FUCHSIAN GROUPS AND COMPACT HYPERBOLIC SURFACES

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Abstract. We present a topological proof of the following theorem of Benoist-Quint: for a finitely generated non-elementary discrete subgroup Γ_1 of $\mathrm{PSL}(2,\mathbb{R})$ with no parabolics, and for a cocompact lattice Γ_2 of $\mathrm{PSL}(2,\mathbb{R})$, any Γ_1 orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.

1. Introduction

Let Γ_1 be a non-elementary finitely generated discrete subgroup with no parabolic elements of $\mathrm{PSL}(2,\mathbb{R})$. Let Γ_2 be a cocompact lattice in $\mathrm{PSL}(2,\mathbb{R})$. The following is the first non-trivial case of a theorem of Benoist-Quint [1].

Theorem 1.1. *Any Γ_1-orbit on $\Gamma_2\backslash \mathrm{PSL}(2,\mathbb{R})$ is either finite or dense.*

The proof of Benoist-Quint is quite involved even in the case as simple as above and in particular uses their classification of stationary measures [2]. The aim of this note is to present a short, and rather elementary proof.

We will deduce Theorem 1.1 from the following Theorem 1.2. Let

- $H_1 = H_2 := \mathrm{PSL}(2,\mathbb{R})$ and $G := H_1 \times H_2$;
- $H := \{(h,h) : h \in \mathrm{PSL}_2(\mathbb{R})\}$ and $\Gamma := \Gamma_1 \times \Gamma_2$.

Theorem 1.2. *For any $x \in \Gamma\backslash G$, the orbit xH is either closed or dense.*

Our proof of Theorem 1.2 is purely topological, and inspired by the recent work of McMullen, Mohammadi and Oh [5] where the orbit closures of the $\mathrm{PSL}(2,\mathbb{R})$ action on $\Gamma_0\backslash \mathrm{PSL}(2,\mathbb{C})$ are classified for certain Kleinian subgroups Γ_0 of infinite co-volume. While the proof of Theorem 1.2 follows closely the sections 8-9 of [5], the arguments in this paper are simpler because of the assumption that Γ_2 is cocompact. We remark that the approach of [5] and hence of this paper is somewhat modeled after Margulis's original proof of Oppenheim conjecture [4]. When Γ_1 is cocompact as well, Theorem 1.2 also follows from [6].

Finally we remark that according to [1], both Theorems 1.1 and 1.2 are still true in presence of parabolic elements, more precisely when Γ_1 is any non-elementary discrete subgroup and Γ_2 any lattice in $\mathrm{PSL}(2,\mathbb{R})$. The topological method presented here could also be extended to this case.

Oh was supported in part by NSF Grant.

2. Horocyclic flow on convex cocompact surfaces

In this section we prove a few preliminary facts about unipotent dynamics involving only one factor H_1.

The group $\mathrm{PSL}_2(\mathbb{R}) := SL_2(\mathbb{R})/\{\pm e\}$ is the group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}^2 := \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$. The isometry corresponding to the element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ is $z \mapsto \dfrac{az+b}{cz+d}$. It is implicit in this notation that the matrices g stand for their equivalence class $\pm g$ in $\mathrm{PSL}_2(\mathbb{R})$. This group $\mathrm{PSL}_2(\mathbb{R})$ acts simply transitively on the unit tangent bundle $T^1(\mathbb{H}^2)$ and we choose an identification of $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{T}^1(\mathbb{H}^2)$ so that the identity element e corresponds to the upward unit vector at i. We will also identify the boundary of the hyperbolic plane with the extended real line $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ which is topologically a circle.

We recall that Γ_1 is a non-elementary finitely generated discrete subgroup with no parabolic elements of the group $H_1 = \mathrm{PSL}_2(\mathbb{R})$, that is, Γ_1 is a convex cocompact subgroup. Let S_1 denote the hyperbolic orbifold $\Gamma_1\backslash\mathbb{H}^2$, and let $\Lambda_{\Gamma_1} \subset \partial\mathbb{H}^2$ be the limit set of Γ_1. Let A_1 and U_1 be the subgroups of H_1 given by

$$A_1 := \{a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R}\} \text{ and } U_1 := \{u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R}\}.$$

Let

$$\Omega_{\Gamma_1} = \{x \in \Gamma_1\backslash H_1 : xA_1 \text{ is bounded}\}. \tag{2.1}$$

As Γ_1 is a convex cocompact subgroup, Ω_{Γ_1} is a compact A_1-invariant subset and one has the equality

$$\Omega_{\Gamma_1} = \{[h] \in \Gamma_1\backslash H_1 : h(0), h(\infty) \in \Lambda_{\Gamma_1}\}.$$

In geometric words, seen as a subset of the unit tangent bundle of S_1, the set Ω_{Γ_1} is the union of all the geodesic lines which stays inside the convex core of S_1.

