Harmonic measures on negatively curved manifolds

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Abstract

We prove that the harmonic measures on the spheres of a pinched Hadamard manifold admit uniform upper and lower bounds.

1 Introduction

Let X be a Hadamard manifold. This means that X is a complete simply connected Riemannian manifold of dimension $k \geq 2$ whose curvature is non-positive $K_X \leq 0$. For x in X and $R > 0$, let $\sigma_{x,R}$ be the harmonic measure on the sphere $S(x,R)$. We refer to Section 3.1 for a precise definition of $\sigma_{x,R}$. The aim of these notes is to give uniform upper and lower bounds for these harmonic measures $\sigma_{x,R}$.

Theorem 1.1. *Let $0 < a < b$ and $k \geq 2$. There exist positive constants M, N depending solely on a, b, k such that for all k-dimensional Hadamard manifolds X with pinched curvature $-b^2 \leq K_X \leq -a^2$, all points x in X, all radius $R > 0$ and all angles $\theta \in [0, \pi/2]$ one has*

$$\tfrac{1}{M}\, \theta^N \leq \sigma_{x,R}(C_x^\theta) \leq M\, \theta^{\frac{1}{N}} \tag{1.1}$$

where C_x^θ stands for any cone with vertex x and angle θ.

These inequalities (1.1) play a crucial role in the extension of the main result of [4] from rank one symmetric spaces to Hadamard manifolds. Indeed, using (1.1), we prove in [3] that *any quasi-isometric map between pinched Hadamard manifolds is within bounded distance of a unique harmonic map.* The key point in (1.1) for this application is the fact that the constants M and N do not depend on x nor R.

The proof of Theorem 1.1 relies on various technical tools of the potential theory on pinched Hadamard manifolds : the Harnack inequality, the barrier functions constructed by Anderson and Schoen in [2], and upper and lower bounds for the Green functions due to Ancona in [1]. Related estimates

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are available, like the one by Kifer–Ledrappier in [6, Theorem 3.1 and 4.1] where (1.1) is proven for the sphere at infinity, or by Ledrappier–Lim in [7, Proposition 3.9] where the Hölder regularity of the Martin kernel is proven. Our approach also gives a non probabilistic proof of the Kifer–Ledrappier estimates.

Here is the organization of this paper. In Chapter 2, we collect basic facts on Hadamard manifolds and their harmonic functions. In Chapter 3, we prove uniform estimates for the normal derivative of the Green functions. In Chapter 4 and 5 we prove successively the upper bound and the lower bound in (1.1). We have postponed to Chapter 6 the proof of a few purely geometric estimates on Hadamard manifolds that were needed in the argument.

2 Pinched Hadamard manifolds

In this chapter, we collect preliminary results on Hadamard manifolds and their harmonic functions.

Let X be a Hadamard manifold. For instance, the Euclidean space $\mathbb{R}^k$ is a Hadamard manifold with zero curvature $K_X = 0$, and the real hyperbolic space $\mathbb{H}^k$ is a Hadamard manifold with constant curvature $K_X = -1$. We will say that X is pinched if there exist constants $a, b > 0$ such that

$$-b^2 \leq K_X \leq -a^2 < 0.$$

For instance, the non-compact rank one symmetric spaces are pinched Hadamard manifolds.

2.1 Laplacian and subharmonic functions

We introduce a few subharmonic functions on X or on open subsets of X which will play the role of *barriers* in the following chapters.

When o is a point in X, we denote by ρ_o the distance function defined by $\rho_o(x) = d(o, x)$ for x in X. When $F : X \to \mathbb{R}$ is a continuous function, we denote by ΔF its Laplacian. In local coordinates $(x_1, \ldots, x_k)$ of X, denoting the coefficients of the metric tensor by g_{ij} and the volume density by $v = \sqrt{\det(g_{ij})}$, one has

$$\Delta F = v^{-1} \partial_{x_i} (v g^{ij} \partial_{x_j} F).$$

A real valued function F on X is *harmonic* if $\Delta F = 0$, *subharmonic* if $\Delta F \geq 0$ and *superharmonic* if $\Delta F \leq 0$.

Lemma 2.1. *Let X be a Hadamard manifold and $o \in X$. The Laplacian of the distance function ρ_o satisfies*

$$\Delta\rho_o \geq (k-1)\rho_o^{-1}\,,$$

where k is the dimension of X.

If $K_X \leq -a^2 < 0$, then one has

$$\Delta\rho_o \geq (k-1)\,a\,\coth(a\,\rho_o) \tag{2.1}$$

and if $-b^2 \leq K_X \leq 0$, then one has

$$\Delta\rho_o \leq (k-1)\,b\,\coth(b\,\rho_o)\,. \tag{2.2}$$

These classical inequalities mean that the difference is a positive measure. See [2, Section 2] and [4, Lemma 3.2].

The following corollary provides useful barriers on X.

Corollary 2.2. *Let X be a Hadamard manifold and $o \in X$.*
a) For $K_X \leq -a^2 < 0$ and $0 < m_0 \leq (k-1)\,a$,
the function $e^{-m_0\rho_o}$ is superharmonic on X.
b) For $-b^2 \leq K_X \leq 0$ and $M_0 \geq (k-1)\,b\,\coth(b/4)$,
the function $e^{-M_0\rho_o}$ is subharmonic on $X \smallsetminus B(o,\frac{1}{4})$.

Proof. For a smooth function $f : [0,\infty[\to \mathbb{R}$ one has

$$\Delta(f\circ\rho_o) = f''\circ\rho_o + (f'\circ\rho_o)\Delta\rho_o.$$

Therefore, for $\tau > 0$, one has

$$\Delta(e^{-\tau\rho_o}) = (\tau - \Delta\rho_o)\,\tau\,e^{-\tau\rho_o}.$$

a) Using (2.1), one gets
$$\Delta(e^{-m_0\rho_o}) \leq (m_0 - (k-1)\,a\,\coth(a\,\rho_o))\,m_0\,e^{-m_0\rho_o} \leq 0.$$
b) Using (2.2), one gets outside the ball $B(o,\frac{1}{4})$,
$$\Delta(e^{-M_0\rho_o}) \geq (M_0 - (k-1)\,b\,\coth(b\,\rho_o))\,M_0\,e^{-M_0\rho_o} \geq 0.$$ □

2.2 Anderson–Schoen barrier

Another very useful barrier is the following function u introduced by Anderson and Schoen in [2].

We denote by ∂X the visual boundary of the Hadamard manifold X. For each point w in X, this boundary is naturally identified with the set of geodesic rays $w\xi$ starting at w. For $0 < \theta \leq \frac{\pi}{2}$ we denote by $C^\theta_{w\xi}$ the closed cone with axis $w\xi$ and angle θ : it is the union of all the geodesic rays $w\eta$ with vertex w and whose angle with the ray $w\xi$ is at most θ. Two geodesic rays with vertex w are said to be *opposite* if their union is a geodesic i.e. if their angle is equal to π.

