CONFORMAL AUTOSIMILARITY

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ABSTRACT. We prove that the conformally autosimilar closed subsets of the Euclidean sphere are exactly the limit sets of convex cocompact Kleinian groups. We prove also other similar results.

CONTENTS

Compacts conformément autosimilaires. Nous prouvons que les compacts conformément autosimilaires de la sphère euclidienne sont exactement les ensembles limites des groupes kleiniens convexes cocompacts.

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1. Introduction

In this article we want to describe the compact subsets of the Euclidean sphere that are *conformally autosimilar*. This means roughly that after zooming in somewhere in this compact subset using a sequence of conformal transformations, what you see looks very much like the compact set you started with. We will see that these conformally autosimilar compact sets are exactly the limit sets of the convex cocompact Kleinian groups.

In this introduction G will denote the group $G := PO_e(d+1,1)$ of conformal transformations of the sphere $X := \mathbb{S}^d$ with $d \geq 1$. Later on, in Section 7.2, we will allow G to be any Gromov hyperbolic group and X will be the boundary of G.

We want to study the dynamics of the group G on the space $\mathcal{K}_1$ of non-empty compact subsets K of X. This space $\mathcal{K}_1$ endowed with the Hausdorff distance is a compact metric space. We first notice that, when $K \neq X$, the orbit closure $\overline{GK}$ in $\mathcal{K}_1$ contains all the subsets $\{x\}$ with only one element. This is why we introduce the space $\mathcal{K}_2$ of compact subsets K of X that contain at least two elements. Let $K \in \mathcal{K}_2$. The main question we address in this preprint is :

When is the orbit GK closed in $\mathcal{K}_2$?

We can now give a precise definition: *A compact subset $K \subset X$ is called conformally autosimilar if GK is closed in $\mathcal{K}_2$.* This definition matches the rough intuition above since *"what you see after a conformal zooming in of K"* is nothing but a limit K_∞ in $\mathcal{K}_1$ of a sequence of images $g_n K$ of K by conformal transformations $g_n \in G$. In our definition, we ask this limit K_∞ to be either a point or the image gK of K by a conformal transformation $g \in G$.

The answer to this question is very simple but seems to be new even for $d = 1$ (see Theorem 3.1 and Lemma 3.3) :

The orbit GK is closed in $\mathcal{K}_2$ if and only if
K is the limit set of a convex cocompact Kleinian group.

We recall that a *Kleinian group* is a discrete subgroup Γ of G, and that such a group is said to be *convex cocompact* if there exists a Γ-invariant closed convex subset C of the hyperbolic space $\mathbb{H}^{d+1}$ such that $\Gamma \backslash C$ is compact. The convex cocompact Kleinian groups have been often studied. They form the simplest class of examples of word-hyperbolic groups (see [10], [15] and more recently [18]).

When looking at the pictures in Mumford, Series and Wright's book, one can guess that some limit sets of Kleinian groups, as the *Schottky dance* [20, p.99-100], are conformally autosimilar while others as the *Indra's necklace* [20, p.158-159], are not. Our Theorem 3.1 corroborates precisely this guess.

Here are a few comments which enlight the diversity of these conformally autosimilar sets. These comments rely on the survey [16, p.512-517] which contains examples of limit sets of convex cocompact Kleinian groups.

When $d = 1$, i.e. when $G = PSL(2, \mathbb{R})$ and $X = \mathbb{P}^1(\mathbb{R})$, a conformally autosimilar compact set $K \subsetneq \mathbb{P}^1(\mathbb{R})$ is always homeomorphic to a Cantor set.

When $d = 2$, i.e. when $G = PSL(2, \mathbb{C})$ and $X = \mathbb{P}^1(\mathbb{C})$, there are much more homeomorphism types of conformally autosimilar compact sets $K \subset \mathbb{P}^1(\mathbb{C})$. There are examples where K is homeomorphic to :
- a Cantor set,
- a Jordan curve,
- a *combination* of Cantor sets and Jordan curves, i.e. a closed subset whose connected components are points and Jordan curves.
- a *Sierpinsky gasket*, i.e. a nowhere dense subset obtained by removing from X a countable union of open disks with disjoint closures,
- a *degenerate Sierpinski gasket* i.e. a nowhere dense subset obtained by removing from X a countable union of disjoint open disks such that the union of their boundaries is connected,...

When $d = 3$, i.e. when $G = PO_e(4, 1)$ and $X = \mathbb{S}^3$, there are even more striking examples. The compact $K \subset \mathbb{S}^3$ may be :
- a wild knot,
- a wild 2-sphere,
- a Menger curve,
- an unknotted curve whose stabilizer in G is not quasi-Fuchsian,...

When $d \geq 4$ more examples arise. In particular the limit sets of the Coxeter subgroups constructed in [11].

Using similar tools, we will <u>also</u> answer the following question :

When does the orbit closure $\overline{GK}$ contains nothing but GK and X?

We will see (Theorem 4.1 and Lemma 3.3) that this happens if and only if

there exists a discrete subgroup Γ of G that preserves K and acts cocompactly on $X \smallsetminus K$.

We will also answer the following question :

When does $\overline{GK}$ contains nothing but GK and pairs of points?

We will see (Theorem 5.1 and Lemma 3.3) that this happens if and only if

K is the union of the limit set of a convex cocompact Kleinian group Γ and of finitely many Γ-orbits.

We will then apply Theorem 3.1 to the study of the dynamics of the group G on this space $\mathcal{K}_2$. In order to emphasize the complexity of this dynamics we will prove the following three facts (Theorem 6.1) :

The union of the closed G-orbits in $\mathcal{K}_2$ is dense.

The group G has a dense orbit in $\mathcal{K}_2$.

There exists G-minimal subsets in $\mathcal{K}_2$ which are not G-orbits.

We recall that a G-minimal subset is a G-invariant non-empty closed subset which is minimal for the inclusion.

We thank M. Kapovich for nice discussions on this topic. These results have been announced and videotaped during the Introductory workshop 740 at MSRI in January 2015.

2. Notations and preliminary results

For the sake of simplicity we will explain our main results in the context of "simple Lie groups of real rank one". In Section 7.2, we will explain how to extend these results to the context of "Gromov hyperbolic groups".

2.1. **Action on the boundary.** Let M be a Riemannian symmetric space of negative curvature and $G = \mathrm{Isom}_e(M)$ be the connected component of the group of isometries of M. In other terms, G is a simple Lie group of real rank one.

Let X be the sphere at infinity or visual boundary of M. We will denote by $d+1$ the dimension of M so that X is homeomorphic to the sphere $\mathbb{S}^d$.

From the geometric point of view X is the set of equivalence classes of geodesic rays $r : [0, \infty) \to M$ where two geodesic rays are said to be equivalent if they stay within bounded distance from one another.

From the Lie group point of view, X is the boundary of G i.e. it is the G-homogeneous space $X = G/P$ whose isotropy group P is a parabolic subgroup of G.

For instance when M is the real hyperbolic space $\mathbb{H}^{d+1}$, with $d \geq 1$, the group G is the conformal group $G = PO_e(d+1, 1)$ and X is the conformal sphere $X := \mathbb{S}^d$. We will define a *conformal transformation of* $\mathbb{S}^d$ to be an element of G.

We recall that the visual compactification $\overline{M}$ of M is a compact space containing M as an open dense set and such that $\overline{M} \smallsetminus M = X$. This space $\overline{M}$ is homeomorphic to the closed $(d+1)$-dimensional ball. When A is a subset of M, we will denote by $\overline{A}$ its closure in $\overline{M}$.

Let K be a closed subset of X. We denote by Γ_K the stabilizer of K in G :

$$\Gamma_K := \{g \in G \mid gK = K\}.$$

This is a closed subgroup of G.

We denote by $\mathrm{Conv}(K)$ the convex hull of K in M. It is the smallest closed convex subset C of M such that $\overline{C} \smallsetminus C = K$. It does exist when K contains at least two points. We will use implicitely the equality $K = X \cap \overline{\mathrm{Conv}(K)}$ (see [1]) and the fact that the map $K \mapsto \mathrm{Conv}(K)$ is continuous (see [8]).