Definition 2.2. *Let $K > 1$. A subset $T \subset \mathbb{R}$ is called K-thick if, for any $t > 0$, T meets $[-Kt, -t] \cup [t, Kt]$.*

Lemma 2.3. *There exists $K > 1$ such that for any $x \in \Omega_{\Gamma_1}$, the subset $T(x) := \{t \in \mathbb{R} : xu_t \in \Omega_{\Gamma_1}\}$ is K-thick.*

Proof. Using an isometry, we may assume without loss of generality that $x = [e]$. Since the element e corresponds to the upward unit vector at i, and since x belongs to Ω_{Γ_1}, both points 0 and ∞ belong to the limit set Λ_{Γ_1}. Since $u_t(\infty) = \infty$ and $u_t(0) = t$, one has the equality

$$T(x) = \{t \in \mathbb{R} : t \in \Lambda_{\Gamma_1}\}.$$

Write $\mathbb{R} - \Lambda_{\Gamma_1}$ as the union $\cup J_\ell$ where J_ℓ's are maximal open intervals. Note that the minimum hyperbolic distance between the convex hulls in $\mathbb{H}^2$

$$\delta := \inf_{\ell \neq m} d(\mathrm{hull}(J_\ell), \mathrm{hull}(J_m))$$

is positive, as 2δ is the length of the shortest closed geodesic of the double of the convex core of S_1. Choose the constant $K > 1$ so that for $t > 0$, one has

$$d(\mathrm{hull}[-Kt, -t], \mathrm{hull}[t, Kt]) = \delta/2.$$

Note that this choice of K is independent of t. If $T(x)$ does not intersect $[-Kt, -t] \cup [t, Kt]$ for some $t > 0$, then the intervals $[-Kt, -t]$ and $[t, Kt]$ must be included in two distinct intervals J_ℓ and J_m, since $0 \in \Lambda_{\Gamma_1}$. This contradicts the choice of K. $\square$

Lemma 2.4. *Let $K > 1$ and let T be a K-thick subset of $\mathbb{R}$. For any sequence h_n in $H_1 \smallsetminus U_1$ converging to e, there exists a sequence $t_n \in T$ such that the sequence $u_{-t_n} h_n u_{t_n}$ has a limit point in $U_1 \smallsetminus \{e\}$.*

Proof. Write $h_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. We compute

$$q_n := u_{-t_n} h_n u_{t_n} = \begin{pmatrix} a_n - c_n t_n & (a_n - d_n - c_n t_n) t_n + b_n \\ c_n & d_n + c_n t_n \end{pmatrix}.$$

Since the element h_n does not belong to U_1, it follows that the $(1, 2)$-entries $P_n(t_n) := (a_n - d_n - c_n t_n) t_n + b_n$ are non-constant polynomial functions of t_n of degree at most 2 whose coefficients converge to 0. Hence, by Lemma 2.5 below, we can choose $t_n \in T$ going to ∞ so that $k \le |P_n(t_n)| \le 1$, for some constant $k > 0$ depending only on K. Since the entry $P_n(t_n)$ is bounded and since h_n converges to e, the product $c_n t_n$ must converge to 0 and the sequence q_n has a limit point in $U_1 - \{e\}$. $\square$

We have used the following basic lemma :

Lemma 2.5. *For every $K > 1$ and $d \ge 1$, there exists $k > 0$ such that, for every non-constant polynomial P of degree d with $|P(0)| \le k$, and for every K-thick subset T of $\mathbb{R}$, there exists t in T such that $k \le |P(t)| \le 1$.*

Proof. Using a suitable homothety in the variable t, we can assume with no loss of generality that P belongs to the set $\mathcal{P}_d$ of polynomials of degree at most d such that $P(1) = \max\limits_{[-1,1]} |P(t)| = 1$.