Lemma 2.3. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. There exist constants $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every two opposite geodesic rays $w\xi^+$ and $w\xi^-$ with the same vertex $w \in X$, there exists a positive superharmonic function u on X such that :*

$$\begin{aligned} u(x) &\geq 1 && \text{for all } x \text{ in the cone } C^{\pi/2}_{w\xi^+} && (2.3) \\ u(x) &\leq C_0\, e^{-\varepsilon_0 d(w,x)} && \text{for all } x \text{ on the ray } w\xi^-. && (2.4) \end{aligned}$$

This function $u = u_{w\xi^+}$ will be called the *Anderson–Schoen barrier* for the ray $w\xi^+$.

Proof. See [2, Proof of Theorem 3.1]. We briefly sketch the construction of the function u. As shown in Figure 1, let o be the point at distance 1 from w on the geodesic ray $w\xi^-$. Choose a non negative continous function u_0

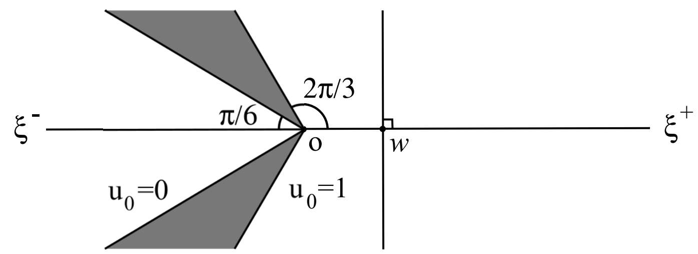

Figure 1: Construction of the Anderson–Schoen barrier.

on $X \smallsetminus o$ which is constant on the rays with vertex o, which is equal to 1 on the cone $C^{2\pi/3}_{o\xi^+}$, and equal to 0 on the cone $C^{\pi/6}_{o\xi^-}$ as in Figure 1. Then, consider a function u_1 obtained by smoothly averaging u_0 on balls of radius 1. It is given by

$$u_1(x) = \frac{\int_X \chi(d(x,y))u_0(y)dy}{\int_X \chi(d(x,y))dy}$$

where dy is the Riemannian measure on X and $\chi \in \mathcal{C}^\infty(\mathbb{R})$ is an even positive function whose support is $[-1,1]$. This function u_1 has the expected behavior (2.3) and (2.4) and its second covariant derivative decays exponentially at infinity. Therefore using the same computation as in Corollary 2.2, one can find explicit constants $\varepsilon_0 > 0$ and $C_0' > 0$ depending only on a, b and k such that the function $u := u_1 + C_0' e^{-\varepsilon_0 \rho_w}$ is superharmonic. This is the required function u. $\square$

2.3 Harnack-Yau inequality

We state without proof a version of Harnack inequality due to Yau in [9].

Lemma 2.4. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq 0$. There exists a constant $C_1 = C_1(k,b)$ such that for every open set $\Omega \subset X$ and every positive harmonic function $u : \Omega \to]0,\infty[$, one has*

$$\|D_x \log u\| \leq C_1 \quad \text{for all } x \text{ in } X \text{ with } d(x,\partial\Omega) \geq 1. \tag{2.5}$$

This lemma is true for any complete Riemannian manifold whose Ricci curvature is bounded below. A short proof has been written by Peter Li and Jiaping Wang in [8, Lemma 2.1].

3 Green functions

We collect in this chapter various estimates for the Green functions on Hadamard manifolds. We also explain why the Green functions are useful to compute the harmonic measures.

3.1 Harmonic measures

We first recall the definition of the harmonic measures.

Since X is a Hadamard manifold, for each x in X, the exponential map $\exp_x : T_xX \to X$ is a $\mathcal{C}^\infty$-diffeomorphism. In particular, for all $R > 0$, the sphere $S(x,R)$ is a $\mathcal{C}^\infty$-submanifold of X. Solving the Dirichlet problem on the closed ball $B(x,R)$ gives rise to a family of Borel probability measures $\sigma^y_{x,R}$ on $S(x,R)$ indexed by $y \in B(x,R)$. These measures are called *harmonic measures*. Indeed, for every continuous function f on the sphere $S(x,R)$, there exists a unique continuous function h_f on the ball $B(x,R)$ such that

$$\Delta h_f = 0 \;\text{ in }\; \mathring{B}(x,R) \quad \text{and} \quad h_f = f \;\text{ in }\; S(x,R). \tag{3.1}$$

The map $f \mapsto h_f(y)$ is then a probability measure $\sigma^y_{x,R}$ on $S(x,R)$. This probability measure is defined by the equality

$$h_f(y) = \textstyle\int_{S(x,R)} f(\eta)\, \mathrm{d}\sigma^y_{x,R}(\eta). \tag{3.2}$$

The harmonic measures that occur in Theorem 1.1 correspond to $y = x$, that is $\sigma_{x,R} := \sigma^x_{x,R}$. Our aim is to prove the bound (1.1) for these measures.

Remark 3.1. *When X is the hyperbolic space $\mathbb{H}^k$, and more generally when X is a rank one symmetric space, the harmonic measure $\sigma_{x,R}$ is a multiple of the Riemannian measure $A_{x,R}$ on the sphere $S(x,R)$. But for a Hadamard manifold X these two measures are not always proportional.*

Remark 3.2. *When solving the Dirichlet problem on the visual compactification $\overline{X} = X \cup \partial X$ one gets a family of probability measures $(\sigma^y_\infty)_{y\in X}$ on ∂X which are also called harmonic measures. See [2, Theorem 3.1].*

3.2 Estimating the Green functions

Before begining the proof of Theorem 1.1, we recall the definition and a few estimates of the Green functions.

For any ball $B(x, R)$ in X and any point y in the interior $\mathring{B}(x, R)$, one denotes by $G^y_{x,R}$ the corresponding Green function. It is the unique function on the ball $B(x, R)$ which is continuous outside y and such that

$$\Delta G^y_{x,R} = -\delta_y \ \text{ in } \ \mathring{B}(x, R) \quad \text{and} \quad G^y_{x,R} = 0 \ \text{ on } \ S(x, R). \tag{3.3}$$

When X is pinched, for any point y in X, one denotes by G^y_∞ the corresponding Green function on X. It is the unique function on X which is continuous outside y and such that

$$\Delta G^y_\infty = -\delta_y \quad \text{and} \quad \lim_{z\to\infty} G^y_\infty(z) = 0.$$

These Green functions $G^y_{x,R}$ and G^y_∞ are positive.

We now state various classical estimates for the Green functions on Hadamard manifolds.

The first lemma gives a uniform estimate for a fixed radius R_0.

