Gear Junior Retreat, Urbana July 2012
Yves Benoist, Divisible Convex Sets A.

Semisimplicity

We gather here preliminary results used to prove Vey's semisimplicity theorem. Exercises with * are more challenging.

Exercise 1. Hilbert metric Let Ω be an open convex set of the real projective space $\mathbb{P}(\mathbb{R}^d)$ which is properly convex i.e. $\overline{\Omega}$ contains no projective lines. For x, y in Ω, we set $d_\Omega(x,y) = \log[x,y,b,a]$ where [ ] stands for the cross-ration and where the points a, x, y, b are collinear in this order with a, b on $\partial\Omega$.
(i) Prove that d_Ω is a distance on Ω which is complete.
(ii) Prove that the straight lines are geodesic for d_Ω. Is the converse true?

Exercise 2. Convex hull of orbits Let Ω be an open properly convex cone of $\mathbb{P}(\mathbb{R}^d)$ and $\Delta \subset \mathrm{PGL}(\mathbb{R}^d)$ be a discrete subgroup which preserves Ω.
(i) Prove that Δ acts properly on Ω.
(ii) Assume that Δ divides Ω i.e. that the quotient $\Delta\backslash\Omega$ is compact. One says then that Ω is divisible. Prove that for every x_0 in Ω the convex hull of the Δ-orbit Δx_0 is equal to Ω. (Hint: If not, construct points of Ω whose Hilbert distance to Δx_0 is arbitrarily large).

Exercise 3. Divisible convex cones and centralizer Let C be an open convex cone of $\mathbb{R}^d$ which is properly convex i.e. $\overline{C}$ contains no lines. Let $\Gamma \subset \mathrm{GL}(\mathbb{R}^d)$ be a discrete subgroup which divides C. One says then that C is divisible. Let H_Γ be the connected component of the centralizer of Γ:
$H_\Gamma = \{h \in \mathrm{GL}(\mathbb{R}^d) \mid h \circ \gamma = \gamma \circ h \text{ for all } \gamma \in \Gamma\}_e$.
(i) Prove that C is H_Γ-invariant.
(ii) Prove that all the elements of H_Γ are diagonalizable over $\mathbb{R}$.
(iii) Prove that the group ΓH_Γ is closed in $\mathrm{GL}(\mathbb{R}^d)$.
(iv) Prove that the discrete group $\Gamma \cap H_\Gamma$ is cocompact in H_Γ.

Exercise 4. Divisible convex cones and divisible convex sets Let C be an open properly convex cone of $\mathbb{R}^d$ and Ω be its image in $\mathbb{P}(\mathbb{R}^d)$.
(i) Prove that if Ω is divisible then C is divisible.
(ii) Prove that if C is divisible then Ω is divisible. (Hint: Use H_Γ).

Exercise 5. Dual cones Let $V = \mathbb{R}^d$, $C \subset V$ be an open properly convex cone and $C^* \subset V^*$ be the dual cone $C^* := \{f \in V^* \mid f(v) > 0 \text{ for all } v \in \overline{C} \smallsetminus \{0\}\}$.
(i) Let $\mathrm{d}f$ be a Lebesgue measure on V^*. Prove that the function $\varphi : C \to \mathbb{R}$ $v \mapsto \varphi(v) = \int_{C^*} e^{-f(v)}\, \mathrm{d}f$ has a positive definite Hessian matrix $(\frac{\partial^2 \varphi}{\partial_i \partial_j})$.
(ii) Prove that the map $C \to C^*; v \mapsto v^* := \int_{C^*} f\, e^{-f(v)}\, \mathrm{d}f$ is a diffeomorphism.
(iii) Prove that, for γ in $\mathrm{SL}(V)$ preserving C and v in C, $(\gamma v)^* = {}^t\gamma^{-1} v^*$.