Assume by contradiction that there exists a sequence P_n of polynomials in $\mathcal{P}_d$ and a sequence of K-thick subsets T_n of $\mathbb{R}$ such that $\sup\limits_{T_n \cap [-1,1]} |P_n(t)|$ converge to 0. After extraction, the sequence T_n converges to a K-thick subset T_∞ and the sequence P_n converges to a polynomial $P_\infty \in \mathcal{P}_d$ which is equal to 0 on the set $T_\infty \cap [-1, 1]$. This is not possible since this set is infinite. $\square$

We record also, for further use, the following classical lemma :

Lemma 2.6. *Let U_1^+ be the semigroup $\{u_t : t \ge 0\}$. If the quotient space $X_1 := \Gamma_1 \backslash H_1$ is compact, any U_1^+-orbit is dense in X_1.*

Proof. For $x \in X_1$, set $x_n := xu_n$. We then have $x_n u_{-n} U_1^+ = xU_1^+$. Hence if z is a limit point of the sequence x_n in X_1, we have $zU \subset \overline{xU_1^+}$. By Hedlund's theorem [3], zU is dense. Hence the orbit xU_1^+ is also dense. $\square$

3. Proof of Theorems 1.1 and 1.2

In this section, using minimal sets and unipotent dynamics on the product space $\Gamma\backslash G$, we provide a proof of Theorem 1.2.

3.1. **Unipotent dynamics.** We recall the notation $G := \mathrm{PSL}_2(\mathbb{R})\times\mathrm{PSL}_2(\mathbb{R})$ and $\Gamma := \Gamma_1 \times \Gamma_2$. Set

- $H_1 = \{(h,e)\}$, $H_2 = \{(e,h)\}$, $H = \{(h,h)\}$;
- $U_1 = \{(u_t,e)\}$, $U_2 = \{(e,u_t)\}$, $U = \{(u_t,u_t)\}$;
- $A_1 = \{(a_t,e)\}$, $A_2 = \{(e,a_t)\}$, $A = \{(a_t,a_t)\}$;
- $X_1 = \Gamma_1\backslash H_1$, $X_2 = \Gamma_2\backslash H_2$, $X = \Gamma\backslash G = X_1 \times X_2$.

Recall that Γ_1 is a non-elementary finitely generated discrete subgroup of H_1 with no parabolic elements and that Γ_2 is a cocompact lattice in H_2.

For simplicity, we write $\widetilde{u}_t$ for (u_t,u_t) and $\widetilde{a}_t$ for (a_t,a_t). Note that the normalizer of U in G is AU_1U_2.

Lemma 3.1. *Let g_n be a sequence in $G \smallsetminus AU_1U_2$ converging to e, and let T be a K-thick subset of $\mathbb{R}$ for some $K > 1$. Then for any neighborhood G_0 of e in G, there exist sequences $s_n \in T$ and $t_n \in \mathbb{R}$ such that the sequence $\widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ has a limit point $q \neq e$ in $AU_2 \cap G_0$.*

Proof. Fix $0 < \varepsilon \le 1$. Write $g_n = (g_n^{(1)}, g_n^{(2)})$ with $g_n^{(i)} = \begin{pmatrix} a_n^{(i)} & b_n^{(i)} \\ c_n^{(i)} & d_n^{(i)} \end{pmatrix}$. Then the products $q_n := \widetilde{u}_{-s_n} g_n \widetilde{u}_{t_n}$ are given by

$$q_n^{(i)} = u_{-s_n} g_n^{(i)} u_{t_n} = \begin{pmatrix} a_n^{(i)} - c_n^{(i)} s_n & (b_n^{(i)} - d_n^{(i)} s_n) - t_n(c_n^{(i)} s_n - a_n^{(i)}) \\ c_n^{(i)} & d_n^{(i)} + c_n^{(i)} t_n \end{pmatrix}.$$