Lemma 3.3. *Let X be a Hadamard manifold with $-b^2 \le K_X \le 0$. For each $R_0 \ge 1$, there exist constants $C_2 > c_2 > 0$ such that, for any x in X :*

$$c_2 \ \le \ G^x_{x,R_0}(z) \ \le \ C_2 \quad \text{for all } \ z \in S(x, \tfrac{1}{2}). \tag{3.4}$$

Proof. This is a special case of [5, Theorem 11.4]. □

The second lemma, due to Ancona in [1], provides estimates for the Green functions which are uniform in the radius R under pinching conditions.

Lemma 3.4. *Let X be a Hadamard manifold with $-b^2 \le K_X \le -a^2 < 0$.*
a) There exist constants $C'_2 > c'_2 > 0$ such that for any $R > 0$, x in X and y in $\mathring{B}(x, R)$ with $d(x, y) \le R-1$, one has :

$$c'_2 \ \le \ G^y_{x,R}(z) \ \le \ C'_2 \quad \text{for all } \ z \in S(y, \tfrac{1}{2}). \tag{3.5}$$

Similarly, for any y in Y, one has :

$$c'_2 \ \le \ G^y_\infty(z) \ \le \ C'_2 \quad \text{for all } \ z \in S(y, \tfrac{1}{2}).$$

b) One can also choose these constants c'_2, C'_2 such that, for any y in X :

$$c'_2\, e^{-M_0 d(y,z)} \ \le \ G^y_\infty(z) \ \le \ C'_2\, e^{-m_0 d(y,z)} \quad \text{for all } \ z \in X \smallsetminus B(y, \tfrac{1}{2}).$$

Proof of Lemma 3.4. a) For the lower bound : by the maximum principle, one has $G^y_{x,R} \ge G^y_{y,2}$ and one uses (3.4). For the upper bound : one has $G^y_{x,R} \le G^y_\infty$ and the bounds for G^y_∞ are in [1, Prop. 7].

b) One uses the estimation of G^y_∞ on the sphere $S(y, \frac{1}{2})$ given in *a*), the barriers given in Corollary 2.2 and the maximum principle. See [1, Remark 2.1 p. 505] for more details. □

3.3 Bounding above the gradient of the Green functions

We explain why we are interested in bounding the gradient of the Green functions, and prove such an upper bound.

Combining equalities (3.1) and (3.3) with Green formula, one gets the equality

$$h_f(y) = \int_{S(x,R)} f(\eta)\, \frac{\partial G^y_{x,R}}{\partial n}(\eta)\ \mathrm{d}A_{x,R}(\eta), \tag{3.6}$$

where $\frac{\partial G}{\partial n} := \mathrm{grad} G.\vec{n}$ denotes the derivative of G in the direction of the inward normal vector $\vec{n}$ to the sphere $S(x,R)$, and where $A_{x,R}$ denotes the Riemannian measure on this sphere. Compared with Formula (3.2), this gives a formula for the harmonic measure :

$$\sigma^y_{x,R} = \frac{\partial G^y_{x,R}}{\partial n}\, A_{x,R}\,. \tag{3.7}$$

The following two lemmas provide a uniform upper bound for this normal derivative when y and η are not too close.

The first lemma gives uniform estimates for a fixed radius R_0.

Lemma 3.5. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq 0$. For each $R_0 > 0$, there exists $C_3 > 0$ such that for any $x \in X$, $\eta \in S(x, R_0)$, one has*

$$\frac{\partial G^x_{x,R_0}}{\partial n}(\eta)\ \leq\ C_3\,. \tag{3.8}$$

The second lemma gives estimates which are uniform in the radius R under a pinching condition.

Lemma 3.6. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. There exists $C'_3 > 0$ such that for $R \geq 1$, $x \in X$, $y \in \mathring{B}(x,R)$, $\eta \in S(x,R)$:*

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta)\ \leq\ C'_3 \qquad \text{as soon as}\ \ d(y,\eta) \geq 1. \tag{3.9}$$

Proof of Lemmas 3.5 and 3.6. The proofs of these two lemmas are the same, except that they rely either on Lemma 3.3 or on Lemma 3.4. We will only prove Lemma 3.6.

The strategy is to construct an explicit superharmonic function F on the ball $B(x,R)$ such that $F(\eta) = 0$, such that $F \geq G^y_{x,R}$ in a neighborhood of η, and whose normal derivative at η is uniformly bounded.

As shown in Figure 2.A, we introduce the point y_0 on the ray $x\eta$ such that $d(x,y_0) = R + \frac{1}{3}$. By construction one has $d(\eta, y_0) = \frac{1}{3}$.

We will choose the function F to be

$$F(z) = C_4(e^{-M_0/3} - e^{-M_0\, d(y_0,z)})\,,$$

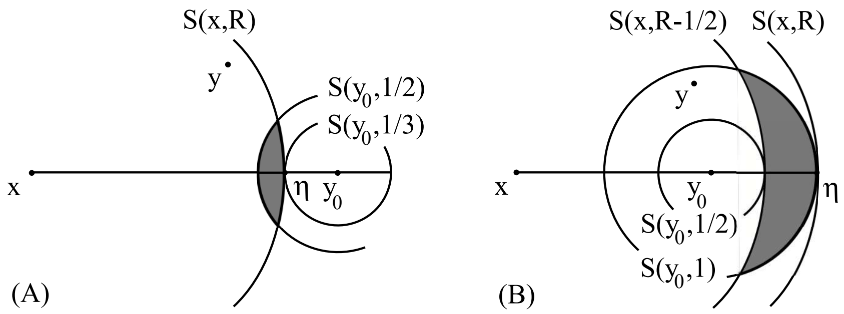

Figure 2: Proofs of uniform majoration and minoration for $\dfrac{\partial G^y_{x,R}}{\partial n}(\eta)$.

for a constant C_4 that we will soon determine.

We first notice that, according to Lemma 2.2,

$$F \text{ is a positive superharmonic function on the ball } B(x,R). \tag{3.10}$$

Moreover, since the point $y \in B(x,R)$ satisfies $d(y,\eta) \geq 1$ and since $K_X \leq 0$, one must have $d(y,y_0) \geq 1$. We now bound uniformly the Green function $G^y_{x,R}$ on the sphere $S(y_0,\frac{1}{2})$. By the maximum principle, one has

$$G^y_{x,R}(z) \leq G^y_{x,R+1}(z) \quad \text{ for all } z \text{ in } B(x,R). \tag{3.11}$$

Moreover, using the bound (3.5) in Lemma 3.4, one gets

$$G^y_{x,R+1}(z) \leq C_2' \quad \text{ for all } z \text{ in } S(y,\tfrac{1}{2}). \tag{3.12}$$

Combining (3.11) and (3.12) with the maximum principle, one infers that

$$G^y_{x,R}(z) \leq C_2' \quad \text{ for all } z \text{ in } B(x,R) \smallsetminus B(y,\tfrac{1}{2}).$$

In particular, one has

$$G^y_{x,R} \leq F \quad \text{on} \quad S(y_0,\tfrac{1}{2}) \cap B(x,R) \tag{3.13}$$

for the choice of the constant

$$C_4 := \frac{C_2'}{e^{-M_0/3} - e^{-M_0/2}}.$$

Combining (3.10), (3.13) and the maximum principle it follows that, on the grey zone of Figure 2.A :

$$G^y_{x,R} \leq F \quad \text{on} \quad B(y_0,\tfrac{1}{2}) \cap B(x,R).$$

Therefore, one has the inequality between the normal derivatives

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta) \;\leq\; \frac{\partial F}{\partial n}(\eta) \;=\; \frac{C_2'\, M_0}{1-e^{-M_0/6}}.$$

This proves the bound (3.9). $\square$

3.4 Bounding below the gradient of the Green functions

We will also need a lower bound for the gradient of the Green functions.

