Positive harmonic functions on the Heisenberg group II

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Abstract
We describe the extremal positive harmonic functions for finitely supported measures on the discrete Heisenberg group: they are proportional either to characters or to translates of induced from characters.

Content

2010 Math. subject class. Primary 31C35 ; Secondary 60B15, 60G50, 60J50
Key words Harmonic function, Martin boundary, Random walk, Nilpotent group

1 Introduction

In this paper, we present the classification of the positive harmonic functions on the discrete Heisenberg group $G = H_3(\mathbb{Z})$.

1.1 Positive harmonic functions

Let $\mu = \sum_{s\in S} \mu_s \delta_s$ be a positive measure on G with finite support $S \subset G$. We recall that a function h on G is said to be μ-harmonic if it satisfies the equality $h = P_\mu h$ where $P_\mu h(g) := \sum_{s\in S} \mu_s\, h(sg)$ for all g in G. We want to describe the cone $\mathcal{H}_\mu^+$ of positive μ-harmonic functions h on G. By Choquet Theorem, it is enough to describe its extremal rays.

The main aim of this paper is to prove that the extremal positive μ-harmonic functions on G are proportional either to a character of G or to a translate of a function which is induced from a character of an abelian subgroup (Theorem 1.1).

The special case where μ is the *southwest measure* was handled in the introductory paper [2]. This case was striking because the classical partition function $h(x, y, z) := p_y(z)$ with

$$p_y(z) := \text{number of partition of } z \text{ by } y \text{ non-negative integers}$$

occurs as one of these extremal positive harmonic functions. This partition function $p_y(z)$ is the simplest instance of a "harmonic function induced from the character of an abelian subgroup" that we will introduce in this paper.

1.2 Construction of harmonic functions

The simplest examples of μ-harmonic functions are μ-harmonic characters. Those are the characters $\chi : G \to \mathbb{R}_{>0}$ such that $\sum_{s\in S} \mu_s \, \chi(s) = 1$. Such a function $h = \chi$ is an extremal positive μ-harmonic function on G which is invariant by the center Z of G, see Lemma 2.1.

We now introduce another construction of extremal positive μ-harmonic functions by inducing harmonic characters. Let $S_0 \subset S$ be a maximal abelian subset and G_0 be the subgroup of G generated by S_0. Denote by $\mu_0 := \sum_{s\in G_0} \mu_s \, \delta_s$ the measure restriction of μ to G_0. Let χ_0 be a μ_0-harmonic character of G_0. We extend χ_0 as a function on G, still denoted χ_0, which is 0 outside G_0. This function χ_0 is μ-subharmonic, so that the sequence $P_\mu^n \chi_0$ is increasing. We set

$$h_{G_0,\chi_0} = \lim_{n\to\infty} P_\mu^n \chi_0.$$

We will tell exactly for which pairs (G_0, χ_0) the function h_{G_0,χ_0} is finite, in Lemma 3.8 and in Propositions 5.1, 5.4 and 5.5. When it is finite, the function h_{G_0,χ_0} is an extremal positive μ-harmonic function on G, see Lemma 3.1. We will call h_{G_0,χ_0} the *harmonic function on G induced from the μ_0-harmonic character χ_0 of G_0.*

For g in G, we denote by $\rho_g : g' \mapsto g'g$ the right translation by g on G. Whenever a function h is μ-harmonic, the function $h_g := h \circ \rho_g$ is also μ-harmonic.

1.3 Main results

Our main theorem tells us that conversely these three constructions are the only possible ones.

Theorem 1.1. *Let $G = H_3(\mathbb{Z})$ be the discrete Heisenberg group and μ be a positive measure on G whose support S is finite and generates the group G. Then every extremal positive μ-harmonic function h on G is proportional either to a character χ of G or to a translate $h_{G_0,\chi_0} \circ \rho_g$ of a function induced from a harmonic character of an abelian subgroup.*

Remark 1.2. - Of course the case where $\mu(G) = 1$ is the major case. However, even when dealing with a probability measure μ, the induction process forces us to work with positive measures μ_0 which are not probability measures.

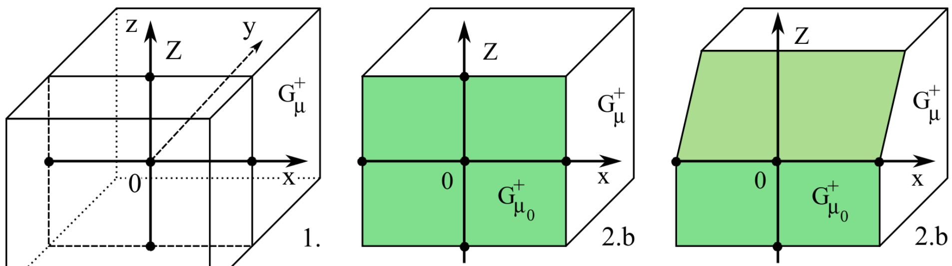

Figure 1: In Case 1 and in Case 2.b of Theorem 5.10, no harmonic function is induced from a character of an abelian subgroup G_0.

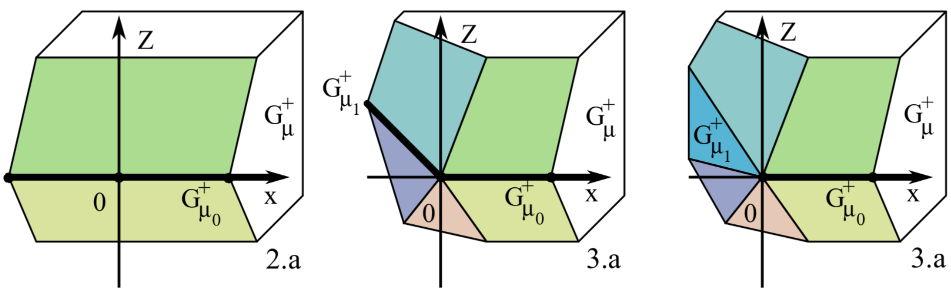

Figure 2: In Case 2.a, exactly two harmonic functions are induced from a character of $G_0 = G_{\mu_0}$ and no other. In Case 3.a, only one harmonic function is induced from a character of $G_0 = G_{\mu_0}$ and one or infinitely many are induced from a character of $G_1 = G_{\mu_1}$.

- Theorem 1.1 can not be extended to all nilpotent groups G. Indeed, the conclusion of Theorem 1.1 is not always valid for a probability measure μ on the nilpotent group G of rank 4 with cyclic center. See Section 5.5.

Theorem 1.1 has been announced in [2]. It will be proven in Chapter 4. Indeed it is a direct consequence of Propositions 4.8 and 4.10. We will give a more precise description of the extremal positive μ-harmonic functions h in Theorem 5.10. In particular, we will say exactly when and how many of these new examples occur. This is illustrated in the schematic Figures 1, 2 and 3. In these figures, we have drawn various cases of semigroup G_μ^+ generated by S that are described in Theorem 5.10. Note that the support of a positive μ-harmonic function h is invariant by the opposite semigroup, i.e. by the semigroup generated by S^{-1}. In particular when $G_\mu^+ = G$, a positive harmonic function h is either identically zero or vanishes nowhere. Here are

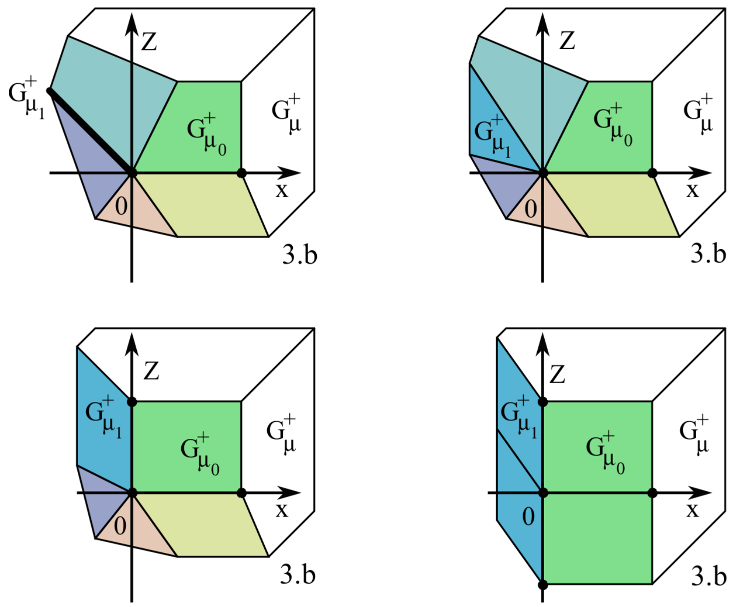

Figure 3: In case 3.b, infinitely many harmonic functions are induced from a character of $G_0 = G_{\mu_0}$ and one or infinitely many are induced from a character of $G_1 = G_{\mu_1}$.

two corollaries of Theorem 5.10 that we will prove in Section 5.4. The first corollary tells us that these new examples always vanish somewhere.

Corollary 1.3. *Same notation. Let h be an extremal positive μ-harmonic function on G which does not vanish. Then h is a character of G.*

The second corollary tells us exactly when no new example occurs. We denote by G_μ^+ the semigroup generated by S.

Corollary 1.4. *Same notation with $\mu(G) = 1$. The following are equivalent:*
(i) Every extremal positive μ-harmonic function h on G is a character of G.
(ii) G_μ^+ contains two non-central elements whose product is in $Z \smallsetminus \{0\}$.

1.4 Previous results

The study of harmonic functions on groups has a very long history. I will just point out the part of it which is relevent for our purposes.

As a general motivation, let us recall that the bounded μ-harmonic functions on a group G are described thanks to bounded functions on the *Poisson boundary* of (G, μ). They are used to study random walks on G-spaces. The extremal positive μ-harmonic functions on G are related to the Martin boundary of (G, μ). They are used to study more precisely the behavior of these random walks, see [1], [9], [13] or [15].

1.4.1 Abelian groups

This part of the history begins with the Choquet–Deny Theorem in [5] :

Let G be a finitely generated abelian group and μ be a positive finite measure on G whose support generates G as a group. Then every extremal positive μ-harmonic function h on G is proportional to a character.

Indeed the proof of this theorem is very short : one notices that the harmonicity equation (2.1) is a decomposition of h as a sum of positive harmonic functions and hence all the terms in this sum are proportional to h.

1.4.2 Bounded harmonic functions

The Choquet-Deny theorem has been extended to nilpotent groups when μ has mass 1 and h is bounded. This is due to Dynkin and Maljutov in [7] :

Let G be a finitely generated nilpotent group and μ be a probability measure on G whose support generates G as a group. Then every bounded μ-harmonic function on G is constant.

1.4.3 When S generates G as a semigroup

The Choquet-Deny theorem has also been extended to nilpotent groups for h unbounded under an extra assumption. This is due to Margulis in [12] :

Let G be a finitely generated nilpotent group and μ be a positive measure on G whose support generates G AS A SEMIGROUP. *Then every extremal positive μ-harmonic function on G is proportional to a character.*

1.4.4 The Heisenberg group

The main significance of our Theorem 1.1 is that even though Choquet-Deny theorem can not be extended to finitely generated nilpotent groups without this extra assumption, for the Heisenberg group one can describe all the positive harmonic functions. Note that, because of Margulis theorem, most of our paper will deal with a positive measure whose support generates G as a group but does not necessarily generate G as a semigroup.

Many recent works focus on the random walks on the discrete Heisenberg group G as in [3], [6] and [8], or on nilpotent groups as in [4] and [10], or on the geometry of words in G as in [11] and [14]. We mention these related results even though we will not use them.

1.5 Strategy of proof

We now explain the strategy of proof of Theorem 1.1 and the organization of the paper.

In Chapter 2, we recall well-known facts on positive harmonic functions and notations for the discrete Heisenberg group G and its positive measures μ with a finite support S.

In Chapter 3, we begin the proof of Theorem 1.1. When h is an extremal μ-harmonic function on G, we focus on the equality $h(g) = P_\mu^n h(g)$ where the right-hand side is written as a weighted sum of values $h(\dot{w}g)$ for words w of length n in S, as in Equation (2.2). In Lemmas 3.1 and 3.2, we check that when the contribution in this sum of the words w whose letters are in a proper subgroup of G, is not negligible, then h is an "induced harmonic function". In Lemma 3.10, we prove a useful generalization: we allow w to be a concatenation of k subwords whose letters are in a proper subgroup with $k \geq 1$ fixed. The proofs are very general and do not assume G to be nilpotent.