2.2. **Convex cocompact subgroups.** The following definitions and properties are classical for discrete subgroups Γ. One can check that they are still valuable for closed subgroups.

Definition 2.1. *Let Γ be a closed subgroup of G. The limit set Λ_Γ is the closed subset of X given by $\Lambda_\Gamma := \overline{\Gamma m} \smallsetminus \Gamma m$ where m belongs to M.*

The domain of discontinuity Ω_Γ is the open set $\Omega_\Gamma := X \smallsetminus \Lambda_\Gamma$.

Note that the limit set does not depend on the choice of m in M, that the limit set is not empty when the subgroup Γ is not bounded, and that the group Γ acts properly on the domain of discontinuity Ω_Γ. Note that we kept the classical name *domain of discontinuity* for Ω_Γ eventhough Γ is not assumed to be discrete.

Note also that, if Γ' is a cocompact subgroup of Γ, they have same limit sets $\Lambda_{\Gamma'} = \Lambda_\Gamma$.

Definition 2.2. *A closed subgroup Γ of G is said to be convex cocompact if there exists a closed nonempty Γ-invariant convex subset C of M on which Γ acts cocompactly.*

When Γ is convex cocompact and unbounded, one can choose C to be the convex hull of the limit set $C = \mathrm{Conv}(\Lambda_\Gamma)$.

A closed subgroup Γ is said to be *non-elementary* when it does not fix a point or a pair of points in $\overline{M}$. Equivalently its limit set Λ_Γ is infinite and minimal.

Remark 2.3. It is well-known that a non-elementary closed subgroup Γ of G is convex cocompact if and only if it acts cocompactly on the set $\Theta_3(\Lambda_\Gamma)$ of distinct triples in Λ_Γ. See [9] for a similar statement in the context of word-hyperbolic groups.

3. Conformally autosimilar sets

In this chapter, we describe the conformally autosimilar compact sets.

3.1. **Closed orbits in $\mathcal{K}_2$.** We keep the notations of Chapter 2 and we denote

$$\mathcal{K}_1 = \mathcal{K}_1(X) := \{K \textit{ non-empty compact subset of } X\}.$$

This space $\mathcal{K}_1$ is a compact metric space with respect to the Hausdorff distance. We introduce also the following open subset $\mathcal{K}_2 = \mathcal{K}_2(X)$ of $\mathcal{K}_1$,

$$\mathcal{K}_2(X) := \{K \textit{ compact subset of } X \textit{ containing at least two points}\}.$$

The first theorem of this article is the following

Theorem 3.1. *Let G be a simple Lie group of real rank one and X be its boundary. Let $K \subset X$ be a compact subset containing at least two points. The following are equivalent.*
(i) The orbit GK is closed in $\mathcal{K}_2(X)$.
(ii) There exists a convex cocompact subgroup Γ of G such that $K = \Lambda_\Gamma$.
(iii) The stabilizer Γ_K is convex cocompact and K is its limit set.

Remark 3.2. In (iii) the subgroup Γ_K of G might be non-discrete. However, in (ii), one can always choose Γ to be discrete since, by Lemma 3.3, the group Γ_K always contains a discrete cocompact subgroup.

Lemma 3.3. *Let G be a simple Lie group of real rank one and H be a non-elementary closed subgroup of G. Then the group H contains a cocompact discrete subgroup Γ.*

Sketch of the proof of Lemma 3.3. Let H_e be the connected component of H and N be the normalizer in G of H_e.

We first notice that, since H is non-elementary, the solvable radical of H_e has to be compact. Then we distinguish two cases.

When H_e is non compact. In this case, since the real rank of G is one, the quotient group N/H_e is compact. We choose Γ to be a cocompact lattice in H_e. Such a group exists by a theorem of Borel in [7], since H_e is a reductive Lie group.

When H_e is compact. In this case, there exists a closed subgroup L of the connected component N_e such that $N_e = LH_e$ and such that the intersection $L \cap H_e$ is finite. We choose Γ to be $\Gamma := L \cap H$. $\square$

Here is a corollary of Theorem 3.1.

Corollary 3.4. *Let $G = \mathrm{PSL}(2, \mathbb{C})$ and $c \subset \mathbb{P}^1(\mathbb{C})$ be a Jordan curve. Then, the orbit Gc is closed in $\mathcal{K}_2(\mathbb{P}^1(\mathbb{C}))$ if and only if c is the limit set of a quasifuchsian subgroup Γ of G.*

We recall that a discrete convex cocompact subgroup Γ of $\mathrm{PSL}(2, \mathbb{C})$ is said to be *quasifuchsian* if its limit set is a Jordan curve.

Remark 3.5. Corollary 3.4 is a boon companion of the main theorem of [4] which says that *a Jordan curve $c \subset \mathbb{P}^1(\mathbb{C})$ is a quasicircle if and only if the orbit closure $\overline{Gc}$ in $\mathcal{K}_2(\mathbb{P}^1(\mathbb{C}))$ contains only Jordan curves.*

Remark 3.6. The key point in the proof of Theorem 3.1 is the fact that closed orbits are always homeomorphic to homogeneous spaces endowed with their quotient topology. Since this key point which is due to Arens will be used at least four times in this article we quote it precisely now (see [21, Thm 2.13]): *Let G be a second countable locally compact group acting continuously on a second countable locally compact space Z. Let $z \in Z$ be a point whose G-orbit is closed and $G_z \subset G$ be the stabilizer of z. Then the orbit map $G \longrightarrow Z$ given by $g \mapsto gz$ induces an homeomorphism between the quotient space G/G_z and the orbit Gz.*

3.2. **Proof of Theorem 3.1.** $(ii) \Longrightarrow (i)$ Let $K = \Lambda_\Gamma$ be the limit set of a convex cocompact subgroup Γ of G. Let $(g_n)_{n\geq 1}$ be a sequence in G such that the limit $K_\infty = \lim\limits_{n\to\infty} g_n K$ exists in $\mathcal{K}_2(X)$. We want to find an element h_∞ in G such that $K_\infty = h_\infty K$.

Note that by [8] the sequence of convex hulls also converges

$$g_n \mathrm{Conv}(K) \xrightarrow[n\to\infty]{} \mathrm{Conv}(K_\infty).$$

Since K_∞ contains at least two points, its convex hull $\mathrm{Conv}(K_\infty)$ contains at least one point m_∞. Thus, there exists a sequence $(m_n)_{n\geq 1}$ in $\mathrm{Conv}(K)$ such that

$$g_n m_n \xrightarrow[n\to\infty]{} m_\infty.$$

Since Γ is convex cocompact, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that the sequence $\gamma_n m_n$ is bounded in M. Then the sequence $h_n := g_n \gamma_n^{-1}$ is bounded in G and, after extraction, it converges to an element h_∞ in G. One then has

$$K_\infty = \lim_{n\to\infty} h_n K = h_\infty K,$$

as required.

$(i) \Longrightarrow (iii)$ We want to prove that the stabilizer Γ_K acts cocompactly on $\mathrm{Conv}(K)$. This will imply simultaneously that Γ_K is convex cocompact and that $K = \Lambda_{\Gamma_K}$. Let $(m_n)_{n\geq 1}$ be a sequence in $\mathrm{Conv}(K)$. We want to find a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that, after extraction, the sequence $\gamma_n m_n$ converges in M.

Since G acts cocompactly -and even transitively- on M, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the sequence $g_n m_n$ converges to a point m_∞ in M. After extraction, the sequence $g_n K$ converges to a compact subset $K_\infty \subset X$ and the sequence of convex hulls also converges

$$g_n \mathrm{Conv}(K) \xrightarrow[n\to\infty]{} \mathrm{Conv}(K_\infty).$$

Since $\mathrm{Conv}(K_\infty)$ contains the point m_∞, the compact set K_∞ contains at least two points. Since the orbit GK is closed in $\mathcal{K}_2(X)$, there exists an element h_∞ in G such that $K_\infty = h_\infty K$. The closedness of the orbit GK in $\mathcal{K}_2(X)$ also implies that the natural bijection

$$\begin{array}{ccc} G/\Gamma_K & \stackrel{\sim}{\longrightarrow} & GK \\ g\Gamma_K & \mapsto & gK \end{array}$$

is a homeomorphism. Hence, by the very definition of the quotient topology on the quotient space G/Γ_K, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that the sequence $h_n := g_n\gamma_n^{-1}$ converges to h_∞. Then the following sequence converges

$$\gamma_n m_n = h_n^{-1} g_n m_n \xrightarrow[n\to\infty]{} h_\infty^{-1} m_\infty,$$

as required.