The following two lemmas provide a uniform lower bound for the normal derivative when y is not too far from η and not too close to the sphere.

The first lemma gives uniform estimates for a fixed radius R_0.

Lemma 3.7. *Let X be a Hadamard manifold with $-b^2 \le K_X \le 0$. For each $R_0 \ge 1$, there exists $c_3 > 0$ such that for $x \in X$, $\eta \in S(x, R_0)$, one has*

$$\frac{\partial G^x_{x,R_0}}{\partial n}(\eta) \;\ge\; c_3\,. \tag{3.14}$$

The second lemma gives estimates which are uniform in the radius R under a pinching condition.

Lemma 3.8. *Let X be a Hadamard manifold with $-b^2 \le K_X \le -a^2 < 0$. There exists $c'_3 > 0$ such that for $R \ge 1$, $x \in X$, $y \in B(x, R-1)$, $\eta \in S(x, R)$:*

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta) \;\ge\; c'_3 \quad \text{as soon as } \; d(y, \eta) \le 4. \tag{3.15}$$

Proof of Lemmas 3.7 and 3.8. The proofs of these two lemmas are the same, except that they rely either on Lemma 3.3 or on Lemma 3.4. We will only prove Lemma 3.8.

The strategy is as in Section 3.3. We construct a subharmonic function f on the ball $B(x, R)$ such that $f(\eta) = 0$, such that $f \le G^y_{x,R}$ in a small ball tangent at η to the sphere $S(x, R)$, and whose normal derivative at η is uniformly bounded below.

As shown in Figure 2.B, we introduce the point y_0 on the ray $x\eta$ such that $d(x, y_0) = R - 1$. By construction one has $d(\eta, y_0) = 1$. We will choose the function f to be

$$f(z) = c_4(e^{-M_0\, d(y_0,z)} - e^{-M_0})\,,$$

for a constant c_4 that we will soon determine.

We first notice that, according to Lemma 2.2,

$$f \text{ is subharmonic outside } B(y_0, \tfrac{1}{2}) \text{ and } f \equiv 0 \text{ on } S(y_0, 1). \tag{3.16}$$

We now give a uniform lower bound for the Green function $G^y_{x,R}(w)$ for all points w in $S(x, R - \frac{1}{2}) \cap B(y_0, 1)$. Since $d(x, y) \le R-1$, we observe that it follows from Lemma 3.4 that, for all z in $S(y, \frac{1}{2})$:

$$G^y_{x,R}(z) \ge c'_2.$$

For $w \in S(x, R - \frac{1}{2}) \cap B(y_0, 1)$, pick $z \in S(y, \frac{1}{2})$ on the segment $[yw]$. Since the segment $[zw]$ is included in the ball $B(x, R - \frac{1}{2})$ and has length at most 6, it follows from Harnack inequality (2.5) that :

$$G^y_{x,R}(w) \geq c'_2 \, e^{-12\, C_1}.$$

This means that

$$G^y_{x,R} \geq f \quad \text{on} \quad S(x, R - \tfrac{1}{2}) \cap B(y_0, 1) \tag{3.17}$$

for the choice of the constant

$$c_4 := \frac{c'_2 \, e^{-12\, C_1}}{e^{-M_0/2} - e^{-M_0}}.$$

Combining (3.16), (3.17) and the maximum principle, one gets the bound on the grey zone of Figure 2.B

$$G^y_{x,R} \geq f \quad \text{on} \quad B(y_0, 1) \smallsetminus B(x, R - \tfrac{1}{2}).$$

Therefore, one has the inequality between the normal derivatives

$$\frac{\partial G^y_{x,R}}{\partial n}(\eta) \;\geq\; \frac{\partial f}{\partial n}(\eta) = \frac{c'_2 \, M_0 \, e^{-12\, C_1}}{e^{M_0/2} - 1}$$

This proves the bound (3.9). $\square$

4 Upper bound for the harmonic measures

The aim of this chapter is to prove the upper bound in (1.1).

We recall that X is a k-dimensional Hadamard manifold satisfying the pinching condition $-b^2 \leq K_X \leq -a^2 < 0$. Let x be a point in X. We will denote by ξ a point on the sphere $S(x, R)$, by $x\xi$ the ray with vertex x that contains ξ, and by $C^\theta_{x\xi}$ the cone with axis $x\xi$ and angle θ. We want to bound

$$\sigma^x_{x,R}(C^\theta_{x\xi}) \leq M \theta^{1/N} \tag{4.1}$$

where the constants M and N depend only on a, b and k. It is not restrictive to assume that $b = 1$. We will distinguish three cases, setting

$$\theta_R := 10^{-3} \, e^{-(R-2)} \quad :$$

⋆ Bounded radius : $R \leq 2$.
⋆ Large angle : $R \geq 2$ and $\theta \geq \theta_R$.
⋆ Small angle : $R \geq 2$ and $\theta \leq \theta_R$.
Without loss of generality, we may assume that $\theta \leq 10^{-3}$.

4.1 Upper bound for a bounded radius

We prove (4.1) when $R \leq 2$.

More precisely, when $R \leq 2$, we will prove the upper bound (4.1) under the weaker pinching condition $-1 \leq K_X \leq 0$. This allows us to multiply the metric by a ratio $2/R$, while preserving this pinching condition. Hence we can assume that the radius is $R_0 = 2$. Using the expression (3.7) for the density of the harmonic measure, the bound (3.8) for this density and the bound (6.7) for the volume $\mathrm{A}_{x,R_0}(C^\theta_{x\xi})$, we get

$$\sigma^x_{x,R_0}(C^\theta_{x\xi}) \;=\; \int_{C^\theta_{x\xi}} \frac{\partial G^x_{x,R_0}}{\partial n}(\eta)\, \mathrm{d}A_{x,R_0}(\eta) \;\leq\; C_3\, \mathrm{A}_{x,R_0}(C^\theta_{x\xi}) \;\leq\; C_3\, V_k\, \theta^{k-1}.$$

This proves (4.1) when $R \leq 2$.

