Recurrence on the space of lattices

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Abstract. This is an introduction to recurrence properties on finite volume homogeneous spaces based on examples.

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1. Introduction

Extending a recurrence theorem due to Eskin and Margulis in [9], We proved with Jean-François Quint in [4] a recurrence theorem for random walks on finite volume homogeneous spaces. This text is an introduction to these two recurrence theorems.

We will explain in this preprint the main ideas of the proof of both recurrence theorems by working on the first non-trivial examples. We will not seek for generality, but we hope that these examples will help the reader to understand the meaning of these theorems. For a complete account on these theorems the reader is refered to [9] and [4].

2. Markov-Feller chains

We first define Markov-Feller operators and their recurrence properties.

A Markov chain on a space X is a mathematical model which describes the evolution of a stochastic process $(x_t)_{t\in\mathbb{N}}$, for which the position x_{t+1} at time $t+1$ is chosen randomly according to a law P_{x_t} which depends only on the position x_t at time t. In this paper we will deal only with Markov-Feller chains on second countable locally compact spaces X, i.e. with Markov chains for which the law P_{x_t} on X depends continuously on the point x_t.

In a more formal way, let X be a second countable locally compact space. A *Markov-Feller chain* on X is a continuous map $x \to P_x$ from X to the space $\mathcal{P}(X)$ of Borel probability measures on X. As usual, this space $\mathcal{P}(X)$ is endowed with the $*$-weak topology. We also denote by P the induced *Markov-Feller operator* on the Banach space $\mathcal{C}_b(X)$ of continuous bounded functions on X. It is given, for f in $\mathcal{C}_b(X)$ and x in X, by $Pf(x) = \int_X f(y)\,\mathrm{d}P_x(y)$.

Iterating n times this Markov chain, one gets a Markov chain $x \to P_x^n$. This probability P_x^n is the law of x_n when you know only the position of the chain at time zero $x_0 = x$. This Markov chain $x \to P_x^n$ is defined inductively by $P_x^1 = P_x$ and, for $n \geq 1$, $P_x^{n+1} = \int_X P_y\,\mathrm{d}P_x^n(y)$. Its associated Markov operator on $\mathcal{C}_b(X)$ is nothing but the n^{th} power P^n.

Here are two very strong recurrence property of P.

Definition 2.1. *We say that P is recurrent on X if, for every $\varepsilon > 0$ and x in X, one can find a compact set $M \subset X$ and an integer n_0 such that, for all $n \geq n_0$, one has $P_x^n(M) \geq 1 - \varepsilon$.*

This means that there is no escape of mass for the laws of the Markov-Feller chain, i.e. any $*$-weak limit of a subsequence of P_x^n will be a probability measure.

Definition 2.2. *We say that P is uniformly recurrent on X if, for every $\varepsilon > 0$, one can find a compact $M \subset X$ such that, for all x in X, one can find an integer n_0 such that, for all $n \geq n_0$, one has $P_x^n(M) \geq 1 - \varepsilon$.*

This means that the compact set M can be chosen independantly of the starting point x.

Most of the Markov chains we will study will be obtained in the following way. Let G be a second countable locally compact group acting continuously on X, and μ be a Borel probability measure on G. The Markov-Feller chain on X will be the corresponding random walk on X, i.e. the transition probability will be $x \to P_{\mu,x} := \mu * \delta_x$. In other words, the corresponding Markov-Feller operator P_μ is given by, for all f in $\mathcal{C}_b(X)$ and x in X, $P_\mu f(x) = \int_G f(gx)\,\mathrm{d}\mu(g)$.

3. Finite volume homogeneous spaces

We introduce now the random walk on finite volume homogeneous spaces and state precisely the two recurrence theorems we want to explain.

3.1. Recurrence on G/Λ.

The reader non familiar with Lie groups may skip this general section. Indeed later on we will mainly focus on examples.

Let G be a connected real algebraic Lie group, let Λ be a lattice in G i.e. Λ is a discrete subgroup of finite covolume in G and $X := G/\Lambda$. Let $\mu \in \mathcal{P}(G)$ be a probability measure on G, with a finite exponential moment, $\int_G \|g\|^\delta\,\mathrm{d}\mu(g) < \infty$, for some $\delta > 0$. Let Γ_μ be the closed subgroup generated by the support of μ and H_μ be the Zariski closure of Γ_μ. We will assume that H_μ is semisimple. We will denote by H_μ^{nc} the smallest algebraic cocompact normal subgroup of H_μ. In [9] Eskin and Margulis proved the following:

Théorème 3.1. (Eskin-Margulis) *Assume that μ has exponential moment, that H_μ is semisimple and that the centralizer of H_μ^{nc} in G is trivial. Then X is uniformly P_μ-recurrent.*

They conjectured in [9, 2.5] the following statement which we proved in [4].

Théorème 3.2. (Benoist-Quint) *Assume that μ has exponential moments and that H_μ is semisimple. Then X is P_μ-recurrent.*

Here is a reformulation of Theorem 3.2.

Corollaire 3.3. *Same assumptions as in Theorem 3.2. Let x be in X. Any weak limit ν_∞ of the sequence $\nu_n := \mu^{*n} * \delta_x$ in the space of finite measures on X, is a probability measure, i.e. $\nu_\infty(X) = 1$.*

Eskin-Margulis recurrence theorem 3.1 is used in [1] as the starting point for the classifications of both the μ-stationary probability measures on X and the Γ_μ-invariant closed subsets of X when G is a simple group and $H_\mu = G$. Benoist-Quint recurrence theorem 3.2 is used in [2] and [3] to extend these classifications to any Lie group G as soon as H_μ is semisimple with no compact factor. We recommend the survey [5] for an introduction to this classification theorem.