$(iii) \Longrightarrow (ii)$ This implication is clear. □

4. Cocompact actions on open sets

In this chapter we modify slightly Theorem 3.1 by allowing the set X to belong to the orbit closure.

4.1. Closed orbits in $\mathcal{K}_2'$.

We now introduce the following open subset $\mathcal{K}_2' = \mathcal{K}_2'(X)$ of $\mathcal{K}_2$,

$$\mathcal{K}_2'(X) := \{K \in \mathcal{K}_2(X) \mid K \neq X\}.$$

The following theorem is an analog of Theorem 3.1.

Theorem 4.1. *Let G be a simple Lie group of real rank one and X be its boundary. Let $K \subsetneq X$ be a compact subset containing at least two points. The following are equivalent.*
(i) The orbit GK is closed in $\mathcal{K}_2'(X)$.
(ii) There exists a closed subgroup Γ of Γ_K acting cocompactly on $X \smallsetminus K$.
(iii) The stabilizer Γ_K acts cocompactly on $X \smallsetminus K$.

In (*ii*) one can always choose Γ to be discrete (see Lemma 3.3).

Remark 4.2. - The compact K is not always equal to the limit set of Γ_K. Indeed, it is the union of the limit set Λ_{Γ_K} and of a Γ_K-invariant family of connected components of the domain of discontinuity Ω_{Γ_K}. For instance, when $G = PSL(2,\mathbb{C})$, a closed half-sphere K of $X = \mathbb{P}^1(\mathbb{C})$ satisfies these equivalent conditions but K is not the limit set of the group $\Gamma_K \simeq PSL(2,\mathbb{R})$.

- Even when K is equal to the limit set Λ_Γ and when both Λ_Γ and Ω_Γ are connected, the group Γ is not always convex cocompact. Indeed Γ might be equal to one of the so-called *singly degenerate* subgroup of $PSL(2,\mathbb{C})$ discovered by Bers in [6] and called degenerate B-groups by Maskit in [19, Th. G.3].

- More generally for any simple real rank one Lie group G there exists a non-elementary discrete subgroup Γ of G which is not convex cocompact but which acts cocompactly on its domain of discontinuity. Let $K := \Lambda_\Gamma$ be the limit set of such a group. As a corollary of both Theorems 3.1 and 4.1, the G-orbit closure is then equal to

$$\overline{G\,K} = G\,K \cup \{X\}.$$

In order to construct Γ, one chooses first two opposite maximal unipotent subgroups U_1 and U_2 of G and, for $i = 1, 2$, a lattice Γ_i in U_i and a compact fundamental domain D_i for the action of Γ_i on Ω_{Γ_i}. One can make these choices so that $\mathring{D}_1 \cup \mathring{D}_2 = X$. In this case the group Γ generated by Γ_1 and Γ_2 is a free product of Γ_1 and Γ_2, it is a discrete subgroup of G, it contains unipotent elements and acts on Ω_Γ with compact fundamental domain $D_1 \cap D_2$ (these facts rely on a ping-pong argument).

The proof of Theorem 4.1 is very similar to the proof of Theorem 3.1.

4.2. **Action on pointed open sets.**

We introduce now the space $\mathcal{L} = \mathcal{L}(X)$

$$\mathcal{L}(X) := \{(K, x) \textit{ where } K \in \mathcal{K}_2'(X) \textit{ and } x \in X \smallsetminus K\}$$

and the diagonal action of G on $\mathcal{L}(X)$. While the proof of Theorem 3.1 used the action of G on M, the proof of Theorem 4.1 will use the action of G on $\mathcal{L}$ which, according to the following lemma, has similar properties.

Lemma 4.3. *The group G acts properly cocompactly on $\mathcal{L}(X)$.*

Restricting this action to the subset of $\mathcal{L}$ where the compact sets K are pairs of points, one gets the following special case of Remark 2.3.

Corollary 4.4. *The group G acts properly cocompactly on the set $\Theta_3(X)$ of triples of distinct points of X.*

We will need the following well-known fact called the *convergence property* : whenever $(g_n)_{n\geq 1}$ is an unbounded sequence of G then, after extraction, there exist two points x_- and x_+ in X, called the repulsing and attracting points, such that the sequence $g_n x$ converges to x_+, uniformly for all x in compact subsets of $X \smallsetminus \{x_-\}$.

Proof of Lemma 4.3. We first prove that the action of G on $\mathcal{L}$ is proper. Let $(g_n)_{n\geq 1}$ be a sequence in G and $(K_n, x_n)_{n\geq 1}$ be a sequence in $\mathcal{L}$ such that

$$\begin{array}{rcl} (K_n, x_n) & \xrightarrow[n\to\infty]{} & (K_\infty, x_\infty) \in \mathcal{L} \\ (g_n K_n, g_n x_n) & \xrightarrow[n\to\infty]{} & (K'_\infty, x'_\infty) \in \mathcal{L}. \end{array}$$

We want to prove that the sequence g_n is bounded. If this is not the case, after extraction, we denote by x_- and x_+ the repulsing and attracting points of the sequence g_n. Since the compact set K'_∞ contains at least two points, the compact K_∞ must contain the repulsing point x_-. Hence one has

$$x_\infty \neq x_- \text{ and } x'_\infty = x_+.$$

Since the compact set K_∞ contains at least two points, one of them, say y_∞, is not equal to x_-. Then any sequence $(y_n)_{n\geq 1}$ with y_n in K_n that converges to y_∞ has an image $g_n y_n$ that converges to x_+. Hence x_+ belongs to K'_∞. Contradiction.

We now prove that the action of G on the space $\mathcal{L}$ is cocompact. Let $(K_n, x_n)_{n\geq 1}$ be a sequence in $\mathcal{L}$. We want to find a sequence $(g_n)_{n\geq 1}$ such that, after extraction, the following sequence converges

$$(g_n K_n, g_n x_n) \xrightarrow[n\to\infty]{} (K'_\infty, x'_\infty) \in \mathcal{L}.$$

Let x_- and x_+ be two distinct points of X. We can find a sequence $(h_n)_{n\geq 1}$ in G such that $x_- \in h_n K_n$ and $x_+ = h_n x_n$. Let g be an element of G such that the sequence of powers $(g^n)_{n\geq 1}$ has x_- and x_+ as repulsing and attracting points. Let F be a compact fundamental domain for the action of the cyclic group $\langle g \rangle$ on $X \smallsetminus \{x_-, x_+\}$, and let B be the compact neighborhood of x_- given by

$$B := \{x_-\} \cup \textstyle\bigcup_{n\leq 0} g^n F.$$

Let $n \geq 1$. Since $h_n K_n$ is compact and contains at least two points, the integer

$$p_n := \max\{p \in \mathbb{Z} \mid g^p h_n K_n \subset B\}$$

is well defined. After extraction, one defines a compact set K'_∞ as the limit $K'_\infty := \lim_{n\to\infty} g^{p_n} h_n K_n$. This compact set K'_∞ contains at least two points : x_- and a point $x_0 \in F$. Moreover K'_∞ does not contain the point $x'_\infty := x_+ = \lim_{n\to\infty} g^{p_n} h_n x_n$. Hence (K'_∞, x'_∞) belongs to $\mathcal{L}$. □

4.3. **Proof of Theorem 4.1.** $(ii) \implies (i)$ Let $(g_n)_{n\geq 1}$ be a sequence in G such that the limit $K_\infty = \lim_{n\to\infty} g_n K$ exists in $\mathcal{K}'_2(X)$. We want to find an element h_∞ in G such that $K_\infty = h_\infty K$.

