Geometric and Arithmetic Aspects of Homogeneous Dynamics

- X. During the spring 2015 at MSRI, one of the two programs focused on *Homogeneous Dynamics*. Can you explain what the participants in this program were interested in?

- Y. *Homogeneous Dynamics* is the study of the stochastic properties of the action of Lie groups on their homogeneous spaces.

- X. This does not look very concrete to me.

- Y. Quite the contrary! One of the aims of this topic is to solve concrete questions coming from arithmetic or geometry by using abstract tools that find their roots in ergodic theory.

- X. What kind of tools?

- Y. Ergodic theorems, mixing properties, invariant measures, entropy and so on. All these tools are applied to an element g of a group G acting on a homogeneous space G/Γ.

- X. This is very abstract! Can you show me a simple example?

- Y. The **first example** is **the continued fraction expansion**.
Start with an irrational real number x_0 and write $x_0 = [a_0, a_1, a_2, a_3, ...]$, where this sequence of integers a_i is constructed as follows: let a_0 be the integer part of x_0, let x_1 be the inverse of the fractional part of x_0, let a_1 be the integer part of x_1 and so on.

- X. I remember that Euler and Lagrange proved that this sequence is periodic if and only if one can write $x_0 = a + b\sqrt{d}$ for some non-square positive integer d and some rational numbers a and b. For instance, one has $\frac{1}{2} + \frac{1}{2}\sqrt{5} = [1, 1, 1, 1, ...]$, $1+\sqrt{2} = [2, 2, 2, 2, ...]$, $\frac{1}{2} + \frac{1}{2}\sqrt{3} = [1, 2, 1, 2, ...]$.

- Y. You are right. But here is an open question. Let d be a non-square positive integer, do there exist two rational numbers a and b for which *the continued fraction of $a + b\sqrt{d}$ contains only 1's and 2's*. Can you guess the answer for $d = 7$?

- X. One moment please... Using my computer, I find that $\frac{5}{8} + \frac{3}{8}\sqrt{7}$ has a periodic continued fraction whose period is $(1, 1, 1, 1, 1, 1, 1, 2, 1, 2)$. But I do not see any homogeneous dynamics in this question.

- Y. This question is related to the excursions of the geodesic flow on the modular surface. This flow is one of the main source of inspiration in *Homogeneous Dynamics*.

- X. Do you have an example with nice pictures?

- Y. The **second example** is **the counting of integer points**.
Do you recognize these four curves?