Here is a straightforward corollary of Theorem 3.2

Corollaire 3.4. *Let Γ be a discrete subgroup of G whose Zariski closure is semisimple. Then any discrete Γ-orbit in G/Λ is finite.*

Proof of Corollary 3.4. By the recurrence property such a Γ-orbit supports a stationary probability measure ν i.e. a measure satisfying $\mu*\nu = \nu$. By the maximum principle, all the points on this Γ-orbit have same mass for ν. Hence this orbit is finite. □

For the sake of simplicity, we always assume from now on that μ has compact support and that H_μ has no compact factor.

3.2. The space of unimodular lattices in $\mathbb{R}^d$.

The main example of finite volume homogeneous space $X = G/\Lambda$ is the space $X_d = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. In this case, the compact subsets are described by the Mahler compactness criterion below.

The space X_d is also the space of unimodular lattices Δ of $\mathbb{R}^d$, i.e. the set of discrete subgroups of $\mathbb{R}^d$ spanned by a basis $v_1, \dots, v_d$ of $\mathbb{R}^d$ of determinant 1.

For $0 \leq i \leq d$, we define the i^{th}-systole function α_i on X_d by

$$\alpha_i(x) = \min\{\|v\| \mid v \in \Lambda^i x \text{ non-zero pure tensor}\}. \tag{3.1}$$

The minimum is taken among tensor v that can be written as $v = v_1 \wedge \cdots \wedge v_i$ with $v_1, \dots, v_i$ linearly independant elements of the lattice x. For instance $\alpha_1(x)$ is the length of the shortest non-zero vector in the lattice $x \subset \mathbb{R}^d$. By convention we set $\alpha_0 \equiv \alpha_d \equiv 1$. These systole functions are continuous. Their relevance lies in the following criterion.

Lemme 3.5. (Mahler compactness criterion) *For $0 < i < d$, the systole functions α_i^{-1} are proper.*

We recall that a real valued function f is said to be proper if the inverse image of a bounded set is relatively compact. When $i = 1$, Lemma 3.5 means that, a sequence x_n in X_d goes to infinity if and only if there exists a sequence of non-zero vectors $v_n \in x_n$ converging to 0.

4. The contraction properties

We give in this section sufficient conditions for the recurrence and for the uniform recurrence of a Markov-Feller operator. These conditions called **CH** and **UCH** are easy to check since they involve only one iteration of the Markov chain.

Let X be a second countable locally compact space and P a Markov-Feller operator on X. We will say that P satisfies the contraction hypothesis if

CH for every compact L of X, there exists a Borel function $f = f_L : X \to [0, \infty]$ such that,
(i) f takes finite values on L,
(ii) for every $M < \infty$, $f^{-1}([0, M])$ is relatively compact in X,
(iii) there exists constants $a < 1$, $b > 0$ such that $Pf \leq af + b$.

Note that f is not assumed to be finite nor continuous.

This **CH** means that there exist on X functions f which have a very strong P-subharmonicity property: the Markov operator contracts f up to an additive constant.

We will say that P satisfies the uniform contraction hypothesis if

UCH There exists a proper function $f : X \to [0, \infty[$ such that $Pf \leq af + b$, where $a < 1$ and $b > 0$.

This **UCH** means that the function f in **CH** can be chosen to be everywhere finite. This **UCH** is a variation of a condition due to Foster that one can find in [10], [13] and [9]. This **UCH** is shown in [13] to be related to the existence of an exponential moment for the first return time in some bounded sets of X.

Lemme 4.1. *Let X be a second countable locally compact space and P a Markov-Feller operator on X.*
a) Assume that P satisfies the contraction hypothesis **CH** *on X, then P is recurrent on X.*
b) Assume that P satisfies the uniform contraction hypothesis **UCH** *on X, then P is uniformly recurrent on X.*

Proof. a) Let x be a point in X and $f = f_x$ be the function given by the hypothesis **CH** for the compact set $L = \{x\}$. Choose for M the closure of the set

$$\{y \in X \mid f(y) \leq \tfrac{2B}{\varepsilon}\}$$

so that the indicator function of the complementary set M^c satisfies $\mathbf{1}_{M^c} \leq \frac{\varepsilon}{2B} f$.

According to the hypothesis **CH**, one has, for every $n \geq 1$

$$P^n f \leq a^n f + b(1 + \cdots + a^{n-1}) \leq a^n f + B$$

with $B = \frac{b}{1-a}$. One then has the inequalities, for all x in X,

$$P^n_x(M^c) = P^n(\mathbf{1}_{M^c})(x) \leq \frac{\varepsilon}{2B} P^n f(x) \leq \frac{\varepsilon a^n}{2B} f(x) + \frac{\varepsilon}{2} \leq \varepsilon$$

as soon as n is sufficiently large so that $f(x) \leq \frac{B}{a^n}$.

b) Same proof with a function f which does not depend on the point x. □

5. Countable spaces

In this section we give basic examples of Markov operators on countable spaces and describe their recurrence properties.

The first example does not satisfy **UCH**.

Example 5.1. (Random walk on groups) *Let G be a discrete infinite group acting on itself by left multiplication, let μ be a probability measure on G whose support spans G, then P_μ is not recurrent on G.*

Proof. There are no ergodic stationary probability measure ν on G. Indeed, the set of element g for which $\nu(g)$ is maximum is finite and G-invariant. □

Remark 5.2. There is a classical notion of recurrence for a Markov chain that we will call here 0-*recurrence*. It says that, for all neighborhood U of the starting point, almost all trajectories of the Markov chain comes back in U. When $G = \mathbb{Z}$ the above Markov chain P_μ is not recurrent on G eventhough it is 0-recurrent.

The second example is very simple but it gives a fairly good picture of what is a Markov chain satisfying **UCH**.