Exercise 6. Dual divisible sets Let $V = \mathbb{R}^d$, Ω be an open properly convex set of $\mathbb{P}(V)$ and $\Omega^* \subset \mathbb{P}(V^*)$ the dual convex set $\Omega^* = \{\mathbb{R}f \in \mathbb{P}(V^*) \mid f(v) \neq 0 \text{ for all } \mathbb{R}v \in \overline{\Omega}\}$. Let $\Delta \subset \mathrm{PGL}(V)$ be a discrete subgroup which divides Ω. Prove that the transpose group ${}^t\Delta \subset \mathrm{PGL}(V^*)$ divides Ω^*. (Hint: Two strategies are possible. 1. Use the diffeomorphism $\Omega \to \Omega^*; \mathbb{R}v \mapsto \mathbb{R}v^*$ from the previous exercise. 2. Use a cohomological dimension argument.)

Exercise 7. Invariant subspaces Let C be an open properly convex cone of $\mathbb{R}^d$ and $\Gamma \subset \mathrm{GL}(\mathbb{R}^d)$ be a discrete subgroup which divides C. Let $W \subset \mathbb{R}^d$ be a non-trivial Γ-invariant subspace.
(i) Prove that $W \cap C$ is empty.
(ii) Prove that $W \cap \overline{C}$ is non-zero.

Exercise 8. Compact metric spaces Prove that a bijective contraction φ of a compact metric space (X, d) is an isometry, i.e. if $d(\varphi(x), \varphi(y)) \leq d(x, y)$ for all x, y in X, then one has $d(\varphi(x), \varphi(y)) = d(x, y)$ for all x, y in X.

Exercise 9. Vey's flow * Let C be an open properly convex cone of $\mathbb{R}^d$ and $\Gamma \subset \mathrm{GL}(\mathbb{R}^d)$ be a discrete subgroup which divides C. Assume that Γ preserves a line $D \subset \mathbb{R}^d$. Prove that D has a Γ-invariant complementary subspace $H \subset \mathbb{R}^d$. (Hint: Introduce the flow whose direction is $D \cap \partial C$ and whose speed for the Hilbert metric is one. Check that this flow is a contraction for the Hilbert metric and apply the previous exercise).
For higher-dimensional Γ-invariant vector subspaces D, one needs more tools.

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Zariski Closure

We gather here preliminary results used to describe the Zariski closures of groups dividing open properly convex cones.

Exercise 1. Proximality and limit sets Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^d)$ acting strongly irreducibly on $\mathbb{R}^d$ i.e. all the finite index subgroups of Γ act irreducibly on $\mathbb{R}^d$. Assume that Γ is proximal i.e. there exists a sequence γ_n in Γ such that the sequence $\frac{\gamma_n}{\|\gamma_n\|}$ converges in $\mathrm{End}(\mathbb{R}^d)$ towards a rank one operator π. Prove that there exists a unique minimal closed non-empty Γ-invariant subset Λ_Γ in $\mathbb{P}(\mathbb{R}^d)$.

Exercise 2. Bounded semigroups Let Δ be a bounded subsemigroup of $\mathrm{SL}(d, \mathbb{R})$. Prove that the group spanned by Δ is also bounded.

Exercise 3. Invariant cones and positive proximality Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^d)$ acting strongly irreducibly on $\mathbb{R}^d$.
$a)$ Assume that Γ preserves an open properly convex cone C of $\mathbb{R}^d$.
(i) Prove that the image of Γ in $\mathrm{PGL}(\mathbb{R}^d)$ is unbounded.
(ii) Prove that Γ is proximal. (Hint: Assume that Γ contains the positive homotheties and introduce semigroups $\overline{\pi\Gamma\pi}\smallsetminus\{0\}$ with $\pi = \lim\limits_{n\to\infty} \frac{\gamma_n}{\|\gamma_n\|}$, $\gamma_n \in \Gamma$).
(iii) Prove that Γ is positively proximal i.e. Γ is proximal and any rank one limit $\pi = \lim\limits_{n\to\infty} \frac{\gamma_n}{\|\gamma_n\|}$ with γ_n in Γ has a positive eigenvalue.
$b)$ Conversely, if Γ is positively proximal, prove that Γ preserves an open properly convex cone C of $\mathbb{R}^d$. (Hint: Use the exercise below).