Set

$$t_n = \tfrac{b_n^{(1)} - d_n^{(1)} s_n}{c_n^{(1)} s_n - a_n^{(1)}}.$$

The differences $q_n - e$ are now rational functions in s_n of the form

$$q_n - e = \tfrac{1}{c_n^{(1)} s_n - a_n^{(1)}} P_n(s_n),$$

where $P_n(s_n)$ is a polynomial function of s_n of degree at most 2 with values in $M_2(\mathbb{R}) \times M_2(\mathbb{R})$. Since the elements g_n do not belong to AU_1U_2, these polynomials P_n are non-constants. In particular, the real valued polynomial functions $s_n \mapsto \|P_n(s_n)\|^2$ are non-constant of degree at most 4. We introduce now the subsets T_n of $\mathbb{R}$ where the denominators are not too small,

$$T_n := \{s \in T \mid |c_n^{(1)} s - a_n^{(1)}| \ge 1/2\}.$$

We claim that for n large these subsets T_n are $4K^2$-thick. Indeed, since T is K-thick, the set $\log|T| \subset \mathbb{R}$ meets every interval of size $\log K$. Since for

n large, one has $|a_n^{(1)} - 1| \leq 1/10$, the set $\log |T_n|$ is obtained by removing from $\log |T|$ an interval of size at most $\log 4$, therefore $\log |T_n|$ meets every interval of size $\log(4K^2)$ and T_n is $4K^2$-thick.

Hence, by Lemma 2.5, we can choose $s_n \in T_n$ going to ∞ so that $k\varepsilon \leq \|P_n(s_n)\| \leq \varepsilon$ for some constant $k > 0$ depending only on K. In particular, using the definition of T_n and the bound on the $(1,1)$-entry of $P_n(s_n)$ we get the inequalities

$$1/2 \leq |c_n^{(1)} s_n - a_n^{(1)}| \leq 3$$

so that

$$k\varepsilon/3 \leq \|q_n - e\| \leq 2\varepsilon.$$

By construction, when ε is small enough, the sequence q_n has a limit point $q \neq e$ in $A_1 A_2 U_2 \cap G_0$.

We claim that this limit $q = (q^{(1)}, q^{(2)})$ belongs to the group AU_2. It suffices to check that the diagonal entries of $q^{(1)}$ and $q^{(2)}$ are equal. If not, the two sequences $c_n^{(i)} s_n$ converge to real numbers $c^{(i)}$ with $c^{(1)} \neq c^{(2)}$, and a simple calculation shows that the $(1,2)$- entries of $q_n^{(2)}$ are comparable to $\frac{c^{(2)} - c^{(1)}}{c^{(1)} - 1} s_n$ which tends to ∞. Contradiction. Hence q belongs to AU_2. $\square$

3.2. **H-minimal and U-minimal subsets.** Let

$$\Omega := \Omega_{\Gamma_1} \times X_2$$

where $\Omega_{\Gamma_1} \subset G_1$ is defined in (2.1). Note that, since Γ_2 is cocompact, one has the equality $\Omega_{\Gamma_2} = X_2$.

Let $x = (x_1, x_2) \in \Gamma \backslash G$ and consider the orbit xH. Note that xH intersects Ω non-trivially. Let Y be an H-minimal subset of the closure $\overline{xH}$ with respect to Ω, i.e., Y is a closed H-invariant subset of $\overline{xH}$ such that $Y \cap \Omega \neq \emptyset$ and the orbit yH is dense in Y for any $y \in Y \cap \Omega$. Since any H orbit intersects Ω, it follows that yH is dense in Y for any $y \in Y$. Let Z be a U-minimal subset of Y with respect to Ω. Since Ω is compact, such minimal sets Y and Z exist. Set

$$Y^* = Y \cap \Omega \quad \text{and} \quad Z^* = Z \cap \Omega.$$

In the following, we assume that

$$\text{the orbit } xH \text{ is not closed}$$

and aim to show that xH is dense in X.

Lemma 3.2. *For any $y \in Y$, the identity element e is an accumulation point of the set $\{g \in G \smallsetminus H : yg \in \overline{xH}\}$.*

Proof. If y does not belong to xH, there exists a sequence $h_n \in H$ such that xh_n converges to y. Hence there exists a sequence $g_n \in G$ converging to e such that $xh_n = yg_n$. These elements g_n do not belong to H; hence proving the claim.

Suppose now that y belongs to xH. If the claim does not hold, then for a sufficiently small neighborhood G_0 of e in G, the set $yG_0 \cap Y$ is included in the orbit yH. This implies that the orbit yH is an open subset of Y. The minimality of Y implies that $Y = yH$, contradicting the assumption that the orbit $yH = xH$ is not closed. $\square$

Lemma 3.3. *There exists an element $v \in U_2 \smallsetminus \{e\}$ such that $Zv \subset \overline{xH}$.*

Proof. Choose a point $z = (z_1, z_2) \in Z^*$. By Lemma 3.2, there exists a sequence g_n in $G \smallsetminus H$ converging to e such that $zg_n \in \overline{xH}$. We may assume without loss of generality that g_n belongs to H_2.