Example 5.3. (Markov chain satisfying UCH) *Consider the Markov chain $x \to P_x$ on $X = \mathbb{N}$ given by $P_x = \frac{1}{3}\delta_{x+1} + \frac{2}{3}\delta_{x-1}$ when $x > 0$ and $P_x = \delta_{x+1}$ when $x = 0$. This Markov chain satisfies the uniform contraction hypothesis* **UCH**. *In particular it is uniformly recurrent.*

Proof. It satisfies **UCH** with the function $f : x \to 2^{x/2}$. Indeed, one has the inequality $Pf \leq \frac{2\sqrt{2}}{3} f + 1$. □

The next example enlights the difference between **UCH** and **CH**.

Example 5.4. (Markov chain satisfying CH but not UCH) *The trivial Markov chain on $X = \mathbb{N}$ given by the transition probabilities $P_x = \delta_x$ satisfies the contraction hypothesis* **CH**. *In particular P is recurrent. However P is not uniformly recurrent.*

Proof. It satisfies **CH** with the functions $f_n : x \to 1$ when $x \leq n$ and $f_n : x \to \infty$ otherwise. Indeed, one has the inequality $Pf_n \leq \frac{1}{2} f_n + 1$. □

A Markov chain P on a countable set X is said to be *transitive* if for all x, y in X, there exists $n \geq 1$ such that $P_x^n(y) > 0$. The following example tells us very roughly that, except for Example 5.4, the conditions **UCH** and **CH** are equivalent.

Example 5.5. (CH + T implies UCH) *Let P be a transitive Markov chain on a countable set X satisfying the contraction hypothesis* **CH**. *Then P satisfies also the uniform contraction hypothesis* **UCH**.

Proof. Since P satisfies **CH**, there exists a stationary probability measure ν on X. Since P is transitive on X, this stationary probability measure has full support on X. By [6, Prop. 1.8], ν is the unique stationary probability measure on X. Hence, for all x in X, one has

$$\lim_{n\to\infty} \tfrac{1}{n} \textstyle\sum_{k\leq n} P_x^k = \nu.$$

For any $\varepsilon > 0$, one can find a finite set F such that $\nu(F^c) \leq \varepsilon/4$. In particular, for all x in X there exists $n_1 > 0$ such that

$$P_x^{n_1}(F^c) \leq \varepsilon/2.$$

Since P satisfies **CH**, there exists a finite set $M \subset X$ and $n_2 \geq 0$ such that, for all $n \geq n_2$, for all y in F, one has $P_y^n(M^c) \leq \varepsilon/2$. Then, for all $n \geq n_1 + n_2$, one has $P_x^n(M^c) \leq \varepsilon$. Hence P satisfies **UCH**. □

6. The uniform contraction hypothesis UCH

In this section we sketch the proof of Margulis-Eskin recurrence theorem. We begin by simpler examples to enlight one by one the ideas entering the proof.

6.1. Linear random walk.

The first idea is a uniform contraction property for the linear random walk on vector spaces which is nothing but a reformulation of the positivity of the first Lyapounov exponent.

Let H be a real algebraic semisimple Lie group with no compact factor. Let μ be a Borel probability measure on H which is Zariski dense, i.e. whose support spans a Zariski dense subgroup of H. For sake of simplicity we will assume from now on that this support is compact. Let V be a real finite dimensional algebraic representation of H. We set V^H for the set of fixed points of H in V.

The following functions on V are contracted by the Markov operators P_μ on V. They will be the building blocks for the construction of f.

Lemme 6.1. ([9, Lemma 4.2]) *Let V be a real algebraic representation of H such that $V^H = \{0\}$. Let φ be the function on V given by $\varphi(v) = \|v\|$. Then there exists $\delta > 0$, $a_0 < 1$ and $n_0 \geq 1$ such that*

$$P_\mu^{n_0} \varphi^{-\delta} \leq a_0 \varphi^{-\delta} \tag{6.1}$$

Proof. This Lemma 6.1 is proven in [9, Lemma 4.2]. We can assume that V is irreducible. The proof relies on Furstenberg theorem on the positivity of the first Lyapounov exponent of μ which tells us that, uniformly for v in $V \smallsetminus 0$, the limit $\lambda_1 = \lim\limits_{n\to\infty} \int_H \log \frac{\|gv\|}{\|v\|} \, \mathrm{d}\mu^{*n}(g)$ exists and is positive. One then write the asymptotic expansion up to order 2 of $e^{-\delta \log(\|hv\|/\|v\|)}$ and computes its image by P_μ^n. $\square$

Since it is harmless to replace μ by the convolution power μ^{*n_0}, we will always assume implicitely that $n_0 = 1$.

6.2. The pointed torus.

Before dealing with the spaces $X = G/\Lambda$, we explain here on a simpler example how Lemma 6.1 is used to prove the uniform contraction hypothesis **UCH**.

Proposition 6.2. *Let μ be a probability measure on $\mathrm{SL}(d, \mathbb{Z})$ with finite support. Assume that H_μ is semisimple with no compact factor and has no non-zero invariant vectors on $\mathbb{R}^d$. Then the Markov operator P_μ on the pointed torus $\mathbb{T}^d \smallsetminus 0$ satisfies* **UCH**.

Proof. We choose for function f on $\mathbb{T}^d \smallsetminus 0$, a small negative power of the distance to 0, i.e. $f(x) = d(x, 0)^{-\delta}$ with δ small enough.

For x in a small neighborhood U of 0, the random walk is linear hence by Lemma 6.1, one has $P_\mu f(x) \leq a_0 f(x)$, for some constant $a_0 < 1$.

For x in the compact set U^c, $P_\mu f$ is bounded by a constant $b > 0$.

In both cases, one has $P_\mu f \leq a_0 f + b$. $\square$

6.3. $H = \mathrm{SL}(2, \mathbb{R})$ and $X = \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$.

We can now give the proof of Margulis-Eskin recurrence theorem in the simplest case.

Proposition 6.3. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(2, \mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$ satisfies* **UCH**.

Proof. We follow the same strategy. We choose for function f on X, a small negative power of the systole, i.e. $f = \alpha_1^{-\delta}$ with δ small enough.