In Chapter 4, we assume that "h is not induced", and we want to prove that h is invariant by the center Z of G. The main idea is to construct a symmetric relation $\mathcal{R}_n$ among the words in S^n such that two related words w and w' have same weight and their image $\dot{w}$ and $\dot{w}'$ in G differ by a non-trivial element z of Z. A key point is to be able to compare the number of words related to w and the number of words related to w', see Lemma 4.4. This allows us to prove that h is proportional to one of its translate h_z, see Proposition 4.3. The last step is to prove that h is indeed equal to its translate h_z. This is done in Propositions 4.8 and 4.10. The key point there, Lemma 4.11 is based on a counting argument that again involves the partition function. This finishes the proof of Theorem 1.1.

In Chapter 5, we give a complete classification of the extremal μ-harmonic functions that are "induced from a character", see Theorem 5.10. Their existence is an important new feature of this article. The proof of this classification in Propositions 5.1, 5.4 and 5.5 uses a transience property for random walks on $\mathbb{Z}$ similar to the large deviation inequality, see Lemma 5.3.

In the last Section 5.5, we explain how to construct, for a rank 4 nilpotent group, new extremal positive μ-harmonic functions that are not induced.

2 Notation and preliminary results

We introduce in this chapter notations that will be used all over this article.

2.1 The cone of μ-harmonic functions

We first recall classical facts on positive μ-harmonic functions.

Let G be a finitely generated group and μ be a positive measure with finite support $S \subset G$. We denote by G_μ^+ the subsemigroup of G generated by S and by G_μ the subgroup of G generated by S.

A positive function $h : G \to [0, \infty[$ is said to be *μ-harmonic* if it satisfies the equality

$$h = P_\mu h \text{ where } P_\mu h : g \to \sum_{s \in S} \mu_s h(sg). \tag{2.1}$$

A non-zero positive μ-harmonic function is said to be *extremal or μ-extremal* if every smaller positive μ-harmonic function $h' \leq h$ is a multiple of h.

A function h is said to be *μ-superharmonic*, respectively *μ-subharmonic*, if it satisfies the inequality $h \geq P_\mu h$, respectively $h \leq P_\mu h$.

We will often write the n^{th} power of the operator P_μ under the form

$$P_\mu^n h(g) \quad = \quad \sum_{w \in S^n} \mu_w h(\dot{w} g), \tag{2.2}$$

where, for a word $w = s_1 \dots s_n \in S^n$ of length $\ell_w = n$, the constant $\mu_w > 0$ is the product $\mu_w := \mu_{s_1} \cdots \mu_{s_n} > 0$ and where the element $\dot{w} \in G$ is the product $\dot{w} := s_1 \cdots s_n$ in G.

Let $\mathcal{H}_\mu^+$ be the convex cone of positive μ-harmonic functions h on G and $\mathcal{E}$ be a Borel set of extremal μ-harmonic functions containing exactly one function in each extremal ray of $\mathcal{H}_\mu^+$. We endow $\mathcal{H}_\mu^+$ with the topology of the pointwise convergence. When $G_\mu^+ = G$ the cone $\mathcal{H}_\mu^+$ *has a compact basis*, this means that there exists a compact subset of $\mathcal{H}_\mu^+$ that meets all rays of $\mathcal{H}_\mu^+$.

In general, the cone $\mathcal{H}_\mu^+$ might not have a compact basis but it is *well-capped*, this means that it is a union of closed convex subcones $\mathcal{H}_{\mu,i}^+$ with compact basis such that $\mathcal{H}_\mu^+ \smallsetminus \mathcal{H}_{\mu,i}^+$ is also convex. This cone $\mathcal{H}_\mu^+$ is also *reticulated*, this means that every two positive μ-harmonic functions h_1 and h_2 admit a maximal μ-harmonic lower bound h_m and also a minimal μ-harmonic upper bound h_M. Indeed one has

$$\begin{aligned} h_m &= \lim_{n\to\infty} P_\mu^n(\min(h_1,h_2)) \geq 0; \quad \text{and} \\ h_M &= \lim_{n\to\infty} P_\mu^n(\max(h_1,h_2)) \ \leq \ h_1 + h_2 < \infty. \end{aligned}$$

By Choquet Theorem, it is enough to describe the extremal rays of this cone $\mathcal{H}_\mu^+$. Indeed, since $\mathcal{H}_\mu^+$ is well-capped, this theorem tells us that *every positive μ-harmonic function h can be written as an integral of non-proportional extremal μ-harmonic functions : $h = \int_{\mathcal{E}} f \, \mathrm{d}\alpha(f)$, for a positive measure α on the set $\mathcal{E}$.*

Since $\mathcal{H}_\mu^+$ is reticulated, this theorem also tells us that *such a measure α is unique.*

In this paper a *character* will always mean a multiplicative morphism $\chi : G \mapsto \mathbb{R}_{>0}$. A character χ is μ-harmonic if and only if it satisfies the equation $\sum_{s\in S} \mu_s \, \chi(s) = 1$.

2.2 Harmonic characters

We discuss here harmonic characters on nilpotent groups.

Let G be a nilpotent finitely generated group and μ be a positive finite measure on G with finite support generating G.

Lemma 2.1. *Every μ-harmonic character of G is an extremal positive μ-harmonic function.*

Proof of Lemma 2.1. Let χ be a μ-harmonic character such that $\chi = h' + h''$ with both h' and h'' positive and μ-harmonic. We want to prove that the function $\widetilde{h}' := \chi^{-1} h'$ is constant. We notice that the measure $\widetilde{\mu} := \chi\mu$ on G is a probability measure and the function $\widetilde{h}'$ is a bounded $\widetilde{\mu}$-harmonic function. Therefore by Dynkin–Maljutov theorem, see Section 1.4, the function $\widetilde{h}'$ is constant. $\square$

2.3 The Heisenberg group

We gather here notation that we will use in this article for the discrete Heisenberg group.

Recall that the discrete Heisenberg group $G := H_3(\mathbb{Z})$ is the set $\mathbb{Z}^3$ of triples seen as matrices $(x, y, z) := \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$. It is endowed with the product

$$(x, y, z)\,(x', y', z') = (x + x', y + y', z + z' + xy')\,. \tag{2.3}$$

We will denote by $0 := (0, 0, 0)$ the identity element of G, and by z_0 the generator $z_0 := (0, 0, 1)$ of the center Z of G.

For two elements $g = (x, y, z)$, $g' = (x', y', z')$ of G, we will denote by $c_{g,g'}$ the integer $c_{g,g'} := xy' - yx'$ so that

$$gg'g^{-1}g'^{-1} = z_0{}^{c_{g,g'}}\,. \tag{2.4}$$

Let $\overline{G} := G/Z \simeq \mathbb{Z}^2$ be the abelianization of G that we embed in the real vector space $V := \overline{G} \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^2$.

Let μ be a positive measure on G with finite support S. We denote by $\overline{\mu}$ the image of μ in $\overline{G}$ and by $\overline{S}$ its support.

We denote by V_μ the vector subspace of V generated by $\overline{S}$ and by V_μ^+ the smallest convex cone of V containing $\overline{S}$. Note that, when $G_\mu = G$, one always has $V_\mu = V$, and, when $G_\mu^+ = G$, one always has $V_\mu^+ = V$.

The description of $\mathcal{H}_\mu^+$, when $G_\mu = G$ will heavily depend on the shape of V_μ^+. We will often distinguish the three cases :

$$V_\mu^+ = \text{the plane, a half-plane, or a properly convex cone.} \tag{2.5}$$

3 Induced harmonic functions

In this chapter we present general facts on μ-harmonic functions on a finitely generated group G. These facts will be particularly useful when G is the Heisenberg group.

3.1 Construction of induced harmonic functions

> The following lemma gives us a method to construct μ-harmonic functions starting from a harmonic function for a smaller measure μ_0. This lemma will be mainly useful when μ_0 is the restriction of μ to a proper subgroup G_0.

Let G be a finitely generated group and μ and μ_0 be positive measures on G with finite support such that $\mu_0 < \mu$, i.e. such that $\mu_1 := \mu - \mu_0$ is also a positive measure.

Lemma 3.1. *Let h_0 be a positive μ_0-harmonic function on G such that the function $h := \sup_{n\geq 1} P_\mu^n h_0$ is finite.*
(i) Then one has $h = \lim_{n\to\infty} P_\mu^n h_0$ and h is a positive μ-harmonic function.
(ii) One can recover h_0 from h as $h_0 = \lim_{n\to\infty} P_{\mu_0}^n h$.
(iii) Moreover when h_0 is μ_0-extremal then h is μ-extremal too.

When it is finite, the function h will be called *induced from the harmonic function h_0*.

Proof of Lemma 3.1. (i) We first notice that, since $h_0 = P_{\mu_0} h_0 \leq P_\mu h_0$, the sequence $P_\mu^n h_0$ is increasing. Hence, when this sequence is bounded it converges to a μ-harmonic function.

(ii) Since $h = P_\mu h \geq P_{\mu_0} h$, the sequence $P_{\mu_0}^n h$ is decreasing. Since $P_{\mu_0}^n h \geq P_{\mu_0}^n h_0 = h_0$, this sequence $P_{\mu_0}^n h$ converges to a μ_0-harmonic function $h_0' := \lim_{n\to\infty} P_{\mu_0}^n h$ such that $h_0' \geq h_0$.

We want to prove that the function $h_0'' := h_0' - h_0$ is zero. Since $h_0 \leq h_0' \leq h$, one has $P_\mu^n h_0 \leq P_\mu^n h_0' \leq h$. Therefore one also has $\lim_{n\to\infty} P_\mu^n h_0' = h$ and hence $\lim_{n\to\infty} P_\mu^n h_0'' = 0$. Since h_0'' is μ_0-harmonic, this last sequence is increasing and hence one has $h_0'' = 0$.

(iii) Assume now that h_0 is μ_0-extremal and assume that h is the sum of two positive μ-harmonic functions $h = h' + h''$. We want to prove that h and h' are proportional. The functions $h_0' = \lim_{n\to\infty} P_{\mu_0}^n h'$ and $h_0'' = \lim_{n\to\infty} P_{\mu_0}^n h''$ are μ_0-harmonic and, by (ii), they give a decomposition $h_0 = h_0' + h_0''$.

Therefore, one has $h_0' = \lambda' h_0$ and $h_0'' = \lambda'' h_0$ for positive constants λ' and λ'' with $\lambda' + \lambda'' = 1$. One has the inequalities $h' \geq \lim_{n\to\infty} P_\mu^n h_0' = \lambda' h$ and $h'' \geq \lim_{n\to\infty} P_\mu^n h_0'' = \lambda'' h$. Since $h = h' + h''$, these inequalities are equalities:

one has $h' = \lambda' h$ and $h'' = \lambda'' h$. This proves that the function h is μ-extremal. $\square$

3.2 Recognizing induced harmonic functions

The following lemma is a converse of Lemma 3.1. It tells us how to recognize a μ-harmonic function that is induced from a μ_0-harmonic function.

Let G be a finitely generated group and $\mu_0 < \mu$ be positive measures on G with finite support.

Lemma 3.2. *Let h be a positive μ-harmonic function on G such that the function $h_0 := \inf_{n \geq 1} P_{\mu_0}^n h$ is non-zero.*
(i) Then one has $h_0 = \lim_{n\to\infty} P_{\mu_0}^n h$ and h_0 is a positive μ_0-harmonic function.
(ii) One has the inequality $h \geq \lim_{n\to\infty} P_\mu^n h_0$.
(iii) Moreover when h is μ-extremal, one has the equality $h = \lim_{n\to\infty} P_\mu^n h_0$ and h_0 is μ_0-extremal too.

In particular, when h is μ-extremal, h_0 is supported by a translate $G_{\mu_0} g$ of the subgroup G_{μ_0}.

Proof of Lemma 3.2. The argument is very similar to Lemma 3.1

(i) Since the function h is positive and μ-harmonic, the sequence $P_{\mu_0}^n h$ is positive and decreasing. Hence it has a limit h_0 which is μ_0-harmonic.

(ii) By assumption, this limit h_0 is non-zero. By construction, one has the inequality $h \geq h_0$. Since h is μ-harmonic, the sequence $P_\mu^n h_0$ is bounded by h and, by Lemma 3.1, the limit $h' := \lim_{n\to\infty} P_\mu^n h_0$ exists, is μ-harmonic and is bounded by h.