Suppose first that at least one g_n belongs to U_2. Set $v = g_n$ be one of those belonging to U_2, so that the point zv belongs to $\overline{xH}$. Since v commutes with U and Z is U-minimal with respect to Ω, one has the equality $Zv = \overline{zvU}$, hence the set Zv is included in $\overline{xH}$.

Now suppose that g_n does not belong to U_2. Then, since the set $T(z_1)$ is K-thick for some $K > 1$ by Lemma 2.3, it follows from Lemma 2.4 that there exist a sequence $t_n \to \infty$ in $T(z_1)$ such that, after extraction, the products $\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}$ converge to an element $v \in U_2 \smallsetminus \{e\}$.

Since the points $z\widetilde{u}_{t_n}$ belong to Ω, this sequence has a limit point $z' \in Z^*$. Since one has the equality

$$z'v = \lim_{n\to\infty} z\widetilde{u}_{t_n}(\widetilde{u}_{-t_n} g_n \widetilde{u}_{t_n}) = \lim_{n\to\infty} (zg_n)\widetilde{u}_{t_n},$$

the point $z'v$ belongs to $\overline{xH}$. We conclude as in the first case that the set $Zv = \overline{z'vU}$ is included in $\overline{xH}$. $\square$

Lemma 3.4. *For any $z \in Z^*$, there exists a sequence g_n in $G \smallsetminus U$ converging to e such that $zg_n \in Z$ for all n.*

Proof. Since the group Γ_2 is cocompact, it does not contain unipotent elements and hence the orbit zU is not compact. By Lemma 2.3, the orbit zU is recurrent in Z^*, hence the set $Z^* \smallsetminus zU$ contains at least one point. Call it z'. Since the orbit $z'U$ is dense in Z, there exists a sequence $\widetilde{u}_{t_n} \in U$ such that $z = \lim z'\widetilde{u}_{t_n}$. Hence one can write $z'\widetilde{u}_{t_n} = zg_n$ with g_n in $G \smallsetminus U$ converging to e. $\square$

Proposition 3.5. *There exists a one-parameter semi-group $L^+ \subset AU_2$ such that $ZL^+ \subset Z$.*

Proof. It suffices to find, for any neighborhood G_0 of e, an element $q \neq e$ in $AU_2 \cap G_0$ such that the set Zq is included in Z; then writing $q = \exp w$ for an element w of the Lie algebra of G, we can take L^+ to be the semigroup $\{\exp(sw_\infty) : s \geq 0\}$ where w_∞ is a limit point of the elements $\frac{w}{\|w\|}$ when the diameter of G_0 shrinks to 0.

Fix a point $z = (z_1, z_2) \in Z^*$. According to Lemma 3.4 there exists a sequence $g_n \in G \smallsetminus U$ converging to e such that $zg_n \in Z$.

Suppose first that g_n belongs to AU_1U_2 for infinitely many n; then one can find $\widetilde{u}_{t_n} \in U$ such that the product $q_n := g_n\widetilde{u}_{t_n}$ belongs to $AU_2 \smallsetminus \{e\}$ and zq_n belongs to Z. Since q_n normalizes U and since Z is U-minimal with respect to Ω, one has the equality $Zq_n = \overline{zU}q_n = \overline{zq_nU}$, hence the set Zq_n is included in Z.

Now suppose that g_n is not in AU_1U_2. By Lemmas 2.3 and 3.1, there exist sequences $s_n \in T(z_1)$ and $t_n \in \mathbb{R}$ such that, after passing to a subsequence, the products $\widetilde{u}_{-s_n}g_n\widetilde{u}_{t_n}$ converge to an element $q \neq e$ in $AU_2 \cap G_0$. Since the elements $z\widetilde{u}_{s_n}$ belong to Z^*, they have a limit point $z' \in Z^*$. Since we have

$$z'q = \lim_{n\to\infty} z\widetilde{u}_{s_n}(\widetilde{u}_{-s_n}g_n\widetilde{u}_{t_n}) = \lim_{n\to\infty}(zg_n)\widetilde{u}_{t_n},$$

the element $z'q$ belongs to Z. We conclude as in the first case that the set $Zq = \overline{z'qU}$ is included in Z. $\square$

Proposition 3.6. *There exist an element $z \in \overline{xH}$ and a one-parameter semi-group $U_2^+ \subset U_2$ such that $zU_2^+ \subset \overline{xH}$.*

Proof. By Proposition 3.5 there exists a one-parameter semigroup $L^+ \subset AU_2$ such that $ZL^+ \subset Z$. This semigroup L^+ is equal to one of the following: U_2^+, A^+ or $v_0^{-1}A^+v_0$ for some element $v_0 \in U_2 \smallsetminus \{e\}$, where U_2^+ and A^+ are one-parameter semigroups of U_2 and A respectively.