We want to bound $P_\mu f$ by $a\, f + b$. The difficulty is that in general an average of maximum is not always bounded by the maximum of the average. We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\alpha_1(x) = \|v\|$. We distinguish two cases.

First case. If all non-collinear vectors w in x satisfy $\|w\| \geq C^2\|v\|$. Then one has $\|gv\| \leq \|gw\|$, and hence $\alpha_1(gx) = \|gv\|$. Using Lemma 6.1, one gets, with $a_0 < 1$,

$$P_\mu \alpha_1^{-\delta}(x) = P_\mu \varphi^{-\delta}(v) \leq a_0 \varphi^{-\delta}(v) = a_0 \alpha_1^{-\delta}(x)\,. \tag{6.2}$$

Second case. If there exists a non-collinear vector w in x with $\|w\| \leq C^2\|v\|$. Then we use the inequality

$$\|v \wedge w\| \leq \|v\|\,\|w\| \tag{6.3}$$

and the fact that, since x has covolume 1, the left-hand side is bounded below by 1. We deduce that $\alpha_1(x) \geq C^{-1}$. Hence, by Mahler criterion, x belongs to a compact subset of X. The continuous function $P_\mu f$ is bounded on this compact set by a constant $b > 0$.

In both cases, one has $P_\mu f \leq a_0 f + b$. □

6.4. $H = \mathrm{SL}(3,\mathbb{R})$ and $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$.

The main new idea needed to prove Margulis-Eskin recurrence theorem in the second simplest case, is the use of all the systole functions α_i.

Proposition 6.4. *Let μ be a Zariski dense compactly supported probability measure on* $\mathrm{SL}(3,\mathbb{R})$*. The Markov operator P_μ on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ satisfies* **UCH**.

Proof. We follow the same strategy as for Proposition 6.3, but we will use negative powers of both systole functions. For $i = 1$ and $i = 2$, we introduce the functions f_i on X given by $f_i(x) = \alpha_i(x)^{-\delta}$ with δ small enough.

We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\alpha_1(x) = \|v\|$. We still distinguish two cases.

First case. If all non-collinear vectors w in x satisfy $\|w\| \geq C^2\|v\|$. Then the same calculation (6.2), gives the bound $P_\mu f_1(x) \leq a_0 f_1(x)$.

Second case. If there exists a non collinear vector w in x with $\|w\| \leq C^2\|v\|$. Then we use the same inequality

$$\|v \wedge w\| \leq \|v\|\,\|w\| \tag{6.4}$$

to deduce $\alpha_1(x) \geq C^{-1}\alpha_2(x)^{\frac{1}{2}}$, and then $\alpha_1(gx) \geq C^{-2}\alpha_2(x)^{\frac{1}{2}}$. One gets the bound $P_\mu f_1(x) \leq C^{2\delta} f_2^{\frac{1}{2}}(x)$.

In both cases, one has

$$P_\mu f_1 \le a_0 f_1 + C^{2\delta} f_2^{\frac{1}{2}}\,, \tag{6.5}$$

for some constants $a_0 < 1$. This is not exactly what we wanted. That is why, we use the companion inequality obtained by using the systole α_2 of the dual lattice

$$P_\mu f_2 \le a_0 f_2 + C^{2\delta} f_1^{\frac{1}{2}}\,. \tag{6.6}$$

Note that for every $\varepsilon_0 > 0$ and $z > 0$, one has $z^{\frac{1}{2}} \le \varepsilon_0\, z + \varepsilon_0^{-1}$. Setting $f = f_1 + f_2$, one deduces from (6.5) and (6.6), the upper bound

$$P_\mu f \le a f + b\,,$$

for the constants $a = a_0 + \varepsilon_0\, C^{2\delta}$ and $b = 2\,\varepsilon_0^{-1} C^{2\delta}$. If ε_0 is small enough one has $a < 1$ as required □

6.5. $H = \mathrm{SL}(d,\mathbb{R})$ and $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$.

For larger d, the main new idea for proving Margulis-Eskin recurrence theorem is Inequality (6.7) which allows us to compare the various systole functions α_i.

Proposition 6.5. *Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(d,\mathbb{R})$. The Markov operator P_μ on $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ satisfies* **UCH**.

The new key point will be to replace Inequality (6.4) by the following key inequality. We recall that an element u in $\Lambda^r\mathbb{R}^d$ is a pure tensor, if one can write $u = u_1 \wedge \cdots \wedge u_r$ with all u_i in $\mathbb{R}^d$.

Lemme 6.6. *For all pure tensors $u \in \Lambda^r\mathbb{R}^d$, $v \in \Lambda^s\mathbb{R}^d$ and $w \in \Lambda^t\mathbb{R}^d$, one has*

$$\|u\|\,\|u \wedge v \wedge w\| \le \|u \wedge v\|\,\|u \wedge w\|\,. \tag{6.7}$$

Proof of Lemma 6.6. Set $\langle u\rangle$ for the vector subspaces spanned by the u_i's. One can reduce to the case where the subspaces $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$ are orthogonal. Then we only have to check the easy inequality $\|v \wedge w\| \le \|v\|\,\|w\|$. □

Proof of Proposition 6.5. We follow the same strategy as for Proposition 6.4, but we will use negative powers of all the systole functions. For $0 \le i \le d$, we introduce the functions $f_i = \alpha_i^{-\delta}$ on X with δ small enough. We fix a constant $C > 0$ such that, for all g in the support of μ and all $i \le d$, one has $\|\Lambda^i g\| \le C$ and $\|\Lambda^i g^{-1}\| \le C$.

Fix i with $0 < i < d$. Using the key inequality (6.7) with $r = i - j$ and $s = t = j$ with $0 < j \le \min(i, d-i)$, instead of using Inequality (6.4), one replace the bounds (6.5) and (6.6) by the following bound.

$$P_\mu f_i \le a_0 f_i + C^{2\delta} \textstyle\sum_{j>0} f_{i-j}^{\frac{1}{2}} f_{i+j}^{\frac{1}{2}}\,, \tag{6.8}$$

for some constants $a_0 < 1$.

