DISCRETE SUBGROUPS OF $\mathrm{SL}_3(\mathbb{R})$ GENERATED BY TRIANGULAR MATRICES

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dedicated to Professor Atle Selberg with deep appreciation

Abstract. Based on the ideas in some recently uncovered notes of Selberg [14] on discrete subgroups of a product of $\mathrm{SL}_2(\mathbb{R})$'s, we show that a discrete subgroup of $\mathrm{SL}_3(\mathbb{R})$ generated by lattices in upper and lower triangular subgroups is an arithmetic subgroup and hence a lattice in $\mathrm{SL}_3(\mathbb{R})$.

1. Introduction

In a locally compact group G, a discrete subgroup Γ of finite co-volume is called a lattice in G. Let $G = \mathrm{SL}_3(\mathbb{R})$, and U_1 and U_2 be the strict upper and lower triangular subgroups of G:

$$U_1 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\} \quad \text{and} \quad U_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Let F_1 and F_2 be lattices in U_1 and U_2 respectively, and set

$$\Gamma_{F_1,F_2} := \langle F_1, F_2 \rangle$$

to be the subgroup of G generated by F_1 and F_2. The main goal of this paper is to determine which F_1 and F_2 can generate a discrete subgroup of G.

Lattices in U_1 can be divided into two classes: a lattice F_1 in U_1 is called irreducible if F_1 does not contain an element of the form $\begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $x \neq 0$ or $\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, $y \neq 0$; F_1 is called reducible otherwise.

For the cases when F_1 is reducible, it is shown in [6] that if Γ_{F_1,F_2} is discrete, then it is commensurable with $\mathrm{SL}_3(\mathbb{Z})$, up to conjugation by an element of $\mathrm{GL}_3(\mathbb{R})$. Recall that two subgroups are called commensurable with each other if their intersection is of finite index in each of them.

The following is our main theorem:

Theorem 1.1. *If F_1 is irreducible and Γ_{F_1,F_2} is discrete, then there exists a real quadratic field K such that Γ_{F_1,F_2} is, up to conjugation by a diagonal element of $\mathrm{GL}_3(\mathbb{R})$, commensurable with the arithmetic subgroup*

$$\{g \in \mathrm{SL}_3(\mathcal{O}_K) : g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} {}^\sigma g^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \}$$

where σ denotes the Galois element of K and $\mathcal{O}_K$ the ring of integers of K.

In a semisimple Lie group G, the unipotent radical of a parabolic subgroup of G is called a horospherical subgroup. The corresponding parabolic subgroup of a horospherical group U is obtained by taking the normalizer subgroup of U. A pair of horospherical subgroups are called opposite if the intersection of the corresponding parabolic subgroups is a common Levi subgroup in both parabolic subgroups.

Theorem 1.1 was the last missing case of the following theorem where all other cases were proved in the Ph. D thesis of the second named author [7]. The formulation of this theorem is due to Margulis, who posed it after hearing Selberg's lecture in 1993 on Theorem 1.4 below.

Theorem 1.2. *Let G be the group of real points of a connected absolutely simple real-split algebraic group $\mathbf{G}$ with real rank at least two. Let F_1 and F_2 be lattices in a pair of opposite horospherical subgroups of G. If the subgroup Γ_{F_1,F_2} generated by F_1 and F_2 is discrete, there exists a $\mathbb{Q}$-form of $\mathbf{G}$ with respect to which U_1 and U_2 are defined over $\mathbb{Q}$ and F_i is commensurable with $U_i(\mathbb{Z})$ for each $i = 1, 2$. Moreover Γ_{F_1,F_2} is commensurable with the arithmetic subgroup $\mathbf{G}(\mathbb{Z})$.*

By a theorem of Borel and Harish-Chandra [1], it follows that Γ_{F_1,F_2} is a lattice in G.

The assumption of G having higher rank (meaning that the real rank of G is at least two) cannot be removed, as one can construct counterexamples in any special orthogonal group $\mathrm{SO}(n,1)$ of rank one. For instance, the subgroup generated by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$ is of infinite index in $\mathrm{SL}_2(\mathbb{Z})$.

To understand the motivation of Theorem 1.2, we recall that Margulis [4, Thm 7.1] and Raghunathan [11] showed independently that any irreducible non-uniform lattice of a semisimple Lie group contains lattices in a pair of opposite horospherical subgroups. This theorem was one of the main steps in Margulis's proof of the arithmeticity of such lattices in higher rank semisimple Lie groups without the use of the super-rigidity theorem of Margulis [5] which had settled the arithmeticity of both uniform and non-uniform lattices at once. The approach of studying lattices in a pair of opposite horospherical subgroups of a non-uniform lattice goes back to Selberg's earlier proof of the arithmeticity of such lattices in a product of $\mathrm{SL}_2(\mathbb{R})$'s [13]. In this context, Theorem 1.2 can be understood as a statement that this property of a non-uniform lattice is sufficient to characterize them among discrete subgroups in higher rank groups.