4.2 Upper bound for a large angle

We prove (4.1) when $R \geq 2$ and $\theta \geq \theta_R$.

As shown in Figure 3.A, we introduce the point w on the ray $x\xi$ such that $d(x,w) = r$ where r is given by

$$\theta = 10^{-3}\, e^{-r}. \tag{4.2}$$

Since $\theta_R \leq \theta \leq 10^{-3}$, one has $0 \leq r \leq R-2$. In particular the point w is at distance at least 2 from every point η on the sphere $S(x,R)$. According to Lemma 6.1, since $4\, e^r\, \theta \leq \frac{\pi}{2}$, one has

$$C^\theta_{x\xi} \cap S(x,R) \subset C^{\pi/2}_{w\xi} \cap S(x,R)\,.$$

We now introduce the Anderson–Schoen barrier $u = u_{w\xi}$ for the ray $w\xi$, as

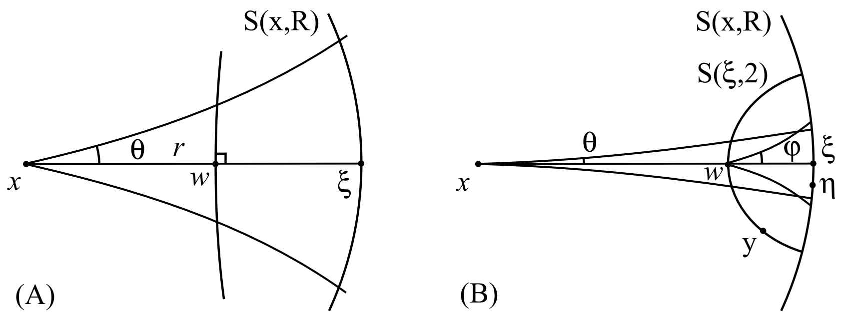

Figure 3: Majoration of $\sigma^x_{x,R}(C^\theta_{x\xi})$ for a large angle θ, and for a small angle θ.

constructed in Lemma 2.3. Since u is superharmonic, since $u \geq 0$ everywhere and since $u \geq 1$ on the cone $C_{w\xi}^{\pi/2}$, one infers from the maximum principle that, for all y in $\mathring{B}(x, R)$:

$$\sigma_{x,R}^{y}(C_{x\xi}^{\theta}) \leq \sigma_{x,R}^{y}(C_{w\xi}^{\pi/2}) \leq u(y).$$

Applying this equality with $y = x$, remembering the exponential decay (2.4) of the Anderson–Schoen barrier on the ray wx, and using (4.2), one gets,

$$\sigma_{x,R}^{x}(C_{x\xi}^{\theta}) \leq u(x) \leq C_0 e^{-\varepsilon_0 r} \leq 10^{3\varepsilon_0}\, C_0\, \theta^{\varepsilon_0}.$$

This proves (4.1) when $R \geq 2$ and $\theta \geq \theta_R$.

4.3 Upper bound for a small angle

We prove (4.1) when $R \geq 2$ and $\theta \leq \theta_R$. The argument will combine both the arguments used in Sections 4.1 and 4.2.

As shown in Figure 3.B, we introduce the point w on the ray $x\xi$ such that $d(x, w) = R - 2$, and the angle φ given by

$$\varphi := 4\, e^{R-2}\, \theta. \tag{4.3}$$

Since $\theta < \theta_R$, one has $\varphi \leq 1/100$. According to Lemma 6.1, one has

$$C_{x\xi}^{\theta} \cap S(x, R) \subset C_{w\xi}^{\varphi} \cap S(x, R)\,.$$

First step : We estimate the measure of the cone $C_{w\xi}^{\varphi}$ for the *harmonic measure* $\sigma_{x,R}^{y}$ *at a point* y within bounded distance from ξ.

Lemma 4.1. *Let X be a Hadamard manifold with $-1 \leq K_X \leq -a^2 < 0$. Keep the above notation $x \in X$, $\xi \in S(x, R)$, $w \in [x\xi]$ with $d(w, \xi) = 2$ and $\varphi \leq 1/100$ as in Figure 3.B. Then there exists a constant $C_5 > 0$ depending only on a, b, k such that for all y in $\mathring{B}(x, R) \cap S(\xi, 2)$ one has*

$$\sigma_{x,R}^{y}(C_{w\xi}^{\varphi}) \leq C_5\, \varphi^{k-1}\,. \tag{4.4}$$

Proof of Lemma 4.1. One uses again the expression (3.7) for the density of the harmonic measure. Since $\varphi \leq 1/100$, it follows from Lemma 6.3.a that, for all η in $C_{w\xi}^{\varphi} \cap S(x, R)$, one has $d(\xi, \eta) \leq 1$ hence $d(y, \eta) \geq 1$. Therefore the bound (3.9) is valid for this density. Hence one computes

$$\sigma_{x,R}^{y}(C_{w\xi}^{\varphi}) \;=\; \int_{C_{w\xi}^{\varphi}} \frac{\partial G_{x,R}^{y}}{\partial n}(\eta)\, \mathrm{d}A_{x,R}(\eta) \;\leq\; C_3'\, \mathrm{A}_{x,R}(C_{w\xi}^{\varphi}) \;\leq\; C_3'\, V_k'\, \varphi^{k-1}\,,$$

thanks to the bound (6.9) for the volume $\mathrm{A}_{x,R}(C_{w\xi}^{\varphi})$. $\square$

Second step : We need again the Anderson–Schoen barrier $u = u_{w\xi}$ for the ray $w\xi$, that we constructed in Lemma 2.3. Since u is superharmonic, since $u \geq 0$ everywhere and since $u \geq 1$ on the sphere $S(\xi, 2) \subset C_{w\xi}^{\pi/2}$, it follows from (4.4) and the maximum principle that, for all y in $\mathring{B}(x, R) \smallsetminus \mathring{B}(\xi, 2)$:

$$\sigma_{x,R}^{y}(C_{x\xi}^{\theta}) \leq \sigma_{x,R}^{y}(C_{w\xi}^{\varphi}) \leq C_5\, \varphi^{k-1}\, u(y) \leq C_5\, \varphi^{\varepsilon_0}\, u(y).$$

Applying this equality with $y = x$, remembering again the exponential decay (2.4) of the Anderson–Schoen barrier on the ray wx and using (4.3), one finally gets :

$$\sigma_{x,R}^{x}(C_{x\xi}^{\theta}) \leq C_5\, \varphi^{\varepsilon_0}\, u(x) \leq C_0 C_5\, \varphi^{\varepsilon_0}\, e^{-\varepsilon_0(R-2)} \leq C_0 C_5\, 4^{\varepsilon_0}\, \theta^{\varepsilon_0}.$$

This proves (4.1) when $R \geq 2$ and $\theta \leq \theta_R$.