When $L^+ = U_2^+$, our claim is proved.

Suppose now $L^+ = A^+$. By Lemma 3.3 there exists an element $v \in U_2 \smallsetminus \{e\}$ such that $Zv \subset \overline{xH}$. Then one has the inclusions

$$ZA^+vA \subset ZvA \subset \overline{xH}A \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and a sequence $\widetilde{a}_{t_n} \in A^+$ going to ∞. Since $z'\widetilde{a}_{t_n}$ belong to Ω, after passing to a subsequence, the sequence $z'\widetilde{a}_{t_n}$ converges to a point $z \in \overline{xH} \cap \Omega$. Moreover, since the Hausdorff limit of the sets $\widetilde{a}_{-t_n}A^+$ is A, one has the inclusions

$$zAvA \subset \lim_{n\to\infty} z'\widetilde{a}_{t_n}(\widetilde{a}_{-t_n}A^+)vA = z'A^+vA \subset \overline{xH}.$$

Now by a simple computation, we can check that the set AvA contains a one-parameter semigroup U_2^+ of U_2, and hence the orbit zU_2^+ is included in $\overline{xH}$ as desired.

Suppose finally $L^+ = v_0^{-1}A^+v_0$ for some v_0 in $U_2 \smallsetminus \{e\}$. We can write $A^+ = \{\widetilde{a}_{\varepsilon t} : t \geq 0\}$ with $\varepsilon = \pm 1$ and $v_0 = (e, u_s)$ with $s \neq 0$. A simple computation shows that the set $U_2' := \{(e, u_{\varepsilon st}) : 0 \leq t \leq 1\}$ is included in $v_0^{-1}A^+v_0A$. Hence one has the inclusions

$$ZU_2' \subset Zv_0^{-1}A^+v_0A \subset ZA \subset \overline{xH}.$$

Choose a point $z' \in Z^*$ and let $z \in \overline{xH}$ be a limit of a sequence $z'\widetilde{a}_{-t_n}$ with t_n going to $+\infty$. Since the Hausdorff limit of the sets $\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n}$ is the semigroup $U_2^+ := \{(e, u_{\varepsilon st}) : t \geq 0\}$, one has the inclusions

$$zU_2^+ \subset \lim_{n\to\infty}(z'\widetilde{a}_{-t_n})\widetilde{a}_{t_n}U_2'\widetilde{a}_{-t_n} \subset \overline{ZU_2'A} \subset \overline{xH}. \qquad \square$$

3.3. **Conclusion.**

Proof of Theorem 1.2. Suppose that the orbit xH is not closed. By Proposition 3.6, the orbit closure $\overline{xH}$ contains an orbit zU_2^+ of a one-parameter subsemigroup of U_2. Since Γ_2 is cocompact in H_2, by Lemma 2.6, this orbit zU_2^+ is dense in zH_2. Hence we have the inclusions

$$X = zG = zH_2H \subset \overline{zU_2^+}H \subset \overline{xH}.$$

This proves the claim. □

Proof of Theorem 1.1. Let $x = [g]$ be a point of $X_2 = \Gamma_2\backslash H_2$. By replacing Γ_1 by $g^{-1}\Gamma_1 g$, we may assume without loss of generality that $g = e$. One deduces Theorem 1.1 from Theorem 1.2 thanks to the following equivalences:
The orbit $[e]H$ is closed (resp. dense) in $\Gamma\backslash G \Longleftrightarrow$
The orbit $\Gamma[e]$ is closed (resp. dense) in $G/H \Longleftrightarrow$
The product $\Gamma_2\Gamma_1$ is closed (resp. dense) in $\mathrm{PSL}_2(\mathbb{R}) \Longleftrightarrow$
The orbit $[e]\Gamma_1$ is closed (resp. dense) in $\Gamma_2\backslash \mathrm{PSL}_2(\mathbb{R})$. □

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