(iii) Assume now that h is μ-extremal. Then one has $h' = \lambda' h$ for some constant $\lambda' \geq 0$. Again by Lemma 3.1, one also has

$$h_0 = \lim_{n\to\infty} P_{\mu_0}^n h' = \lambda' \lim_{n\to\infty} P_{\mu_0}^n h = \lambda' h_0 \,. \tag{3.1}$$

Therefore one has $\lambda' = 1$.

It remains to check that h_0 is μ_0-extremal. Assume that $h_0 = h_0' + h_0''$ with both h_0' and h_0'' positive μ_0-harmonic. The limit $h'' := \lim_{n\to\infty} P_\mu^n h_0''$ is a μ-harmonic function bounded by h. Hence one has $h'' = \lambda'' h$ and by the same computation as (3.1), one gets $h_0'' = \lambda'' h_0$. This proves that h_0 is extremal. $\square$

The following definition relies on the previous lemmas :

Definition 3.3. A μ-harmonic function h on G is said to be *induced from a subgroup* G_0 if

$$\lim_{n\to\infty} P^n_{\mu_0} h \neq 0. \tag{3.2}$$

where μ_0 is the restriction of μ to G_0.

By Lemma 3.2, when h is μ-extremal this limit (3.2) is equal to $h_0 \circ g$ where g is in G and h_0 is an extremal μ_0-harmonic function supported on G_0. Therefore one has $h = h_{G_0,h_0} \circ \rho_g$ where $h_{G_0,h_0} := \lim_{n\to\infty} P^n_\mu h_0$. In this case *the function h is a translate of the harmonic function induced from h_0.* Equivalently, *the function h is induced from $h_0 \circ \rho_g$.*

Definition 3.4. A μ-harmonic function is said to be induced, if there exists a subgroup G_0 of infinite index in G such that h is induced from G_0. It is said to be *non-induced* otherwise.

Remark 3.5. The reason why we require in this definition G_0 to have infinite index will be explained in Lemma 4.1.

A posteriori, for an extremal positive μ-harmonic function h on the Heisenberg group G with $G_\mu = G$, this requirement is not so useful. Indeed, by Corollary 3.6, the characters of G are not induced from proper finite index subgroups. Moreover, by Definition 3.3, if h is induced from an infinite index subgroup G_0, it is also induced from all the finite index subgroup of G that contain G_0.

Corollary 3.6. *Let G be a finitely generated group and μ a positive measure on G with finite support such that $G_\mu = G$. A μ-harmonic character χ of G is never induced from a proper subgroup $G_0 \subset G$.*

Proof. Since $G_\mu = G$, the restriction μ_0 of μ to G_0 satisfies $\mu_0 < \mu$. Since χ is a character, one has $P_{\mu_0}\chi = \alpha\chi$ with some constant $\alpha > 0$. Since $P_\mu \chi = \chi$, one has $\alpha < 1$. Therefore, one has $\lim_{n\to\infty} P^n_{\mu_0}\chi = 0$, and the μ-harmonic function χ is not induced from G_0. $\square$

3.3 Double induction

The following lemma tells us that two successive inductions of a positive harmonic function is equivalent to a direct induction.

Let G be a finitely generated group.

Lemma 3.7. *Let $\mu_0 < \mu'_0 < \mu$ be positive measures on G with finite support. Let h_0 be a positive μ_0-harmonic function on G. The following are equivalent:*
(i) the function $h := \lim_{n\to\infty} P^n_\mu h_0$ is finite.
(ii) the functions $h'_0 := \lim_{n\to\infty} P^n_{\mu'_0} h_0$ and $h' := \lim_{n\to\infty} P^n_\mu h'_0$ are finite.

In this case, the two induced μ harmonic functions are equal $h = h'$.

Proof of Lemma 3.7. $(i) \implies (ii)$ Since $h_0 \leq h$, one has the inequalities $P^n_{\mu'_0} h_0 \leq P^n_{\mu'_0} h \leq P^n_\mu h = h$ and $h'_0 \leq h$. Therefore, one also has the inequalities $P^n_\mu h'_0 \leq P^n_\mu h = h$ and $h' \leq h$.

$(ii) \Longrightarrow (i)$ Since $h_0 \leq h'_0$, one has $P^n_\mu h_0 \leq P^n_\mu h'_0$ and $h \leq h'$. $\square$

3.4 Induction of characters

We give now a few conditions that have to be satisfied in order for the induction of a harmonic character to be a finite function.

Let G be a finitely generated group and μ be a positive measure on G with finite support S such that $G = G_\mu$. We write $\mu = \mu_0 + \mu_1$ as a sum of two positive measures and set $S_0 := \mathrm{supp}\mu_0$ and $G_0 := G_{\mu_0}$. Let χ_0 be a μ_0-harmonic character of G_0 that we extend by 0 as a function on G. We denote by

$$Z_G(G_0) := \{g \in G \mid gg_0 = g_0 g \text{ for all } g_0 \text{ in } G_0\}$$

the centralizer of G_0 in G, and by

$$N_G(G_0, \chi_0) := \{g \in G \mid gg_0g^{-1} \in G_0 \text{ and } \chi_0(gg_0g^{-1}) = \chi_0(g_0) \text{ for all } g_0 \text{ in } G_0\}$$

the normalizer of (G_0, χ_0) in G.

Lemma 3.8. *If the induced μ-harmonic function h_{G_0,χ_0} is finite, then :*
(i) The measure μ_0 is the restriction of μ to G_0 and $S_0 = S \cap G_0$.
(ii) The subgroup G_0 has infinite index in G.
(iii) One has $G^+_{\mu_1} \cap G_0 = \emptyset$.
(iv) One has $G^+_{\mu_1} \cap Z_G(G_0) = \emptyset$.
(v) One has $G^+_{\mu_1} \cap N_G(G_0, \chi_0) = \emptyset$.

Remark 3.9. - In particular, the supports S_0 of μ_0 and S_1 of μ_1 are disjoint and the semigroup $G^+_{\mu_1}$ does not meet the center Z of G.

- Note also that if one wants h_{G_0,χ_0} to be μ-extremal, the group G_0 must be generated by S_0. Indeed if this is not the case, the μ_0-harmonic character χ_0 is not μ_0-extremal and, by Lemma 3.2, the function h_{G_0,χ_0} is not μ-extremal.

- The above conditions are not the only necessary conditions, as we will see in Chapter 5.

Proof of Lemma 3.8. (i) This is equivalent to $\mu_1(G_0) = 0$ which follows from (iii).

(ii) This follows from (iii). Indeed pick an element s_1 in the support of μ_1, if the index were finite, there would exist a positive power s_1^d belonging to G_0.

(iii) This follows from (v) because $G_0 \subset N_G(G_0, \chi_0)$.

(iv) This follows from (v) because $Z_G(G_0) \subset N_G(G_0, \chi_0)$.

(v) This point is the main content of Lemma 3.8. We proceed by contraposition. Let S_1 be the support of μ_1 and $w_1 = s_1 \dots s_\ell \in S_1^\ell$, with $\ell \geq 1$ be a word such that $\dot{w}_1$ belongs to $N_G(G_0, \chi_0)$.

The proof relies on a cautious analysis of the words that occur in Equality (2.2). We recall the notation $\mu_{1,w_1} := \mu_{1,s_1} \cdots \mu_{1,s_\ell} > 0$. We will denote P_{w_1} for the operator of left translation by $\dot{w}_1 := s_1 \cdots s_\ell \in G$; it is given by $P_{w_1} h(g) = h(\dot{w}_1 g)$ for all function h on G and all g in G. One computes

$$\begin{aligned}
P_\mu^{n+\ell}\chi_0(\dot{w}_1^{-1}) &\geq \sum_{1\leq i\leq n} \mu_{1,w_1} P^i_{\mu_0} P_{w_1} P^{n-i}_{\mu_0} \chi_0(\dot{w}_1^{-1}) \\
&= \sum_{1\leq i\leq n} \mu_{1,w_1} P^i_{\mu_0} P_{w_1} \chi_0(\dot{w}_1^{-1}) \qquad \text{because } \chi_0 \text{ is } \mu_0\text{-harmonic} \\
&= \sum_{1\leq i\leq n} \mu_{1,w_1} \sum_{w_0 \in S_0^i} \mu_{0,w_0} \chi_0(\dot{w}_1 \dot{w}_0 \dot{w}_1^{-1}) \quad \text{by definition of } P_{\mu_0} \\
&= \sum_{1\leq i\leq n} \mu_{1,w_1} \sum_{w_0 \in S_0^i} \mu_{0,w_0} \chi_0(\dot{w}_0) \quad \text{because } \dot{w}_1 \text{ normalizes } \chi_0 \\
&= \sum_{1\leq i\leq n} \mu_{1,w_1} \chi_0(0) \;=\; n\mu_{1,w_1} \quad \text{because } \chi_0 \text{ is } \mu_0\text{-harmonic.}
\end{aligned}$$

This goes to infinity with n, and the induced function is not finite. □

3.5 Negligible trajectories

We now discuss a lemma on non-induced extremal positive μ-harmonic functions. This lemma will be useful for the proof of the Z-semiinvariance of these functions on the Heisenberg group.

Let G be a finitely generated group and μ be a positive measure on G with finite support S generating G.

For every word $w = s_1 \dots s_n \in S^n$, we define $k_w \geq 0$ to be the smallest integer k for which we can write $w = w_0 \dots w_k$ as a concatenation of *strongly non-generating* subwords w_j. *Strongly non-generating* means that there exists an infinite index subgroup G_j of G containing all the letters s_i occuring in the subword w_j. The following lemma tells us that the words with k_w bounded are negligible in the sum (2.2) for a non-induced μ-harmonic function.

Lemma 3.10. *Let h be a non-induced positive μ-harmonic function on G Then, for all $k \geq 0$, and g in G, the partial sums*

$$I_{n,k}(g) := \sum_{w \in S^n \, , \, k_w \leq k} \mu_w \, h(\dot{w}g) \, . \tag{3.3}$$

converge to 0 *when* $n \to \infty$.

Proof of Lemma 3.10. Fix g in G. For w in S^n we introduce the maximal *strongly non-generating* suffix σ of w. Suffix means that one can write $w = w'\sigma$. We denote by $S_{0,w}$ the set of letters of σ and by $\ell_{0,w}$ the length of σ. Since there are only finitely many subsets S_0 of S, we can write $I_{n,k}(g)$ as a finite sum $I_{n,k}(g) = \sum I_{n,k,S_0}(g)$ where $I_{n,k,S_0}(g)$ involves the words w for which $S_{0,w} = S_0$. Here this finite sum is indexed by the subsets S_0 of S that generates an infinite index subgroup of G. We argue by induction on k.

First assume $k = 0$. For such $S_0 \subset S$ one has

$$I_{n,0,S_0}(g) \quad \leq \quad \sum_{w_0 \in S_0^n} \mu_{w_0} \, h(\dot{w}_0 g) \quad = \quad P^n_{\mu_0} h(g),$$

where μ_0 is the restriction of μ to S_0. By Definitions 3.3 and 3.4, since h is non-induced and since S_0 generates an infinite index subgroup of G, the sequence $P^n_{\mu_0} h(g)$ converges to 0 when $n \to \infty$, and the claim (3.3) is true for $k = 0$.

Now assume $k \geq 1$. Fix $\varepsilon_0 > 0$. Since h is non-induced, as above, we can choose ℓ_0 such that, for any subset S_0 of S that generates an infinite index subgroup of G, one has $P^{\ell_0}_{\mu_0} h(g) \leq \varepsilon_0$ where μ_0 is the restriction of μ to S_0. We decompose the sum $I_{n,k,S_0}(g)$ as a sum of two terms

$$I_{n,k,S_0}(g) = I'_{n,k,S_0,\ell_0}(g) + I''_{n,k,S_0,\ell_0}(g)$$

where $I'_{n,k,S_0,\ell_0}(g)$ involves the words w for which $\ell_{0,w} \geq \ell_0$ and $I''_{n,k,S_0,\ell_0}(g)$ involves the words w for which $\ell_{0,w} < \ell_0$.