5 Lower bound for the harmonic measures

The aim of this chapter is to prove the lower bound in (1.1).

The structure of this chapter is very similar to the structure of Chapter 4. We recall that X is a k-dimensional Hadamard manifold satisfying the pinching condition $-b^2 \leq K_X \leq -a^2 < 0$, x is a point on X, ξ a point on the sphere $S(x, R)$ and $C_{x\xi}^{\theta}$ the cone with axis $x\xi$ and angle θ. We want to prove that

$$\sigma_{x,R}^{x}(C_{x\xi}^{\theta}) \geq \tfrac{1}{M}\theta^N \tag{5.1}$$

where the constants M and N depend only on a, b and k. It is not restrictive to assume that $b = 1$. Fix a length $l_0 \geq 2$ such that

$$\tfrac{1}{2} \geq C_0\, e^{-\varepsilon_0(l_0-1)}\,. \tag{5.2}$$

We will distinguish three cases, setting

$$\theta'_R = 2\pi e^{-a(R-l_0)}\ :$$

⋆ Bounded radius : $R \leq l_0$.
⋆ Large angle : $R \geq l_0$ and $\theta \geq \theta'_R$.
⋆ Small angle : $R \geq l_0$ and $\theta \leq \theta'_R$.

5.1 Lower bound for a bounded radius

We prove (5.1) when $R \leq l_0$.

As in Section 4.1, when $R \le l_0$, we will prove the lower bound (5.1) under the weaker pinching condition $-1 \le K_X \le 0$. This allows us to multiply the distance by a ratio l_0/R, while preserving this pinching condition. Hence we can assume that the radius is $R_0 = l_0$. Using the expression (3.7) for the density of the harmonic measure, the bound (3.14) for this density, and the bound (6.7) for the volume $\mathrm{A}_{x,R_0}(C^\theta_{x\xi})$, one estimates

$$\sigma^x_{x,R_0}(C^\theta_{x\xi}) \;=\; \int_{C^\theta_{x\xi}} \frac{\partial G^x_{x,R_0}}{\partial n}(\eta)\,\mathrm{d}A_{x,R_0}(\eta) \;\ge\; c_3\,\mathrm{A}_{x,R_0}(C^\theta_{x\xi}) \;\ge\; c_3\,v_k\,\theta^{k-1}.$$

This proves (5.1) when $R \le l_0$.

5.2 Lower bound for a large angle

We prove (5.1) when $R \ge l_0$ and $\theta \ge \theta'_R$.

As shown in Figure 4.A, we introduce the point w on the ray $x\xi$ such that $d(x,w) = r$ where r is given by

$$\theta = 2\pi\, e^{-ar}. \tag{5.3}$$

Since $\theta'_R \le \theta \le \pi/2$, one has $0 \le r \le R - l_0$.

In particular the point w is at distance at least l_0 from every point η on the sphere $S(x,R)$. Since $\frac{1}{4}\,e^{ar}\,\theta \ge \pi/2$, it follows from Lemma 6.1 that

$$C^\theta_{x\xi} \cap S(x,R) \supset C^{\pi/2}_{w\xi} \cap S(x,R)\,.$$

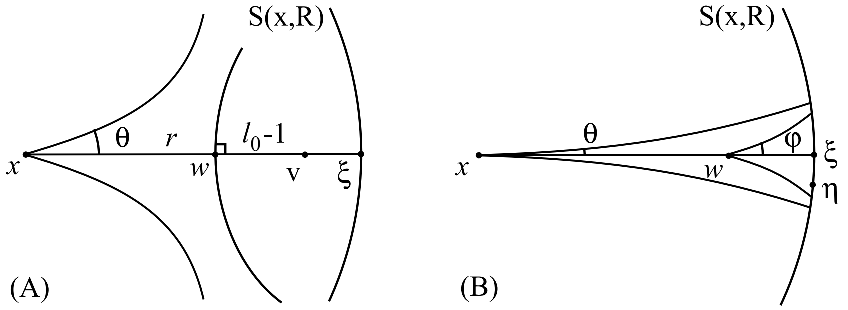

Figure 4: Minoration of $\sigma^x_{x,R}(C^\theta_{x\xi})$ for a large angle θ, and for a small angle θ.

First step : We first estimate the measure of the cone $C^\theta_{x\xi}$ for the *harmonic measure* $\sigma^v_{x,R}$ *at a point* v suitably chosen on the ray $x\xi$.

Here we need the Anderson–Schoen barrier $u = u_{wx}$ for the ray wx i.e. the ray opposite to $w\xi$. Since u is superharmonic, since $u \ge 0$ everywhere and since $u \geq 1$ on the cone $C_{wx}^{\pi/2}$, it follows from the maximum principle that, for all y in $\mathring{B}(x,R)$:

$$\sigma_{x,R}^{y}(C_{x\xi}^{\theta}) \geq \sigma_{x,R}^{y}(C_{w\xi}^{\pi/2}) = 1 - \sigma_{x,R}^{y}(C_{wx}^{\pi/2}) \geq 1 - u(y).$$

Applying this equality with the point $y = v$ on the ray $w\xi$ such that $d(w,v) = l_0 - 1$ and remembering the exponential decay (2.4) of the Anderson–Schoen barrier on the ray $w\xi$, one gets, using (5.2) :

$$\sigma_{x,R}^{v}(C_{x\xi}^{\theta}) \geq 1 - u(v) \geq 1 - C_0 e^{-\varepsilon_0(l_0-1)} \geq \tfrac{1}{2}. \tag{5.4}$$

Second step : We now apply Harnack inequality (2.5) to the positive harmonic function $y \mapsto \sigma_{x,R}^{y}(C_{x\xi}^{\theta})$ on the ball $\mathring{B}(x,R)$. Since the segment $[xv]$ stays at distance at least 1 from the sphere $S(x,R)$ and has length bounded by $r + l_0$, it follows from (5.3) and (5.4) that :

$$\sigma_{x,R}^{x}(C_{x\xi}^{\theta}) \geq \sigma_{x,R}^{v}(C_{x\xi}^{\theta}) e^{-C_1 l_0 - C_1 r} \geq \tfrac{1}{2} e^{-C_1 l_0} \left(\tfrac{\theta}{2\pi}\right)^{C_1/a}.$$

This proves (5.1) when $R \geq l_0$ and $\theta \geq \theta'_R$.

5.3 Lower bound for a small angle

We prove (5.1) when $R \geq l_0$ and $\theta \leq \theta'_R$. The argument will be similar to those in Section 4.3.