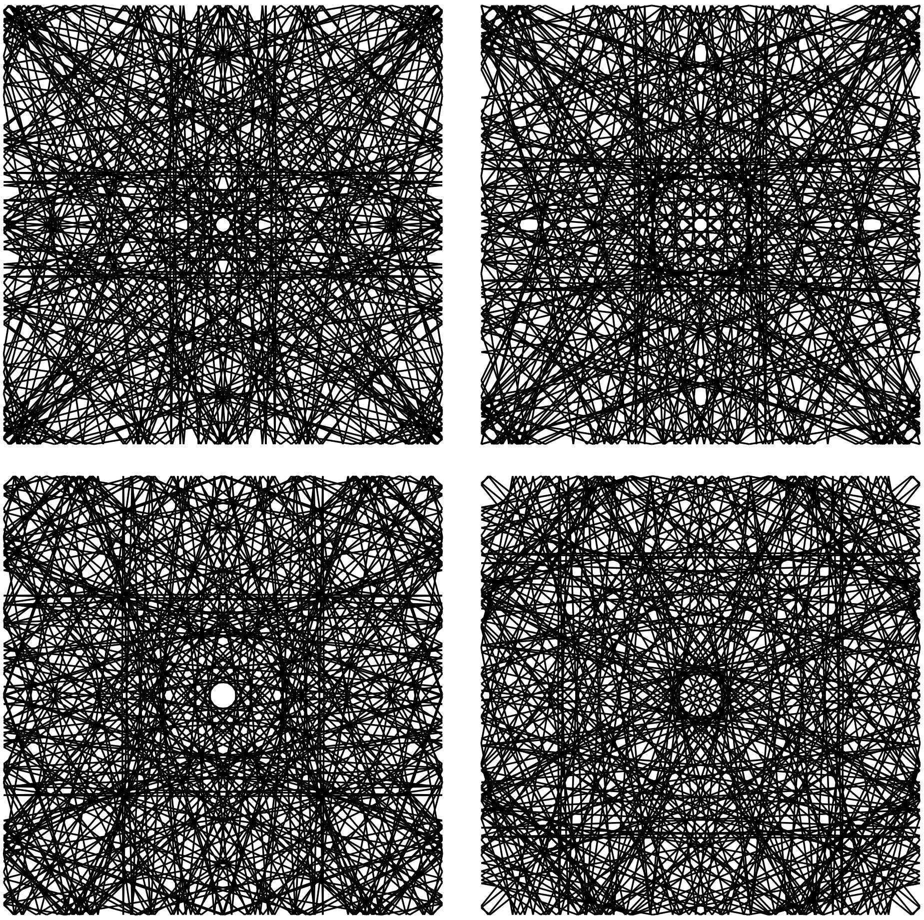

Figure 1: Can you see the circle in these four pictures?

- X. ???? Those are both nice and messy. The equation might be both subtle and complicated :o(

- Y. Not at all. Each one of these four curves is just a circle, whose radius R is approximately 50! I overlaid all that one sees through a square window of side length 1 successively centered at the integer points in the plane :o)

- X. This reminds me that Gauss proved that the number of integer points inside this circle is approximately πR^2 with an error term bounded by $2\pi R$.

- Y. You are right. But it is not known whether this error term is $O(R^{\frac{1}{2}+\varepsilon})$. One knows that these circles become equidistributed in the square and one controls their speed of equidistribution.

- X. Does this equidistribution property help to find the best error term in Gauss approximation?

- Y. Not quite. But for the analogous question with tilings in the hyperbolic space, the equidistribution of large spheres allows also to obtain *counting results similar to those of Gauss*. The proof relies on the ergodic properties of the horocyclic flow which is the second main source of inspiration in *Homogeneous Dynamics*.

- X. Do you have a simpler example?

- Y. The **third example** is the $\times 2 \times 3$ **question**.
One starts with an irrational number x and one denotes by $\{x\}$ its fractional part which is x minus its integer part. One looks at the n^2 points $\{2^p 3^q x\}$ where p and q vary between 1 and n. Here is an open question. Do *these sets of points become equidistributed in the interval* $[0,1]$ for large n.

- X. The word *equidistributed* seems important. What does it mean?

- Y. Here it means that the proportion of points in these sets that belong to a given interval $I \subset [0,1]$ is approximatively equal to the volume of I, with an error term going to 0 when n grows.

- X. You mean the length of I not the volume?

- Y. Correct! But in the interval $[0,1]$ the words *length*, *volume*, *mass*, and *probability* are synonymous! Don't tell that to a physicist!

- X. This reminds me of a theorem of Borel : almost all real numbers x are *normal*, which means that the sets of points $\{10^p x\}$ with $p \leq n$ become equidistributed in the interval $[0,1]$ for large n.

- Y. Here we insist on the equidistribution being true for all irrational numbers x. This question is also an important source of inspiration in *homogeneous dynamics* where pairs of commuting transformations often occur.

- X. I guess that in your example the pair is the two maps $x \mapsto \{2x\}$ and $x \mapsto \{3x\}$. But these two maps are not invertible transformations of the circle $\mathbb{R}/\mathbb{Z}$. Isn't it a problem?

- Y. We force them to be invertible. We replace this circle by a *solenoid*: each point x of the circle is replaced by the Cantor set of all its possible predecessors!

- X. You mean that this guy x knows its future but has forgotten its past and you force him to remember it!

- Y. Kind of. Mathematically this solenoid is a compact homogeneous space G/Γ where G is the product $G = \mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_3$, the $\mathbb{Q}_p$'s being the p-adic fields and Γ the diagonal subgroup $\mathbb{Z}[\frac{1}{6}]$. The use of these local fields is another important feature in *Homogeneous Dynamics*.

- X. Do you have another example with nice pictures?

- Y. The **fourth example** is **the equidistribution of lattices**.

- X. By a lattice you mean a subgroup of $\mathbb{R}^d$ generated by a basis of $\mathbb{R}^d$ as for instance the lattice $\mathbb{Z}^d$ of integer points in $\mathbb{R}^d$?

- Y. Exactly. We will assume $d = 2$ to make it simple. We will focus on the shape of the lattices, not on their size. Thus we consider as equal two lattices which are images of one another by a homothety. Do you know how to parametrize the set X of these lattices Λ?