Bounding I'_n. One computes, using the μ-harmonicity of h,

$$\begin{aligned} I'_{n,k,S_0,\ell_0}(g) &\leq \sum_{w_0 \in S_0^{\ell_0}} \mu_{w_0} \sum_{w_1 \in S^{n-\ell_0}} \mu_{w_1}\, h(\dot{w}_1 \dot{w}_0 g) \\ &\leq \sum_{w_0 \in S_0^{\ell_0}} \mu_{w_0}\, h(\dot{w}_0 g) \;\leq\; \varepsilon_0 \end{aligned}$$

Bounding I''_n. One decomposes I''_{n,k,S_0,ℓ_0} as a finite sum

$$I''_{n,k,S_0,\ell_0}(g) = \textstyle\sum_\sigma \mu_\sigma I''_{n,k,\sigma}(g)$$

over the finitely many words σ of length $\ell < \ell_0$ where

$$\begin{aligned} I''_{n,k,\sigma}(g) &\leq \sum_{w' \in S^{n-\ell},\, k_{w'} \leq k-1} \mu_{w'}\, h(\dot{w}'\dot{\sigma} g) \\ &\leq I_{n-\ell,k-1}(\dot{\sigma} g)\,. \end{aligned}$$

Therefore by the induction hypothesis one has $\lim\limits_{n\to\infty} I''_{n,k,\sigma}(g) = 0$. Since ε_0 can be chosen arbitrarily small, one deduces that $\lim\limits_{n\to\infty} I_{n,k}(g) = 0$. $\square$

3.6 When G_μ^+ meets the center

There is a simple case where the semiinvariance of μ-harmonic functions is easy to prove, namely when G_μ^+ meets the center.

Let G be a finitely generated group, Z be the center of G and μ be a finite positive measure on G.

Lemma 3.11. *Assume that an element z of Z belongs to the semigroup G_μ^+. Then, for every extremal positive μ-harmonic function h on G there exists a constant $q > 0$ such that $h_z = qh$.*

We recall that h_z is the function $g \mapsto h(gz)$.

Proof of Lemma 3.11. This is a slight generalization of the Choquet–Deny Theorem. Let $n \geq 1$ be an integer such that z is in the support of μ^{*n}. The equality $h = P_\mu^n h$ is of the form $h = \alpha h_z + h'$ where $\alpha > 0$ and h' is a positive function. Since the function h_z is also μ-harmonic, the extremality of h implies that h_z is proportional to h. $\square$

4 Z-Invariance of harmonic functions

In all this chapter we keep the following notation :

$$\begin{array}{l} G \textit{ is a finite index subgroup in } H_3(\mathbb{Z}),\ Z \textit{ is the center of } G, \\ \mu \textit{ is a positive measure with finite support } S \textit{ such that } G_\mu = G, \\ h \textit{ is a positive } \mu\textit{-harmonic function on } G. \end{array} \tag{4.1}$$

In this chapter we will mainly focus on non-induced μ-harmonic functions (see Definitions 3.3 and 3.4) and we will prove that they are Z-invariant.

We begin by a lemma that explain our choices in Definition 3.4.

Lemma 4.1. *The positive μ-harmonic function h is non-induced if and only if $\lim\limits_{n\to\infty} P^n_{\mu_0} h = 0$, for all restriction μ_0 of μ to an abelian subset S_0 of S.*

Proof. By Definition 3.4, "h non-induced" means that h is non-induced from an infinite index subgroup G_0 of G. Note that the subgroups $G_0 \subset G$ of infinite index are exactly the abelian subgroups. Indeed any two non-commuting elements of $H_3(\mathbb{Z})$ generate a finite index subgroup of $H_3(\mathbb{Z})$. □

Remark 4.2. A finite index subgroup G of $H_3(\mathbb{Z})$ is not always isomorphic to $H_3(\mathbb{Z})$, but it contains a finite index subgroup that is isomorphic to $H_3(\mathbb{Z})$. Extending our theorem 1.1 to these groups G would be straightforward but not so interesting.

The main reason we want to work with this slightly larger class of group G in this chapter is that, in the "proof by induction" of Proposition 4.10, we need to apply the "induction hypothesis" to a finite index subgroup of G.

4.1 Semiinvariance of harmonic functions

In this section we prove that h is semiinvariant by one central element. The proofs below are self-contained. They are inspired by the more intuitive proofs for the south-west measure in [2] that rely on Young diagrams.

Proposition 4.3. *Keep notation* (4.1) *and assume that h is μ-extremal and non-induced. Then there exist $z \neq 0$ in Z and $q > 0$ such that $h_z = qh$.*

Proof of Proposition 4.3. By Lemma 3.11, we can assume $S \cap Z = \emptyset$.

For $n \geq 2$, we introduce a symmetric relation on S^n given by

$$\begin{aligned}\mathcal{R}_n : \ = \ &\{(w,w') \in S^n \times S^n \mid w = w_0 s s' w_0' \text{ and } w' = w_0 s' s w_0' \text{ where} \\ &\qquad w_0 \in S^i,\ w_0' \in S^{n-i-2}\ ,\ s \in S,\ s' \in S \text{ with } ss' \neq s's\}.\end{aligned}$$

This means that w and w' are obtained from one another by switching two consecutive non-commuting letters. For a word $w \in S^n$ we let

k_w = the number of pairs of consecutive non-commuting letters in w.

Since G is the Heisenberg group $H_3(\mathbb{Z})$ and since $S \cap Z = \emptyset$, this number k_w is the same as the one occuring in Lemma 3.10. Indeed, there exists a unique partition $S = S_0 \cup \ldots \cup S_\ell$ of S such that two elements s, s' of S commute if and only if they belong to the same S_i. To go on the proof of Proposition 4.3, we will need the following two lemmas. *Proof to be continued.*

We denote by $p_0 := \max_{s,s' \in S} |c_{s,s'}|$ where the integers $c_{s,s'}$ are defined in (2.4).

Lemma 4.4. *For $(w,w') \in \mathcal{R}_n$, one has*
(i) $\dot{w} = \dot{w}' z_0^p$ *for some integer p with $0 < |p| \leq p_0$,*
(ii) $\mu_{w'} = \mu_w$ *and*
(iii) $|k_{w'} - k_w| \leq 2$.

Proof of Lemma 4.4. (*i*) This follows from the equality $ss' = s's\, z_0^{c_{s,s'}}$.

(*ii*) The same letters occur in w and w'.

(*iii*) The pairs of adjacent letters in w and w' are the same except for at most two of them. □

Lemma 4.5. *For g in G, one has* $h(g) \leq \sum_{0<|p|\leq p_0} h(z_0^p g)$.

Proof of Lemma 4.5. Replacing h by its translate h_g, we can assume that $g = 0$. We want to prove that the following difference is non-positive :

$$D \ := \ h(0) - \sum_{0<|p|\leq p_0} h(z_0^p) \ \leq \ 0\,.$$

Using notations (2.2), we compute D as

$$D \ = \ \sum_{w \in S^n} \mu_w h(\dot{w}) - \sum_{0<|p|\leq p_0} \sum_{w' \in S^n} \mu_{w'} h(\dot{w}' z_0^p)\,.$$

We fix $\varepsilon_0 > 0$ and $k_0 \geq 2 + 2\varepsilon_0^{-1}$. By Lemma 3.10, one can find an integer $n \geq 1$ such that the first sum limited at the trajectories w for which $k_w < k_0$ is bounded by ε_0. Using the fact that, for w in S^n, the fiber

$$\{(w, w') \mid w' \in S^n \ , \ (w, w') \in \mathcal{R}_n\}$$

of the maps $\mathcal{R}_n \mapsto S^n$; $(w, w') \mapsto w$ has cardinality k_w, one gets

$$D \ \leq \ \varepsilon_0 + \sum_{\substack{(w,w') \in \mathcal{R}_n, \\ k_w \geq k_0}} \left(\frac{\mu_w}{k_w} h(\dot{w}) - \frac{\mu_{w'}}{k_{w'}} \sum_{0<|p|\leq p_0} h(\dot{w}' z_0^p) \right) .$$

By Lemma 4.4, the element $\dot{w}$ is equal to at least one of those $\dot{w}' z_0^p$, therefore one gets

$$D \ \leq \ \varepsilon_0 + \sum_{\substack{(w,w') \in \mathcal{R}_n, \\ k_w \geq k_0}} \mu_w \frac{k_{w'} - k_w}{k_w k_{w'}} h(\dot{w}) \ .$$

By Lemma 4.4, one has $|k_{w'} - k_w| \leq 2$, and $2/k_{w'} \leq 2/(k_0 - 2) \leq \varepsilon_0$, and

$$D \ \leq \ \varepsilon_0 + \varepsilon_0 \sum_{(w,w') \in \mathcal{R}_n} \frac{\mu_w}{k_w} h(\dot{w}) \, .$$

Using again that k_w is the cardinality of the fiber and using the harmonicity of h, one gets

$$D \ \leq \ \varepsilon_0 + \varepsilon_0 \sum_{w \in S^n} \mu_w h(\dot{w}) \ = \ \varepsilon_0 + \varepsilon_0 \, h(0) \, .$$

Since ε_0 can be chosen arbitrarily small, this gives $D \leq 0$ as expected. $\square$

End of proof of Proposition 4.3. Lemma 4.5 tells us that there exists a finite subset $F \subset Z \smallsetminus \{0\}$ and a positive μ-harmonic function h' such that

$$\textstyle\sum_{z \in F} h_z = h + h'.$$

Since the cone $\mathcal{H}_\mu^+$ is well-capped and reticulated, both the function h' and the sum $\sum_{z \in F} h_z$ admit a unique desintegration in μ-extremal functions (see Section 2.1). Hence, since all the positive μ-harmonic functions h and h_z are μ-extremal, the function h has to be proportional to one of these translates h_z. $\square$

Remark 4.6. We now want to deduce from the semi-invariance of h proven in Proposition 4.3, the Z-invariance of h. This is not a general fact. Indeed, the harmonic function h in Case 3.b) of Theorem 5.10 can be Z-semiinvariant but is not Z-invariant. Hence, we have to use once more the assumption that h is not induced. One technical difficulty comes from the fact that, when $G_\mu^+ \neq G$, the cone $\mathcal{H}_\mu^+$ often does not have a compact basis. This prevents us from using the same arguments as in [12].

4.2 z-invariance and Z-invariance

We first notice that in order to prove the Z-invariance of a positive μ-harmonic function h on the Heisenberg group G, it is enough to check that it is invariant under one non trivial element of Z.

Lemma 4.7. *Keep notation* (4.1) *and assume that there exists $z \neq 0$ in Z such that $h_z = h$. Then h is Z-invariant. In particular, if h is μ-extremal, it is proportional to a μ-harmonic character of G.*

Note that in this lemma the positive μ-harmonic function h is not assumed to be μ-extremal.

Proof of Lemma 4.7. We write $z = z_0^p$. We can assume that p is the smallest positive integer for which $h_z = h$. We can also assume that h is extremal in the convex cone

$$\mathcal{H}_{\mu,z}^+ := \{\text{positive, } \mu\text{-harmonic and } z\text{-invariant functions on } G\}.$$

Therefore the functions $h_{z_0^i}$, for $i = 1, \ldots, p$, are non-proportional functions which are extremal in this cone, and the function $f := h_{z_0} + \cdots + h_{z_0^p}$ is μ-harmonic and Z-invariant.

We claim that f is extremal among the μ-harmonic functions on G/Z. Indeed, assume that one can write $f = f' + f''$ with both f' and f'' positive, μ-harmonic and Z-invariant. We argue as in the proof of Proposition 4.3 with the well-capped and reticulated cone $\mathcal{H}_{\mu,z}^+$. Both the function f' and f admit a unique desintegration in extremal functions in this cone (see Section 2.1). Hence, since all the functions h_z are extremal in this cone, one must have $f' = \sum_{1\leq i\leq p} \lambda_i h_{z_0^i}$ for some constants $\lambda_i \geq 0$. Since f' is z_0-invariant, all these constants are equal to some $\lambda \geq 0$ and one has $f' = \lambda f$. This proves that f is extremal among the μ-harmonic functions on G/Z.

Since G/Z is abelian, by the Choquet–Deny Theorem, this function f is a μ-harmonic character of G. Therefore, by Lemma 2.1, this function f is μ-extremal and one has $p = 1$. This means that h is Z-invariant. $\square$

4.3 Z-invariance when V_μ^+ contains a line

In this section, we finish the proof of our main Theorem 1.1 when the cone V_μ^+ is the plane or a half-plane, see (2.5).