Theorem 1.2 was conjectured by Margulis without the assumption of G being real-split (see [6] for the statement of the full conjecture). Most of these general cases were proved in [7], while the essential missing cases are when the real rank of $\mathbf{G}$ is precisely two. We believe that the new ideas presented in this paper combined with the techniques from [7] should lead us to resolving these missing cases.

As shown in [8], any Zariski dense discrete subgroup of G containing a lattice of a horospherical subgroup necessarily intersects a pair of opposite horospherical subgroups as lattices. Therefore we deduce:

Corollary 1.3. *Let G be as in Theorem 1.2 and Γ be a Zariski dense discrete subgroup of G. If Γ contains a lattice of a horospherical subgroup of G, then Γ is an arithmetic lattice of G.*

The main tool of Theorem 1.2, except for the case of Theorem 1.1, given in [7] is Ratner's theorem on orbit closures of unipotent flows in $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ [12]. This approach is not available in the situation of Theorem 1.1. We learned a new idea from Selberg's proof of the following theorem.

Letting G be the product of n-copies of $\mathrm{SL}_2(\mathbb{R})$, a Hilbert modular subgroup of G is defined to be

$$\{(x^{(1)}, \cdots, x^{(n)}) : x^{(1)} \in \mathrm{SL}_2(\mathcal{O}_K)\}$$

where K is a totally real number field of degree n with the ring of integers $\mathcal{O}_K$, and $x^{(i)}$'s are n-conjugates of $x^{(1)}$. A lattice F in the strict upper-triangular subgroup of G is called *irreducible* if any non-trivial element of F has a non-trivial projection to each $\mathrm{SL}_2(\mathbb{R})$-component of G.

Theorem 1.4 (Selberg)**.** [14] *Let $n \geq 2$ and $G = \prod_n \mathrm{SL}_2(\mathbb{R})$. Let Γ be a Zariski dense discrete subgroup of G which contains an irreducible lattice F_1 in the upper triangular subgroup. Then there exists an element $g \in \prod_n \mathrm{GL}_2(\mathbb{R})$ such that a subgroup of $g\Gamma g^{-1}$ of finite index is contained in a Hilbert modular subgroup.*

It follows from a result of Vaserstein [15] that $g\Gamma g^{-1}$ is commensurable with a Hilbert modular subgroup. Theorem 1.4 also resolves Conjectures 1.1 and 1.2 in [9], which was written without being aware of this theorem.

Only in December of 2008, the second named author received Selberg's lecture notes [14] written in the early 90's from Hejhal, who found them while going through Selberg's papers in the previous summer. The lecture notes contained an ingenious proof of the discreteness criterion on a Zariski dense subgroup containing F_1 as above, in particular implying Theorem 1.4.

While trying to understand together these beautiful lecture notes, the authors realized the main idea of Selberg, studying the double cosets of the group Γ under the multiplications by F_1 in both sides and using the Bruhat decomposition to detect them, can be used to resolve the case of $\mathrm{SL}_3(\mathbb{R})$ which the techniques in [7] and [9] didn't apply to. More precisely Proposition 2.6 in the next section is the new key ingredient which was missing in the approach of [9]. Using the fact that in $\mathrm{SL}_3(\mathbb{R})$ the group of diagonal elements preserving the co-volume of lattices in U_1 commutes with the longest Weyl element w_0, our proof for $\mathrm{SL}_3(\mathbb{R})$ is much simpler than Selberg's proof of Theorem 1.4 for the product of $\mathrm{SL}_2(\mathbb{R})$'s.

The general framework of the proof, investigating the action of the common normalizer subgroup on the space of lattices in U_1 and U_2, goes in the same spirit as in [7], which had been strongly influenced by the original work of Margulis [4].

Finally we mention that we recently extended Theorem 1.4 to a product of $\prod_{\alpha \in I} \mathrm{SL}_2(k_\alpha)$ for any local field k_α of characteristic zero.

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[1]We understand that there are plans to make [14] and several other unpublished lectures of Selberg available on a webpage at IAS.