Since $K_\infty \neq X$, there exists a point x_∞ in the open set $X \smallsetminus K_\infty$ and hence a sequence $(x_n)_{n\geq 1}$ in the open set $\Omega := X \smallsetminus K$ such that

$$g_n x_n \xrightarrow[n\to\infty]{} x_\infty.$$

Since Γ acts cocompactly on Ω, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that the sequence $\gamma_n x_n$ converges to a point y_∞ in Ω. Then the following two sequences converge

$$\begin{aligned} (K, \gamma_n x_n) &\xrightarrow[n\to\infty]{} (K, y_\infty) \in \mathcal{L} \\ g_n\gamma_n^{-1}(K, \gamma_n x_n) &\xrightarrow[n\to\infty]{} (K_\infty, x_\infty) \in \mathcal{L}. \end{aligned}$$

Since, by Lemma 4.3, the action of G on $\mathcal{L}$ is proper, the sequence $h_n := g_n\gamma_n^{-1}$ is bounded in G and, after extraction, converges to an element h_∞ in G. One has then

$$K_\infty = \lim_{n\to\infty} h_n K = h_\infty K,$$

as required.

$(i) \implies (iii)$ Let $(x_n)_{n\geq 1}$ be a sequence in the open set $\Omega := X \smallsetminus K$. We want to find a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that, after extraction, the sequence $\gamma_n x_n$ converges in Ω.

Since, by Lemma 4.3, the group G acts cocompactly on $\mathcal{L}$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the following sequence converges

$$(g_n K, g_n x_n) \xrightarrow[n\to\infty]{} (K_\infty, x_\infty) \in \mathcal{L}.$$

The compact set K_∞ contains at least two points and is not equal to X. Since the orbit GK is closed in $\mathcal{K}'_2$, there exists an element h_∞ in G such that $K_\infty = h_\infty K$. The closedness of the orbit GK in $\mathcal{K}'_2(X)$ also implies that the natural bijection $G/\Gamma_K \xrightarrow{\sim} GK$ is a homeomorphism. Hence there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that the sequence $h_n := g_n\gamma_n^{-1}$ converges to h_∞. Then the following sequence converges

$$\gamma_n x_n = h_n^{-1} g_n x_n \xrightarrow[n\to\infty]{} h_\infty^{-1} x_\infty,$$

as required.

$(iii) \Longrightarrow (ii)$ This implication is clear. $\square$

5. Cocompact actions on triples

In this chapter we modify again Theorem 3.1 by allowing pairs of points to belong to the orbit closure.

5.1. **Closed orbits in $\mathcal{K}_3$.** We now introduce the following open subset $\mathcal{K}_3 = \mathcal{K}_3(X)$ of $\mathcal{K}_2$,

$$\mathcal{K}_3(X) := \{K \textit{ compact subset of } X \textit{ containing at least three points}\}.$$

We also introduce, for any K in $\mathcal{K}_3(X)$ the set,

$$\Theta_3(K) := \{(x, y, z) \textit{ triple of distinct points of } K\}.$$

The following theorem is also an analog of Theorem 3.1.

Theorem 5.1. *Let G be a simple Lie group of real rank one and X be its boundary. Let $K \subset X$ be a compact subset containing at least three points. The following are equivalent.*
(i) The orbit GK is closed in $\mathcal{K}_3(X)$.
(ii) There exists a closed subgroup Γ of G acting cocompactly on $\Theta_3(K)$.
(iii) The stabilizer Γ_K acts cocompactly on $\Theta_3(K)$.
(iv) The stabilizer Γ_K is a convex cocompact subgroup of G and K is the union of its limit set Λ_{Γ_K} and of a Γ_K-invariant discrete subset of the domain of discontinuity Ω_{Γ_K}.

In (ii) one can always choose Γ to be discrete (see Lemma 3.3).

Remark 5.2. A finite set K always satisfies these equivalent conditions. In this case, the group Γ_K is finite and its limit set is empty.

The group Γ_K might be infinite and cyclic. In this case, the limit set Λ_{Γ_K} contains exactly two points and K is the union of Λ_{Γ_K} and of finitely many Γ_K-orbits.

The proof of Theorem 5.1 is also similar to the proof of Theorem 3.1. We will need the following well-known lemma on the barycenter map in a Riemannian symmetric space M of negative curvature. We first define, for m_1, m_2, m_3 in M the barycenter $\beta(m_1, m_2, m_3) \in M$ as the unique point where the sum of the distance functions $m \mapsto \sum_{i \leq 3} d(m, m_i)$ achieves its minimum.

Lemma 5.3. *$a)$ The barycenter map $\beta : M^3 \to M$ is continuous and has a unique continuous extension to the visual compactification $\overline{\beta} : \overline{M}^3 \to \overline{M}$.*
$b)$ Moreover, for x, y, z in $\overline{M}$, one has the equivalence

$$\overline{\beta}(x, y, z) \in X \Longleftrightarrow \text{two points among } x,\, y,\, z \text{ are equal and are in } X.$$

Proof of Lemma 5.3. Fix a point m_0 in M and define the Busemann function $b : \overline{M} \times M \to \mathbb{R}$ by

$$b_x(m) = \lim_{q \to x} d(q, m) - d(q, m_0), \quad \text{for all } x \text{ in } \overline{M} \text{ and } m \text{ in } M.$$

The function b is continuous on $\overline{M} \times M$ and is a convex function of m.

For all x, y, z in $\overline{M}$, we consider the convex function on M

$$m \mapsto \Phi_{x,y,z}(m) := b_x(m) + b_y(m) + b_z(m).$$

The function Φ is also continuous on $\overline{M}^3 \times M$.

Here is the definition of the map $\overline{\beta}$:

First case When the three points x, y, z are distinct or when at most one of them is in X, the function $\Phi_{x,y,z}$ is a convex and proper function on M which has a unique minimum point ξ. We define $\overline{\beta}(x, y, z)$ to be this minimum point $\xi \in M$.

Second case When at least two points among x, y, z are equal to a point ξ in X, we define $\overline{\beta}(x, y, z)$ to be this point $\xi \in X$.

a) We now check the continuity of the map $\overline{\beta}$ at a point $(x, y, z) \in \overline{M}^3$. Let (x_n, y_n, z_n) be a sequence in $\overline{M}^3$ converging to (x, y, z). We want to prove that the sequence $\xi_n := \overline{\beta}(x_n, y_n, z_n)$ converges to the point $\xi := \overline{\beta}(x, y, z)$. The case where ξ is in M is rather easy. The case where all ξ_n are in X is also rather easy. Let us focus on the case where ξ is in X and all ξ_n are in M. In this case, one has, for instance, $\xi = x = y \in X$. We know that the sequence of proper convex functions $\Phi_n := \Phi_{x_n,y_n,z_n}$ converges to the convex function $\Phi_\infty := \Phi_{x,y,z}$. Let V_ξ be a neighborhood of ξ in $\overline{M}$. Notice that, the functions Φ_n are uniformly bounded below in $M \smallsetminus V_\xi$. Notice also that the limit function Φ_∞ is not bounded below in $M \cap V_\xi$. Hence for n large, the minima ξ_n must belong to V_ξ. Therefore the sequence ξ_n converges to ξ as required.

b) This equivalence follows directly from the definition of $\overline{\beta}$. □

5.2. **Proof of Theorem 5.1.** $(ii) \implies (i)$ Let $(g_n)_{n\geq 1}$ be a sequence in G such that the limit $K_\infty = \lim_{n\to\infty} g_n K$ exists in $\mathcal{K}_3(X)$. We want to find an element h_∞ in G such that $K_\infty = h_\infty K$.

Since K_∞ contains at least three points, there exists a sequence $(x_n, y_n, z_n)_{n\geq 1}$ in $\Theta_3(K)$ such that the sequence of images converges

$$(g_n x_n, g_n y_n, g_n z_n) \xrightarrow[n\to\infty]{} (x_\infty, y_\infty, z_\infty) \in \Theta_3(K_\infty).$$

Since Γ acts cocompactly on $\Theta_3(K)$, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that, after extraction, the sequence $(\gamma_n x_n, \gamma_n y_n, \gamma_n z_n)$ converges to a triple $(x'_\infty, y'_\infty, z'_\infty) \in \Theta_3(K)$. Since, by Corollary 4.4, the action of G on $\Theta_3(X)$ is proper, the sequence $h_n := g_n\gamma_n^{-1}$ is bounded in G and, after extraction, converges to an element h_∞ in G. One has then

$$K_\infty = \lim_{n\to\infty} h_n K = h_\infty K,$$

as required.