Proposition 4.8. *Keep notation* (4.1)*, assume that V_μ^+ contains a line and that the μ-harmonic function h is not induced. Then h is Z-invariant.*

Proof of Proposition 4.8. We can assume that h is μ-extremal and apply Proposition 4.3. Then our claim follows from the following slightly stronger Proposition 4.9. This stronger version will also be useful in Chapter 5. $\square$

Proposition 4.9. *Keep notation* (4.1) *and assume that the cone V_μ^+ contains a line. Assume also that there exists $z \neq 0$ in Z and $q > 0$ such that $h_z = qh$. Then the function h is Z-invariant.*

Proof of Proposition 4.9. According to Lemma 4.7, it is enough to prove that $q = 1$. Replacing h by a multiple of a suitable translate, we can assume that $h(0) = 1$. Replacing z by its inverse if necessary, we can also assume that $q \geq 1$. Since the cone V_μ^+ contains a line, there exists two words w_0 in S^{n_0} and w_0' in $S^{n_0'}$ whose product is in the center:

$$\dot{w}_0 \dot{w}_0' = z^a \text{ for some } a \text{ in } \mathbb{Z}.$$

Since the cone V_μ^+ is not a line, there exists also a word w_1 in S^{n_1} such that

$$\dot{w}_0 \dot{w}_1 \dot{w}_0^{-1} \dot{w}_1^{-1} = z^b \text{ for some } b \geq 1. \tag{4.2}$$

Note that one might have to switch w_0 and w_0' to ensure that $b \geq 1$.

Assume, for a contradiction, that $q \neq 1$, so that $q > 1$. Choose an integer $\ell \geq 1$ such that $C := \mu_{w_0} \mu_{w_0'}\, q^{a+b\ell} > 1$. Notice the equality, for all $k \geq 1$,

$$\dot{w}_0^k\, \dot{w}_1^\ell\, {\dot{w}_0'}^k\, \dot{w}_1^{-\ell} = z^{ak+b\ell k}. \tag{4.3}$$

Note that both Equations (4.2) and (4.3) rely on the bilinear formula (2.4) for the commutators in the Heisenberg group. Now we can compute with $n := kn_0+\ell n_1+kn_0'$,

$$\begin{array}{rcl} h(\dot{w}_1^{-\ell}) & = & P_\mu^n h(\dot{w}_1^{-\ell}) \\ & \geq & \mu_{w_0}^k \mu_{w_1}^\ell \mu_{w_0'}^k \, h(\dot{w}_0^k, \dot{w}_1^\ell \, \dot{w}_0'^{\,k} \, \dot{w}_1^{-\ell}) \\ & \geq & \mu_{w_0}^k \mu_{w_1}^\ell \mu_{w_0'}^k \, q^{ak+b\ell k} \;\; = \;\; \mu_{w_1}^\ell C^k \end{array}$$

Since $C > 1$ and since this inequality is valid for all integer $k \geq 1$ one gets a contradiction. This proves that $q = 1$. $\square$

4.4 Z-invariance when V_μ^+ contains no line

In this section, we finish the proof of our main Theorem 1.1 when the cone V_μ^+ is properly convex, see (2.5).

Proposition 4.10. *Keep notation* (4.1)*, assume that V_μ^+ contains no line and that the μ-harmonic function h is not induced. Then h is Z-invariant.*

Beginning of proof of Proposition 4.10. The proof is by induction on the cardinality of the support S of μ, simultaneously for all the finite index subgroups G of $H_3(\mathbb{Z})$. We will use the induction hypothesis inside the proof of Lemma 4.12.

First step We begin the proof by a few reduction steps.

We can assume that h is μ-extremal. Indeed by Definitions 3.3 and 3.4, almost all the μ-extremal μ-harmonic positive functions f that occur in the desintegration $h = \int_{\mathcal{E}} f \, \mathrm{d}\alpha(f)$ of h are non-induced. In this case, by Proposition 4.3, there exist $z = z_0^p \neq 0$ in Z and $q > 0$ such that $h_z = qh$. According to Lemma 4.7, it is enough to prove that $q = 1$.

a. We can assume $z = z_0$. Because we can replace h by the function $f := q_0^{-1} h_{z_0} + \cdots + q_0^{-p} h_{z_0^p}$ where $q_0 > 0$ is chosen so that $q_0^p = q$. This function f is μ-harmonic and Z-semiinvariant. It might not be μ-extremal, but this property will not be used in the argument below.

b. We can assume $S \cap Z = \emptyset$. Indeed, by **a**, if μ_Z is the restriction of μ to the center, one has $P_{\mu_Z} h = \lambda h$ for a constant $0 \leq \lambda < 1$. But then the function h is harmonic for the measure $(1-\lambda)^{-1}(\mu - \mu_Z)$. It might not be extremal for this measure, but, as we just said, this is not important.

c. We can assume $h(0) = 1$. Because we can replace h by a multiple of a suitable translate.

d. We can assume $q < 1$. Because we can replace the generator z_0 by its inverse. We are now looking for a contradiction.

Second step We now can enter the key argument of the proof. Since the cone V_μ^+ is properly convex and since $S \cap Z = \emptyset$, we can find a partition of the support of μ in two non-empty subsets

$$S = S_1 \cup S_2 \,, \tag{4.4}$$

such that

$$c_{s_1,s_2} \geq 1 \quad \text{for all } s_1 \text{ in } S_1 \text{ and } s_2 \text{ in } S_2. \tag{4.5}$$

The partition (4.4) is given by a suitable decomposition $V_\mu^+ = V_1^+ \cup V_2^+$ of the properly convex cone V_μ^+ in two cones V_1^+ and V_2^+ of disjoint interior so that the inequalities (4.5) will follow from the bilinear formula (2.4) for the commutators in the Heisenberg group.

We will use the decomposition $\mu = \mu_1 + \mu_2$ where $\mu_1 := \mathbf{1}_{S_1}\, \mu$ and where $\mu_2 := \mathbf{1}_{S_2}\, \mu$. The proof again starts with the equality (2.2) which tells us that, for all $n \geq 1$,

$$1 = h(0) \;\; = \sum_{w \in S^n} \mu_w h(\dot{w}) \,. \tag{4.6}$$

We will cut this sum into pieces parametrized by pairs $(w_1, w_2) \in S_1^{n_1} \times S_2^{n_2}$, with $n_1 + n_2 = n$. We define

$$B_{w_1,w_2} \;\; = \;\; \{w \in S^n \;\text{ containing } w_1 \text{ and } w_2 \text{ as subwords}\}$$

For instance when $w_1 = 11$ and $w_2 = 23$, one has

$$B_{w_1,w_2} \;\; = \;\; \{1123, 1213, 1231, 2113, 2131, 2311\}.$$

This allows us to write the above sum (4.6) as

$$1 \;\; = \sum_{n_1+n_2=n} \; \sum_{w_1 \in S_1^{n_1}} \; \sum_{w_2 \in S_2^{n_2}} \; \sum_{w \in B_{w_1,w_2}} \mu_w h(\dot{w}) \,. \tag{4.7}$$

For every w in B_{w_1,w_2}, we write, using iteratively (2.4) and (4.5),

$$\dot{w} = \dot{w}_2 \dot{w}_1 z_0^{n_w} \;\text{ for some integer } n_w \geq 1. \tag{4.8}$$

Then Equality (4.7) becomes

$$1 \;=\; \sum_{n_1+n_2=n} \; \sum_{w_1\in S_1^{n_1}} \; \sum_{w_2\in S_2^{n2}} \; \mu_{w_1}\mu_{w_2} h(\dot w_2 \dot w_1)\,(\sum_{w\in B_{w_1,w_2}} q^{n_w}\;)\,. \qquad (4.9)$$

To pursue our analysis, we will need the following lemma which bounds this last sum. *Proof to be continued.*

Lemma 4.11. *For all w_1 in $S_1^{n_1}$ and w_2 in $S_2^{n_2}$, one has*

$$\sum_{w\in B_{w_1,w_2}} q^{n_w} \;\;\le\;\; \eta(q)^{-1} < \infty. \qquad (4.10)$$

where $\eta(q) := \prod_{i\geq 1}(1-q^i) > 0$.

Note that this upper bound does not depend on (w_1, w_2).

Proof of Lemma 4.11. For each word $w = s_1 \dots s_n$ in B_{w_1,w_2}, we set

$$m_w := |\{(i,j) \mid 1 \le i < j \le n \;\text{ and }\; s_i \in S_1,\; s_j \in S_2\}|.$$

Condition (4.5) implies that

$$m_w \le n_w \;\text{ for all } w \text{ in } B_{w_1,w_2}$$

A word $w = s_1 \dots s_n$ in B_{w_1,w_2} is determined by the increasing sequence $1 \le i_1 < i_2 < \dots < i_{n_2} \le n$ of places i where s_i belongs to S_2, and m_w is given by

$$m_w \;\;=\;\; (i_{n_2}-n_2) + \cdots + (i_2-2) + (i_1-1)\,.$$

Therefore, for all $m \ge 1$, the number

$$p(n_1, n_2, m) := |\{w \in B_{w_1,w_2} \mid m_w = m\}|$$

is equal to the number of partitions of m by n_2 non-increasing integers $a_1, \dots, a_{n_2}$ bounded by n_1 :

$$p(n_1, n_2, m) = |\{n_1 \ge a_1 \ge \dots \ge a_{n_2} \ge 0 \;\text{ and }\; m = a_1 + \cdots + a_{n_2}\}|\,.$$

This quantity is bounded by the partition function

$$p(m) = |\{a_1 \ge \dots \ge a_k \ge \dots \ge 0 \;\text{ and }\; m = a_1 + \cdots + a_k + \cdots\}|.$$

The generating function of the partition function is

$$\sum_{m\geq 0} p(m)q^m \;=\; \prod_{i>0}(1+q^i+q^{2i}+\cdots) \;=\; \prod_{i>0}(1-q^i)^{-1} \;=\; \eta(q)^{-1}\,.$$

We now collect the sequence of inequalities we have just proven

$$\sum_{w\in B_{w_1,w_2}} q^{n_w} \leq \sum_{w\in B_{w_1,w_2}} q^{m_w} = \sum_{m\geq 0} p(n_1,n_2,m)q^m \leq \sum_{m\geq 0} p(m)q^m = \eta(q)^{-1}$$

and we obtain the bound (4.10) we were looking for. □

End of proof of Proposition 4.10. We plug Inequality (4.10) in Formula (4.9) and we obtain, for all $n\geq 1$

$$\sum_{n_1+n_2=n} P_{\mu_1}^{n_1}P_{\mu_2}^{n_2}h(0) \geq \eta(q) > 0\,. \tag{4.11}$$

This contradicts the following Lemma 4.12 □

Lemma 4.12. *With the same notation. In particular $\mu=\mu_1+\mu_2$ with S_1 and S_2 disjoint, and h is a non-induced μ-harmonic function on G.*
a) One has $\lim\limits_{n\to\infty} P_{\mu_1}^n h = 0$ and $\lim\limits_{n\to\infty} P_{\mu_2}^n h = 0$.
b) One also has

$$\lim_{n\to\infty}\sum_{n_1+n_2=n} P_{\mu_1}^{n_1}P_{\mu_2}^{n_2}h = 0\,. \tag{4.12}$$

Proof of Lemma 4.12. a) Let us prove it for μ_1.

If S_1 is abelian, this follows from the assumption that h is non-induced.

If S_1 is not abelian, we will use our induction hypothesis. Assume, for a contradiction, that the μ_1-harmonic function $h' := \lim\limits_{n\to\infty} P_{\mu_1}^n h$ is non-zero. By Lemma 3.2, this function h' is μ_1-extremal and satisfies

$$\lim_{n\to\infty} P_\mu^n h' = h. \tag{4.13}$$

By Lemma 3.7, this μ_1-harmonic function h' is not induced and, since S_1 is smaller than S, the function h' is a μ_1-harmonic character of the group G_{μ_1}. Since this group G_{μ_1} has finite index in the group G_μ, Lemma 3.8.*ii* tells us that the function $\lim\limits_{n\to\infty} P_\mu^n h'$ is not finite. This contradicts (4.13).

b) The argument is the same as for Lemma 3.10, but is simpler. We fix g in G and $\varepsilon_0>0$. According to point a), there exists $N_1\geq 1$ such that

$P_{\mu_1}^{N_1} h(g) \leq \varepsilon_0$. Let I_n be the left-hand side of (4.12). We decompose $I_n(g)$ as the sum of two terms

$$I_n(g) = I'_n(g) + I''_n(g)$$

where $I'_n(g)$ involves the terms with $n_1 \geq N_1$ and $I''_n(g)$ involves the terms with $n_1 < N_1$

Bounding I'_n. One computes, using the μ-harmonicity of h,

$$\begin{aligned} I'_n(g) &= \sum_{n'_1+n_2=n-N_1} P_{\mu_1}^{N_1} P_{\mu_1}^{n'_1} P_{\mu_2}^{n_2} h(g) \\ &\leq P_{\mu_1}^{N_1} P_{\mu}^{n-N_1} h(g) \;=\; P_{\mu_1}^{N_1} h(g) \;\leq\; \varepsilon_0 \,. \end{aligned}$$

Bounding I''_n. One decomposes $I''_n(g)$ as a finite sum

$$I''_n(g) = \sum_{n_1<N_1} \sum_{w_1 \in S_1^{n_1}} \mu_{w_1} P_{\mu_2}^{n-n_1}(\dot{w}_1 g)$$

over the finitely many words w_1 of length $n_1 < N_1$. By point a), all terms of the sum go to 0 so that one has $\lim\limits_{n\to\infty} I''_n(g) = 0$.