Exercise 4. Invariant cones and limit sets Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^d)$ acting strongly irreducibly on $V = \mathbb{R}^d$. Assume that Γ is proximal. Prove that the following two assertions are equivalent:
(i) The group Γ preserves an open properly convex set Ω of $\mathbb{P}(V)$.
(ii) For any lines $\mathbb{R}v_1$, $\mathbb{R}v_2$ in the limit set $\Lambda \subset \mathbb{P}(V)$ of Γ and any lines $\mathbb{R}f_1$, $\mathbb{R}f_2$ in the limit set $\Lambda^* \subset \mathbb{P}(V^*)$ of ${}^t\Gamma$, one has $f_1(v_1)f_2(v_2)f_1(v_2)f_2(v_1) \geq 0$. (Hint: Construct dense sequences $x_i = \mathbb{R}v_i$ in Λ, and $y_j = \mathbb{R}f_j$ in Λ^* such that $f_j(v_i) \neq 0$ for all i, j).

Exercise 5. Minimal and maximal invariant cones Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^d)$ acting strongly irreducibly on $\mathbb{R}^d$. Assume that Γ preserves an open properly convex cone C of $\mathbb{R}^d$.
(i) Prove that the image of ∂C in $\mathbb{P}(\mathbb{R}^d)$ contains the limit set Λ_Γ.
(ii) Prove that there exists a Γ-invariant open properly convex cone C_{min} of $\mathbb{R}^d$ which, up to sign, is included in any Γ-invariant open properly convex cone C of $\mathbb{R}^d$. Prove that there also exists a maximal one C_{max}.
(iv) Assume that Γ divides C. Prove that $C_{min} = C = C_{max}$.

Exercise 6. Invariant cones and complex structures Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^{2d})$ acting strongly irreducibly on $\mathbb{R}^{2d}$. Assume that Γ preserves an open properly convex cone of $\mathbb{R}^{2d}$. Prove that Γ does not preserve a complex structure on $\mathbb{R}^{2d}$. (Hint: Use a previous exercise)

Exercise 7. Invariant cones and symplectic structures Let Γ be a subgroup of $\mathrm{GL}(\mathbb{R}^{2d})$ acting strongly irreducibly on $\mathbb{R}^{2d}$. Assume that Γ preserves an open properly convex cone of $\mathbb{R}^{2d}$. Prove that Γ does not preserve a symplectic structure ω on $\mathbb{R}^{2d}$. (Hint: If not, prove that for any three points v_1, v_2, v_3 of ∂C the products $\omega(v_1, v_2)\omega(v_1, v_3)$ has to be non-negative).

Exercise 8. Invariant cones and quadratic forms * (i) Prove that for any integers $p \geq q \geq 1$, there exists a subgroup Δ of $\mathrm{SO}(p, q)$ acting strongly irreducibly on $\mathbb{R}^{p+q}$ and preserving an open properly convex set Ω of $\mathbb{P}(\mathbb{R}^{p+q})$. (Hint: Denote by b the associated bilinear form and construct a sequence of isotropic vectors $v_i \in \mathbb{R}^{p+q}$ such that $b(v_i, v_j) > 0$ for all $i \neq j$).
(ii) Prove that, if moreover Δ divides Ω then $q = 1$ and Ω is an ellipsoid. (Hint: Prove that the boundary $\partial\Omega$ is included in the set of isotropic lines).

Exercise 9. Zariski closure * Let $d = 3$, 4 or 5. Let Δ be a discrete subgroup of $\mathrm{SL}(d, \mathbb{R})$ which acts strongly irreducibly on $\mathbb{R}^d$ and divides an open properly convex set Ω in $\mathbb{P}(\mathbb{R}^d)$. Assume that Ω is not an ellipsoid. Prove that Δ is Zariski dense in $\mathrm{SL}(d, \mathbb{R})$. For $d \geq 6$, one needs more tools.

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Closedness

We gather here preliminary results used to prove the closedness of the moduli space of properly convex projective structures on a compact manifold.