2. Proof of Theorem 1.1

Let U_1 and U_2 be as in Theorem 1.1 with lattices F_1 and F_2 respectively. We assume that F_1 is irreducible. The normalizers $N(U_1)$ and $N(U_2)$ are upper and lower triangular subgroups (with diagonals) respectively.

Let $A := N(U_1) \cap N(U_2)$, that is, the diagonal subgroup of $\mathrm{SL}_3(\mathbb{R})$. Set

$$w_0 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By the Bruhat decomposition of G, the map $U_2 \times A \times U_2 \to G$ given by

$$(u_1, a, u_2) \mapsto u_1 a w_0 u_2$$

is a diffeomorphism onto a Zariski-dense open subset, say, Ω of G. One can check that

$$\Omega = \{(g_{ij}) \in G : g_{13} \neq 0,\, g_{12}g_{23} - g_{13}g_{22} \neq 0\}$$

and that the A-component, say, $\mathrm{diag}(a_1, a_2, a_3)$ of $(g_{ij}) \in \Omega$ of the decomposition is determined by

$$a_1 = g_{13}, \quad a_1 a_2 = g_{12}g_{23} - g_{13}g_{22}, \quad a_1 a_2 a_3 = 1. \tag{2.1}$$

In particular for a given $g \in \Omega$, the two quantities g_{13} and $g_{12}g_{23} - g_{13}g_{22}$ are invariant by the multiplications of elements of U_2 in either side. This is an important observation which will be used later.

For simplicity, we will often write $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in U_1$ as $(x, y, z) \in U_1$, and similarly, $\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \in U_2$ as $(x, y, z) \in U_2$.

Note that the center $Z(U_1)$ of U_1 is given by the one-dimensional group $\{(0, 0, z) \in U_1 : z \in \mathbb{R}\}$, and that the center $Z(F_1)$ of F_1 is a lattice in $Z(U_1)$. Hence $F_1/Z(F_1)$ is a lattice in $U_1/Z(U_1)$ with its image $F_1/Z(U_1)$ identified as $\{(x, y) \in \mathbb{R}^2 : (x, y, *) \in F_1\}$.

Lemma 2.2. *There exists $u \in U_1$ such that $u^{-1}\Gamma_{F_1,F_2}u$ contains lattices F_1' and F_2' in U_1 and U_2 respectively such that for some non-zero $\alpha_0, \beta_0 \in \mathbb{R}$,*

$$F_1/Z(U_1) = F_1'/Z(U_1) \quad \text{and} \quad F_2'/Z(U_2) = \{(\alpha_0 x, \beta_0 y) : (y, x) \in F_1/Z(U_1)\}.$$

Proof. Since Γ_{F_1,F_2} is Zariski dense, it intersects the Zariski open subset $U_1 A w_0 U_1$ non-trivially. Let $\gamma = u a w_0 v \in U_1 A w_0 U_1$. Then $u^{-1}\Gamma_{F_1,F_2}u$ contains $F_1' := u^{-1}F_1 u$ as well as $F_2' := a(w_0 v F_1 v^{-1} w_0^{-1})a^{-1}$. Observe that both $u^{-1}F_1 u$ and vF_1v^{-1} are equal to F_1, modulo the center of U_1 and that

$$w_0 \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} w_0^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -y & 1 & 0 \\ z & -x & 1 \end{pmatrix}.$$

Hence the claim follows by setting $\alpha_0 = a_2/a_1$ and $\beta_0 = a_3/a_2$ for $a = \mathrm{diag}(a_1, a_2, a_3)$. □

Therefore by replacing Γ_{F_1,F_2} by its conjugation by an element of U_1, we may henceforth assume that the lattice F_2 satisfies

$$F_2/Z(U_2) = \{(\alpha_0 x, \beta_0 y) : (y,x) \in F_1/Z(U_1)\} \tag{2.3}$$

for some non-zero $\alpha_0, \beta_0 \in \mathbb{R}$.

Lemma 2.4. *For a compact subset C of A, we have*

$$\# F_2 \backslash (U_2 C w_0 U_2 \cap \Gamma)/F_2 < \infty.$$

Proof. Let C_2 be a compact subset such that $U_2 = F_2 C_2$. Since $F_2 \subset \Gamma$ and hence $U_2 C w_0 U_2 \cap \Gamma \subset F_2(C_2 C w_0 C_2 \cap \Gamma) F_2$, we have

$$\# F_2 \backslash (U_2 C w_0 U_2 \cap \Gamma)/F_2 \leq \#(C_2 C w_0 C_2 \cap \Gamma)$$

which clearly implies the claim, as a discrete subgroup contains only finitely many elements in a given compact subset. □

Denote by B the subgroup of A consisting of elements whose conjugation actions on U_1 and U_2 are volume preserving, that is,

$$B = \{\tilde{u} := \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-2} & 0 \\ 0 & 0 & u \end{pmatrix} : u \in \mathbb{R}^*\}.$$

The restriction of the adjoint action of B on the Lie algebra $\mathrm{Lie}(U_1)$ of U_1 induces the action of B on the space of lattices in $\mathrm{Lie}(U_1)$, which will be identified as $\mathbb{R}^3$.