Since ε_0 can be chosen arbitrarily small, one deduces $\lim\limits_{n\to\infty} I_n(g) = 0$. □

This ends the proof of Proposition 4.10.

We can now complete the proof of our main theorem 1.1.

Proof of Theorem 1.1. Let h be an extremal positive μ-harmonic function on G. By Propositions 4.8 and 4.10, either h is Z-invariant or h is induced.

Assume first that h is Z-invariant, then h is an extremal positive harmonic function on the abelian group G/Z and, by Choquet-Deny theorem (see Section 1.4), h is proportional to a character of G.

Assume now that h is induced. Since h is extremal, as we have seen in Lemma 3.2 and Definitions 3.3 and 3.4, there exist an infinite index subgroup G_0 of G and an extremal μ_0-harmonic function on G_0 where μ_0 is the restriction of μ to G_0, such that the function h is a translate of the function h_{G_0,h_0} induced from h_0. Since G is the Heisenberg group, this group G_0 is abelian and, by Choquet-Deny theorem, the extremal μ_0-harmonic function h_0 is proportional to a character of G_0. □

5 Existence of induced harmonic functions

In this chapter, except for Section 5.5, we will keep the following notations :

$$\begin{array}{l} G = H_3(\mathbb{Z}) \textit{ is the Heisenberg group, } Z \textit{ is the center of } G, \\ \mu \textit{ is a positive measure with finite support } S \textit{ such that } G_\mu = G, \\ S_0 \subset S \textit{ is a maximal abelian subset, } \mu_0 := \mathbf{1}_{S_0}\mu,\ G_0 := G_{\mu_0}, \\ \chi_0 \textit{ is a } \mu_0\textit{-harmonic character of } G_0 \textit{ and } \mu_1 := \mu - \mu_0. \end{array} \tag{5.1}$$

By Theorem 1.1, we know that an extremal positive μ-harmonic functions on G which is not proportional to a character is proportional to a translate of an induced μ-harmonic function of the form h_{G_0,χ_0}. Note that the maximality of S_0 is guaranteed by Lemma 3.8.*iv*.

We will give in this chapter a necessary and sufficient condition for the induced μ-harmonic function h_{G_0,χ_0} to be finite.

In Lemma 3.8.*iv*, we have already found that the following condition is necessary : $G^+_{\mu_1} \cap Z_G(G_0) = \emptyset$. Since V_{μ_0} is a line, one can check that this condition is equivalent to :

$$S_1 \cap Z = \emptyset \quad \text{and} \quad V^+_{\mu_1} \cap V_{\mu_0} = \{0\}\,. \tag{5.2}$$

We will assume that it is satisfied.

We distinguish two cases according to the rank of the abelian group G_0.

5.1 Induction of characters when $\operatorname{rank} G_0 = 1$

In this section we give the necessary and sufficient condition for the induced function h_{G_0,χ_0} to be finite when $\operatorname{rank} G_0 = 1$.

Note that, since S_0 is maximal abelian in S, one has the equivalence :

$$\operatorname{rank} G_0 = 1 \Longleftrightarrow G_0 \cap Z = \{0\}.$$

Proposition 5.1. *Keep notation* (5.1). *Assume* (5.2) *and* $\operatorname{rank} G_0 = 1$. *Then the induced harmonic function* $h := h_{G_0,\chi_0}$ *is finite if and only if the probability measure* $\widetilde{\mu}_0 := \chi_0\mu_0$ *on* G_0 *is not centered.*

Remark 5.2. - The measure $\widetilde{\mu}_0 = \chi_0\mu_0$ is a probability measure because χ_0 is a μ_0-harmonic character.
- The condition $\widetilde{\mu}_0$ *centered* means, as usual, that $\sum_{s\in S_0} \widetilde{\mu}_{0,s}\,\overline{s} = 0$ in V, where $\overline{s}$ is the image of s in V.
- This condition $\widetilde{\mu}_0$ *non-centered* is always satisfied when V^+_μ contains no line.

Proof of Proposition 5.1. Using (5.2) and $\operatorname{rank} G_0 = 1$, we can assume that

$$S_0 \subset \{(x,0,0) \mid x \in \mathbb{Z}\} \text{ and } S_1 \subset \{(x,y,z) \in G \mid y \geq 1\}.$$

Let $\tau : G_0 \mapsto \mathbb{Z}$ be the morphism given by $\tau(g_0) = x$ for $g_0 = (x,0,0)$.

First case When $\widetilde{\mu}_0$ is centered.
We fix s_1 in S_1 and we compute, as in Lemma 3.8, for $n \geq 1$,

$$\begin{aligned} h(s_1^{-1}) &\geq P_\mu^{n+1}\chi_0(s_1^{-1}) \\ &\geq \mu_{s_1} \sum_{k \leq n} P_{\mu_0}^k P_{s_1} P_{\mu_0}^{n-k}\chi_0(s_1^{-1}) \\ &= \mu_{s_1} \sum_{k \leq n} P_{\mu_0}^k P_{s_1}\chi_0(s_1^{-1}) \\ &= \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w}\, \chi_0(s_1 \dot{w} s_1^{-1}) \end{aligned} \tag{5.3}$$

The words w that contribute to this sum are those for which $s_1 \dot{w} s_1^{-1} \dot{w}^{-1} \in G_0$, i.e. $\dot{w} = 0$ or, equivalently, $\tau(\dot{w}) = 0$. Hence letting n go to ∞, one gets

$$h(s_1^{-1}) \geq \mu_{s_1} \sum_{k \geq 0} \sum_{w \in S_0^k} \widetilde{\mu}_{0,w} \mathbf{1}_{\{\tau(\dot{w})=0\}}.$$

If we write $w = s_1 \ldots s_n$ and $x_i := \tau(s_i)$, and if we think of these letters s_i as independent random variables with same law $\widetilde{\mu}_0$, this inequality can be rewritten as

$$h(s_1^{-1}) \geq \mu_{s_1} \sum_{k \geq 0} \mathbb{P}(x_1 + \cdots + x_k = 0).$$

But since the random variables $x_i \in \mathbb{Z}$ are centered, the expected number of passage at 0 of the walk $x_1 + \cdots + x_k$ is infinite, and the function h is not finite.

Second case When $\widetilde{\mu}_0$ is not centered.
The computation is similar but more involved since we want to prove finiteness of $h(g)$ at every point g in G.

We want a uniform upper bound for

$$P_\mu^n \chi_0(g) = \sum_{w \in S^n} \mu_w\, \chi_0(\dot{w} g).$$

The only words w that contribute to this sum are those for which $\dot{w}g$ is in G_0. By assumption (5.2), if we extract from w the maximal subword $\sigma = s_1 \dots s_\ell$ whose letters are in S_1, the length ℓ of σ is uniformly bounded by an integer ℓ_0. Therefore we can split the above sum into a finite sum

$$P_\mu^n \chi_0(g) \;=\; \sum_{\ell \le \ell_0} \sum_{\sigma \in S_1^\ell} \mu_\sigma \, Q_{\sigma, n-\ell} \, \chi_0(g),$$

where

$$\begin{aligned} Q_{\sigma,n} \, \chi_0(g) &= \sum_{k_0+\cdots+k_\ell=n} P_{\mu_0}^{k_\ell} P_{s_\ell} \cdots P_{\mu_0}^{k_1} P_{s_1} P_{\mu_0}^{k_0} \, \chi_0(g) \\ &= \sum_{k_1+\cdots+k_\ell \le n} P_{\mu_0}^{k_\ell} P_{s_\ell} \cdots P_{\mu_0}^{k_1} P_{s_1} \, \chi_0(g) \end{aligned} \tag{5.4}$$

We want to bound the limit

$$\begin{aligned} Q_\infty(g) &:= \lim_{n\to\infty} Q_{\sigma,n} \, \chi_0(g) \\ &= \sum_{k_1 \ge 0} \cdots \sum_{k_\ell \ge 0} \sum_{w_1 \in S_0^{k_1}} \cdots \sum_{w_\ell \in S_0^{k_\ell}} \mu_{w_1} \cdots \mu_{w_\ell} \, \chi_0(s_1 \dot{w}_1 \cdots s_\ell \, \dot{w}_\ell \, g). \end{aligned} \tag{5.5}$$

For $i \le \ell$, let $\sigma_i := s_1 \cdots s_i \in G$ and $b_i \ge 1$ be the integer given by

$$\sigma_i g_0 \sigma_i^{-1} g_0^{-1} = z_0^{-b_i \tau(g_0)} \text{ for all } g_0 \text{ in } G_0,$$

so that one has

$$s_1 \dot{w}_1 \cdots s_\ell \, \dot{w}_\ell \, g \;=\; \dot{w}_1 \cdots \dot{w}_\ell \, \sigma_\ell \, g \, z_0^{-b_1 \tau(\dot{w}_1) - \cdots - b_\ell \tau(\dot{w}_\ell)}. \tag{5.6}$$

Writing $\sigma_\ell \, g = g_0 \, z_0^c$ with g_0 in G_0 and c in $\mathbb{Z}$ one gets

$$Q_\infty(g) = \chi_0(g_0) \sum_{k_1 \ge 0} \cdots \sum_{k_\ell \ge 0} \sum_{w_1 \in S_0^{k_1}} \cdots \sum_{w_\ell \in S_0^{k_\ell}} \widetilde{\mu}_{0,w_1} \cdots \widetilde{\mu}_{0,w_\ell} \, \mathbf{1}_{\{b_1 \tau(\dot{w}_1) + \cdots + b_\ell \tau(\dot{w}_\ell) = c\}}.$$

If we think of all the letters occuring in one of the words $w_1, \dots, w_\ell$ as independent random variables with same law $\widetilde{\mu}_0$, this equality can be written as

$$Q_\infty(g) \;=\; \chi_0(g_0) \sum_{k_1 \ge 0} \cdots \sum_{k_\ell \ge 0} \mathbb{P}(b_1 S_{1,k_1} + \cdots + b_\ell S_{\ell,k_\ell} = c)$$

where $S_{i,k_i} := \tau(\dot{w}_i)$. Then the finiteness of $Q_\infty(g)$ follows from the following Lemma 5.3. $\square$

Lemma 5.3. *Let $(X_{i,k})_{i\leq \ell,\, k\geq 1}$, be independent real variables with same law. Assume this law has finite support and is not centered. Let $(b_i)_{i\leq \ell}$ be positive numbers and c be a real number. Set $S_{i,k} := X_{i,1} + \cdots + X_{i,k}$. Then one has*

$$\sum_{k_1\geq 0} \cdots \sum_{k_\ell \geq 0} \mathbb{P}(b_1 S_{1,k_1} + \cdots + b_\ell S_{\ell,k_\ell} = c) \;\; < \;\; \infty\,. \tag{5.7}$$

Proof of Lemma 5.3. We adapt the classical proof of the large deviation inequality. We set $X = X_{1,1}$. Assume for instance that $\mathbb{E}(X) > 0$. One can choose $\varepsilon > 0$ so that all the expectations

$$\alpha_i := \mathbb{E}(e^{-\varepsilon b_i X})$$

are smaller than 1. Then one computes

$$\begin{aligned} \mathbb{P}(b_1 S_{1,k_1} + \cdots + b_\ell S_{\ell,k_\ell} = c) \;\; &\leq \;\; \mathbb{E}(e^{\varepsilon\,(c - b_1 S_{1,k_1} - \cdots - b_\ell S_{\ell,k_\ell})}) \\ &= \;\; e^{\varepsilon c}\, \mathbb{E}(e^{-\varepsilon b_1 X})^{k_1} \cdots \mathbb{E}(e^{-\varepsilon b_\ell X})^{k_\ell} \\ &= \;\; e^{\varepsilon c}\, \alpha_1^{k_1} \cdots \alpha_\ell^{k_\ell} \end{aligned}$$

and therefore, summing all these inequalities, we find the following upper bound for the left-hand side L of (5.7)

$$L \;\; \leq \;\; e^{\varepsilon c}\,(1-\alpha_1)^{-1} \cdots (1-\alpha_\ell)^{-1} < \infty\,.$$

This ends the proof of the lemma and of Proposition 5.1. □

5.2 Induction of characters when $\operatorname{rank} G_0 = 2$

In this section we give the necessary and sufficient condition for the induced function h_{G_0,χ_0} to be finite when $\operatorname{rank} G_0 = 2$ or equivalently when $G_0 \cap Z \neq \{0\}$.