$(i) \implies (iii)$ Let $(x_n, y_n, z_n)_{n\geq 1}$ be a sequence in $\Theta_3(K)$. We want to find a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that, after extraction, the image sequence $(\gamma_n x_n, \gamma_n y_n, \gamma_n z_n)$ converges in $\Theta_3(K)$.

Since, by corollary 4.4, the group G acts cocompactly on $\Theta_3(X)$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that, after extraction, the sequence $(g_n x_n, g_n y_n, g_n z_n)$ converges to a triple $(x_\infty, y_\infty, z_\infty) \in \Theta_3(X)$. After reextraction, the sequence $g_n K$ converges to a compact subset $K_\infty \subset X$ and this compact set K_∞ contains at least three points. Since the orbit GK is closed in $\mathcal{K}_3(X)$, there exists an element h_∞ in G such that $K_\infty = h_\infty K$. The closedness of the orbit GK in $\mathcal{K}_3(X)$ implies also that the natural bijection $G/\Gamma_K \xrightarrow{\sim} GK$ is a homeomorphism. Hence there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that the sequence $h_n := g_n\gamma_n^{-1}$ converges to h_∞. Then the following sequence converges

$$\gamma_n(x_n, y_n, z_n) = h_n^{-1} g_n(x_n, y_n, z_n) \xrightarrow[n\to\infty]{} h_\infty^{-1}(x_\infty, y_\infty, z_\infty),$$

as required.

$(iii) \implies (ii)$ This implication is clear.

$(iv) \implies (iii)$ Let $(x_n, y_n, z_n)_{n\geq 1}$ be a sequence in $\Theta_3(K)$. We want to find a sequence $(\gamma_n)_{n\geq 1}$ in $\Gamma := \Gamma_K$ such that, after extraction, the image sequence $(\gamma_n x_n, \gamma_n y_n, \gamma_n z_n)$ converges in $\Theta_3(K)$.

When K is a finite set and hence Γ_K is a finite group and its limit set Λ_{Γ_K} is empty, this assertion is clear.

We may assume that K and hence Γ is infinite. Let $q_n \in \overline{M}$ be the barycenter $q_n := \overline{\beta}(x_n, y_n, z_n)$. Since these three points are distinct, by Lemma 5.3, the barycenter q_n belongs to M. Let p_n be the projection of q_n on the convex hull $\mathrm{Conv}(\Lambda_\Gamma)$. Since Γ is convex cocompact, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that, after extraction, the sequence $\gamma_n p_n$ converges to a point $p_\infty \in M$. After reextraction, the sequences $\gamma_n x_n$, $\gamma_n y_n$, and $\gamma_n z_n$ converge in K respectively to points x_∞, y_∞, and z_∞. We want to prove that these three points are distinct.

Assume by contradiction that this is not the case and that, for instance, one has $y_\infty = z_\infty$. We first notice that the sequence $\gamma_n q_n$ converges to the barycenter $y_\infty = \overline{\beta}(x_\infty, y_\infty, z_\infty)$ and that p_∞ is the projection of y_∞ on $\mathrm{Conv}(\Lambda_\Gamma)$. In particular y_∞ does not belong to the limit set Λ_Γ. Since $K \smallsetminus \Lambda_\Gamma$ is discrete in the domain of discontinuity, one must have for n large enough, $y_n = y_\infty = z_\infty = z_n$. Contradiction.

$(iii) \implies (iv)$ Since the compact K is invariant under the group $\Gamma := \Gamma_K$, it contains the limit set Λ_Γ. Since Γ acts cocompactly on $\Theta_3(K)$, it also acts cocompactly on $\Theta_3(\Lambda_\Gamma)$ and, by Remark 2.3, the group Γ is convex cocompact. It only remains to prove that the set $K \smallsetminus \Lambda_\Gamma$ is discrete in the domain of discontinuity Ω_Γ.

Assume, by contradiction, that this is not the case. Then there exists a sequence $(x_n)_{n\geq 1}$ of distinct points of K that converges to a point x_∞ in Ω_Γ. Since Γ acts cocompactly on $\Theta_3(K)$, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that the three sequences $\gamma_n x_{3n}$, $\gamma_n x_{3n+1}$, and $\gamma_n x_{3n+2}$ converge to three distinct points of K. After reextraction, we denote by x_- and x_+ the repulsing and attracting points of the sequence γ_n. Both points belong to Λ_Γ. Since the point x_∞ belongs to Ω_Γ, it is not equal to x_- and the three sequences $\gamma_n x_{3n}$, $\gamma_n x_{3n+1}$, and $\gamma_n x_{3n+2}$ converge to the same point x_+. Contradiction. □

6. Dynamics of G on $\mathcal{K}_2$

In this chapter we apply Theorem 3.1 to the study of the dynamics of the group G on the space $\mathcal{K}_2(X)$.

6.1. **Closed, dense and minimal orbits.** The following theorem tells us that this dynamics has features similar to those of Anosov flows.

Theorem 6.1. *Let G be a simple Lie group of real rank one and X be its boundary. Then*
a) The union of closed G-orbits in $\mathcal{K}_2(X)$ is dense in $\mathcal{K}_2(X)$.
b) The space $\mathcal{K}_2(X)$ contains a dense G-orbit.
c) $\mathcal{K}_2(X)$ contains non-closed G-orbits whose closures are G-minimal.

The proof of Theorem 6.1 will occupy this whole chapter.

6.2. **Closed orbits.** We first check that *the union of closed G-orbits in $\mathcal{K}_2(X)$ is dense in $\mathcal{K}_2(X)$.*

Proof of Theorem 6.1.a. Use Theorem 3.1 and Lemmas 6.2, 6.3. □

Lemma 6.2. *Let G be a simple Lie group of real rank one and X be its boundary. For every finite subset $F \subset X$ and every $\varepsilon > 0$, there exists a free convex cocompact subgroup Γ of G whose limit set Λ_Γ contains F and is contained in the ε-neighborhood F_ε of F.*

Proof. We may assume that F has even cardinality $n = 2p$. We then choose p hyperbolic elements g_1,..., g_p in G such that

$$F = \bigcup_{i \le p} \{x_{g_i}^-, x_{g_i}^+\}$$

where $x_{g_i}^-$ and $x_{g_i}^+$ are the repulsing and attracting points of g_i. Then, for n large, the group

$$\Gamma_n := \langle g_i^n \mid i \le p \rangle$$

is a Schottky subgroup and its limit set satisfies

$$F \subset \Lambda_\Gamma \subset F_\varepsilon .$$

Indeed, by the classical ping-pong argument, this is the case as soon as, for any $g \neq h$ in the finite set $S := \bigcup_{i \le p} \{g_i, g_i^{-1}\}$, one has both

$$4\varepsilon < d(x_g^+, x_h^+),$$

and

$$g^n(B(x_g^-, \varepsilon)^c) \subset B(x_g^+, \varepsilon)$$

where $B(x, \varepsilon)$ is the open ball of center x and radius ε. □

Lemma 6.3. *Let X be a compact metric space. Then the set $\mathcal{F}$ of finite subsets of X is dense in the set $\mathcal{K}(X)$ of compact subsets of X.*

Proof. This is well-known. One approximates any compact set $K \subset X$ by the set F of centers of a finite cover of K by balls of radius ε. □

6.3. **Dense orbits.** We now check that *the space $\mathcal{K}_2(X)$ contains a dense G-orbit.*

Proof of Theorem 6.1.b. Fix two distinct points x_- and x_+ in X, let g be an element of G with x_- and x_+ as repulsing and attracting fixed points, let F be a compact fundamental domain for the action of the cyclic group $\langle g \rangle$ on $X \smallsetminus \{x_-, x_+\}$ chosen so that the sequence of compact sets

$$F_n := \bigcup_{-n<k<n} g^k F$$

which covers $X \smallsetminus \{x_-, x_+\}$ also satisfies $F_n \subset \mathring{F}_{n+1}$. Let $\mathcal{K}_{x_-,x_+}$ be the set of compact subsets $K \subset X$ containing both x_- and x_+, and let $(K_n)_{n\geq 1}$ be a dense sequence in $\mathcal{K}_{x_-,x_+}$. Then the compact set K defined as a disjoint union

$$K := \bigcup_{n\geq 1} g^{n^2}(K_n \cap F_n) \,\cup\, \{x_-, x_+\}$$

has a dense $\langle g\rangle$-orbit in $\mathcal{K}_{x_-,x_+}$.