As shown in Figure 4.B, we introduce the point w on the ray $x\xi$ such that $d(x,w) = R - 2$. Let φ be the angle given by

$$\varphi := 10^{-3}\, e^{a(R-l_0)}\, \theta. \tag{5.5}$$

Since $\theta \leq \theta'_R$, one has $\varphi \leq \frac{1}{100}$, and, since $\frac{1}{4} e^{a(R-2)}\theta \geq \varphi$, according to Lemma 6.1, one has

$$C_{x\xi}^{\theta} \cap S(x,R) \supset C_{w\xi}^{\varphi} \cap S(x,R)\,.$$

First step : We estimate the measure of the cone $C_{w\xi}^{\varphi}$ for the *harmonic measure* $\sigma_{x,R}^{w}$ *seen from the point* w.

Lemma 5.1. *Let X be a Hadamard manifold with $-1 \leq K_X \leq -a^2 < 0$. Keep the above notation $x \in X$, $\xi \in S(x,R)$, $w \in [x\xi]$ with $d(w,\xi) = 2$ and $\varphi \leq 1/100$ as in Figure 4.B. Then there exists a constant $c_5 > 0$ depending only on a, b, k such that*

$$\sigma_{x,R}^{w}(C_{w\xi}^{\varphi}) \geq c_5\, \varphi^{k-1}\,. \tag{5.6}$$

Proof of Lemma 5.1. One again uses the expression (3.7) for the density of the harmonic measure. Since $\varphi \leq 1/100$, by Lemma 6.3.a, for all η in $C^{\varphi}_{w\xi} \cap S(x,R)$, one has $d(\xi,\eta) \leq 1$. Therefore one has $2 \leq d(w,\eta) \leq 3$ and the bound (3.15) with $y = v$ is valid for this density. Hence one computes

$$\sigma^{w}_{x,R}(C^{\varphi}_{w\xi}) \;=\; \int_{C^{\varphi}_{w\xi}} \frac{\partial G^{w}_{x,R}}{\partial n}(\eta)\, \mathrm{d}A_{x,R}(\eta) \;\geq\; c'_3\, \mathrm{A}_{x,R}(C^{\varphi}_{w\xi}) \;\geq\; c'_3\, v'_k\, \varphi^{k-1},$$

thanks to the bound (6.9) for the volume $\mathrm{A}_{x,R}(C^{\varphi}_{w\xi})$. $\square$

Second step : We apply again Harnack inequality (2.5) to the positive harmonic function $y \mapsto \sigma^{y}_{x,R}(C^{\varphi}_{w\xi})$ on the ball $\mathring{B}(x,R)$. Since the segment $[xw]$ stays at distance at least 1 from the sphere $S(x,R)$ and has length smaller than R, this gives, using (5.6),

$$\sigma^{x}_{x,R}(C^{\theta}_{x\xi}) \geq \sigma^{x}_{x,R}(C^{\varphi}_{w\xi}) \;\geq\; \sigma^{w}_{x,R}(C^{\varphi}_{w\xi})e^{-C_1 R} \geq c_5\, \varphi^{k-1} e^{-C_1 R}$$

Increasing C_1, one can assume $C_1/a \geq k$. Hence one gets, using also (5.5),

$$\sigma^{x}_{x,R}(C^{\theta}_{x\xi}) \geq c'_5\, \varphi^{C_1/a}\, (\tfrac{\theta}{\varphi})^{C_1/a} \;=\; c'_5\, \theta^{C_1/a}\,,$$

with $c'_5 := 10^{-3C_1/a} e^{-C_1 l_0} c_5$. This proves (5.1) when $R \geq l_0$ and $\theta \leq \theta'_R$.

6 Geometry of Hadamard manifold

This last chapter is self-contained. We collect here two basic geometric estimations in Hadamard manifolds that we used in the previous chapters.

6.1 Geometry of triangles

We first compare the angles in a triangle.

We will denote by $\mathbb{H}^2(-a^2)$ the real hyperbolic plane with curvature $-a^2$.

Lemma 6.1. *Let X be a Hadamard manifold with $-b^2 \leq K_X \leq -a^2 < 0$. Let r, R, L be the side lengths of a geodesic triangle in X and let θ, φ be the two angles as in Figure 5. Assume that $0 \leq \varphi \leq \pi/2$ and $bL \geq 2$. Then one has the following angle estimates*

$$\tfrac{1}{4}\, e^{ar} \leq \frac{\varphi}{\theta} \leq 4\, e^{br}. \tag{6.1}$$

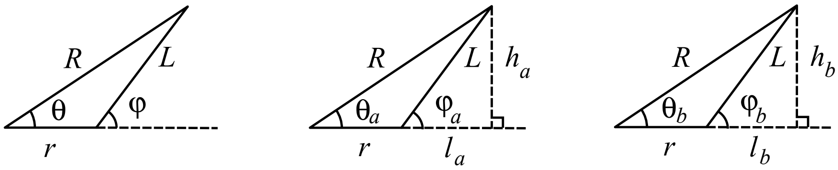

Figure 5: A triangle in X and its comparison triangles in $\mathbb{H}^2(-a^2)$ and $\mathbb{H}^2(-b^2)$.

Proof. The proof relies on comparison triangles in the hyperbolic planes $\mathbb{H}^2(-a^2)$ and $\mathbb{H}^2(-b^2)$ i.e. the triangles with same side lengths r, R and L.

We denote by θ_a and φ_a the angles and by h_a, l_a the lengths seen in $\mathbb{H}^2(-a^2)$ as in Figure 1. We use similar notations in $\mathbb{H}^2(-b^2)$. The pinching assumption tells us that

$$\theta_b \leq \theta \leq \theta_a \quad \text{and} \quad \varphi_a \leq \varphi \leq \varphi_b. \tag{6.2}$$

We will use the following equalities for the right triangle of $\mathbb{H}^2(-a^2)$ with side lengths L, l_a and h_a,

$$\sinh(aL)\sin\varphi_a = \sinh(ah_a) \quad \text{and} \quad \cosh(aL) = \cosh(al_a)\cosh(ah_a). \tag{6.3}$$

Taking the ratio of these two equalities and repeating this computation for the right triangle with side lengths R, $r + l_a$ and h_a, one gets

$$\frac{\sin\varphi_a}{\sin\theta_a} = \frac{\tanh(aR)}{\tanh(aL)} \frac{\cosh(ar+al_a)}{\cosh(al_a)}. \tag{6.4}$$

We will also use the easy inequalities for $t \geq 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$,

$$\tfrac{1}{2}e^t \leq \cosh(t) \leq e^t \quad \text{and} \quad \tfrac{2}{\pi}\alpha \leq \sin\alpha \leq \alpha. \tag{6.5}$$