Setting

$$f = \sum_{0<i<d} \psi_i \;\text{ where }\; \psi_i = \varepsilon_0^{i(d-i)} f_i\,,$$

with ε_0 very small, one deduces the upper bounds

$$\begin{aligned} P_\mu \psi_i &\leq a_0 \psi_i + \varepsilon_0\, C^{2\delta} \sum_{j>0} \psi_{i-j}^{\frac{1}{2}} \psi_{i+j}^{\frac{1}{2}} \\ &\leq a_0 \psi_i + \varepsilon_0\, C^{2\delta} \sum_{0\leq k\leq d} \psi_k\,, \end{aligned}$$

and hence

$$P_\mu f \leq a f + b\,,$$

for the constant $a = a_0 + d\,\varepsilon_0\, C^{2\delta}$ and for $b = 2\,\varepsilon_0\, C^{2\delta}$. If ε_0 is small enough one has $a < 1$ as required □

6.6. H irreducible on $\mathbb{R}^d$ and $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$.

This case is not more difficult than the previous one.

Proposition 6.7. *Let μ be a compactly supported probability measure on* $\mathrm{SL}(d,\mathbb{R})$ *such that H_μ is a semisimple group with no compact factors which acts irreducibly on $\mathbb{R}^d$. Then the Markov operator P_μ on $X = \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$ satisfies* **UCH**.

In this case, the vector space $V = \Lambda^i\mathbb{R}^d$ is the sum $V = V_+ \oplus V_0$ with V_0 the set of H_μ-invariant vectors and V_+ the H_μ-invariant supplementary subspace. We write $v = v_+ + v_0$ for the corresponding decomposition of a vector v in V. The new feature is that this subspace V_0 might be non-trivial. This is harmless because of the following lemma.

We will write $f \ll g$ for $f \leq C\, g$ where C is a constant.

Lemme 6.8. *Keep these notations, in particular, H_μ is irreducible on $\mathbb{R}^d$. For $0 < i < d$ and all pure tensor v in $\Lambda^i\mathbb{R}^d$, one has $\|v\| \ll \|v_+\|$.*

Proof of Lemma 6.8. This follows from a compacity argument, since by the irreducibility assumption, the space V_0 does not contain non-zero pure tensors. □

Proof of Proposition 6.7. The proof is exactly the same as for Proposition 6.5. We just notice that, by Lemmas 6.1 and 6.8, the function $\varphi : v \mapsto \|v\|$ on $V = \Lambda^i\mathbb{R}^d$ still satisfies Inequality (6.1) on the set of pure tensors of $\Lambda^i\mathbb{R}^d$. □

A proof of a more general case of Eskin-Margulis recurrence theorem will be given in section 7.5.

7. The contraction hypothesis CH

In this section, we want to explain the proof of Benoist-Quint recurrence theorem.

7.1. $H = \mathrm{SL}(2,\mathbb{R})$ and $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$.

We begin by the simplest case. The main new idea is a modification of the systole function in which one replaces the lattice x by its intersection with an ε_0-neighborhood of the expanding space.

Proposition 7.1. *Let μ be a Zariski dense compactly supported probability measure on* $\mathrm{SL}(2,\mathbb{R})$*. The Markov operator P_μ on $X = \mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ satisfies* **CH**.

Remark 7.2. We consider the group $H = \mathrm{SL}(2,\mathbb{R})$ as a subgroup of $G = \mathrm{SL}(3,\mathbb{R})$ fixing the last vector e_3 of the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$. Since the centralizer of H in G is non trivial, Margulis-Eskin recurrence theorem does not apply to this case. Indeed P_μ does not satisfy **UCH**, because, by Mahler criterion, the closed H-invariant subsets

$$Y_\varepsilon := \{x \in X \mid \varepsilon\, e_3 \in x\}$$

are going away from any compact subsets of X when $\varepsilon \searrow 0$.

The vector space $V = \mathbb{R}^3$ is the sum $V = V_+ \oplus V_0$ with $V_+ = \mathbb{R}^2$ and $V_0 = \mathbb{R}$. We still write $v = v_+ + v_0$ for the corresponding decomposition of a vector v in V. Same for $V^* = \Lambda^2 V$. A new key point will be to replace Inequality (6.4) by the following inequality (7.1).

Lemme 7.3. *For every v, w in $\mathbb{R}^3$, one has*

$$\|(v \wedge w)_+\| \leq \|v_+\|\,\|w_0\| + \|v_0\|\,\|w_+\| \tag{7.1}$$

Proof of Lemma 7.3. One has $(v \wedge w)_+ = v_+ \wedge w_0 + v_0 \wedge w_+$. □

Proof of Proposition 7.1. We follow the same strategy as for Proposition 6.4. Since the positivity of the Lyapounov exponent occurs only in the V_+ direction and since the projection of a lattice in V_+ might be dense, we have to introduce the following modification of the systole functions. We fix $\varepsilon_0 > 0$ small, and we set, for x in X,

$$\alpha_{\varepsilon_0,1}(x) = \min\{\|v_+\| \mid v \in x \smallsetminus \{0\}\ ,\ \|v_0\| < \varepsilon_0\}. \tag{7.2}$$

The minimum is taken among all non zero vectors v of x belonging to the ε_0-neighborhood of the plane V_+. The new feature is that this quantity $\alpha_{\varepsilon_0,1}(x)$ is not always positive, indeed

$$\alpha_{\varepsilon_0,1}(x) = 0 \Longleftrightarrow x \in Y_\varepsilon \ \text{ for some } \ \varepsilon < \varepsilon_0.$$

Similarly using the dual lattice $x^* = \Lambda^2 x$ in the dual space $V^* = V_+^* \oplus V_0^*$, we set

$$\alpha_{\varepsilon_0,2}(x) = \min\{\|v_+\| \mid v \in x^* \smallsetminus \{0\}\ ,\ \|v_0\| < \varepsilon_0\}. \tag{7.3}$$

We introduce the functions $f_{\varepsilon_0,i} = \alpha_{\varepsilon_0,i}^{-\delta}$ with δ small enough. We fix a constant $C > 0$ such that, for all g in the support of μ, one has $\|g\| \leq C$ and $\|g^{-1}\| \leq C$. Let v be a vector of x such that $\|v_0\| < \varepsilon_0$ and $\alpha_{\varepsilon_0,1}(x) = \|v_+\|$.