Indeed, for any $Y \in \mathcal{K}_{x_-,x_+}$, one can find a sequence $S \subset \mathbb{N}$ such that the subsequence $(K_n)_{n\in S}$ of compact sets converges to Y. But then, since, for all $n \geq 1$, the compact set $g^{-n^2}(K)$ coincides with K_n on F_n and contain $\{x_-, x_+\}$, the sequence of compact sets $(g^{-n^2}(K))_{n\in S}$ converges also to Y.

This proves that K has a dense G-orbit in $\mathcal{K}_2(X)$. $\square$

6.4. **Minimal subsets for the free group.** Finally, we check that $\mathcal{K}_2(X)$ *contains non-closed orbits* GK *whose closure are minimal.* We want to construct a compact set $K \in \mathcal{K}_2(X)$ such that the orbit GK is not closed but the orbit closure $\overline{GK}$ is G-minimal. This means that, for every K_∞ in the orbit closure $\overline{GK}$, the element K belongs to the orbit closure $\overline{GK_\infty}$.

We begin with a special case. Let G_0 be the free group with three generators $S_0 = \{a, b, c\}$, i.e. the set of reduced finite words

$$G_0 := \{w = s_1 \cdots s_n \mid n \geq 0,\ s_i \in S_0^{\pm 1} \text{ and } s_{i+1} \neq s_i^{-1}\}.$$

Let X_0 be the Gromov boundary of G_0, i.e. the set of reduced infinite words

$$X_0 := \{w = s_1 \cdots s_n \cdots \ \mid\ s_i \in S_0^{\pm 1} \text{ and } s_{i+1} \neq s_i^{-1}\}.$$

We set as before

$$\mathcal{K}_2(X_0) := \{K \textit{ compact subsets of } X_0 \textit{ containing at least two points.}\}$$

Lemma 6.4. *The space $\mathcal{K}_2(X_0)$ contains a non-closed G_0-orbit G_0K whose closure is G_0-minimal.*

We need to introduce more notations. Let Z be the compact set

$$Z = \{0,1\}^{\mathbb{Z}}, \text{ and } \Sigma = \langle \sigma \rangle$$

be the infinite cyclic group generated by the shift $\sigma : Z \to Z$, which is defined, for $\varepsilon = (\varepsilon_n)_{n\in\mathbb{Z}}$, by $(\sigma\varepsilon)_n = \varepsilon_{n+1}$. Let

$$G(0) := \langle b \rangle \text{ and } G(1) := \langle c \rangle$$

be the cyclic subgroups of G_0 generated respectively by b and c. For each ε in Z we define a compact subset $K_\varepsilon \in \mathcal{K}_2(X_0)$ by

$$\begin{aligned} K_\varepsilon \; := \; & \{\textit{reduced infinite words } w \textit{ in } X_0 \textit{ of the form} \\ & a^{n_1}\gamma_1 a^{n_2}\gamma_2 \cdots a^{n_k}\gamma_k \cdots \quad \textit{with } n_k \in \mathbb{Z} \\ & \textit{and } \gamma_k \in G(\varepsilon_{n_1+\cdots+n_k}) \textit{ for all } k \geq 1.\} \end{aligned}$$

It is useful to notice that *the compact sets K_ε depend continuously on their parameter $\varepsilon \in Z$*. Indeed, by definition of K_ε, for all $N \geq 1$, the set $S_{\varepsilon,N}$ of reduced word of length N that appear at the beginning of a word $w \in K_\varepsilon$ depends only on the finite sequence $(\varepsilon_n)_{-N\leq n\leq N}$.

Remark 6.5. Before going on, we give an intuitive and hand-waving way to recognize that an infinite reduced word w belongs to K_ε. I think of ε as a biinfinite sequence of bits written on the line $\mathbb{Z}$. Meanwhile I am reading the word w from the beginning, I ask a friend to move on this line starting from the spot $s = 0$. When my friend is at the spot s he can read the bit ε_s.
- If I read the letter a, I ask my friend to move one step forward.
- If I read the letter a^{-1}, I ask him to move one step backward.
- If I read the letter b or b^{-1}, I check that he reads the bit 0.
- If I read the letter c or c^{-1}, I check that he reads the bit 1.
The word w belongs to K_ε if and only if it passes all these checks.

Proof of Lemma 6.4. We will choose the compact K to be K_ε for an element ε in Z whose orbit $\Sigma\varepsilon$ is not closed in Z but whose orbit closure $\overline{\Sigma\varepsilon}$ is Σ-minimal. The claim of Lemma 6.4 follows then from Lemma 6.6 below. □

Lemma 6.6. *Let G_0 be the free group on three generators a, b, c, X_0 be its boundary, and ε, ε' be elements of Z.*
a) The compact $K_{\varepsilon'}$ belongs to G_0K_ε if and only if ε' belongs to $\Sigma\varepsilon$.
b) The orbit closure of K_ε in $\mathcal{K}_2(X_0)$ is equal to

$$\overline{G_0K_\varepsilon} = \{gK_{\varepsilon'} \mid g \in G_0 \textit{ and } \varepsilon' \in \overline{\Sigma\varepsilon}\}.$$

Proof of Lemma 6.6. We first notice the following facts :
- One always has the equality $aK_{\sigma(\varepsilon)} = K_\varepsilon$.
- When $\varepsilon_0 = 0$, one has the equality $bK_\varepsilon = K_\varepsilon$, the compact K_ε contains no word beginning with c or c^{-1}, and hence all reduced words of $c^{\pm1}K_\varepsilon$ begin with the letter $c^{\pm1}$.
- When $\varepsilon_0 = 1$, one has the equality $cK_\varepsilon = K_\varepsilon$, the compact K_ε contains no word beginning with b or b^{-1}, and hence all reduced words of $b^{\pm1}K_\varepsilon$ begin with the letter $b^{\pm1}$.

a) Let $g \in G_0$ be an element such that $K_\varepsilon = gK_{\varepsilon'}$. The previous remarks prove that the element g belongs to the set

$$(6.1)\quad S_\varepsilon := \{g = a^{n_1}\gamma_1 \cdots a^{n_p}\gamma_p \ \ \textit{with}\ n_k \in \mathbb{Z}\ \textit{and}\ \gamma_k \in G(\varepsilon_{n_1+\cdots+n_k})\ \}$$

and that, for such a g, one has $\varepsilon = \sigma^{n_1+\cdots+n_p}(\varepsilon')$.

b) Since the map $Z \to \mathcal{K}_2(X_0); \varepsilon \mapsto K_\varepsilon$ is continuous, all compact sets $gK_{\varepsilon'}$ with ε' in $\overline{\Sigma\varepsilon}$ belong to the orbit closure $\overline{G_0K_\varepsilon}$.

Conversely, let $K_\infty \in \mathcal{K}_2(X_0)$ be a compact set belonging to the orbit closure $\overline{G_0K_\varepsilon}$. We want to prove that $K_\infty = h_\infty K_{\varepsilon'}$, for some h_∞ in G_0 and ε' in $\overline{\Sigma\varepsilon}$. Let g_n be a sequence of elements of G_0 such that

$$g_nK_\varepsilon \xrightarrow[n\to\infty]{} K_\infty.$$

Write the reduced word g_n as a product $g_n = h_nj_n$ where j_n is the longest possible word such that j_n^{-1} belongs to the set S_ε. According to the previous discussion, one has the equality

$$g_nK_\varepsilon = h_nK_{\varepsilon^{(n)}},$$

where $\varepsilon^{(n)}$ belongs to $\Sigma\varepsilon$.