The logarithm map $\log : U_1 \to \mathrm{Lie}(U_1)$ is given by

$$\log \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & x & z - \frac{1}{2}xy \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

For simplicity, we write $\log(x,y,z) = (x, y, z - \frac{1}{2}xy)$.

We denote by Δ_{F_1} the additive subgroup of $\mathbb{R}^3$ generated by $2\log(F_1)$, which is clearly a lattice in $\mathbb{R}^3$. Since

$$2\log(X) + 2\log(X') = \log(X(X')^2 X)$$

for any $X = (x,y,z)$ and $X' = (x', y', z')$ in U_1, we deduce that

$$\Delta_{F_1} \subset \log(F_1).$$

Consider the orbit of Δ_{F_1} under B:

$$\begin{aligned} B.\Delta_{F_1} &= \{\mathrm{Ad}(\tilde{u})(x,y,z) : (x,y,z) \in \Delta_{F_1}, \tilde{u} \in B\} \\ &= \{(u^3 x, u^{-3} y, z) : (x,y,z) \in \Delta_{F_1}, u \in \mathbb{R}^*\}. \end{aligned}$$

Similarly for Δ_{F_2} defined to be the additive subgroup of $\mathbb{R}^3$ generated by $2\log F_2$, the orbit of Δ_{F_2} under B is of the form

$$\begin{aligned} B.\Delta_{F_2} &= \{\mathrm{Ad}(\tilde{u})(x,y,z) : (x,y,z) \in \Delta_{F_2}, \tilde{u} \in B\} \\ &= \{(u^{-3} x, u^3 y, z) : (x,y,z) \in \Delta_{F_2}, u \in \mathbb{R}^*\}. \end{aligned}$$

Lemma 2.5. *The orbit $B.\Delta_{F_1}$ is relatively compact in the space of lattices in $\mathbb{R}^3$.*

Proof. (cf. proof [9, Thm 2.2]) Below we use freely several theorems due to Zassenhaus, Minkowski, and Mahler which are standard in the geometry of numbers (see [10], [3]). Let $\epsilon_0 > 0$ be such that $\Gamma \cap W_{\epsilon_0}$ generates a nilpotent subgroup, where W_{ϵ_0} is the ϵ_0-neighborhood of e in G and the commutators $ghg^{-1}h^{-1}$, $g, h \in W_{\epsilon_0}$ are contained again in $W_{\epsilon_0/2}$. Such a neighborhood W_{ϵ_0} is called a neighborhood of Zassenhaus. By Minkowski's theorem, there exists $c > 1$, depending on the covolume of Δ_{F_2}, such that any lattice in $B.\Delta_{F_2}$ contains a non-zero vector of norm at most c. Or equivalently, for some $c' > 1$, any lattice in bF_2b^{-1}, $b \in B$ has a non-zero element in $W_{c'}$. Take $a \in A$ which contracts U_2, so that $a.\log(X) = \log(aXa^{-1})$ is of norm less than ϵ_0 for all $x \in U_2 \cap W_{c'}$. Now suppose the sequence $b_n.\Delta_{F_1}$ is unbounded; so is $b_n.\Delta_{aF_1a^{-1}}$. It follows from Mahler's compactness criterion that there exists a sequence $\delta_n = (x_n, y_n, z_n) \in F_1$ such that $b_n a\delta_n a^{-1} b_n^{-1} \in W_{\epsilon_0}$. Since b_n acts on $(0, 0, \mathbb{R})$ trivially, it follows that $(x_n, y_n) \neq 0$. By the irreducibility assumption on F_1, we have $x_n \neq 0$ and $y_n \neq 0$.

Choose $b_n\delta'_n b_n^{-1} \in b_nF_2b_n^{-1} \cap W_{c'}$, so that $ab_n\delta'_n b_n^{-1}a^{-1} \in W_{\epsilon_0}$. This implies that δ_n and δ'_n together must generate a nilpotent subgroup, and even a unipotent subgroup, as any nilpotent subgroup generated by unipotent elements is unipotent. This is a contradiction as $x_n \neq 0$ and $y_n \neq 0$. ☐

The following is a main proposition, the key idea of whose proof was learned from Selberg's lecture notes [14].