First case. If all the non collinear vector $w = w_+ + w_0$ in x with $\|w_0\| < \varepsilon_0$ satisfy $\|w_+\| \geq C^2\|v_+\|$. The same arguments as in (6.2), gives the bound $P_\mu f_{\varepsilon_0,1}(x) \leq a_0 f_{\varepsilon_0,1}(x)$ with $a_0 < 1$.

Second case There exists a non collinear vector w with $\|w_0\| < \varepsilon_0$ satisfying $\|w_+\| \leq C^2\|v_+\|$.

In case $\|v_+\| < \varepsilon_0$, we have $\|(v \wedge w)_0\| < \varepsilon_0$, and we use Inequality (7.1) to deduce $2\varepsilon_0 C^2 \alpha_{\varepsilon_0,1}(x) \geq \alpha_{\varepsilon_0,2}(x)$ and get the bound $P_\mu f_{\varepsilon_0,1}(x) \leq (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,2(x)}$.

In case $\|v_+\| \geq \varepsilon_0$, one has the bound $P_\mu f_{\varepsilon_0,1}(x) \leq \varepsilon_0^{-\delta} C^\delta$.

In all these three cases, one has

$$P_\mu f_{\varepsilon_0,1} \leq a_0 f_{\varepsilon_0,1} + (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,2} + \varepsilon_0^{-\delta} C^\delta\,, \tag{7.4}$$

for some constant $a_0 < 1$. Similarly, one has

$$P_\mu f_{\varepsilon_0,2} \leq a_0 f_{\varepsilon_0,2} + (2\varepsilon_0 C^3)^\delta f_{\varepsilon_0,1} + \varepsilon_0^{-\delta} C^\delta\,. \tag{7.5}$$

Setting $f_{\varepsilon_0} = f_{\varepsilon_0,1} + f_{\varepsilon_0,2}$, one deduces then from (7.4) and (7.5), the upper bound

$$P_\mu f_{\varepsilon_0} \leq a\, f_{\varepsilon_0} + b\,,$$

for the constants $a = a_0 + (2\varepsilon_0\, C^3)^\delta$ and $b = 2\,\varepsilon_0^{-\delta} C^\delta$. If ε_0 is small enough one has $a < 1$ as required. □

7.2. $H = \mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$ and $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$.

In this case the main new idea is to replace the norm by a function φ_{ε_0} which takes into account suitable powers of the norm in the irreducible subrepresentations of H_μ.

Proposition 7.4. *Let $d = d_1 + d_2$. Let μ be a Zariski dense compactly supported probability measure on $\mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$. Then the Markov operator P_μ on $X = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ satisfies* **CH**.

We will need a stronger inequality generalizing both (6.7) and (7.1).

Let $\mathbb{R}^d := \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$ be the associated orthogonal decomposition. For any couple $\lambda = (\lambda_1, \lambda_2) \in \mathbb{N}^2$, we denote by $u \to u_\lambda$ the projector of $\Lambda^*\mathbb{R}^d$ on the component $\Lambda^{\lambda_1}\mathbb{R}^{d_1} \otimes \Lambda^{\lambda_2}\mathbb{R}^{d_2}$. We endow $\mathbb{N}^2$ with the partial order

$$\lambda \leq \mu \Longleftrightarrow (\lambda_1 \leq \mu_1 \ \text{ and } \ \lambda_2 \leq \mu_2). \tag{7.6}$$

For any λ, μ in $\mathbb{N}^2$ we denote by $m_{\lambda,\mu}$ the minimum and $M_{\lambda,\mu}$ the maximum of λ and μ, that is $m_{\lambda,\mu} = (\min(\lambda_1, \mu_1), \min(\lambda_2, \mu_2))$ and similarly for the maximum. We denote by

$$R(\lambda, \mu) := \{\nu \in \mathbb{N}^2 \mid m_{\lambda,\mu} \leq \nu \leq M_{\lambda,\mu}\}$$

the "rectangle" between $m_{\lambda,\mu}$ and $M_{\lambda,\mu}$, and by R the rectangle

$$R := \{\nu \in \mathbb{N}^2 \mid \nu \leq (d_1, d_2)\}.$$

Lemme 7.5. (Mother Inequality for $SL \times SL$) *For any pure tensors u, v, w in $\Lambda^*(\mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2})$, and λ, μ in R, one has*

$$\|u_\lambda\| \, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu,\rho \in R(\lambda,\mu) \\ \nu+\rho = \lambda+\mu}} \|(u \wedge v)_\nu\| \, \|(u \wedge w)_\rho\|. \tag{7.7}$$

The only proof of Lemma 7.5 that I know relies on representation theory. We will explain this proof in Section 7.3.