We first check that the length of h_n is bounded. By construction, for every x in $K_{\varepsilon^{(n)}}$, the concatenated word h_nx is a reduced word. Hence the diameter of $h_nK_{\varepsilon^{(n)}}$ goes to zero when the length of h_n goes to ∞. Since the limit compact set K_∞ contains at least two points, the length of h_n must be bounded.

Since the length of h_n is bounded, after extracting, the sequence h_n is constant $h_n = h_\infty$, the sequence $\varepsilon^{(n)}$ converges to an element $\varepsilon' \in \overline{\Sigma\varepsilon}$, and one has $K_\infty = h_\infty K_{\varepsilon'}$ as required. $\square$

Remark 6.7. One can prove that the compact set K_ε is equal to the limit set of the subgroup of G_0

$$H_\varepsilon := \{g = a^{n_1}\gamma_1 \cdots a^{n_p}\gamma_p \in S_\varepsilon \mid n_1 + \cdots + n_p = 0\ \}.$$

However using a similar construction, one can also produce compact subsets K of X_0 that satisfy the conditions of Lemma 6.4 and whose stabilizer in G_0 is trivial.

6.5. **Minimal subsets.** We now relate orbit closures in $\mathcal{K}_2(X_0)$ and $\mathcal{K}_2(X)$ thanks to the following lemma.

Lemma 6.8. *Let G be a simple Lie group of real rank one and X be its boundary. Let G_0 be a closed convex cocompact subgroup of G and X_0 be its limit set. Let $K \subset X_0$ be a compact subset containing at least two points.*

a) If the orbit G_0K is closed in $\mathcal{K}_2(X_0)$, then the orbit GK is also closed in $\mathcal{K}_2(X)$.

b) If the orbit closure $\overline{G_0K}$ is G_0-minimal in $\mathcal{K}_2(X_0)$, then the orbit closure $\overline{GK}$ is also G-minimal in $\mathcal{K}_2(X)$.

Proof of Lemma 6.8. Both $a)$ and $b)$ will follow from the following claim : *For every limit $K_\infty = \lim\limits_{n\to\infty} g_nK$ in $\mathcal{K}_2(X)$ with g_n in G, there exist $h_\infty \in G$ and $K'_\infty \in \overline{G_0K}$ such that $K_\infty = h_\infty K'_\infty$.*

We first prove this claim. After extraction, the sequence g_nX_0 converges to a compact subset $X_{0,\infty} \subset X$ that contains at least two points. Since, by Theorem 3.1, the orbit GX_0 is closed in $\mathcal{K}_2(X)$, the limit $X_{0,\infty}$ belongs to the orbit GX_0. By remark 3.6, the closedness of the orbit GX_0 in $\mathcal{K}_2(X)$ implies also that the map $G/G_0 \xrightarrow{\sim} GK$ is proper. Hence there exists a sequence $(\gamma_n)_{n\geq 1}$ in G_0 such that the sequence $h_n := g_n\gamma_n^{-1}$ converges to an element h_∞ in G. Then the following sequence converges

$$\gamma_nK = h_n^{-1}g_nK \xrightarrow[n\to\infty]{} h_\infty^{-1}K_\infty.$$

This proves our claim.

a) We assume that G_0K is closed. Let $K_\infty \in \overline{GK}$. We want to check that $K_\infty \in GK$. According to our claim, one can write $K_\infty = h_\infty K'_\infty$ with $h_\infty \in G$ and $K'_\infty \in \overline{G_0K}$. Since G_0K is closed, the compact set K'_∞ belongs to the orbit G_0K and hence the compact set K_∞ belongs to the orbit GK.

b) We assume that $\overline{G_0K}$ is minimal. Let $K_\infty \in \overline{GK}$. We want to check that $K \in \overline{GK_\infty}$. According to our claim, one can write $K_\infty = h_\infty K'_\infty$ with $h_\infty \in G$ and $K'_\infty \in \overline{G_0K}$. Since $\overline{G_0K}$ is minimal, the compact set K belongs to $\overline{G_0K'_\infty}$, and hence the compact set K also belongs to $\overline{GK_\infty}$. □

Finally, we can check that $\mathcal{K}_2(X)$ *contains non-closed orbits GK whose closure $\overline{GK}$ are G-minimal.*

Proof of Theorem 6.1.c. We keep the notations of both Lemmas 6.6 and 6.8. We choose $G_0 \subset G$ to be a Schottky subgroup on three generators $S_0 := \{a, b, c\}$. This implies that G_0 is a free group on these generators, that G_0 is a discrete convex cocompact subgroup of G and that the boundary X_0 of G_0 is nothing but the limit set $X_0 = \Lambda_{G_0} \subset X$.

We can choose the group G_0 and its generators so that

(6.2) the centralizer in G of the group $\langle b, aca^{-1}\rangle$ is trivial.

We choose the compact K to be K_ε for an element ε in Z whose orbit $\Sigma\varepsilon$ is not closed in Z but whose orbit closure $\overline{\Sigma\varepsilon}$ is Σ-minimal.

According to Lemma 6.6, the orbit closure $\overline{G_0 K_\varepsilon}$ in $\mathcal{K}_2(X_0)$ is G_0-minimal. Hence by Lemma 6.8, the orbit closure $\overline{GK_\varepsilon}$ in $\mathcal{K}_2(X)$ is G-minimal.

It remains to check that the orbit GK_ε is not closed in $\mathcal{K}_2(X)$. Assume, by contradiction, that this orbit GK_ϵ is closed. Since K_ϵ is totally disconnected, the stabilizer G_{K_ϵ} is a discrete subgroup of G. Since this orbit GK_ϵ is closed there exists for any $\eta \in \overline{\Sigma\varepsilon}$, an element g_η in G such that

$$K_\eta = g_\eta K_\epsilon.$$

Choose an integer $n = n(\eta)$ such that $\eta_n = 0$ and $\eta_{n+1} = 1$. By definition of K_η, the following elements belong to its stabilizer,

$$a^n b a^{-n} \in G_{K_\eta} \text{ and } a^{n+1} c a^{-n-1} \in G_{K_\eta}.$$

Hence their conjugates belong to the stabilizer of K_ε,

$$g_\eta^{-1} a^n b a^{-n} g_\eta \in G_{K_\varepsilon} \text{ and } g_\eta^{-1} a^{n+1} c a^{-n-1} g_\eta \in G_{K_\varepsilon}.$$

According to (6.2), the element g_η is determined by the value of n and by these two elements of G_{K_ε}. This leaves only countably many possibilities for the element g_η, and hence for the compact set K_η and also for the parameter η. Since the orbit closure $\overline{\Sigma\varepsilon}$ is uncountable, this gives a contradiction. $\square$

7. Concluding remarks

In this chapter, we briefly discuss without proof various analogs of our results.

7.1. **Closed orbits in $\mathcal{K}_i$.** In the context of Theorems 3.1 and 4.1, it is natural to look for a similar characterization of closed G-orbits GK in the space

$$\mathcal{K}_i(X) := \{\textit{compact subset } K \textit{ of } X \textit{ containing at least } i \textit{ points}\}$$

when $i \geq 4$. Unfortunately, one can check that there are many closed orbits in $\mathcal{K}_i(X)$ which are not related to discrete subgroups of G. Indeed, *for any finite subset $F_0 \subset X$ containing at least three points, there exist compact subsets K in $\mathcal{K}_i(X)$ whose orbit closure in $\mathcal{K}_3(X)$ is equal to $\overline{GK} = GK \cup GF_0$ and whose stabilizer Γ_K is trivial.* For example, using the notations of Section 6.3, one can assume that $\{x_+, x_-\} \subset F_0$ and choose $K = \bigcup_{n\geq 1} g^{n^2} F_0$.