Exercise 1. Dimension 2 Let Γ be an infinite non-solvable subgroup of $\mathrm{SL}(2,\mathbb{R})$.
(i) Prove that the group Γ is Zariski dense in $\mathrm{SL}(2,\mathbb{R})$.
(ii) Prove that the group Γ is either discrete or dense.
(iii) Prove that the group Γ contains a matrix whose eigenvalues are real and positive.
(iv) Prove that the group Γ contains a matrix whose eigenvalues are real and negative.

Exercise 2. Invariant convex cone and positive semiproximality Let Γ be a subgroup of $\mathrm{GL}(d,\mathbb{R})$ which preserves an open properly convex cone C of $\mathbb{R}^d$. Prove that every element g of Γ is positively semiproximal i.e. the spectral radius of g is an eigenvalue of g.

Exercise 3. Cohomological dimension Let Δ be a subgroup of $\mathrm{SL}(d,\mathbb{R})$ which divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$.
(i) Prove that Δ is finitely generated.
(ii) Prove that Δ contains a torsion-free finite index subgroup.
(iii) Prove that the cohomological dimension of Δ is $d-1$.

Exercise 4. Normal subgroups in Zariski dense groups Let Γ be a Zariski dense subgroup of $\mathrm{SL}(d,\mathbb{R})$. Prove that the group Γ does not contain any infinite abelian normal subgroup.

Exercise 5. Zassenhaus neighborhoods Prove that every Lie group G contains a neighborhood U of e such that, for every discrete subgroup Γ of G, the intersection $\Gamma \cap U$ is included in a connected nilpotent subgroup of G.

Exercise 6. Limits of discrete faithful morphisms Let Γ be a finitely generated group which does not contain any infinite abelian normal subgroup.
(i) Prove that the set of faithful morphisms is closed in $Hom(\Gamma, \mathrm{SL}(d,\mathbb{R}))$.
(ii) Prove that the set of faithful morphisms with discrete image is also closed in $Hom(\Gamma, \mathrm{SL}(d,\mathbb{R}))$.

Exercise 7. Auslander projection theorem Let Γ be a discrete subgroup of $\mathrm{SL}(d,\mathbb{R})$ which does not contain any infinite abelian normal subgroup. Let G be the Zariski closure of Γ and N a normal abelian subgroup of G.
(i) Prove that the intersection $N \cap \Gamma$ is finite.
(ii) Prove that the image of Γ in G/N is also discrete. (Hint: Prove this first when N is the center Z of G. Then notice that the adjoint group $\mathrm{Ad}(G)$ is included in the group $P := \{\varphi \in \mathrm{GL}(\mathfrak{g}) \mid \varphi(\mathfrak{n}) = \mathfrak{n}\}$, use a Zassenhaus neighborhood in P and use an automorphism of P which contracts $\mathrm{Ad}(N)$).

Exercise 8. Positive semiproximality and invariant convex sets Let Δ be a Zariski dense subgroup of $\mathrm{SL}(d,\mathbb{R})$ all of whose elements are positively semiproximal.
(i) Prove that Δ preserves a properly convex open subset Ω of $\mathbb{P}(\mathbb{R}^d)$.
(ii) Prove that the cohomological dimension of Δ is at most $d-1$.

Exercise 9. Limits of groups dividing convex sets * Let Δ be a finitely generated group. Assume that the centers of the finite index subgroups of Δ are trivial. Let $\rho_n \in Hom(\Delta, \mathrm{SL}(d,\mathbb{R}))$ be a sequence of faithful discrete morphisms such that $\rho_n(\Delta)$ divides an open properly convex set Ω_n of $\mathbb{P}(\mathbb{R}^d)$. Assume that the sequence ρ_n converges to a morphism $\rho_\infty \in Hom(\Delta, \mathrm{SL}(d,\mathbb{R}))$. Let Δ_∞ be the discrete group $\Delta_\infty := \rho_\infty(\Delta)$. We want to prove the assertion

$$(A): \qquad \Delta_\infty \text{ divides an open properly convex set } \Omega_\infty \text{ of } \mathbb{P}(\mathbb{R}^d).$$

(i) Prove that all the elements of Δ_∞ are positively semiproximal.
(ii) Prove that Δ_∞ does not contain infinite abelian normal subgroups.
(iii) Prove that if Δ_∞ acts irreducibly on $\mathbb{R}^d$ then (A) is true.
(iv) Prove (A) when $d = 3$. (due to Goldman and Choi in this case).
(v) Prove (A) when $d = 4$. For $d \geq 5$, one needs more tools.