Exemple 7.1. For u, v, w vectors in $\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2}$, one has

$$\begin{aligned}
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{3,0}\| \ll \|(u\wedge v)_{2,0}\| \, \|(u\wedge w)_{2,0}\|, \\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{2,1}\| \ll \|(u\wedge v)_{2,0}\| \|(u\wedge w)_{1,1}\| + \|(u\wedge v)_{1,1}\| \|(u\wedge w)_{2,0}\|, \\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{1,2}\| \ll \|(u\wedge v)_{1,1}\| \|(u\wedge w)_{1,1}\|, \\
&\|u_{1,0}\| \, \|(u\wedge v\wedge w)_{0,3}\| \ll \|(u\wedge v)_{0,2}\| \|(u\wedge w)_{1,1}\| + \|(u\wedge v)_{1,1}\| \|(u\wedge w)_{0,2}\|.
\end{aligned}$$

Among these inequalities, the most one inequality is the third one since the terms $\|(u \wedge v)_{0,2}\| \, \|(u \wedge w)_{2,0}\| + \|(u \wedge v)_{2,0}\| \, \|(u \wedge w)_{0,2}\|$ do not occur on the right hand side.

For $\lambda \in R$ we set

$$|\lambda| := (d_1 - \lambda_1)\lambda_1 + (d_2 - \lambda_2)\lambda_2.$$

Let $\varepsilon_0 > 0$. For v in $\Lambda^i E$, with $0 < i < d$, we define

$$\varphi_{\varepsilon_0}(v) = \max_{\lambda \in R, |\lambda| \neq 0} \varepsilon_0^{\frac{-(d-i)i}{|\lambda|}} \|v_\lambda\|^{\frac{1}{|\lambda|}}. \tag{7.8}$$

Note that this function is the inverse of the function denoted φ_{ε_0} in [4].

Lemme 7.6. *There exists $\delta > 0$, $a_0 < 1$ and $n_0 \geq 1$ such that,*

$$P_\mu^{n_0} \varphi_{\varepsilon_0}^{-\delta} \leq a_0 \varphi_{\varepsilon_0}^{-\delta} \quad \text{for any } \varepsilon_0 > 0. \tag{7.9}$$

Proof of Lemma 7.6. This follows from Lemma 6.1. □

Proof of Proposition 7.4. The proof is the same as for Proposition 7.1, replacing Inequality (6.1) by (7.9) and Inequality (7.1) by (7.7). We define for x in X

$$\alpha_{\varepsilon_0}(x) = \min\{\varphi_{\varepsilon_0}(v) \mid v \in \Lambda^\bullet x \smallsetminus 0 \text{ , pure tensor with } \|v_0\| < \varepsilon_0\},$$

where the minimum is taken over all the non-zero pure tensor v in some $\Lambda^i x$ for which $\|v_0\| < \varepsilon_0$. We also introduce the function on X

$$f_{\varepsilon_0}(x) = \alpha_{\varepsilon_0}(x)^{-\delta}.$$

If δ and ε_0 are small enough, this function f_{ε_0} satisfies

$$P_\mu f_{\varepsilon_0} \leq a \, f_{\varepsilon_0} + b$$

for some constants $a < 1$ and $b > 0$. Moreover, for x in X, one has the equivalence : $f_{\varepsilon_0}(x) = \infty$ if and only if, for some i, $\Lambda^i x$ contains an H-invariant pure tensor v with $\|v\| < \varepsilon_0^{(d-i)i}$.

This proves that P_μ satisfies **CH** on X. □

7.3. Mother Inequality.

In this section, we sketch the proof of Inequality (7.7). We will see that it is a special case of the Mother Inequality (7.11) based on Representation Theory.

Let $H \subset \mathrm{SL}(\mathbb{R}^d)$ be a semisimple algebraic subgroup, $A \subset H$ be a maximal split subtorus of H, $\Sigma = \Sigma(A, H)$ be the set of (restricted) roots, i.e. Σ is the set of non-zero weights of A in the Lie algebra $\mathfrak{h}$ of H. We choose a system $\Sigma^+ \subset \Sigma$ of positive roots. Let P be the set of algebraic characters of A. We endow P with the partial order given, for λ, μ in P, by

$$\lambda \leq \mu \Longleftrightarrow \mu - \lambda \text{ is a sum of positive roots.} \tag{7.10}$$

For any real algebraic irreducible representation of H, the set of weights of A in this representation has a unique maximal element λ called the (restricted) highest weight of the representation. Let P^+ be the set of all these highest weights. For any algebraic representation of H in a real finite dimensional vector space V, for λ in P^+, we denote by $v \mapsto v_\lambda$ the H-equivariant projection on the sum of all the irreducible subrepresentations of V whose highest weight is equal to λ.

Lemme 7.7. (Mother Inequality) *Let $H \subset \mathrm{SL}(\mathbb{R}^d)$ be a semisimple algebraic subgroup. For pure tensors u, v, w in $\Lambda^*\mathbb{R}^d$ and λ, μ in P^+, one has*

$$\|u_\lambda\| \, \|(u \wedge v \wedge w)_\mu\| \ll \max_{\substack{\nu, \rho \in P^+ \\ \nu + \rho \geq \lambda + \mu}} \|(u \wedge v)_\nu\| \, \|(u \wedge w)_\rho\|. \tag{7.11}$$

Proof of Lemma 7.7 ⇒ Lemma 7.5. Let $H = \mathrm{SL}(d_1, \mathbb{R}) \times \mathrm{SL}(d_2, \mathbb{R})$, $d = d_1 + d_2$. We choose the Lie algebra $\mathfrak{a}$ to be the set of diagonal matrices in $\mathfrak{h}$, and we choose the positive roots of $\mathfrak{h}$ to be the linear forms $e_i^* - e_j^*$ with either $1 \leq i < j \leq d_1$ or $d_1 < i < j \leq d$. We can embed the rectangle R as a subset of the set P^+ of dominant weights. Indeed, for λ in R, the representation of H in $\Lambda^{\lambda_1}\mathbb{R}^{d_1} \otimes \Lambda^{\lambda_2}\mathbb{R}^{d_2}$ is irreducible with highest weight

$$\widetilde{\lambda} = e_1^* + \cdots + e_{\lambda_1}^* + e_{d_1+1}^* + \cdots + e_{d_1+\lambda_2}^*.$$