7.2. **Gromov hyperbolic groups.** Let (M, d) be a proper geodesic Gromov hyperbolic space. Let X be the Gromov boundary of M and $G \subset \mathrm{Isom}(M)$ be a non-elementary closed subgroup that acts cocompactly on M (see [15], [13], or [24] for precise definitions). Such a locally compact group G will be called a *Gromov hyperbolic group*. In this setting the notion of convex cocompact subgroup of G has been replaced by the notion of *quasiconvex subgroup*. This is a closed subgroup Γ of G that acts cocompactly on the union of geodesics joining points of its limit set (see the main theorem of [22] for various equivalent definitions of a quasiconvex subgroup). With this modification, one can check that *Theorems 3.1, 4.1, 5.1, and 6.1 are still valid for such a group G and its boundary X.* For instance, one has

Theorem 7.1. *Let G be a Gromov hyperbolic group and X be its boundary. Let $K \subset X$ be a compact subset containing at least two points. The following are equivalent.*
(i) *The orbit GK is closed in $\mathcal{K}_2(X)$.*
(ii) *There exists a quasiconvex subgroup Γ of G such that $K = \Lambda_\Gamma$.*
(iii) *The stabilizer Γ_K is a quasiconvex subgroup and K is its limit set.*

Indeed the proof of Theorem 3.1 still work in this general context even though the "convex cocompact" subgroups Γ_K might not contain any cocompact discrete subgroup. We do not repeat this proof.

Important examples of such Gromov hyperbolic spaces M are
(a) Cayley graphs of finitely generated word-hyperbolic groups G.
(b) Universal covers of compact Riemannian manifolds with negative curvature.

In all this paper we have used implicitely the point of view of "convergence groups" (see [12] or [23]). Note that according to theorem 8.1 of [9], the words "Gromov hyperbolic groups" are synonymous to the words "uniform convergence groups".

7.3. **Divisible Convex sets.** Let $G = \mathrm{SL}(d{+}1, \mathbb{R})$, $X = \mathbb{P}(\mathbb{R}^{d+1})$ and

$$\mathcal{K}_c(X) := \{K \textit{ properly convex compact subset of } X\}.$$

Recall that a compact convex subset K of X is said to be properly convex if K is included in an affine chart of X and if its interior $\Omega := \mathring{K}$ is not empty. The following theorem is an analog of Theorem 4.1.

Theorem 7.2. *Let $G = \mathrm{SL}(d{+}1, \mathbb{R})$, $X = \mathbb{P}(\mathbb{R}^{d+1})$, $K \subset X$ be a properly convex compact subset and $\Omega = \mathring{K}$. The following are equivalent.*
(i) *The orbit GK is closed in $\mathcal{K}_c(X)$.*

(ii) There exists a closed subgroup Γ of Γ_K acting cocompactly on Ω.
(iii) The stabilizer Γ_K acts cocompactly on Ω.

In (ii) the group Γ can not always be chosen to be discrete.

The implication $(ii) \Rightarrow (i)$ is due to Benzecri (see [2, Cor. 2.4] or [5]). Proving the converse implication was indeed the starting point of our investigations.

The proof of Theorem 7.2 is the same as the proof of Theorem 4.1, replacing Lemma 4.3 by the following analog fact which is due to Benzecri (see loc. cit. or [3]) : *the group G acts properly cocompactly on the set $\mathcal{L}_c(X)$ of pointed open properly convex subsets (Ω, x) of X.*

Since a few readers asked us to repeat this same proof in this new context, here it is:

Proof of Theorem 7.2. $(ii) \Longrightarrow (i)$ Let K be the closure of a properly convex open set Ω of X on which a subgroup Γ of G acts cocompactly. Let $(g_n)_{n\geq 1}$ be a sequence in G such that the limit $K_\infty = \lim\limits_{n\to\infty} g_n K$ exists in $\mathcal{K}_c(X)$. We want to find an element h_∞ in G such that $K_\infty = h_\infty K$. The open set $\Omega_\infty := \mathring{K}_\infty$ contains at least one point x_∞. Thus, there exists a sequence $(x_n)_{n\geq 1}$ in Ω such that

$$g_n x_n \xrightarrow[n\to\infty]{} x_\infty.$$

Since Γ acts cocompactly on Ω, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ such that the sequence $\gamma_n x_n$ is bounded in Ω. Then, since the action of G on $\mathcal{L}_c(X)$ is proper, the sequence $h_n := g_n\gamma_n^{-1}$ is bounded in G and, after extraction, it converges to an element h_∞ in G. One then has

$$K_\infty = \lim_{n\to\infty} h_n K = h_\infty K.$$

$(i) \Longrightarrow (iii)$ We want to prove that the stabilizer Γ_K acts cocompactly on $\Omega := \mathring{K}$. Let $(x_n)_{n\geq 1}$ be a sequence in Ω. We want to find a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that, after extraction, the sequence $\gamma_n x_n$ converges in Ω. Since G acts cocompactly on $\mathcal{L}_c(X)$, there exists a sequence $(g_n)_{n\geq 1}$ in G such that the sequence $g_n(\Omega, x_n)$ converges to a couple $(\Omega_\infty, x_\infty)$ in $\mathcal{L}_c(X)$. Since the orbit GK is closed in $\mathcal{K}_c(X)$, there exists an element h_∞ in G such that $\Omega_\infty = h_\infty\Omega$. The closedness of the orbit GK in $\mathcal{K}_c(X)$ also implies that the natural bijection $G/\Gamma_K \xrightarrow{\sim} GK$ is a homeomorphism. Hence, there exists a sequence $(\gamma_n)_{n\geq 1}$ in Γ_K such that the sequence $h_n := g_n\gamma_n^{-1}$ converges to h_∞. Then the following sequence converges

$$\gamma_n x_n = h_n^{-1} g_n x_n \xrightarrow[n\to\infty]{} h_\infty^{-1} x_\infty \in \Omega.$$

$(iii) \Longrightarrow (ii)$ This implication is clear. $\square$

7.4. **Higher rank.** For a higher rank semisimple Lie group G, its flag variety X, and a well-chosen G-invariant open subset $\mathcal{K}' \subset \mathcal{K}(X)$, the analogous question is appealing: *can one describe the compact subset $K \in \mathcal{K}'$ such that GK is closed in $\mathcal{K}'$?*

Note that in this setting the notion of convex cocompact subgroups is replaced with the notion of Anosov subgroups. See [14] and [17].

Here is an answer to this question when G is a product of rank-one Lie groups $G = G_1 \times G_2$, and hence X is the product $X = X_1 \times X_2$ of their boundaries. We will say that a compact subset of X is *degenerate* if it is included in a *cross* $\{x_1\}\times X_2 \cup X_1 \times \{x_2\}$. We first notice that, when $K \neq X$, the orbit closure $\overline{GK}$ in $\mathcal{K}(X)$ always contains a degenerate compact subset. This is why it is natural to introduce the set $\mathcal{K}'$ of nondegenerate compact subset of X:

$$\mathcal{K}' := \{K \in \mathcal{K}(X) \mid K \not\subset \{x_1\}\times X_2 \cup X_1 \times \{x_2\} \;\; \forall x_1 \in X_1\,,\, x_2 \in X_2\}.$$

We just state here the final result without proof.

Theorem 7.3. *Let $G = G_1 \times G_2$ be the product of two rank one Lie groups and $K \in \mathcal{K}'$. If the orbit GK is closed in $\mathcal{K}'$, we are in one of the following four cases, modulo exchange of the factors :*
(i) K is elementary i.e. K is included in a finite union of crosses $\{x_1\}\times X_2 \cup X_1 \times \{x_2\}$.
(ii) There exist a convex cocompact subgroup $\Gamma_1 \subset G_1$, a Γ_1-invariant compact set $K_1 \subset X_1$ and $x_2 \in X_2$ such that $K = \Lambda_{\Gamma_1}\times X_2 \cup K_1 \times \{x_2\}$.
(iii) There exist convex cocompact subgroups $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ such that K is the product of their limit sets $K = \Lambda_{\Gamma_1} \times \Lambda_{\Gamma_2}$.
(iv) There exist convex cocompact subgroups $\Gamma_1 \subset G_1$ and $\Gamma_2 \subset G_2$ such that K is the limit set of the graph of an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$.

One can also list the compact sets K that occur in case (i). In particular, either K is finite, or it is contained in the union of two crosses.

In case (iii), it is interesting to notice that the stabilizer Γ_K is not an Anosov subgroup of G.

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