 is Zariski connected. Then G_1 decomposes as a semi-direct product $G_1 = U_1 \rtimes T_1$, where T_1 is an algebraic torus in G_1. As F is an isomorphism of algebraic groups, we have that $F(U_1) = U_2$. Letting $T_2 = F(T_1)$, we also have that $G_2 = U_2 \rtimes T_2$. As T_i is an algebraic torus, for $i = 1, 2$, there exists an element v_i in $\mathbb{R}^n$ which is fixed for the action of T_i (Lemma 4.1). By Proposition 5.4, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that for all u_1 in U_1, one has $p(u_1 \cdot v_1) = F(u_1) \cdot v_2$. Hence for all g_1 in G_1, one has $p(g_1 \cdot v_1) = F(g_1) \cdot v_2$. ¿From that we get our conclusion $p \circ g_1 = F(g_1) \circ p$.

b) Let us now deal with the general case. One can still write, for $i = 1, 2$, $G_i = U_i \rtimes S_i$ where S_i is the stabilizer in G_i of a point v_i of $\mathbb{R}^n$. As in a), it is enough to prove that the group $S_2' := F(S_1)$ has a fixed point in

$\mathbb{R}^n$. Otherwise stated, it is enough to prove that S_2 and S_2' are conjugated subgroups of G_2.

The Zariski connected component T_2 of S_2' is a maximal torus of G_2 and, by Lemma 4.1, has a fixed point in $\mathbb{R}^n$. We can suppose that this point is v_2 so that T_2 is also the Zariski connected component of S_2. Let N_2 be the normalizer in G_2 of T_2. The group $V_2 := N_2 \cap U_2$ is a unipotent group. Since N_2 contains both S_2 and S_2', one has the equalities $N_2 = V_2 \rtimes S_2 = V_2 \rtimes S_2'$. Let us divide these equalities by the normal subgroup T_2. The groups S_2/T_2 and S_2'/T_2 are maximal compact subgroups of N_2/T_2 and hence they are conjugated. We conclude that S_2 and S_2' are conjugated inside N_2. □

In the following theorem we show that any isomorphism between simply transitive subgroups of $\mathrm{P}(\mathbb{R}^n)$ is induced by an inner automorphism.

Theorem 6.2 *Let G be a Lie group. Any two continuous and transitive polynomial actions ρ_1, ρ_2 of G with finite isotropy groups are polynomially conjugated.*

Proof: First notice that, by Lemma 2.1, the connected component G_e of G is a linear group and, since G_e is homeomorphic to $\mathbb{R}^n$, G_e is solvable. Then notice that the kernels of ρ_1 and ρ_2 coincide with the maximal finite normal subgroup of G, hence we can assume that ρ_1 and ρ_2 are injective. The subgroups $\rho_1(G)$ and $\rho_2(G)$ are isomorphic via the map $\rho_2 \circ \rho_1^{-1}$. By [10, Lemma 4.41], this isomorphism extends to an isomorphism $F : A(\rho_1(G)) \to A(\rho_2(G))$ of their algebraic hulls (notice that in [10], this assertion is proved only for solvable groups but that it extends directly to virtually solvable groups). By Propositions 4.2 and 6.1, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all a in $A(\rho_1(G))$, one has $F(a) \circ p = p \circ a$. This map p is the conjugation we are looking for. □

The same proof works for polynomial crystallographic actions of bounded degree of polycyclic-by-finite groups:

Proof of the Theorem 1.1 : Let us denote by $\rho_1, \rho_2 : \Gamma \to \mathrm{P}(\mathbb{R}^n)$ these two actions. First of all recall that the kernel of ρ_i $(i = 1, 2)$ is the unique maximal finite normal subgroup F_Γ of Γ. Therefore, we can assume, without loss of generality, that $F_\Gamma = 1$ and that ρ_1 and ρ_2 are injective. Moreover, by [10, Lemma 4.41], we know that the isomorphism $\rho_2 \circ \rho_1^{-1} : \rho_1(\Gamma) \to \rho_2(\Gamma)$ extends to an isomorphism of algebraic groups $F : A(\rho_1(\Gamma)) \to A(\rho_2(\Gamma))$. By Propositions 4.3 and 6.1, there exists an element p in $\mathrm{P}(\mathbb{R}^n)$ such that, for all a in $A(\rho_1(\Gamma))$, one has $F(a) \circ p = p \circ a$. This map p is again the one we are looking for. □

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Y.B. : Ecole Normale Supérieure, 45 rue d'Ulm 75230 Paris, France

K.D. : Katholieke Universiteit Leuven, Campus Kortrijk, B–8500 Kortrijk, Belgium