One can describe the restriction to the subset $\widetilde{R} + \widetilde{R} \subset P^+$ of the partial order (7.10). Indeed, one has the equivalence, for λ, μ, ν, ρ in R,

$$\widetilde{\nu} + \widetilde{\rho} \geq \widetilde{\lambda} + \widetilde{\mu} \Longleftrightarrow (\ \nu + \rho = \lambda + \mu \text{ and } \min(\lambda, \mu) \leq \nu \leq \max(\lambda, \mu)\ ).$$

In the left-hand side, the inequality is defined by (7.10) while, in the right-hand side, it is defined by (7.6). This proves that the bound (7.11) can be reformulated as the bound (7.7). □

Proof of Lemma 7.7. Follows directly from the next two lemmas. □

Lemme 7.8. *Let H be a real algebraic reductive group, V be a real algebraic representation of H. For λ, μ in P^+ and v, w in V, one has*

$$\|v_\lambda\| \, \|w_\mu\| \ll \|(v \otimes w)_{\lambda+\mu}\| \, .$$

Proof. This bound follows by a compacity argument, once one has noticed that Equality $(v \otimes w)_{\lambda+\mu} = 0$ implies $v_\lambda \otimes w_\mu = 0$. $\square$

Lemme 7.9. *Let $V = \mathbb{R}^d$ and $r, s, t \geq 0$. There exists a linear map*

$$\Psi : \Lambda^{r+s}V \otimes \Lambda^{r+t}V \to \Lambda^r V \otimes \Lambda^{r+s+t}V \text{ such that}$$
$$(u \wedge v) \otimes (u \wedge w) \mapsto u \otimes (u \wedge v \wedge w)\ ,$$

for all pure tensors $u \in \Lambda^r V$, $v \in \Lambda^s V$ and $w \in \Lambda^t V$.

This map Ψ is unique and is $\mathrm{GL}(V)$ equivariant.

Proof. This exercise in exterior algebra is left to the reader. See [4]. $\square$

7.4. Benoist-Quint recurrence theorem.

We show how Theorem 3.2 can be deduced from the previous ideas.

We will only deal with the following case which, thanks to Margulis Arithmeticity Theorem (see [12]), is the most important one.

Proposition 7.10. *Let $G \subset \mathrm{SL}(d, \mathbb{R})$ be a semisimple algebraic subgroup defined over $\mathbb{Q}$ and $\Lambda = G \cap \mathrm{SL}(d, \mathbb{Z})$. Let μ be a Zariski dense compactly supported probability measure on a semisimple subgroup H with no compact factors. Then the Markov operator P_μ on $X = G/\Lambda$ satisfies* **CH**.

Proof. We recall that the quotient $X = G/\Lambda$ is closed in $X_d = \mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$ (see [7]). Hence, we can assume that $G = \mathrm{SL}(d, \mathbb{R})$ and $X = X_d$. We keep the notation of Section 7.3. We choose an element H_0 in the interior of the Weyl chamber, and set, for λ in P^+, $|\lambda| = \lambda(H_0)$. Let $\varepsilon_0 > 0$. Exactly as in Formula (7.8), for v in $\Lambda^i \mathbb{R}^d$ with $0 < i < d$, we define

$$\varphi_{\varepsilon_0}(v) = \max_{\lambda \in P^+ \smallsetminus 0} \varepsilon_0^{\frac{-(d-i)i}{|\lambda|}} \|v_\lambda\|^{\frac{1}{|\lambda|}}$$

so that Lemma 7.6 is still true with this function φ_{ε_0}. As in the proof of Proposition 7.4, we define, for x in X,

$$\alpha_{\varepsilon_0}(x) = \min\{\varphi_{\varepsilon_0}(v) \mid v \in \Lambda^\bullet x \smallsetminus 0\ ,\ \text{pure tensor with } \|v_0\| < \varepsilon_0\}.$$

and check the condition **CH** with the same functions $f_{\varepsilon_0} = \alpha_{\varepsilon_0}^{-\delta}$ provided that δ and ε_0 are small enough. $\square$

7.5. Eskin-Margulis recurrence theorem.

We show how Theorem 3.1 can be deduced from Theorem 3.2.

We will again only deal with the most important case.

Proposition 7.11. *Let $G \subset \mathrm{SL}(d, \mathbb{R})$ be a semisimple algebraic subgroup defined over $\mathbb{Q}$ and $\Lambda = G \cap \mathrm{SL}(d, \mathbb{Z})$. Let μ be a Zariski dense compactly supported probability measure on a semisimple subgroup H with no compact factors and with trivial centralizer in G. Then the Markov operator P_μ on $X = G/\Lambda$ satisfies* **UCH**.

Proof. We consider the function f_{ε_0} of Section 7.4 restricted to the finite volume G-orbit $G/\Lambda \subset \mathrm{SL}(d,\mathbb{R})/\mathrm{SL}(d,\mathbb{Z})$. We only have to check that, for ε_0 small enough, f_{ε_0} is everywhere finite on G/Λ.

Assume by contradiction that this is not the case, then there exists a sequence of H-invariant non-zero vectors $v_n \in \Lambda^i\mathbb{R}^d$ such that $\|v_n\| \searrow 0$ and $g_n \in G$ such that $g_n v_n$ belongs to the lattice $\Lambda^i\mathbb{Z}^d$. As a consequence, there exists n_0 such that, for $n \geq n_0$, every G-invariant polynomial F on $\Lambda^i\mathbb{R}^d$ with $F(0) = 0$ satisfies also $F(v_n) = 0$. This means that v_n is an unstable vector. By Kempf Theorem in [11], the stabilizer of an unstable vector is a parabolic subgroup $P \neq G$. Hence the semisimple group H is included in P. As a consequence H has a non-trivial centralizer. Contradiction. $\square$

We conclude this survey by an open question: it is very likely that Theorems 3.1 and 3.2 are still true without any moment assumption on μ.

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