We first prove the lower bound in (6.1). We notice that, by (6.2) the angle φ_a is acute, and hence the angle θ_a is also acute. One computes using (6.2), (6.4), (6.5) and the bound $L \leq R$,

$$\frac{\varphi}{\theta} \geq \frac{\varphi_a}{\theta_a} \geq \frac{2}{\pi}\frac{\sin\varphi_a}{\sin\theta_a} = \frac{2}{\pi}\frac{\tanh(aR)}{\tanh(aL)}\frac{\cosh(ar+al_a)}{\cosh(al_a)} \geq \frac{e^{ar}}{\pi} \geq \tfrac{1}{4}\, e^{ar}.$$

We now prove the upper bound in (6.1) when the angle φ_b is acute. The computation is similar, using also the assumption $bL \geq 2$,

$$\frac{\varphi}{\theta} \leq \frac{\varphi_b}{\theta_b} \leq \frac{\pi}{2}\frac{\sin\varphi_b}{\sin\theta_b} = \frac{\pi}{2}\frac{\tanh(bR)}{\tanh(bL)}\frac{\cosh(br+bl_b)}{\cosh(bl_b)} \leq \frac{\pi e^{br}}{\tanh(2)} \leq 4\, e^{br}.$$

Finally, we prove the upper bound in (6.1) when the angle φ_b is obtuse. We notice that since the angle φ is acute, one has $r \leq R$ and therefore

$$2h_b \geq R + L - r \geq L \tag{6.6}$$

and $bh_b \geq 1$. The computation is similar using equalities analog to (6.3),

$$\frac{\varphi}{\theta} \leq \frac{\pi}{2\theta_b} \leq \frac{\pi}{2}\frac{1}{\sin\theta_b} = \frac{\pi}{2}\frac{\tanh(bR)}{\tanh(bh_b)}\cosh(br-bl_b) \leq \frac{\pi e^{br}}{2\tanh(1)} \leq 4\,e^{br}.$$

Note that the constants in (6.1) are not optimal. □

6.2 Volume estimates

We now estimate the Riemannian measures on spheres.

As before, for x in X and $R > 0$, we denote by $A_{x,R}$ the Riemannian measure of the sphere $S(x,R)$.

The first lemma gives volume estimates for a fixed radius $R_0 \geq 1$.

Lemma 6.2. *Let X be a Hadamard manifold with $-1 \leq K_X \leq 0$ and fix $R_0 \geq 1$. There exist constants $V_k > v_k > 0$ depending only on k and R_0, such that for every x in X, ξ in $S(x,R)$ and $\varphi \leq \frac{\pi}{2}$, one has*

$$v_k\,\varphi^{k-1} \;\leq\; \mathrm{A}_{x,R_0}(C^{\varphi}_{x\xi}) \;\leq\; V_k\,\varphi^{k-1}. \tag{6.7}$$

Proof. Because of the pinching assumption, the exponential map $\exp_x : T_xX \to X$ is a diffeomorphism and its restriction to the ball $B(0,R_0) \subset T_xX$ induces a diffeomorphism $\Phi_x : B(0,R_0) \to B(x,R_0)$ whose derivatives are uniformly bounded

$$\|D\Phi_x\| \leq e^{R_0} \;\text{ and }\; \|D\Phi_x^{-1}\| \leq 1\,. \tag{6.8}$$

The bounds (6.7) follow. □

The second lemma gives volume estimates which are uniform in R.

Lemma 6.3. *Let X be a Hadamard manifold with $-1 \leq K_X \leq 0$. Let $R \geq 2$, $x \in X$, $\xi \in S(x,R)$, $w \in [x\xi]$ with $d(w,\xi) = 2$ and $\varphi \leq \varphi_0 := 1/100$ as in Figure 6.*
a) One has the inclusion $C^{\varphi}_{w\xi} \cap S(x,R) \subset B(\xi,1)$.
b) There exist constants $V'_k > v'_k > 0$ depending only on k such that

$$v'_k\,\varphi^{k-1} \;\leq\; \mathrm{A}_{x,R}(C^{\varphi}_{w\xi}) \;\leq\; V'_k\,\varphi^{k-1}. \tag{6.9}$$

Proof. *a*) Let η be a point on $S(x,R)$ such that the angle φ between $w\xi$ and $w\eta$ is bounded by 1/100. The triangle $(w,\,\xi,\,\eta)$ satisfies also the following properties :

$d(w,\xi) = 2$, $d(w,\eta) \geq 2$, and the angle between ξw and $\xi\eta$ is acute.

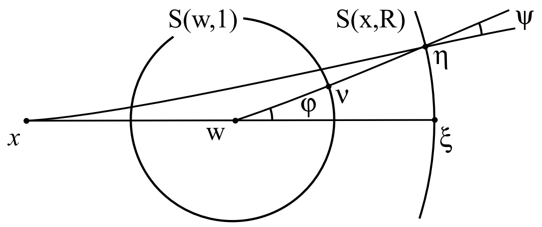

Figure 6: Estimation of the volume $A_{x,R}(C^{\varphi}_{w\xi})$.

Since $-1 \leq K_X \leq 0$, the comparison triangle (w', ξ', η') in $\mathbb{H}^2$ satisfies the same properties. A direct computation in $\mathbb{H}^2$ gives then $d(\eta', \xi') \leq 1$. Therefore one also has $d(\eta, \xi) \leq 1$.

b) As shown in Figure 6, since X is a Hadamard manifold, the intersection $S(x,R) \cap C^{\varphi_0}_{w\xi}$ is a hypersurface that can be parametrized in polar coordinates seen from w : there exists a $\mathcal{C}^\infty$ diffeomorphism

$$\begin{array}{rcl} \Psi_w \ : \ S(w,1) \cap C^{\varphi_0}_{w\xi} & \longrightarrow & S(x,R) \cap C^{\varphi_0}_{w\xi} \\ \nu & \mapsto & \eta = \Psi_w(\nu) = \exp_w(\rho_\nu \exp_w^{-1} \nu)\,, \end{array}$$

where ρ is a $\mathcal{C}^\infty$ function on $S(w,1) \cap C^{\varphi_0}_{w\xi}$ with values in the interval $[2,3]$.

Since X is a Hadamard manifold, at every point of this hypersurface $S(x,R) \cap C^{\varphi_0}_{w\xi}$ the angle ψ between the normal vector to $S(x,R)$ and the radial vector seen from w is at most φ_0. Therefore, using Jacobi fields, one checks that the derivatives of Ψ_w and its inverse are uniformly bounded

$$\|D\Psi_w\| \leq \tfrac{e^2}{\cos(\varphi_0)} \leq 10 \quad \text{and} \quad \|D\Psi_w^{-1}\| \leq 1\,.$$

Therefore, one has

$$\mathrm{A}_{w,1}(C^{\varphi}_{w\xi}) \ \leq \ \mathrm{A}_{x,R}(C^{\varphi}_{w\xi}) \ \leq \ 10^{k-1} \mathrm{A}_{w,1}(C^{\varphi}_{w\xi})$$

The bounds (6.9) follow then from (6.7). □

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