Proposition 2.6. *There are infinitely many distinct $(x_n, y_n, z_n) \in F_1$ such that for all n, k,*

$$\mathbb{R}(x_n, y_n) \neq \mathbb{R}(x_k, y_k), \quad x_ny_n = x_ky_k, \quad z_n = z_k.$$

Proof. By Lemma 2.5, we can find a sequence $b_n \in B$ tending to infinity such that $b_n\Delta_{F_1}$ converges to a lattice in $\mathbb{R}^3$. In particular, there exists a sequence $\delta_n \in F_1$ and a non-identity element $\delta \in U_1$ such that $b_n.\log(\delta_n) = \log(b_n\delta_nb_n^{-1})$ converges to $\log(\delta)$.

Therefore, replacing δ by δX for a suitable $X \in Z(F_1)$, we may assume without loss of generality that $\delta \in \Omega = U_2Aw_0U_2$, and consequently $\delta_n \in \Omega$ for all large n.

Write $\delta_n = (x_n, y_n, z_n) \in F_1$ and $\delta = (x, y, z) \in U_1$. We claim that for any k, there exist only finitely many n such that (x_n, y_n) is a scalar multiple of (x_k, y_k). Suppose on the contrary that $(x_n, y_n) = \lambda_n(x_k, y_k)$ for infinitely many n's. If $b_n = \mathrm{diag}(c_n, c_n^{-2}, c_n)$, then $(c_n^3x_n, c_n^{-3}y_n, z_n)$ converges to (x, y, z). Note that $(x_n, y_n) \neq 0$ and hence neither x_n nor y_n is zero for each n. Therefore we deduce that an infinite subsequence of $(c_n^3\lambda_n, c_n^{-3}\lambda_n)$ converges to (xx_k^{-1}, yy_k^{-1}), which is a contradiction as $c_n \to \infty$ as $n \to \infty$. Therefore by passing to a subsequence, we may assume that $(x_n, y_n) \notin \mathbb{R}(x_k, y_k)$ for all k, n.

Observing that B commutes with w_0 and normalizes U_2, it is easy to check that that the A-component of δ_n in the decomposition $U_2Aw_0U_2$ is equal to that of $b_n\delta_nb_n^{-1}$. Therefore the A-components of δ_n converge to the A-component of δ.

By Lemma 2.4, it follows that

$$\#F_2\backslash\{\delta_n\}/F_2 < \infty.$$

However as can be seen from (2.1), the multiplications by F_2 from either side on δ_n change neither z_n nor $x_ny_n - z_n$. Therefore we have infinitely many distinct elements $(x_n, y_n, z_n) \in F_1$ with the same x_ny_n as well as the same z_n for all n. ☐

As we will see in the following lemma 2.7, Proposition 2.6 associates to F_1 a real quadratic field; hence a lattice F_1 which is arbitrary a priori becomes a very special one coming from a quadratic field.

Selberg [14] proved a claim analogous to 2.6 for the product of n-copies of $\mathrm{SL}_2(\mathbb{R})$ in order to obtain a totally real number field of degree n associated to an irreducible lattice F_1 in the upper triangular subgroup. As mentioned in the introduction, our proof of Proposition 2.6 is simpler than Selberg's proof due to the structure of $\mathrm{SL}_3(\mathbb{R})$ which provides a Weyl element w_0 commuting with B. Such an element does not exist in Selberg's situation, and his proof uses more of the geometry of numbers.

Lemma 2.7. *There exist $\alpha, \beta \neq 0$ such that*

$$F_1/Z(U_1) \subset \{(\alpha(p + q\sqrt{d}), \beta(p - q\sqrt{d}) : p, q \in \mathbb{Q}\}$$

for some quadratic number $d > 0$.

Proof. Consider a sequence $\{\delta_n = (x_n, y_n, z_n) \in F_1\}$ given by the above proposition 2.6. Setting $\alpha = x_1$, $\beta = y_1$, and $a_0 = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$, we define

$$F_1' = a_0 F_1 a_0^{-1}.$$

Observe that $F_1'/Z(U_1)$ contains $(1,1)$, $(x_2', y_2') := (\alpha^{-1}x_2, \beta^{-1}y_2)$ and $(x_3', y_3') := (\alpha^{-1}x_3, \beta^{-1}y_3)$. Since $F_1'/Z(U_1)$ is a lattice in $\mathbb{R}^2$, it must be contained in the $\mathbb{Q}$-span of $(1,1)$ and (x_2', y_2'). Consequently,