We split the statement into two cases depending on the shape of the convex cone V_μ^+.

Proposition 5.4. *Keep notation* (5.1)*. Assume* (5.2) *and* $\operatorname{rank} G_0 = 2$*. Assume moreover that the cone V_μ^+ contains a line. Then the induced harmonic function $h := h_{G_0,\chi_0}$ is not finite.*

Proof of Proposition 5.4. This follows from Proposition 4.9. Indeed, let z be a non-zero element of $G_0 \cap Z$ and $q := \chi_0(z)$. Assume, for a contradiction, that the function h is finite. By Lemmas 2.1 and 3.1, this function h is μ-extremal. By construction this function h is semiinvariant : one has $h_z = qh$. Hence by Proposition 4.9, one has $q = 1$ and by Lemma 4.7 the μ-harmonic function h is Z-invariant. Therefore, by the Choquet–Deny Theorem, this function h is a μ-harmonic character of G. But by Corollary 3.6, a μ-harmonic character is never induced. Contradiction. □

Proposition 5.5. *Keep notation* (5.1). *Assume* (5.2) *and* $\operatorname{rank} G_0 = 2$. *Assume moreover that the cone* V_μ^+ *contains no line. Then the induced harmonic function* $h := h_{G_0,\chi_0}$ *is finite if and only if*

$$\textit{there exist } s_0 \textit{ in } S_0 \textit{ and } s_1 \textit{ in } S_1 \textit{ such that } \ \chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) > 1\,. \tag{5.8}$$

Remark 5.6. Eventhough we will not use this remark, it is interesting to notice that, since Assumption (5.2) is satisfied and since the cone V_μ^+ is properly convex, this condition (5.8) is equivalent to

$$\textit{for all } s_0 \textit{ in } S_0 \smallsetminus Z \textit{ and } s_1 \textit{ in } S_1 \textit{ one has } \ \chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) > 1\,. \tag{5.9}$$

Proof of Proposition 5.5. The calculation is the same as for Proposition 5.1, but the interpretation is different. Using (5.2) and the proper convexity of the cone V_μ^+, we can assume that

$$S_0 \subset \{(x,0,z) \in G \mid x \geq 0\} \ \text{ and } \ S_1 \subset \{(x,y,z) \in G \mid y \geq 1\}. \tag{5.10}$$

Let $\tau : G_0 \mapsto \mathbb{Z}$ be the morphism given by $\tau(g_0) = x$ for $g_0 = (x,0,z)$.

Proof of $\Longrightarrow$ By (5.10), we know that the half-line $V_{\mu_0}^+$ is extremal in the properly convex cone V_μ^+. Assume by contraposition, that for all s_0 in S_0 and s_1 in S_1 one has $\chi_0(s_0 s_1 s_0^{-1} s_1^{-1}) \leq 1$. In particular, one has

$$\chi_0(s_1 \dot{w} s_1^{-1}) \ \geq \ \chi_0(\dot{w}) \ \ \text{for all } s_1 \in S_1 \text{ and } w \in S_0^k. \tag{5.11}$$

We fix s_1 in S_1 and, using (5.11), we compute, for $n \geq 1$, as in (5.3),

$$\begin{aligned} h(s_1^{-1}) \ &\geq \ \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w}\, \chi_0(s_1 \dot{w} s_1^{-1}) \\ &\geq \ \mu_{s_1} \sum_{k \leq n} \sum_{w \in S_0^k} \mu_{0,w}\, \chi_0(\dot{w}) \\ &\geq \ \mu_{s_1} \sum_{k \leq n} \chi_0(0) \ \geq \ n\, \mu_{s_1} \end{aligned}$$

Letting n go to ∞, one gets $h(s_1^{-1}) = \infty$.

Proof of $\Longleftarrow$ As for Proposition 5.1, one can find an integer ℓ_0 and one can split $P_\mu^n \chi_0(g)$ as a sum parametrized by words $\sigma = s_1 \dots s_\ell$ with letters in S_1 and $\ell \leq \ell_0$:

$$P_\mu^n \chi_0(g) \quad = \quad \sum_{\ell \leq \ell_0} \sum_{\sigma \in S_1^\ell} \mu_\sigma \, Q_{\sigma, n-\ell} \, \chi_0(g), \quad \text{where, as in (5.4),}$$

$$Q_{\sigma,n} \, \chi_0(g) \quad = \quad \sum_{k_1 + \cdots + k_\ell \leq n} P_{\mu_0}^{k_\ell} P_{s_\ell} \cdots P_{\mu_0}^{k_1} P_{s_1} \, \chi_0(g) \, .$$

As in (5.5) , we want to bound the limit

$$Q_\infty(g) \quad := \quad \lim_{n \to \infty} Q_{\sigma,n} \, \chi_0(g) \, .$$

The only words w that contribute to this sum are those for which $\dot{w} g$ is in G_0. For $i \leq \ell$, let $\sigma_i := s_1 \cdots s_i \in G$ and $b_i \geq 1$ be the integer given by

$$\sigma_i \, g_0 \, \sigma_i^{-1} g_0^{-1} = z_0^{-b_i \tau(g_0)} \ \text{ for all } g_0 \text{ in } G_0,$$

so that one has

$$s_1 \dot{w}_1 \cdots s_\ell \, \dot{w}_\ell \, g \quad = \quad \sigma_1 \dot{w}_1 \sigma_1^{-1} \cdots \sigma_\ell \, \dot{w}_\ell \, \sigma_\ell^{-1} \, \sigma_\ell \, g \, . \tag{5.12}$$

Hence, one gets

$$Q_\infty(g) \quad = \quad \mu_\sigma \, \chi_0(\sigma_\ell \, g) \, F_1 \cdots F_\ell$$

where, for all $i \leq \ell$,

$$F_i \quad := \quad \sum_{k \geq 0} \sum_{w \in S_0^k} \mu_{0,w} \, \chi_0(\sigma_i \, \dot{w} \, \sigma_i^{-1}) \, .$$

We want to prove that the sums F_i are finite. We will denote by $q_0 > 0$ the real number such that for all i in $\mathbb{Z}$ such that z_0^i is in G_0, one has $\chi_0(z_0^i) = q_0^i$. By assumption, one has $q_0 > 1$. One computes then

$$F_i \quad = \quad \sum_{k \geq 0} \sum_{w \in S_0^k} \widetilde{\mu}_{0,w} \, q_0^{-b_i \tau(\dot{w})} \, ,$$

where, as before, $\widetilde{\mu}_0$ is the probability measure $\chi_0\mu_0$. Let p_w be the number of letters of w that belong to $S_0 \smallsetminus Z$ and $\alpha := \widetilde{\mu}_0(S_0 \smallsetminus Z) < 1$. One goes on :

$$\begin{aligned} F_i &\leq \sum_{k\geq 0}\sum_{w\in S_0^k} \widetilde{\mu}_{0,w}\, q_0^{-p_w} \\ &= \sum_{k\geq 0}\sum_{j\leq k} \tbinom{j}{k}\alpha^j(1-\alpha)^{k-j}q_0^{-j} \\ &= \sum_{k\geq 0}(1-\alpha+\alpha q_0^{-1})^k \;=\; \alpha^{-1}(1-q_0^{-1})^{-1} \;<\; \infty\,. \end{aligned}$$

This proves the finiteness of F_i, of $Q_\infty(g)$ and of the function h_{G_0,χ_0}. □

5.3 Existence of harmonic characters

We explain in this section when μ_0-harmonic characters on abelian groups do exist.

Let $G_0 = \mathbb{Z}^d$ and μ_0 be a positive measure with finite support S_0 generating G_0 as a group. For a character χ_0 of G_0 we set

$$\mathbb{E}(\chi_0) := \sum_{s\in S_0} \mu_{0,s}\chi_0(s).$$

The map $\chi_0 \to \mathbb{E}(\chi_0)$ is the Laplace transform of μ_0. We denote by

$$\lambda(\mu_0) := \inf_{\chi_0} \mathbb{E}(\chi_0) \tag{5.13}$$

the minimum of this Laplace transform. Here is an example where it is easy to compute $\lambda(\mu_0)$.

Remark 5.7. If S_0 is included in a properly convex cone of $\mathbb{R}^d$, one has $\lambda(\mu_0) = \mu_0(0)$.
More generally, if S_0 is included in a half-space bounded by a hyperplane H_0, one has $\lambda(\mu_0) = \lambda(\mu_0|_{H_0})$.

Lemma 5.8. *There exists a μ_0-harmonic character if and only if $\lambda(\mu_0) \leq 1$. We can choose it so that $\widetilde{\mu}_0 := \chi_0\mu_0$ is not centered if and only if $\lambda(\mu_0) < 1$.*

Proof. Lemma 5.8 follows from the following three remarks:
- A character χ_0 is μ-harmonic if and only if $\mathbb{E}(\chi_0) = 1$.
- The group of characters is isomorphic to $\mathbb{R}^d$, hence it is connected.
- Since S_0 contains non-zero elements one has $\sup_{\chi_0} \mathbb{E}(\chi_0) = \infty$. □

Corollary 5.9. *a) If $\mu_0(S_0) \leq 1$, μ_0-harmonic characters exist.*
b) If $\mu_0(S_0) < 1$, we can choose it so that $\widetilde{\mu}_0 := \chi_0\mu_0$ is not centered.
c) If $\mu_0(S_0) > 1$ and μ_0 is centered, μ_0-harmonic characters do not exist.

Proof. This follows from Lemma 5.8 and the inequality $\lambda(\mu_0) \leq \mu_0(S_0)$. □

5.4 Conclusion

We sum up in the following theorem the main results we have obtained in this paper.

Let $G = H_3(\mathbb{Z})$ be the Heisenberg group, Z be the center of G, μ be a positive measure on G with finite support S such that $G_\mu = G$. We use the notation of Section 2.3.

Theorem 5.10. *The extremal positive μ-harmonic functions on G are proportional either to a character of G or to a translate of a function h induced from a character on an abelian subgroup. Here is the list when $\mu(G)=1$.*
1) **When V_μ^+ is the plane**. *There is no induced μ-harmonic function.*
2) **When V_μ^+ is a half-plane**. *Let V_0 be the boundary line of V_μ^+ and $G_0 \subset G$ be the subgroup generated by the elements of S above V_0 and $\mu_0 := \mu|_{G_0}$. Then h is equal to a function h_{G_0,χ_0} induced from a μ_0-harmonic character χ_0 of G_0.*
a) If $G_0 \cap Z = \{0\}$ there are exactly two such h_{G_0,χ_0}.
b) If $G_0 \cap Z \neq \{0\}$ there is no such h_{G_0,χ_0}.
3) **When V_μ^+ is properly convex**. *Let V_i^+, $i = 0, 1$, be the two extremal rays of V_μ^+, let $G_i \subset G$, be the two subgroups generated by the elements of S above V_i^+ and $\mu_i := \mu|_{G_i}$. Then h is equal to a function h_{G_i,χ_i} induced from a μ_i-harmonic character χ_i of G_i, $i = 0$ or 1.*
a) If $G_i \cap Z = \{0\}$ there is exactly one such h_{G_i,χ_i}.
b) If $G_i \cap Z \neq \{0\}$ there are uncountably many such h_{G_i,χ_i}.

Remark 5.11. Theorem 5.10 is illustrated in the schematic Figures 1, 2 and 3 of the introduction. We have drawn a rough approximation of the shape of the semigroup $G_\mu^+ \subset G$ and its subsemigroups $G_{\mu_0}^+$ and $G_{\mu_1}^+$, in order to illustrate the different cases that occur in Theorem 5.10. In these pictures the center Z is the vertical axis.

Proof of Theorem 5.10. The first claim follows from Proposition 4.8 and Proposition 4.10. Moreover, Case 1) and the first claims of Cases 2) and 3) follow from Lemma 3.8.iv.