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Hyperbolicity

We gather here preliminary results used to describe the geometry of an open divisible properly convex set.

Exercise 1. Affine zooming semigroup Let $\mathcal{P}$ be the set of Borel probability measures on $[0, 1]$ endowed with the weak topology. Let $\mathcal{P}'$ be the subset of probability measures with dense support. For $0 \leq a < b \leq 1$ and μ in $\mathcal{P}'$, we denote $\Phi_{a,b}(\mu)$ the measure $\varphi \mapsto \mu([a, b])^{-1} \int_0^1 \varphi(a+(b-a)t)\, \mathrm{d}\mu(t)$ for φ Borel function on $[0, 1]$. The set $S := \{\Phi_{a,b}\}$ is a semigroup of transformations of $\mathcal{P}'$. Let $C > 0$. A probability measure μ on $[0, 1]$ is C-doubling if for every $x \in [0, 1]$ and $\varepsilon > 0$, one has $\mu(B(x, 2\varepsilon)) \leq C\mu(B(x, \varepsilon))$.
(i) Prove that a C-doubling measure μ is atom-free i.e. $\mu(\{x\}) = 0$ for all point x.
(ii) Let Q be a closed subset of $\mathcal{P}$ which is included in $\mathcal{P}'$ and is S-invariant. Prove that there exists $C > 0$ such that for any μ in Q, the measure μ is C-doubling.
(iii) Conversely, the set Q_C of C-doubling measures on $[0, 1]$ is a closed S-invariant subset of $\mathcal{P}$ which is included in $\mathcal{P}'$.

Exercise 2. Right-angle pentagons Let P be a convex pentagon in the projective plane $\mathbb{P}(\mathbb{R}^3)$. For $i = 1, \ldots, 5$, let $\sigma_i \in \mathrm{GL}(3, \mathbb{R})$ be the projective reflection fixing pointwise the i^{th}-side of P and preserving the two lines supporting the adjacent sides.

Let Γ be the group generated by these five reflections. Prove that the set $\Omega := \cup_{\gamma\in\Gamma}\gamma P$ is an open divisible convex subset of $\mathbb{P}(\mathbb{R}^3)$.

Exercise 3. Benzecri compactness theorem Let $G_d = \mathrm{PGL}(d+1, \mathbb{R})$, X_d be the set of open properly convex subset of $\mathbb{P}(\mathbb{R}^{d+1})$ endowed with the Hausdorff topology and, $Y_d = \{(\Omega, x) \mid \Omega \in X_d\ ,\ x \in \Omega\}$. Prove that the group G_d acts properly and cocompactly on Y_d. (Hint: Prove first, using John ellipsoid, that the group $H_d = \mathrm{Aff}(\mathbb{R}^d)$ of affine transformations of $\mathbb{R}^d$ acts properly and cocompactly on the set Z_d of open bounded convex subsets of $\mathbb{R}^d$).

Exercise 4. Triangles in the orbit closure Assume $d = 2$ and let $\Omega \in X_2$.

(i) Prove that if Ω is not strictly convex, i.e. if $\partial\Omega$ contains open segments, then the orbit closure $\overline{G_d\Omega}$ contains a triangle.

(ii) Prove that if $\partial\Omega$ is not C^1 then the orbit closure $\overline{G_d\Omega}$ also contains a triangle.

Exercise 5. Ellipsoids in the orbit closure Let $\Omega \in X_d$.

(i) Prove that if $\partial\Omega$ is C^2 then the orbit closure $\overline{G_d\Omega}$ contains an ellipsoid.

(ii) Describe $\overline{G_d\Omega}$ when $d = 2$ and Ω is either a polygon or a quarter disk.