$$(x_3', y_3') = (p_0 + q_0 x_2', p_0 + q_0 y_2')$$

for some $p_0, q_0 \in \mathbb{Q}$. By the properties of (x_n, y_n) as given by Proposition 2.6, we have $p_0 \neq 0$ and $q_0 \neq 0$. Moreover as $x_3'y_3' = x_2'y_2' = 1$, it follows that

$$(p_0 + q_0 x_2')(p_0 + q_0 (x_2')^{-1}) = 1,$$

or equivalently

$$(x_2')^2 + \frac{1}{p_0 q_0}(p_0^2 + q_0^2 - 1)x_2' + 1 = 0.$$

This implies that x_2' is either a rational number or a real quadratic number with its reciprocal being its conjugate. The former cannot happen as it would imply F_1 is reducible.

Therefore we have obtained that

$$F_1'/Z(U_1) \subset \{(p + q\sqrt{d}, p - q\sqrt{d}) : p, q \in \mathbb{Q}\}$$

for some quadratic number $d > 0$, which implies the claim. □

Consider the $\mathbb{Q}$-points of the special unitary group defined by the hermitian matrix w_0 and $K = \mathbb{Q}(\sqrt{d})$:

$$\mathrm{SU}(w_0)_{\mathbb{Q}} := \{g \in \mathrm{SL}_3(K) : g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} {}^{\sigma}g^t = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\}.$$

Denote by Λ the corresponding arithmetic subgroup:

$$\Lambda := \mathrm{SU}(w_0)_{\mathbb{Q}} \cap \mathrm{SL}_3(\mathcal{O}_K).$$

It is easy to check that $\log(\Lambda \cap U_1)$ is given by

$$\{(x, \sigma(x), r\sqrt{d}) : x \in \mathcal{O}_K,\ r \in \mathbb{Z}\}$$

with $\log(Z(\Lambda \cap U_1))$ given by $\{(0, 0, \mathbb{Z} \cdot \sqrt{d})\}$.

Proposition 2.8. *For* $a_0 = \begin{pmatrix} \alpha^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$, *the lattice* $a_0 F_1 a_0^{-1}$ *is commensurable with the subgroup* $U_1 \cap \Lambda$.

Proof. By replacing F_1 by $a_0 F_1 a_0^{-1}$, we may assume that $F_1/Z(U_1)$ is a $\mathbb{Z}$-module of the real quadratic field $\mathbb{Q}(\sqrt{d})$ of rank two. As we concern only the commensurability class, we will further assume that

$$F_1/Z(U_1) = \{(p + q\sqrt{d}, p - q\sqrt{d}) : p, q \in \mathbb{Z}\}.$$

Note that the commutator of $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ in U_1 is of the form:

$$X_1 X_2 X_1^{-1} X_2^{-1} = (0, 0, x_1 y_2 - x_2 y_1) \in U_1.$$

Therefore the center $Z(F_1)$ is contained in the $\mathbb{Q}$-multiple of $\sqrt{d}$.

It is also easy to see that for any $(p, q) \in \mathbb{Z}^2$, there exists an element $\phi(p, q) \in \mathbb{R}$, unique up to the addition by $\log(Z(F_1))$ such that

$$(p + q\sqrt{d}, p - q\sqrt{d}, \phi(p, q)) \in \Delta_{F_1}.$$

By the uniqueness, we may choose ϕ so that the map $(p, q) \in \mathbb{Z}^2 \mapsto \phi(p, q)$ is a $\mathbb{Z}$-linear map.

Since

$$(p + q\sqrt{d}, p - q\sqrt{d}, \phi(p, q) + \frac{1}{2}(p^2 - q^2 d)) \in F_1,$$

Proposition 2.6 implies that there are infinitely many $p_n, q_n \in \mathbb{Z}$ such that $\phi(p_n, q_n)$ is a constant modulo the center of F_1. This implies that $\phi = 0$, modulo the center of F_1. This proves that Δ_{F_1} is commensurable with $\log(\Lambda \cap U_1)$. Hence F_1 is commensurable with the lattice $U_1 \cap \Lambda$. □

Corollary 2.9. *The stabilizer of* Δ_{F_1} *in* B *is commensurable with*

$$\{\mathrm{diag}(u, u^{-2}, u) : u \in \mathcal{U}_K\}$$

where $\mathcal{U}_K$ *denotes the units of* $\mathcal{O}_K$. *In particular, the orbit* $B.\Delta_{F_1}$ *is compact.*

Proof. As $a_0 F_1 a_0^{-1}$ is commensurable with $U_1 \cap \Lambda$, there exist an ideal $\mathfrak{a}$ of $\mathcal{O}_K$ and $k_0 \in \mathbb{Z}$ such that $\Delta_{a_0 F_1 a_0^{-1}}$ contains

$$\{(x, \sigma(x), z) : x \in \mathfrak{a}, z \in (k_0\mathbb{Z}) \cdot \sqrt{d}\}.$$

Now $\mathfrak{a}^* := \{u \in \mathcal{O}_K : u\mathfrak{a} = \mathfrak{a}\}$ is an infinite subgroup of the unit group $\mathcal{U}_K$. Clearly $\{\mathrm{diag}(u, u^{-2}, u) : u \in \mathfrak{a}^*\}$ is contained in the stabilizer of Δ_{F_1} in B. As B is a one-dimensional group, having an infinite stabilizer clearly implies that the orbit $B.\Delta_{F_1}$ is compact. □

The following lemma is stated in [4, Lem 2.1.4].

Lemma 2.10. *Let* Γ *be a discrete subgroup of a Lie group* G *and* H_1, H_2 *closed subgroups of* G. *If* $H_i \cap \Gamma$ *is co-compact in* H_i *for* $i = 1, 2$, *then* $H_1 \cap H_2 \cap \Gamma$ *is co-compact in* $H_1 \cap H_2$.

Proof. Let $g_m \in H_1 \cap H_2$ be any sequence. By the assumption, there exist sequences $\gamma_m \in H_1 \cap \Gamma$ and $\gamma'_m \in H_2 \cap \Gamma$ such that $g_m\gamma_m \to h_1 \in H_1$ and $g_m\gamma'_m \to h_2 \in H_2$.

Then $\gamma_m^{-1}\gamma'_m \to h_1^{-1}h_2$ and hence $\gamma_m^{-1}\gamma'_m$ is a constant sequence for all large m. It follows that for some $m_0 > 1$,

$$\delta_m := \gamma_m\gamma_{m_0}^{-1} \in H_1 \cap H_2 \cap \Gamma$$

for all large m. Hence $g_m\delta_m \to h_1\gamma_{m_0}^{-1}$, showing that any sequence in $(H_1 \cap H_2)/(H_1 \cap H_2 \cap \Gamma)$ has a convergent subsequence. This implies our claim. □

Proposition 2.11. *The lattice $a_0F_2a_0^{-1}$ is commensurable with the subgroup $U_2 \cap \Lambda$.*

Proof. Since the argument proving Corollary 2.9 is symmetric for F_1 and F_2, the stabilizer of Δ_{F_2} of B is commensurable with $\{\mathrm{diag}(u, u^{-2}, u) : u \in \mathcal{U}_{K'}\}$ for some real quadratic field K'. By (2.3), the stabilizer of $F_2/Z(F_2)$ in B is equal to that of $F_1/Z(F_1)$ and hence contains the stabilizers of Δ_{F_1} and Δ_{F_2} in B. Therefore the two quadratic fields K and K' must coincide. Hence for some $a_1 \in A$, the lattice $a_1F_2a_1^{-1}$ is commensurable with $U_2 \cap \Lambda$, while $a_0F_2a_0^{-1}$ is commensurable with $U_1 \cap \Lambda$. By conjugating Γ_{F_1,F_2} with a_0, we may assume without loss of generality that $a_0 = e$. Therefore F_1 is commensurable with $\Lambda \cap U_1$ and F_2 is commensurable with $a_1(\Lambda \cap U_2)a_1^{-1}$.

We now claim that F_2 is commensurable with $\Lambda \cap U_2$. The proof below is adapted from [9, Prop. 2.4]. As noted before, there exists an infinite subgroup of $\{\mathrm{diag}(u, u^{-2}, u) : u \in \mathcal{U}_K\}$ which stabilizes Δ_{F_1} and Δ_{F_2} simultaneously. We denote this subgroup of B by Λ_B.

Let Γ_0 denote the normalizer of Γ_{F_1,F_2}. Then Γ_0 is discrete as the normalizer of a discrete Zariski dense subgroup is discrete. Clearly, Γ_0 contains $\Lambda_B \ltimes F_1$.

Take a non-trivial element $\gamma = (x', y', z') \in F_2$ with $x'y'z' \neq 0$. Then $\gamma(B \ltimes U_1)\gamma^{-1} \cap (B \ltimes U_1)$ is a conjugate of B; in particular, it is non-trivial. As $\Delta_B \ltimes F_1$ is a co-compact subgroup of $B \ltimes U_1$, it follows from Lemma 2.10 that $\gamma(\Delta_B \ltimes F_1)\gamma^{-1} \cap (\Delta_B \ltimes F_1)$ is a co-compact subgroup of $\gamma(B \ltimes U_1)\gamma^{-1} \cap (B \ltimes U_1)$, and hence non-trivial.

Therefore there exist

$$\delta_1 = \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_1^{-2} & 0 \\ 0 & 0 & u_1 \end{pmatrix}\begin{pmatrix} 1 & x_1 & z_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \delta_2 = \begin{pmatrix} u_2 & 0 & 0 \\ 0 & u_2^{-2} & 0 \\ 0 & 0 & u_2 \end{pmatrix}\begin{pmatrix} 1 & x_2 & z_2 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

in $\Delta_B \ltimes F_1$ satisfying

$$\gamma\delta_1 = \delta_2\gamma. \tag{2.12}$$

We claim that

$$x', y' \in K \quad \text{and} \quad y' = \sigma(x'). \tag{2.13}$$

It follows from (2.12) that $u_1 = u_2$, $z_1 = z_2 \neq 0$ and

$$\begin{pmatrix} 1 & x_1 & z_1 \\ u_1^3x' & u_1^3x'x_1 + 1 & u_1^3x'z_1 + y_1 \\ z' & z'x_1 + u_1^{-3}y' & z'z_1 + u_1^{-3}y'y_1 + 1 \end{pmatrix} = \begin{pmatrix} 1 + x_2x' + z_2z' & x_2 + z_2y' & z_2 \\ x' + y_2z' & 1 + y_2y' & y_2 \\ z' & y' & 1 \end{pmatrix}.$$

By comparing the $(2,3)$ and $(1,2)$ entries of the matrix equation (2.12), we deduce

$$x' = \frac{y_2 - y_1}{u_1^3z_1} \quad \text{and} \quad y' = \frac{x_1 - x_2}{z_2}$$

and hence

$$x', y' \in K.$$

In order to show $y' = \sigma(x')$, it suffices to show

$$\sigma(u_1^3) = -\frac{z_1}{\sigma(z_1)},$$

as $\sigma(x_i) = y_i$ for each $i = 1, 2$.

By comparing $(3, 2)$ and $(3, 3)$ entries, we obtain

$$z_1 = \frac{x_1 \sigma(x_1)}{1 - u_1^3}.$$

Using $u_1^{-1} = \sigma(u_1)$, we deduce $\frac{z_1}{\sigma(z_1)} = -\sigma(u_1^3)$, proving the claim (2.13). As F_2 is commensurable with $a_1(\Lambda \cap U_2)a_1^{-1}$, the existence of $\gamma \in F_2$ with $\gamma = (x', \sigma(x'), *)$ implies that F_2 is commensurable with $\Lambda \cap U_2$. □

Theorem 2.14. *For an upper triangular matrix $g \in \mathrm{GL}_3(\mathbb{R})$, $g\Gamma_{F_1,F_2}g^{-1}$ is commensurable with Λ.*

Proof. We have shown so far that for some upper triangular matrix $g \in \mathrm{GL}_3(\mathbb{R})$, $g\Gamma_{F_1,F_2}g^{-1}$ contains subgroups of finite indices in $\Lambda \cap U_1$ and $\Lambda \cap U_2$. By a theorem of Venkataramana [16], this implies that $g\Gamma_{F_1,F_2}g^{-1}$ is commensurable with Λ, finishing the proof. □

Proof of Theorem 1.1: Let g be an upper triangular matrix of $\mathrm{GL}_3(\mathbb{R})$ given by Theorem 2.14. Since $gF_2g^{-1} \subset gU_2g^{-1} \cap \mathrm{SU}(w_0)_{\mathbb{Q}}$, it follows that gU_2g^{-1} is defined over $\mathbb{Q}$ with respect to the $\mathbb{Q}$-form of G given by $\mathrm{SU}(w_0)$. Since both subgroups gU_2g^{-1} and U_2 are opposite to U_1 and defined over $\mathbb{Q}$, there exists $h \in U_1(\mathbb{Q})$ such that $hgU_2g^{-1}h^{-1} = U_2$ by [2]. Hence hg belongs to the intersection of the normalizers of U_1 and U_2 in $\mathrm{GL}_3(\mathbb{R})$. Consequently $d := hg$ is a diagonal element. Since $h \in \mathrm{SU}(w_0)_{\mathbb{Q}}$, $h\Lambda h^{-1}$ is commensurable with Λ, and hence $d\Gamma_{F_1,F_2}d^{-1}$ is commensurable with Λ, finishing the proof.

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