- **Case 2.a)** Since $rank\, G_0 = 1$, by Proposition 5.1, χ_0 must be a μ_0-harmonic character of G_0 with $\chi_0\mu_0$ non centered. Since $\mu_0(G_0) < 1$ and since μ_0 is not supported by a half-line, there are exactly two such χ_0.
- **Case 2.b)** Since $rank\, G_0 = 2$, this follows from Proposition 5.4.
- **Case 3.a)** Since $rank\, G_i = 1$, by Proposition 5.1, χ_i must be a μ_i-harmonic character of G_i with $\chi_i\mu_i$ non centered. Since $\mu_i(G_i) < 1$ and since μ_i is supported by a half-line, there is exactly one such χ_i.
- **Case 3.b)** Since $rank\, G_i = 2$, by Proposition 5.5, χ_i must be a μ_i-harmonic character of G_i satisfying (5.8). Since $\mu(G_i) < 1$, and since μ_i is supported by a half-space delimited by Z, there are uncountably many such χ_i. □

Remark 5.12. - When μ is not assumed to be a probability measure, the formulation of Theorem 5.10 has to be modified. Indeed, if $\mu(\{0\}) \geq 1$, positive μ-harmonic functions cannot exist. More precisely, each of the three cases 2.a), 3.a) and 3.b) has to be split into two subcases :
2.a') *If* $G_0 \cap Z = \{0\}$ *and* $\lambda(\mu_0) < 1$*, there are exactly two such* h_{G_0,χ_0}.
2.a'') *If* $G_0 \cap Z = \{0\}$ *and* $\lambda(\mu_0) \geq 1$*, there are no such* h_{G_0,χ_0}.
3.a') *If* $G_i \cap Z = \{0\}$ *and* $\mu(\{0\}) < 1$ *there is exactly one such* h_{G_i,χ_i}.
3.a'') *If* $G_i \cap Z = \{0\}$ *and* $\mu(\{0\}) \geq 1$ *there are no such* h_{G_i,χ_i}.
3.b') *If* $G_i \cap Z \neq \{0\}$ *and* $\lambda(\mu_Z)_i < 1$*, there are uncountably many such* h_{G_i,χ_i}.
3.b'') *If* $G_i \cap Z \neq \{0\}$ *and* $\lambda(\mu_Z)_i \geq 1$*, there are no such* h_{G_i,χ_i}.
- Here is the definition, motivated by Condition (5.8), of the constants $\lambda(\mu_Z)_i$. For instance, for $i = 0$, we choose $s_j \in \mathrm{supp}(\mu_{\mathrm{j}}) \smallsetminus Z$, $j = 0$ or 1, and set
$\lambda(\mu_Z)_0 := \inf\{\sum_{s\in Z} \mu_s\chi_Z(s) \mid \chi_Z \text{ character of } Z,\ \chi_Z(s_0s_1s_0^{-1}s_1^{-1}) > 1\}$.
- Note that Cases 2.a''), 3.a'') and 3.b'') do not occur when μ is a probability measure.

We now can deduce from Theorem 5.10 the corollaries in the introduction :

Proof of Corollary 1.3. The support of h_{G_0,χ_0} is the semigroup generated by G_0 and S^{-1}. In the cases where a μ-harmonic function h_{G_0,χ_0} induced from a character is finite, by Lemma 3.8.iv, one has $V^+_{\mu_1} \cap V_{\mu_0} = \{0\}$, and this semigroup is never equal to G. □

Proof of Corollary 1.4. Both conditions (i), (ii) are true in Cases 1) and 2.b). Both conditions are not true in Cases 2.b), 3.a) and 3.b). □

One also has the following variation of Corollary 1.4.

Corollary 5.13. *Same notation and $\mu(G) = 1$. The following are equivalent:*
(i) Extremal positive μ-harmonic functions are Z^p-semiinvariant for a $p \geq 1$.
(ii) $G_\mu^+ \cap Z \not\subset \{0\}$.

Condition (i) means that there exist $q > 0$ and a non-zero element z in Z such that $h_z = qh$.

Proof of Corollary 5.13. Both conditions (i), (ii) are true in Cases 1), 2.b) and 3.b). Both conditions are not true in Cases 2.a) and 3.a). $\square$

5.5 A nilpotent group of rank 4

In this section, we explain why Theorem 1.1 can not be extended to all nilpotent groups.

In this section $G = N_4(\mathbb{Z})$ will be the nilpotent group equal to the set $\mathbb{Z}^4$ of quadruples seen as matrices $(t, x, y, z) := \begin{pmatrix} 1 & t & t^2/2 & z \\ 0 & 1 & t & y \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The product is

$$(t, x, y, z)\,(t', x', y', z') = (t + t', x + x', y + y' + tx', z + z' + ty' + \tfrac{1}{2}t^2x')\,.$$

The center Z of G is generated by $z_0 := (0, 0, 0, 1)$. Let μ be the measure

$$\mu := \frac{1}{2}(\delta_a + \delta_b) \ \text{ where } a := (1, 0, 0, 0) \text{ and } b = (0, 1, 0, 0).$$

A μ-harmonic function h on G is a function that satisfies, for all g in G,

$$2\,h(g) = h(ag) + h(bg) \ \text{ or, equivalently,}$$

$$2\,h(t, x, y, z) = h(t{+}1, x, y + x, z{+}y{+}x/2) + h(t, x{+}1, y, z).$$

We now construct extremal positive μ-harmonic functions on G.

Fix a sequence σ of rapidly increasing integers $1 \leq \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots$. We introduce the left-infinite word w_σ in a and b of the form

$$w_\sigma = \cdots a^{(\sigma_\lambda)} b a^{(\sigma_{\lambda-1})} b \cdots b a^{(\sigma_1)} b a^{(\sigma_0)}$$

where the notation $a^{(m)}$ means that the letter a is repeated m-times. For each $k \geq 0$ we denote by $w_{\sigma,k}$ the suffix of length k of w_σ, i.e. the word given by the k last letters of w_σ. We introduce the functions on G

$$\psi_\sigma := \sum_{k\geq 0} 2^k \mathbf{1}_{\dot{w}_{\sigma,k}} \ \text{ and } \ h_\sigma := \sup_{n\geq 1} P_\mu^n \psi_\sigma\,.$$

As before, the dot means that we replace the word by its image in G.

Lemma 5.14. *Let $G = N_4(\mathbb{Z})$ and μ and σ be as above. Assume that*

$$\sigma_{\lambda+1} \geq 2\,\sigma_\lambda^2 \text{ for all } \lambda \geq 0. \tag{5.14}$$

a) The function ψ_σ is subharmonic and the sequence $P_\mu^n \psi_\sigma$ is increasing.
b) The limit $h_\sigma = \lim\limits_{n\to\infty} P_\mu^n \psi_\sigma$ is finite.
c) The function h_σ is an extremal positive μ-harmonic function on G.
d) The function h_σ is not Z-invariant.
e) The function h_σ is not induced.

Note that Condition (5.14) is not optimized.

Proof. *a*) One has $\psi_\sigma \leq P_\mu \psi_\sigma$ and therefore $P_\mu^n \psi_\sigma \leq P_\mu^{n+1}\psi_\sigma$.

b) This is the key point. Fix $g = (t, x, y, z) \in G$. Fix also two words u and v in a and b such that $g = \dot{u}\dot{v}^{-1}$. Such words always exist. By definition of h_σ, one has

$$h_\sigma(g) = 2^{t+x} \lim_{\lambda\to\infty} (\text{number of words } w \text{ such that } \dot{w}\dot{u} = \dot{w}_{\sigma,n_\lambda}\dot{v}), \tag{5.15}$$

where w_{σ,n_λ} is the suffix of w_σ of length $n_\lambda := \lambda + \sum_{i\leq\lambda} \sigma_i$, i.e.

$$w_{\sigma,n_\lambda} = a^{(\sigma_\lambda)} b a^{(\sigma_{\lambda-1})} b \cdots b a^{(\sigma_1)} b a^{(\sigma_0)}. \tag{5.16}$$

We also write

$$w := a^{(k_\ell)} b a^{(k_{\ell-1})} b \cdots b a^{(k_1)} b a^{(k_0)}, \tag{5.17}$$

with all $k_i \geq 0$. We denote by $\ell(w)$ the length of a word w.

We want to prove, using Condition (5.14), that the quantity (5.15) is finite. We will see that there exists $\lambda_0 = \lambda_0(g)$ such that the number of words w such that

$$\dot{w}\dot{u} = \dot{w}_{\sigma,n_\lambda}\dot{v} \tag{5.18}$$

does not depend on λ for $\lambda \geq \lambda_0(g)$. More precisely, we will see below that, for $\lambda \geq \lambda_0(g)$, Equality (5.18) implies that $k_\ell = \sigma_\lambda$, so that we could remove the prefix $a^{(\sigma_\lambda)} b$ in both words and replace λ by $\lambda - 1$.

Equality (5.18) gives four equations. We could write them down but we will not need to. The first two equations tell us that the same number of a's and the same number of b's occur in the words wu and $w_{\sigma,n_\lambda} v$. In particular, those words have same length. The third equation tells us that the sum of the positions of the b's in these words are the same. Once these three equations are satisfied, one can write $\dot{w}\dot{u} = \dot{w}_{\sigma,n_\lambda}\dot{v}z_0^{N_w}$ with $N_w \in \mathbb{Z}$. The fourth equation tells us that this integer N_w is zero.

We claim that, for $\lambda \geq \lambda_0(g)$, if $k_\ell \neq \sigma_\lambda$ then $N_w \neq 0$. The reason is that we can go from the word wu to the word $w_{\sigma,n_\lambda}v$ by a (minimal) succession of "moves" that changes only the central component. These "moves" are

$$w_1abw_2baw_3 \quad \longleftrightarrow \quad w_1baw_2abw_3.$$

The images in G are modified by a factor $z_0^{\ell_a(w_2)+1}$, where $\ell_a(w_2)$ is the number of a's occuring in the word w_2:

$$\dot{w}_1ba\dot{w}_2ab\dot{w}_3 \quad = \quad \dot{w}_1ab\dot{w}_2ba\dot{w}_3z_0^{\ell_a(w_2)+1}.$$

Therefore, by (5.14), the largest contributions to N_w come from the "moves" that involve the first b on the left of the word w_{σ,n_λ}. Hence, when $k_\ell \neq \sigma_\lambda$ one has

$$|N_w| \quad \geq \quad \sigma_{\lambda-1} - (\ell(v) + \sum_{i\leq\lambda-2}\sigma_i)^2$$

which is non zero for λ large enough by Condition (5.14).

c) By construction h_σ is μ-harmonic. We want to prove that h_σ is extremal. Assume that $h_\sigma = h' + h''$ with h' and h'' positive μ-harmonic. It follows also from the previous computations that

$$2^{-k}h_\sigma(\dot{w}_{\sigma,k}) = 1 \ \text{ for all } k \geq 0.$$

Introduce the two limits

$$\alpha' := \lim_{k\to\infty} 2^{-k}h'(\dot{w}_{\sigma,k}) \ \text{ and } \ \alpha'' := \lim_{k\to\infty} 2^{-k}h''(\dot{w}_{\sigma,k}).$$

These limits exist since by harmonicity of h' and h'' these sequences are non-increasing. Moreover, one has $\alpha' + \alpha'' = 1$ and

$$h' \geq \alpha'\psi_\sigma \ \text{ and } \ h'' \geq \alpha''\psi_\sigma.$$

Using again the harmonicity of h' and h'', one deduces

$$h' \geq \alpha'h_\sigma \ \text{ and } \ h'' \geq \alpha''h_\sigma.$$

Since $\alpha' + \alpha'' = 1$, these inequalities must be equalities. This proves that h is extremal.

d) The above computation also tells us that $\mathrm{supp}(h_\sigma) \cap Z = \{0\}$. This prevents h_σ to be Z-invariant.

e) If the function h_σ were induced, it would be the translate by an element $g \in G$ of a function induced from a character of the cyclic group G_a generated by a or of the cyclic group G_b generated by b. Since $h_\sigma(w_k) \neq 0$, for all $k \geq 1$, all the sets $G_\mu^+ w_k$ would meet $G_a g^{-1}$ or $G_b g^{-1}$, which is impossible since both $\lim\limits_{k\to\infty} \ell_b(w_k) = +\infty$ and $\lim\limits_{k\to\infty} \ell_a(w_k) = +\infty$. $\square$

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