Exercise 6. Closed subsets and Gromov hyperbolicity (i) Let F be a closed G_d-invariant subset of X_d all of whose elements Ω are strictly convex. Prove that there exists $\delta > 0$ such that for all Ω in F, the Hilbert metric d_Ω on Ω is δ-hyperbolic i.e. all geodesic triangles in (Ω, d_Ω) are δ-thin.

(ii) Conversely the set $F_\delta := \{\Omega \in X_d \mid d_\Omega \text{ is } \delta\text{-hyperbolic}\}$ is a closed G_d-invariant subset of X_d all of whose elements Ω are strictly convex.

Exercise 7. Closed orbits Let $\Omega \in X_d$. Prove that if Ω is divisible then the orbit $G_d\Omega$ in X_d is closed.

Exercise 8. Strictly convex divisible sets Let Δ be a subgroup of $\mathrm{SL}(d, \mathbb{R})$ which divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$. Prove that the following statement are equivalent.

(i) Ω is strictly convex.

(ii) The Hilbert metric d_Ω on Ω is Gromov hyperbolic.

(iii) The group Δ is Gromov hyperbolic.

(iv) The boundary $\partial\Omega$ is C^1.

Exercise 9. Hyperbolicity and doubling measure * Assume $d = 2$ and let $\Omega \in X_d$. Prove that the Hilbert metric d_Ω is Gromov hyperbolic if and only if the curvature measure on $\partial\Omega$ is locally doubling.

For $d \geq 3$, one needs more tools.

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Open Questions

Problem 1. Divisible non convex sets Let $d = 3$. Describe all the open subsets U of $\mathbb{R}^d$ for which there exists a discrete group Γ of affine transformations of $\mathbb{R}^d$ preserving U and acting properly on U with a compact quotient $\Gamma\backslash U$. When $d = 2$, the sets U are known to be either the plane, the half-plane, the quarter-plane or the punctured plane.

Problem 2. Fundamental groups of surfaces Describe the possible Zariski closures of discrete subgroups Γ of $\mathrm{GL}(\mathbb{R}^d)$ which act irreducibly on $\mathbb{R}^d$, which are isomorphic to the fundamental group of a compact surface and which preserve an open properly convex cone C of $\mathbb{R}^d$. Can Γ preserve a quadratic form of signature (p, q) with $p \geq q \geq 2$? [Partial results obtained by Danciger-Guéritaud-Kassel in 2017]

Problem 3. Real projective and real hyperbolic structures Prove that every connected component of the moduli space of strictly convex projective structures on a compact 3-dimensional manifold contains an hyperbolic structure. Equivalently, deform continuously any 3-dimensional divisible strictly convex set Ω to an ellipsoid through a family of divisible convex sets. The same statement is true in dimension 2 and false in dimension 4. [A counterexample has been announced to exist by Ballas-Danciger-Lee-Marquis in 2023]

Problem 4. Real projective and complex hyperbolic structures Prove that a group isomorphic to a lattice of SU(2,1) can not divide an open properly convex set Ω of the 4-dimensional real projective space $\mathbb{P}(\mathbb{R}^5)$. Equivalently Ω is not quasiisometric to the complex hyperbolic space $H^2_{\mathbb{C}}$. One knows that Ω is not always quasiisometric to the real hyperbolic space $\mathbb{H}^4_{\mathbb{R}}$.

Problem 5. Density of the limit set Let Δ be a discrete subgroup of $\mathrm{GL}(\mathbb{R}^d)$ which acts irreducibly on $\mathbb{R}^d$ and divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$. Assume that Ω is not homogeneous. Prove that the limit set Λ_Ω is equal to the boundary $\partial\Omega$. This is known only when Ω is strictly convex or when Ω is 3-dimensional. [Solved positively by Piere-Louis Blayac in 2022, relying on the rank dichotomy due to Andrew Zimmer in 2020]