- X. Yes, one has $X = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. After a homothety and a rotation by the angle $a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, one can assume that the vector $(1,0)$ is one of the shortest non-zero vectors of Λ and one let (x,y) with $y > 0$ be a shortest non-horizontal vector of Λ. This vector is in the strip given by $|x| \leq \frac{1}{2}$ and $x^2 + y^2 \geq 1$.

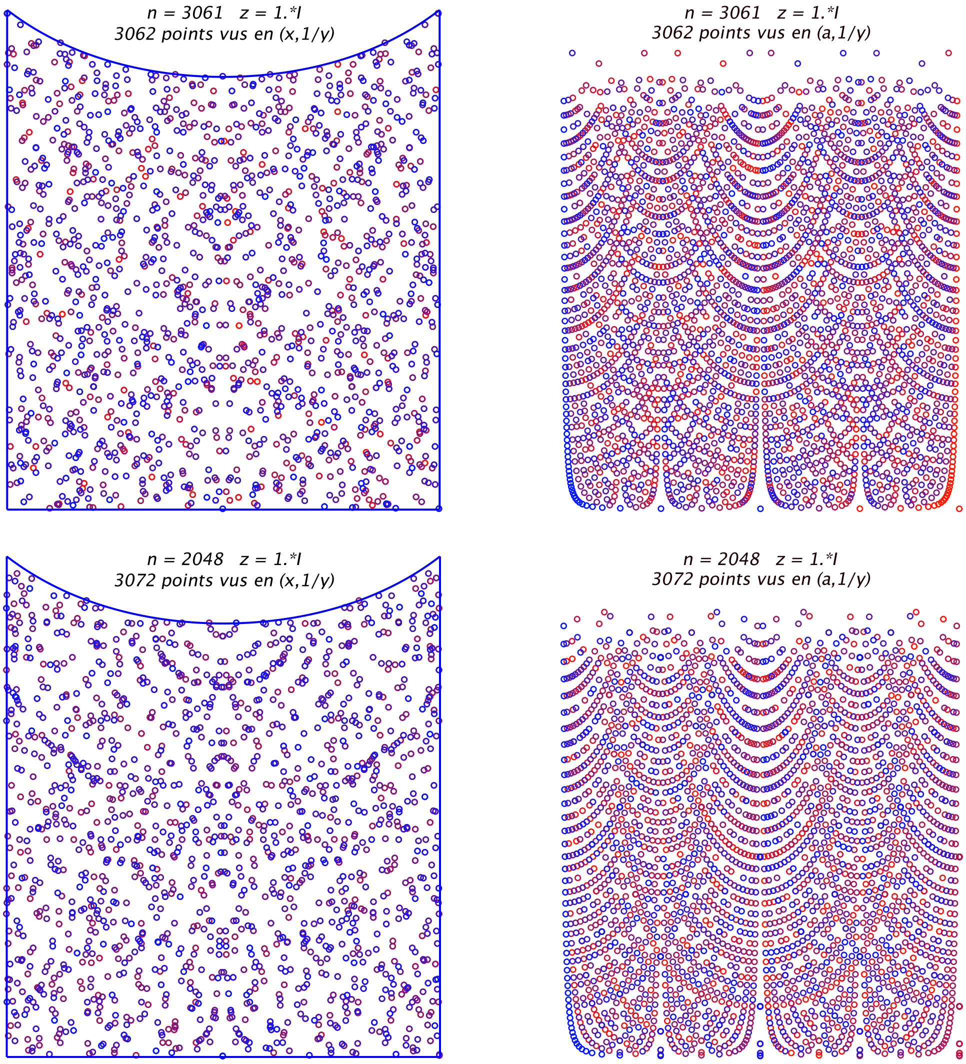

Figure 2: Equidistribution of finite index lattices

- Y. Very good. We will use (a, x, y') with $y' = 1/y$ as parameters for Λ so that our set of parameters is bounded. Now we consider the set F_n of sublattices Λ of index n in $\mathbb{Z}^2$. This is a finite set. Do you know why?

- X. Yes, Λ must contain $n\mathbb{Z}^2$ and is determined by its image in $(\mathbb{Z}/n\mathbb{Z})^2$.

- Y. The theorem is that *these sets F_n become equidistributed in X for large n*, and, more precisely, one can bound effectively the error term.

- X. What does *equidistributed* mean here?

- Y. The same as before: the proportion of points of F_n that belong to a given ball B in X converges to $m(B)/m(X)$ where m is the measure $m = \mathrm{d}a\,\mathrm{d}x\,\mathrm{d}y'$. This is illustrated in the two upper pictures where n is the prime number $n = 3061$. First for the parameters (x, y') then for (a, y').

- X. Why did you draw almost the same pictures twice?

- Y. The lower pictures show that this equidistribution occurs also with a non-prime integer n. Here $n = 2048$ is a power of 2. To be precise, we have only drawn here the sublattices $\Lambda \in F_n$ that are not included in $2\mathbb{Z}^2$.

- X. Can you explain the nice structures on the right-hand side pictures?

- Y. Here is a hint: the vectors $\sqrt{ny'}(\cos a, \sin a)$ belong to $\mathbb{Z}^2$.

- X. I guess the homogeneous dynamics hidden in this example is again the geodesic flow on the modular surface...

- Y. Not quite! The homogeneous space here is $\mathrm{SL}(2, \mathbb{A})/\mathrm{SL}(2, \mathbb{Q})$ where $\mathbb{A}$ is the ring of adèles of $\mathbb{Q}$.

- X. Why does one need such a strange ring?

- Y. The ring of adèles is a very natural object: it is a locally compact ring that contains $\mathbb{Q}$ as a discrete subring and such that the quotient $\mathbb{A}/\mathbb{Q}$ is both compact and connected.

- X. It behaves like the field $\mathbb{R}$ of real numbers for the ring $\mathbb{Z}$ of integers!

- Y. Precisely. Another key tool in the proof is the *uniform mixing* property, also called *spectral gap* or *uniform decay of matrix coefficients.*

- X. This looks tough... Do you have a simpler example?

- Y. The **fifth example** is **the normal subgroup theorem**.
I will just describe a special case of this theorem. Consider a finite dimensional division algebra L over $\mathbb{Q}$ whose center is equal to $\mathbb{Q}$.

- X. You mean like the quaternion algebras over $\mathbb{Q}$.

- Y. The quaternion algebras are those L for which $\dim_{\mathbb{Q}} L = 4$. The dimension $\dim_{\mathbb{Q}} L$ is always a square d^2. Here we will assume $d \geq 3$.

- X. But the quaternion algebras are the only examples I know!

- Y. Yet, there are many others. Indeed, these division algebras L are described by the so-called Brauer group of $\mathbb{Q}$.

- X. The very Brauer group which plays a role in the class field theory?

- Y. Yes. Now choose a basis of L in which the multiplication of L has integer coefficients and let Γ be the multiplicative subgroup of $L \smallsetminus \{0\}$ whose elements and their inverses have integer coordinates.

- X. This group Γ is a non-commutative analogue of the group of units in a number field. Is this group Γ infinite as in Dirichlet's units theorem?

- Y. Yes for $d \geq 3$. Indeed, Γ is a discrete cocompact subgroup of the group $G = \mathrm{SL}(d, \mathbb{R})$. One wants to describe the normal subgroups of Γ.

- X. This group Γ cannot be simple because a congruence condition like being equal to 1 modulo n defines a finite index normal subgroup of Γ.

- Y. Exactly. The theorem says that *the normal subgroups of* Γ *are either finite or have finite index in* Γ.

- X. You mean Γ is almost simple! What happens for the quaternion division algebras?

- Y. In this case the group Γ is either finite or a finite extension of the fundamental group of a higher genus surface. It has lots of normal subgroups.

- X. I guess the homogeneous space in this example is G/Γ.

- Y. Yes. But another important homogeneous space in this context is the so-called flag variety $\mathcal{F}$ of G. One of the key points in the proof is to classify the Γ-invariant sub-σ-algebras of the Lebesgue σ-algebra of $\mathcal{F}$.

- X. Do you have another example with nice pictures?

- Y. Yes, many! The Apollonian circles, the integer points on spheres, the gaps in $\sqrt{n}$ modulo one, the random walks on tori, the space of quasicristals, the irrational quadratic forms ... but we are running out of time.

- X. Thanks for your answers. How can I learn more on this topic?

- Y. It's up to you to decide. Some PhD students first study either the Margulis arithmeticity theorem or the Ratner classification theorem. Others focus directly on one of the many concrete remaining open questions.

- X. Like the ones you explained to me. Where did you find these five examples?

- Y. The first one is due to McMullen, the second and fifth to Margulis, the third to Furstenberg, and the fourth to Clozel, Oh and Ullmo.

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