Problem 6. Curvature of the boundary Let $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ be an open properly convex set which is divisible. Assume that Ω is not the ellipsoid. Prove that the curvature of $\partial\Omega$ is supported by a subset of Lebesgue measure zero. This is known only when Ω is strictly convex. This is not known even for Ω in dimension 3. [Solved positively by M. Crampon in 2013]

Problem 7. Dynamics of the geodesic flow Let Δ be a discrete subgroup of $\mathrm{GL}(\mathbb{R}^d)$ which acts irreducibly on $\mathbb{R}^d$ and divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$. Prove that the geodesic flow of the Hilbert metric in the quotient $\Delta\backslash\Omega$ has a dense orbit. Does this flow have a measure of maximum entropy? A unique one? All this is known only when Ω is strictly convex. This is not known even for Ω in dimension 3. [Solved positively in dimension 3 by Harrison Bray in 2018 and in general by Pierre-Louis Blayac in 2021]

Problem 8. Property T Let Δ be a discrete subgroup of $\mathrm{GL}(\mathbb{R}^d)$ which acts irreducibly on $\mathbb{R}^d$ and divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$ which is not homogeneous. Is it true that Δ does not satisfy Kazhdan property T? This is not known even when Ω is strictly convex. There are examples of homogeneous divisible convex open sets Ω, for which Δ has property T.

Problem 9. $\mathbb{Z}^2$-subgroups Let Δ be a discrete subgroup of $\mathrm{GL}(\mathbb{R}^d)$ which acts irreducibly on $\mathbb{R}^d$ and divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$ which is not strictly convex. Is it true that Δ contains a subgroup isomorphic to $\mathbb{Z}^2$? This is known only for 3-dimensional Ω. When Ω is strictly convex, Δ does not contain subgroups isomorphic to $\mathbb{Z}^2$ since Δ is Gromov hyperbolic.

Problem 10. Construction of divisible convex sets Prove that for any integer $d \geq 4$ there exists a discrete subgroup Δ of $\mathrm{GL}(\mathbb{R}^d)$ which acts strongly irreducibly on $\mathbb{R}^d$ and divides an open properly convex subset Ω of $\mathbb{P}(\mathbb{R}^d)$ which is not homogeneous and which is not strictly convex. Many examples are known with Ω of low dimension 3, 4, 5, 6, ... [Solved positively by Blayac-Viaggi in 2023]

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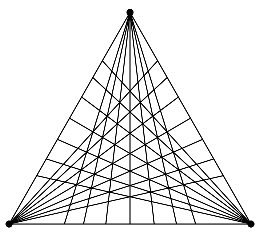
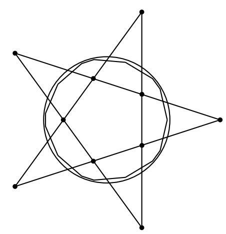
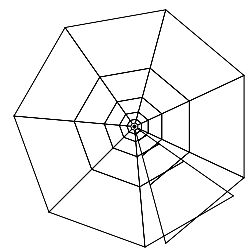
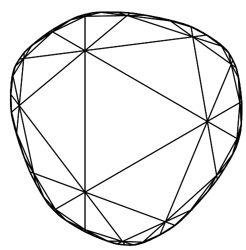
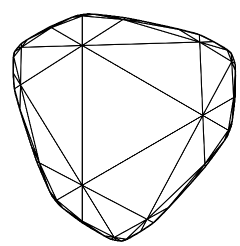
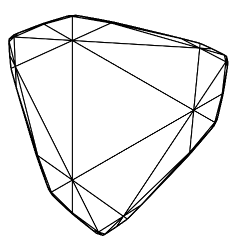
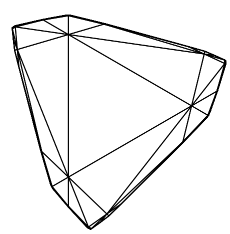
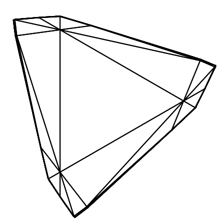
Pictures

We draw first a few 2-dimensional divisible sets:

We draw now various views of a 3-dimensional divisible convex set associated to the following Coxeter group and the following prismatic fundamental domain: