On the rate of exponential decay of coefficients on homogeneous spaces

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Abstract

For any homogeneous space of a real semisimple algebraic group G, we define an exponent with multiple interpretations from representation theory and group theory. As an application, we give a temperedness criterion for $L^2(G/H)$ for any closed subgroup H of G, which extends the existing ones of Benoist-Kobayashi for connected subgroups and Lutsko-Weich-Wolf for discrete subgroups.

Contents

1 Introduction

As a cornerstone in the harmonic analysis on semisimple groups, the notion of tempered representations was pioneered by Harish-Chandra in the 1960s and has later become a major tool in establishing uniform decay of coefficients which has found a multitude of applications.

Compared with the long-standing development of tempered representations, the notion of tempered homogeneous spaces is rather recent, initiated by Benoist and Kobayashi in [BK15] and further developed in a series of subsequent works [BK22, BK21, BK23]. More recently, this line of research has been extended by other works including [EO23, LWW24, FO24] which studied temperedness in the complementary context of Riemannian locally symmetric spaces.

Let G be a real semisimple algebraic group and H be a closed subgroup in the analytic topology. The homogeneous space G/H admits a G-quasi-invariant Radon measure, giving rise to the unitary representation $\lambda_{G/H}$ of G on the Hilbert space $\mathrm{L}^2(G/H)$ of square-integrable functions on G/H.

In this paper, we study the exponential decay property of $\lambda_{G/H}$ after [BK15, BK22, LWW24] and extend their results to any closed subgroup H. More precisely, we establish a direct relationship between the following four quantities associated with the homogeneous space G/H.

Definition 1.1. The *uniform decay* exponent $\theta_{G/H}$ is defined to be the infimum of $\theta \in [0,1]$ such that for all $f_1, f_2 \in C_c(G/H)$, there exists a constant $C > 0$ such that uniformly for all $g \in G$, we have

$$\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \leq C \exp\{2(\theta - 1)\rho\kappa(g)\},$$

where the map $\kappa : G \to \mathfrak{a}^+$ is the Cartan projection to the positive Weyl chamber $\mathfrak{a}^+$ of a Cartan subspace $\mathfrak{a}$ of $\mathfrak{g}$, and the linear form $\rho : \mathfrak{a} \to \mathbb{R}$ is the half sum of positive restricted roots of $(\mathfrak{a};\mathfrak{g})$. This number does not depend on the choice of the Cartan decomposition of G.

Remark. For a discrete subgroup Γ, the exponent $\theta_{G/\Gamma}$ is related to the number $\theta(\mathrm{L}^2(G/\Gamma))$ defined in [LWW24, §1] via

$$\theta\big(\mathrm{L}^2(G/\Gamma)\big) = \max\big\{2\theta_{G/\Gamma} - 1,\, 0\big\}.$$

Definition 1.2. The *optimal integrability* exponent $p_{G/H}$ of the homogeneous space G/H is defined by the infimum of $p \in [1, \infty]$ such that for any compact subset B of G/H, we have

$$\big\langle \lambda_{G/H}(\cdot)\mathbb{1}_B, \mathbb{1}_B \big\rangle \in \mathrm{L}^p(G).$$

Remark. Our definition of $p_{G/H}$ coincides with the number $p_{G/H}$ defined in [BK15, §4.2] when H is a reductive subgroup. Compared with the exponent $q(G; G/H)$ in [Kob25, Def 7.12], our definition is more natural when G/H only admits quasi-invariant measures.

Definition 1.3 (Definition 4.11)**.** The *volume growth* exponent $\delta_{G/H}$ of H inside G is defined by

$$\delta_{G/H} := \max\left\{0,\ \sup_{B \Subset G} \limsup_{g \to \infty} \frac{\log \nu_H(H \cap BgB)}{\log \nu_G(BgB)}\right\},$$

where B ranges over compacta of nonempty interior, the measure ν_G is the Haar measure on G, and $\mathrm{d}\nu_H(h) = (\det \mathrm{Ad}_H h)^{1/2}\,\mathrm{d}h$ is the symmetric measure on H (cf. Section 2.1). This number also equals (the positive part of) the abscissa of convergence for the following analogue of Dirichlet series (cf. Proposition 4.13)

$$t \mapsto \int_H e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h).$$

Definition 1.4 (Definition 4.8)**.** The *local volume decay* exponent $\beta_{G/H}$, which is mainly of interest for connected subgroups, is defined by

$$\beta_{G/H} := \sup_{\mathfrak{h}} \frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}}},$$

where the functions $\rho_{\mathfrak{h}}, \rho_{\mathfrak{g}} : \mathfrak{h} \to \mathbb{R}^+$ are respectively the half sum of absolute values of the real parts of complex eigenvalues for the adjoint action of $\mathfrak{h}$ on the Lie algebras $\mathfrak{h}, \mathfrak{g}$, as defined in [BK22]. By convention, we set $0/0 = 0$.

Related results. By definition, a unitary representation of the semisimple group G is *tempered* if it is weakly contained in the regular representation of G. The fundamental work [CHH88] characterized the tempered representations for semisimple groups via the uniform decay property and uniform integrability, which in particular implies that

$$\mathrm{L}^2(G/H) \text{ is tempered} \iff \theta_{G/H} \leq \frac{1}{2} \iff p_{G/H} \leq 2. \tag{1.5}$$

In [BK15], Benoist-Kobayashi proved for any reductive subgroup H that

$$\theta_{G/H} = \beta_{G/H} = 1 - \frac{1}{p_{G/H}}. \tag{1.6}$$

Going further in [BK22], they then proved for any closed subgroup H with finitely many connected components that

$$\mathrm{L}^2(G/H) \text{ is tempered} \iff \beta_{G/H} \leq \frac{1}{2}. \tag{1.7}$$

In a complementary direction, other authors studied the case when H is a discrete subgroup. First by Edwards and Oh in [EO23] for Anosov subgroups and eventually by Lutsko, Weich, and Wolf in [LWW24] for the general case, it has been proven for any discrete subgroup Γ of G that

$$\max\left\{\theta_{G/\Gamma}, \frac{1}{2}\right\} = \max\left\{\sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho}, \frac{1}{2}\right\}, \tag{1.8}$$

where $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is the growth indicator function first introduced in [Qui02], so in particular $\mathrm{L}^2(G/\Gamma)$ is tempered iff $\psi_\Gamma \leq \rho$.

Statement of results. Let G be a real semisimple algebraic group. Our first main result contains a response to the optimal integrability problem [Kob25, Prob 7.13] for all homogeneous spaces of G.

Theorem A. *Let H be a closed subgroup of G. Then*

$$\theta_{G/H} = \delta_{G/H} = 1 - \frac{1}{p_{G/H}}.$$

As an immediate consequence of Theorem A and the uniform decay characterization (1.5) of temperedness, we obtain the following temperedness criterion, in response to [Kob25, Prob 7.18].

Corollary B. *Let H be a closed subgroup of G. Then*

$$\mathrm{L}^2(G/H) \text{ is tempered } \iff \delta_{G/H} \leq \frac{1}{2}.$$

Meanwhile, Theorem A unifies the previous results (1.6) of Benoist-Kobayashi and (1.8) of Lutsko-Weich-Wolf in that it recovers them as the following corollaries.

Corollary C. *Let H be a reductive subgroup of G. Then*

$$\theta_{G/H} = \delta_{G/H} = \beta_{G/H}.$$

Corollary D. *Let Γ be a discrete subgroup of G. Then*

$$\theta_{G/\Gamma} = \delta_{G/\Gamma} = \max\left\{\sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho},\, 0\right\}.$$

Our second main result extends the equivalence in Theorem A and Corollary C to all algebraic subgroups.

Proposition E. *Let H be an algebraic subgroup of G. Then*

$$\max\left\{\theta_{G/H},\, \frac{1}{2}\right\} = \max\left\{\delta_{G/H},\, \frac{1}{2}\right\} = \max\left\{\beta_{G/H},\, \frac{1}{2}\right\}.$$

In particular,

$$\max\left\{p_{G/H},\, 2\right\} = \max\left\{\frac{1}{1-\beta_{G/H}},\, 2\right\}.$$

Proposition E extends the criterion (1.7) of Benoist-Kobayashi. Indeed, given any closed subgroup H with finitely many components, Chevalley's *théorie des répliques* yields the existence of two algebraic subgroups H_1, H_2 of G which satisfy

$$H_1 \subset H \subset H_2, \quad \mathfrak{h}_1 = [\mathfrak{h}, \mathfrak{h}] = [\mathfrak{h}_2, \mathfrak{h}_2].$$

Then Herz's *principe de majoration* implies that the unitary representations $\mathrm{L}^2(G/H), \mathrm{L}^2(G/H_1), \mathrm{L}^2(G/H_2)$ are all tempered as long as one of them is so; cf. [BK22, §2.4]. As a result, Proposition E implies (1.7).

Remark 1.9. The results above carry through to G being a real reductive group without modification for $\theta_{G/H}, \delta_{G/H}, \beta_{G/H}$, but for $p_{G/H}$ one needs to replace $\mathrm{L}^p(G)$ by $\mathrm{L}^p(G_{\mathrm{ss}})$ in the definition.

Outline of the paper. In Section 2, we recall some fundamental elements in the analysis on real semisimple groups. In Section 3, we recall some basic definitions and facts about unitary representations. In Section 4, we establish the fundamental tools to address the growth and decay of volume in real semisimple groups, which are indispensable to the proofs of the main results. In Section 5, we prove Theorem A by following the strategy of [LWW24] and then deduce Corollary C and Corollary D. In Section 6, we prove Proposition E by following the strategy of [BK22] with input from the method of [LWW24].

2 Analysis on semisimple Lie groups

The general references for this section include [Kna86, Hel01].

2.1 Measures on homogeneous spaces

In this subsection, let G be a locally compact group and $\mathrm{d}x$ be a left Haar measure on G. The modular function $\Delta_G : G \to \mathbb{R}_{>0}$ is a continuous group morphism defined by

$$(R_{g^{-1}})_* \,\mathrm{d}x = \Delta_G(g)\,\mathrm{d}x, \quad \text{or} \quad \mathrm{d}(xg) = \Delta_G(g)^{-1}\,\mathrm{d}x.$$

If the group G is a Lie group and $\mathfrak{g}$ its Lie algebra, then

$$\Delta_G(g) = \det \mathrm{Ad}_G(g)^{-1}.$$

A locally compact group G is said to be unimodular if $\Delta_G \equiv 1$. In general, a right Haar measure on G can be defined by

$$\mathrm{d}(x^{-1}) = \Delta_G(x)^{-1}\,\mathrm{d}x.$$

What will play a role later is the *symmetric measure* ν_G on G defined by

$$\mathrm{d}\nu_G(x) = \Delta_G(x)^{-\frac{1}{2}}\,\mathrm{d}x.$$

The symmetry can be seen from the fact that $\mathrm{d}\nu_G(x^{-1}) = \mathrm{d}\nu_G(x)$.

Now let H be a closed subgroup of G and G/H be the associated homogeneous space. The invariant measures on G/H are characterized as follows. It should be noted that the integration formula holds up to normalization of Haar measures. On locally compact groups, we take the left Haar measures by default.

Lemma 2.1 ([BdlHV08, Lem B.1.3])**.** *The homogeneous space G/H always admits a G-quasi-invariant Radon measure. More precisely, the following data are equivalent:*

(1) a function $\delta : G \to \mathbb{R}_{>0}$ which is continuous and satisfies

$$\delta(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\delta(g) \tag{2.2}$$

for all $g \in G$ and $h \in H$;

(2) a quasi-invariant Radon measure μ on G/H.

The connection between these two items is given by

$$\int_G f(g)\delta(g)\,\mathrm{d}g = \int_{G/H}\int_H f(gh)\,\mathrm{d}h\,\mathrm{d}\mu(gH) \tag{2.3}$$

for all $f \in C_{\mathrm{c}}(G)$. Moreover, the Radon-Nikodym derivative is given explicitly by

$$\frac{\mathrm{d}(g_*\mu)}{\mathrm{d}\mu}(xH) = \frac{\delta(g^{-1}x)}{\delta(x)}$$

for all $g \in G$ and $x \in G$. □

In particular, when there exists a G-invariant Radon measure on G/H, i.e. when $\Delta_G|_H \equiv \Delta_H$, such a measure is unique up to scalar. The following lemma is a key tool to produce integration formulae on Lie groups, while being general itself.

Lemma 2.4. *Let S, T be closed subgroups of G so that the complement of ST in G has zero Haar measure, while $K = S \cap T$ is a compact subgroup. Then we can normalize the Haar measures so that*

$$\mathrm{d}g = \frac{\Delta_G(t)}{\Delta_T(t)}\,\mathrm{d}s\,\mathrm{d}t.$$

In other words, for all $f \in C_{\mathrm{c}}(G)$ we have

$$\int_G f(g)\,\mathrm{d}g = \int_S\int_T f(st)\frac{\Delta_G(t)}{\Delta_T(t)}\,\mathrm{d}s\,\mathrm{d}t.$$

Proof. Let $\delta(st) = \Delta_T(t)/\Delta_G(t)$. We claim that this is well defined on G. Indeed, since the group $K = S \cap T$ is compact, the restrictions of both Δ_G and Δ_T to K are trivial. If $s_1t_1 = s_2t_2$, then $t_2^{-1}t_1 \in K$; hence $\Delta_T(t_1) = \Delta_T(t_2)$ and $\Delta_G(t_1) = \Delta_G(t_2)$. Since δ satisfies (2.2), we obtain a quasi-invariant Radon measure μ on $G/T = S/K$ with

$$\int_G f(g)\delta(g)\,\mathrm{d}g = \int_{S/K}\int_T f(st)\,\mathrm{d}t\,\mathrm{d}\mu(sK).$$

Since δ is S-left-invariant, the left hand side is invariant if we replace $f(\cdot)$ by $f(s_0\cdot)$ for any $s_0 \in S$, whence the measure μ is S-left-invariant. With K compact, another application of Lemma 2.1 yields

$$\begin{aligned}\int_{S/K}\int_T f(st)\,\mathrm{d}t\,\mathrm{d}\mu(sK) &= \int_{S/K}\int_K\int_T f(skt)\,\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}\mu(sK)\\ &= \int_S\int_T f(st)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$ $\square$

2.2 Decomposition and integration

From now on, G will always denote a semisimple real algebraic group. The results carry through to real reductive groups with mild modification. The rich decomposition theory of these groups gives rise to a variety of integration formulae.

The semisimple group G comes with an analytic involution $\Theta : G \to G$ so that its differential $\theta : \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution. Let the corresponding Cartan decomposition be $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then $\mathfrak{g}$ is naturally equipped with an adjoint-invariant inner product $B_\theta(X, Y) = -B_0(X, \theta Y)$, where B_0 is the Killing form of $\mathfrak{g}$.

Let K be the maximal compact subgroup of G with $\mathrm{Lie}(K) = \mathfrak{k}$. Let $\mathfrak{a}$ be a Cartan subspace (i.e. a maximal split abelian subspace) of $\mathfrak{p}$ and $A = \exp \mathfrak{a}$ the Cartan subgroup. Let $\Sigma = \Sigma(\mathfrak{a}; \mathfrak{g})$ denote the set of restricted roots. The corresponding root space decomposition can be written as

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m},$$

where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. Fix a positive system $\Sigma^+ \subset \Sigma$. Then the closed positive Weyl chamber is given by

$$\mathfrak{a}^+ := \left\{X \in \mathfrak{a} : \alpha(X) \geq 0,\ \forall\alpha \in \Sigma^+\right\}.$$

and denote by $A^+ = \exp \mathfrak{a}^+$ its exponential. The Killing form induces an inner product on $\mathfrak{a}$ so that $\mathfrak{a}^*$ is identified with $\mathfrak{a}$ and we denote by $|\cdot|$ the induced Euclidean norm on $\mathfrak{a}$. The linear form $\rho \in \mathfrak{a}^*$ is defined to be the half sum of positive roots:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+} (\dim \mathfrak{g}_\alpha)\alpha.$$

Let M and M' be respectively the centralizer and normalizer subgroup of A in K. Then $\mathrm{Lie}(M) = \mathrm{Lie}(M') = \mathfrak{m}$ and the finite quotient group $M'/M = W(\mathfrak{a}; \mathfrak{g}) =: W_G$ is the (restricted) Weyl group which acts simply transitively on the set of Weyl chambers of $\mathfrak{a}^+$.

2.2.1 Integration from Iwasawa decomposition Let $G = KP$ be the Iwasawa decomposition, where $P = MAN$ is a minimal parabolic subgroup. Denote by $\eta : G \to \mathfrak{a}$ the Iwasawa projection so that $g \in Ke^{\eta(g)}N$. Then we have the following result.

Proposition 2.5. *For any $f \in C_c(G)$, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_P f(kp)\, e^{2\rho\eta(p)}\,\mathrm{d}k\,\mathrm{d}p. \qquad \square$$

2.2.2 Integration from Cartan decomposition

Proposition 2.6. *For any $f \in C_c(G)$, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_{\mathfrak{a}^+} \int_K f(k_1 e^X k_2) \left(\prod_{\alpha\in\Sigma^+} \sinh^{\dim \mathfrak{g}_\alpha} \alpha(X) \right) \mathrm{d}k_1\,\mathrm{d}X\,\mathrm{d}k_2. \quad \square$$

2.2.3 The Bruhat decomposition For each group element $w \in W_G$, fix a representative $m_w \in M'$. Let w^* be the unique element of W which maps $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. Denote by $\overline{N} := N^{w^*} = \Theta N$.

Theorem 2.7. *Write $N^w := m_w N m_w^{-1}$. Then*

$$G = \bigsqcup_{w\in W_G} B m_w B = \bigsqcup_{w\in W_G} MANN^w m_w.$$

Moreover, the term $Bm_{w^*}B = \overline{N}MAN$ is an open submanifold of G, while the other terms are submanifolds of strictly lower dimensions. $\square$

Hence, $\overline{N}MAN$ is an open submanifold of G whose complement has zero Lebesgue measure. Moreover, multiplication map $\overline{N} \times MAN \to \overline{N}MAN$ is a bijection. We also have the following formula.

Proposition 2.8. *For any $f \in C_c(G)$, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_{\overline{N}} \int_M \int_A \int_N f(\bar{n}man)\, e^{2\rho \log a}\,\mathrm{d}\bar{n}\,\mathrm{d}m\,\mathrm{d}a\,\mathrm{d}n. \qquad \square$$

2.3 Parabolic subgroups

The closed subgroup $P = MAN$ is a minimal parabolic subgroup of G. Let Q be a parabolic subgroup of G containing P and $Q = M_Q A_Q N_Q$ be the Langlands decomposition of Q. Then M_Q is a reductive subgroup. Set $L = M_Q A_Q$ which is the Levi factor of Q so that $Q = L \ltimes N_Q$ gives the Levi decomposition.

2.3.1 Subgroups and subalgebras The choice of $G = KAN$ fixes the set Σ of restricted roots, the subset Σ^+ of the positive ones, and the subset Π of the simple ones. Write

$$\mathfrak{g}, \mathfrak{k}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}, \mathfrak{q}, \mathfrak{l}, \mathfrak{m}_Q, \mathfrak{a}_Q, \mathfrak{n}_Q$$

respectively for the Lie algebras of the Lie subgroups

$$G, K, M, A, N, Q, L, M_Q, A_Q, N_Q.$$

By the classification of parabolic subgroups, there exists a subset $\Pi' \subset \Pi$ such that

$$\mathfrak{a}_Q = \{X \in \mathfrak{a} : \alpha(X) = 0, \forall \alpha \in \Pi'\}.$$

Write $\langle \Pi' \rangle$ for the span of Π'. Define $\Sigma_Q^+ := \Sigma^+ \setminus \langle \Pi' \rangle$ and $\Sigma_M^+ := \Sigma^+ \cap \langle \Pi' \rangle$. Recall that the space $\mathfrak{a}$ has a Euclidean structure induced by the Killing form. Let

$$\mathfrak{a}_M := \mathfrak{a}_Q^\perp \text{ in } \mathfrak{a}, \quad \mathfrak{n}_M := \bigoplus_{\alpha \in \Sigma_M^+} \mathfrak{g}_\alpha.$$

Then as vector spaces, we have

$$\mathfrak{m}_Q = \mathfrak{m} \oplus \mathfrak{a}_M \oplus \mathfrak{n}_M \oplus \theta \mathfrak{n}_M, \quad \mathfrak{n}_Q = \bigoplus_{\alpha \in \Sigma_Q^+} \mathfrak{g}_\alpha, \quad \mathfrak{a} = \mathfrak{a}_M \oplus \mathfrak{a}_Q, \quad \mathfrak{n} = \mathfrak{n}_M \oplus \mathfrak{n}_Q.$$

Let $K_M = K \cap M_Q$ and A_M, N_M be the analytic subgroups corresponding to $\mathfrak{a}_M, \mathfrak{n}_M$. Then $M_Q = K_M A_M N_M$ is an Iwasawa decomposition of M_Q, $A = A_M A_Q \cong A_M \times A_Q$, and $N = N_M N_Q \cong N_M \ltimes N_Q$. We remark that all the groups discussed here are closed subgroups of G.

Notation. For $\alpha \in \Sigma$, let $m_\alpha := \dim \mathfrak{g}_\alpha$. Define

$$\rho_Q = \frac{1}{2} \sum_{\alpha \in \Sigma_Q^+} (\dim \mathfrak{g}_\alpha) \alpha, \quad \rho_M = \frac{1}{2} \sum_{\alpha \in \Sigma_M^+} (\dim \mathfrak{g}_\alpha) \alpha.$$

Sometimes ρ_M is denoted by ρ_L. For $X \in \mathfrak{a}$, write X_Q, X_M respectively for the orthogonal projection of X to the subspaces $\mathfrak{a}_Q, \mathfrak{a}_M$.

Lemma 2.9. *For every $X \in \mathfrak{a}$, we have $\rho_Q(X_M) = \rho_M(X_Q) = 0$.*

Proof. That $\rho_M(X_Q) = 0$ follows directly from the definition. To prove $\rho_Q(X_M) = 0$, let us assume that $\Pi' \neq \emptyset$; otherwise, there is nothing to prove. Dually, this is equivalent to $\alpha \perp \rho_Q$ for all $\alpha \in \Pi'$. But any $\alpha \in \Pi'$ is a simple root, so the α-reflection s_α preserves setwise $\Sigma \cap \langle \Pi' \rangle$ and hence also $\Sigma^+ \setminus \langle \Pi' \rangle$. But that means $s_\alpha(\rho_Q) = \rho_Q$, i.e. $\alpha \perp \rho_Q$. $\square$

2.3.2 Integration Let the map $\eta : G \to \mathfrak{a}$ denote the Iwasawa projection so that $g \in Ke^{\eta(g)}N$ for all $g \in G$. The modular function of Q is given by $\Delta_Q(q) = \exp\{-2\rho_Q \eta(q)\}$, whence the symmetric measure is given by

$$\mathrm{d}\nu_Q(q) = \Delta_Q(q)^{-\frac{1}{2}}\,\mathrm{d}q = e^{\rho_Q \eta(q)}\,\mathrm{d}q.$$

Following Lemma 2.4, we have the following analogue of Proposition 2.5.

Proposition 2.10. *For any $f \in C_c(G)$, we have*

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_Q f(kq)\,e^{2\rho_Q \eta(q)}\,\mathrm{d}k\,\mathrm{d}q. \qquad \square$$

2.4 The Cartan projection

Let $\kappa : G \to \mathfrak{a}^+$ denote the Cartan projection, so that $g \in Ke^{\kappa(g)}K$ for all $g \in G$. Recall that the Euclidean norm $|\cdot|$ on $\mathfrak{a}$ is invariant by the Weyl group W_G. Denote by $\mathfrak{a}(r)$ the closed metric ball centered at 0 of radius r.

We say that a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in G *go to infinity* (write $g_n \to \infty$), if they eventually leave every compactum of G. This is equivalent to saying that $|\kappa(g_n)| \to +\infty$ as $n \to \infty$.

Lemma 2.11 ([Ben96, Prop 5.1])**.** *For any compact subset B of G, there exists $r > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$.* $\square$

2.5 Spherical functions

To each linear form $\chi \in \mathfrak{a}^*$, we associate the following function on G:

$$\Xi^G_\chi(g) := \int_K e^{-(\chi+\rho)\eta(g^{-1}k)}\,\mathrm{d}k,$$

where $\eta : G \to \mathfrak{a}$ is the Iwasawa projection. Such functions are called *spherical functions*. They are K-bi-invariant, smooth, and decay exponentially fast at infinity. In fact, using tools from hypergeometric functions, one can obtain precise information of their asymptotics, but for us the following results suffice.

Lemma 2.12 ([Kna86, Prop 7.15])**.** *For any $\chi \in \mathfrak{a}^*$ and $w \in W_G$, we have*

$$\Xi^G_\chi = \Xi^G_{w\chi}. \qquad \square$$

Recall that the Cartan subspace $\mathfrak{a}$ is naturally identified with its dual space $\mathfrak{a}^*$. Let the positive Weyl chamber $\mathfrak{a}^+$ correspond to $(\mathfrak{a}^*)^+$.

Lemma 2.13 ([NPP14, Thm 3.4])**.** *For each $\chi \in (\mathfrak{a}^*)^+$, there exists a polynomial $p(\cdot)$ on $\mathfrak{a}$ such that for all $g \in G$, we have*

$$\exp\{(\chi - \rho)\kappa(g)\} \le \Xi^G_\chi(g) \le p(\kappa(g))\exp\{(\chi - \rho)\kappa(g)\}\,. \qquad \square$$

3 Unitary representations

The general references for this section include [Kna86, BdlHV08].

3.1 Unitary representations

Let G be a locally compact group. A *unitary representation* of G is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a complex Hilbert space and $\pi : G \to \mathrm{U}(\mathcal{H})$ is a group morphism from G to the group $\mathrm{U}(\mathcal{H})$ of unitary operators on $\mathcal{H}$, such that π is *strongly continuous* in the sense that for any $v \in \mathcal{H}$, the map $G \to \mathcal{H}$, $g \mapsto \pi(g)v$ is continuous. A *matrix coefficient* of π is a map of the form

$$G \to \mathbb{C}, \quad g \mapsto \langle \pi(g)v_1, v_2 \rangle ,$$

where $v_1, v_2 \in \mathcal{H}$. By strong continuity, matrix coefficients are bounded continuous functions on G.

Two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are *equivalent* if there exists a G-intertwining isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$. Equivalent unitary representations are automatically *unitarily equivalent*, i.e. we can additionally require the intertwining isomorphism to be a unitary operator. We will not distinguish equivalent representations.

Example 3.1. Let $\mathrm{d}x$ be a left Haar measure on G. The convention for the L^2-scalar product of functions is

$$\langle f_1, f_2 \rangle_{\mathrm{L}^2} = \int_G f_1(x)\overline{f_2(x)}\,\mathrm{d}x.$$

The *left regular representation* λ_G of G on the Hilbert space $\mathrm{L}^2(G)$ acts by

$$\lambda_G(g)f : x \mapsto f(g^{-1}x), \quad \text{for } f \in \mathrm{L}^2(G).$$

Then $(\lambda_G, \mathrm{L}^2(G))$ is a unitary representation of G. The unitarity of λ_G follows from the left invariance of $\mathrm{d}x$, and the strong continuity follows from the fact that $C_{\mathrm{c}}(G)$ is dense in $\mathrm{L}^2(G)$. The right regular representation comes with an extra factor from the modular function and turns out equivalent to the left one.

3.2 Induced representations

Let G be a locally compact group, H be a closed subgroup of G, and $(\sigma, \mathcal{V})$ be a unitary representation of H. Let μ be a quasi-invariant Radon measure on the homogeneous space G/H and δ be the associated function satisfying (2.2) (cf. Lemma 2.1).

We describe the induced unitary representation $(\pi, \mathcal{H}) = \mathrm{Ind}_H^G(\sigma, \mathcal{V})$. Elements of $\mathcal{H}$ are measurable vector-valued functions $f : G \to \mathcal{V}$ with σ-equivariance

$$f(xh) = \sigma(h)^{-1} f(x)$$

for all $x \in G$ and $h \in H$, and L^2-integrability

$$\|f\|^2 := \int_{G/H} \langle f(x), f(x) \rangle_{\mathcal{V}} \,\mathrm{d}\mu(xH) < +\infty,$$

where $\langle f(x), f(x) \rangle_{\mathcal{V}}$ does not depend on the representative of xH since σ is unitary. The induced action of G is given by

$$\pi(g) f(xH) = f(g^{-1}xH) \left(\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(xH) \right)^{\frac{1}{2}} = f(g^{-1}xH) \left(\frac{\delta(g^{-1}x)}{\delta(x)} \right)^{\frac{1}{2}}.$$

Here, the cocycle term ensures that $\pi(g)$ is a unitary operator. Although *a priori* this definition depends on the measure μ, it turns out that different choices of μ will give unitarily equivalent representations. In particular, if $\sigma = 1_H$, then π is the *quasi-regular representation* $\lambda_{G/H}$ on $\mathrm{L}^2(G/H)$.

There is a simple way to produce elements in $\mathcal{H}$. For $\varphi \in C_{\mathrm{c}}(G)$ and $v \in \mathcal{V}$, define the map $\mathrm{I}_H^G(\varphi, v) = \mathrm{I}(\varphi, v) : G \to \mathcal{V}$ by

$$\mathrm{I}(\varphi, v)(x) := \int_H \varphi(xH)\sigma(h)v \,\mathrm{d}h.$$

We only specify I_H^G when necessary. The equivariance property follows from

$$\mathrm{I}(\varphi, v)(xh_0) = \int_H \varphi(xh_0h)\sigma(h)v \,\mathrm{d}h = \sigma(h_0)^{-1}\mathrm{I}(\varphi, v)(x)$$

for all $x \in G$ and $h_0 \in H$, and the L^2-integrability from

$$\begin{aligned} \|\mathrm{I}(\varphi, v)\|^2 &= \int_{G/H} \left\| \int_H \varphi(xH)\sigma(h)v \,\mathrm{d}h \right\|_{\mathcal{V}}^2 \mathrm{d}\mu(xH) \\ &\leq \|v\|_{\mathcal{V}}^2 \int_{G/H} \int_H |\varphi(xH)|^2 \,\mathrm{d}h \,\mathrm{d}\mu(xH) \\ &= \|v\|_{\mathcal{V}}^2 \int_G |\varphi(g)|^2 \delta(g) \,\mathrm{d}g < +\infty. \end{aligned}$$

Hence, the function $\mathrm{I}(\varphi, v)$ belongs to $\mathcal{H}$. We have the following fact.

Lemma 3.2 ([BdlHV08, Lem B.1.2])**.** *Let $\sigma = 1_H$. Then the map $C_{\mathrm{c}}(G) \to C_{\mathrm{c}}(G/H)$ given by $\varphi \mapsto \mathrm{I}(\varphi, 1)$ is surjective.* □

Lemma 3.3. *Given any neighborhood B_G of e in G and any $\psi \in C_{\mathrm{c}}(G)$, there exists finitely many $\varphi_i \in C_{\mathrm{c}}(G)$ with $(\operatorname{supp} \varphi_i)(\operatorname{supp} \varphi_i)^{-1} \subset B_G$ such that $\psi = \sum_i \varphi_i$.*

Proof. Since B_G is a neighborhood of e, we can find a relatively compact open neighborhood B of e with $BB^{-1} \subset B_G$. Now $\{Bg : g \in \operatorname{supp}\psi\}$ gives an open cover of $\operatorname{supp}\psi$. By the compactness of $\operatorname{supp}\psi$, there exists a finite subcover $\{Bg_i : i \in I\}$ for some finite index set I. Then we can find a finite partition of unity $\{\chi_i \in C_c(Bg_i)\}_{i\in I}$ subordinated to this cover, so that $1 = \sum_{i\in I} \chi_i$ over $\operatorname{supp}\psi$. Then $\varphi_i := \psi\chi_i \in C_c(G)$ satisfies $(\operatorname{supp}\varphi_i)(\operatorname{supp}\varphi_i)^{-1} \subset B_G$ and $\psi = \sum_i \varphi_i$. □

Corollary 3.4. *Let $\sigma = 1_H$. For any neighborhood B_G of e in G, the set*

$$\left\{\mathrm{I}(\varphi, 1) : \varphi \in C_c(G) \text{ with } (\operatorname{supp}\varphi)(\operatorname{supp}\varphi)^{-1} \subset B_G\right\}$$

spans $C_c(G/H)$ and hence is a total subset in $\mathrm{L}^2(G/H)$.

Proof. The corollary follows from Lemma 3.2 and Lemma 3.3. □

3.3 Weak containment

For unitary representations of noncompact groups, the notion of containment is too strong to work with. Instead, we introduce the notion of weak containment. Let G be a locally compact group.

Definition 3.5. Let $(\sigma, \mathcal{V})$ and $(\pi, \mathcal{H})$ be two unitary representations of G. Say that σ is *weakly contained* in π, if every diagonal matrix coefficient $\langle \sigma(\cdot)v, v\rangle$ can be approximated, uniformly on compacta, by convex combinations of diagonal matrix coefficients of π.

Fact 3.6. Weak containment is preserved under induction and restriction of unitary representations.

Fact 3.7. A locally compact group G is amenable iff the trivial representation 1_G is weakly contained in the regular representation λ_G.

Example 3.8. Let G be a real semisimple algebraic group and P be a minimal parabolic group. As P is amenable, the trivial representation 1_P is weakly contained in the regular representation λ_P. Hence, the quasi-regular representation $\lambda_{G/P} = \operatorname{Ind}_P^G 1_P$ is weakly contained in $\operatorname{Ind}_P^G \lambda_P = \lambda_G$.

3.4 Tempered representations

Now let G be a real semisimple algebraic group.

Definition 3.9. A unitary representation π of G is *tempered* if π is weakly contained in the regular representation $(\lambda_G, \mathrm{L}^2(G))$.

Example 3.8 implies that $(\lambda_{G/P}, \mathrm{L}^2(G/P))$ is tempered. The tempered representations of G admit nice equivalent characterizations.

Theorem 3.10 ([CHH88]). *Let G be a real semisimple algebraic group and K be a maximal compact subgroup of G. Then for any unitary representation $(\pi, \mathcal{H})$ of G, the following statements are equivalent:*

(i) *π is tempered;*

(ii) *for all K-finite vectors v_1, v_2 in $\mathcal{H}$, we have*

$$|\langle \pi(g)v_1, v_2\rangle| \leq \sqrt{\dim\langle Kv_1\rangle \dim\langle Kv_2\rangle}\, \|v_1\|\, \|v_2\|\, \Xi_0^G(g);$$

(iii) *there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$, such that for all $v_1, v_2 \in \mathcal{H}_0$, the coefficients $\langle \pi(\cdot)v_1, v_2\rangle \in \mathrm{L}^{2+\varepsilon}(G)$ for any $\varepsilon > 0$.* □

In view of Lemma 2.13, the optimal decay of spherical functions is given by Ξ_0^G. Meanwhile, it follows from Lemma 2.13 and Proposition 2.6 that we have $\Xi_0^G \in \mathrm{L}^{2+\varepsilon}(G)$ for any $\varepsilon > 0$. For a closed subgroup H of G, we have the following consequence.

Corollary 3.11. *The quasi-regular representation $(\lambda_{G/H}, \mathrm{L}^2(G/H))$ is tempered iff we have $\theta_{G/H} \leq 1/2$.* □

4 Volume growth and volume decay

The goal of this section is to establish some fundamental tools to address the growth and decay of volume in real semisimple groups, which will play a key role in the proof of the main theorems. Let G be a real semisimple algebraic group and ν_G be its Haar measure.

4.1 Local volume decay in G

Recall that the Bruhat decomposition (Theorem 2.7) asserts that the multiplication map $\overline{N} \times M \times A \times N \to G$ is a diffeomorphism onto an open subset of full measure.

Lemma 4.1. *For any compact subset B of $\overline{N}MAN$, there exists a constant $C > 0$ such that for all $a \in A^+$ we have $\nu_G(aBa^{-1} \cap B) \leq C\exp\{-2\rho \log a\}$.*

Proof. By the Bruhat decomposition, there exist compact sets

$$B_{\overline{N}} \subset \overline{N}, B_A \subset A, B_N \subset N$$

such that $B \subset B_{\overline{N}} M B_A B_N$. From Proposition 2.8 we deduce

$$\nu_G\big(aBa^{-1} \cap B\big) \leq \int_{B_{\overline{N}}} \int_M \int_{B_A} \int_{B_N} \mathbb{1}_B(a^{-1}\bar{n} m a_1 n a)\, e^{2\rho \log a_1}\, \mathrm{d}\bar{n}\, \mathrm{d}m\, \mathrm{d}a_1\, \mathrm{d}n.$$

Since a normalizes both $\overline{N}$ and N, and since MA centralizes a, we have furthermore

$$\begin{aligned}
&\int_{B_{\overline{N}}}\int_M\int_{B_A}\int_{B_N} \mathbb{1}_B(a^{-1}\bar{n}ma_1na)\, e^{2\rho\log a_1}\,\mathrm{d}\bar{n}\,\mathrm{d}m\,\mathrm{d}a_1\,\mathrm{d}n\\
&= \int_{B_{\overline{N}}}\int_M\int_{B_A}\int_{B_N} \mathbb{1}_B\big((a^{-1}\bar{n}a)ma_1(a^{-1}na)\big)\; e^{2\rho\log a_1}\,\mathrm{d}\bar{n}\,\mathrm{d}m\,\mathrm{d}a_1\,\mathrm{d}n\\
&\leq \int_{B_{\overline{N}}\cap aB_{\overline{N}}a^{-1}} \mathrm{d}\bar{n}\int_M \mathrm{d}m\int_{B_A} e^{2\rho\log a_1}\,\mathrm{d}a_1\int_{B_N}\mathrm{d}n\\
&\leq \int_{a^{-1}B_{\overline{N}}a\cap B_{\overline{N}}} e^{-2\rho\log a}\,\mathrm{d}\bar{n}'\, C(B_A,B_N)\\
&\leq C(B_{\overline{N}},B_A,B_N)\exp\{-2\rho\log a\}\,,
\end{aligned}$$

where we set $\bar{n}' = a^{-1}\bar{n}a$ and get $\mathrm{d}\bar{n}' = e^{2\rho\log a}\,\mathrm{d}\bar{n}$. $\square$

Lemma 4.2. *There exists an open neighborhood B_G of e in G, such that for all $k\in K$ we have $kB_Gk^{-1}\subset \overline{N}MAN$.*

Proof. Since $\overline{N}MAN$ is an open neighborhood of e, and since the map

$$K\times G\to G,\quad (k,g)\mapsto kgk^{-1}$$

is continuous, we deduce that for each $k\in K$, there exists a neighborhood V_k of k in K and an open neighborhood U_k of e in G, such that

$$k_0xk_0^{-1}\in\overline{N}MAN,\quad \forall k_0\in V_k,\ \forall x\in U_k.$$

By the compactness of K, there exists a covering by a finite collection $V_{k_1},\dots,V_{k_m}$ for some $k_1,\dots,k_m\in K$. Let B_G be the intersection of all the U_{k_i} for $i=1,\dots,m$. Then B_G is an open neighborhood of e in G which satisfies the statement of the lemma. $\square$

Now we can upgrade the statement of Lemma 4.1 from A^+ to all of G.

Proposition 4.3. *Let B_G be given as in Lemma 4.2. Then for any $\varphi_1,\varphi_2\in C_c(B_G)$, there exists a constant $C=C(\varphi_1,\varphi_2)$ such that*

$$\left|\int_G \varphi_1(g^{-1}xg)\varphi_2(x)\,\mathrm{d}x\right| \leq C\exp\{-2\rho\kappa(g)\}$$

uniformly for all $g\in G$.

Proof. For $i=1,2$, define $\tilde{\varphi}_i(x) := \sup_{k\in K}|\varphi_i(kxk^{-1})|$. Then $\tilde{\varphi}_1,\tilde{\varphi}_2$ are continuous functions compactly supported in $\overline{N}MAN$ by Lemma 4.2. For

$g \in G$, we can write $g = k_2 e^X k_1$ for $X = \kappa(g)$ and some $k_1, k_2 \in K$. By the unimodularity of G, we have

$$\begin{aligned}\left|\int_G \varphi_1(g^{-1}xg)\varphi_2(x)\,\mathrm{d}x\right| &= \left|\int_G \varphi_1(k_1^{-1}e^{-X}k_2^{-1}xk_2e^Xk_1)\varphi_2(x)\,\mathrm{d}x\right| \\ &= \left|\int_G \varphi_1(k_1^{-1}e^{-X}ye^Xk_1)\varphi_2(k_2yk_2^{-1})\,\mathrm{d}y\right| \\ &\leq \int_G \tilde{\varphi}_1(e^{-X}ye^X)\tilde{\varphi}_2(y)\,\mathrm{d}y,\end{aligned}$$

which is bounded from above by $C\exp\{-2\rho(X)\}$ by Lemma 4.1. □

4.2 The rho-function and volume decay

Let H be a real algebraic group, $\mathfrak{h}$ be its Lie algebra, and $\tau : H \to \mathrm{GL}(V)$ be an algebraic representation on a d-dimensional real vector space V. By abusing the notation, denote by $\tau : \mathfrak{h} \to \mathrm{End}(V)$ the differential map which is a representation of the Lie algebra $\mathfrak{h}$. To these data we associate the following rho-function $\rho_V : \mathfrak{h} \to \mathbb{R}^+$.

Definition 4.4. For each $Y \in \mathfrak{h}$, the action of $\tau(Y)$ on $V \otimes \mathbb{C}$ admits a Jordan normal form over $\mathbb{C}$ with diagonal elements $\lambda_1, \ldots, \lambda_d$. We define

$$\rho_V(Y) := \frac{1}{2}\sum_{i=1}^{d} |\mathrm{Re}\,\lambda_i|\,.$$

It follows from the definition that ρ_V is a continuous homogeneous function which is invariant by the adjoint action of H.

Let $\mathfrak{a}_\mathfrak{h}$ be a maximal split abelian subalgebra of $\mathfrak{h}$. Since $\tau(\mathfrak{a}_\mathfrak{h})$ is a split abelian subalgebra of $\mathrm{End}(V)$, the action of $\mathfrak{a}_\mathfrak{h}$ is jointly diagonalizable over $\mathbb{R}$. Then the restriction $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$ is the half sum of the absolute values of the eigenvalues and therefore is a piecewise linear, continuous, convex, homogeneous function. As V is finite-dimensional, the function ρ_V is uniformly Lipschitz on $\mathfrak{a}_\mathfrak{h}$. If τ is faithful, then $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$ is a polyhedral norm on $\mathfrak{a}_\mathfrak{h}$.

By the Jordan decomposition, every element $Y \in \mathfrak{h}$ splits uniquely as a sum of commuting elements $Y = Y_e + Y_h + Y_n$ and Y_h conjugates into $\mathfrak{a}_\mathfrak{h}$. Since $\rho_V(Y) = \rho_V(Y_h)$, the function ρ_V is determined by $\rho_V|_{\mathfrak{a}_\mathfrak{h}}$.

Example 4.5. Let H be a reductive group and $(\tau, V) = (\mathrm{Ad}, \mathfrak{h})$ be the adjoint representation. Fix a positive system $\Sigma^+(\mathfrak{a}_\mathfrak{h}; \mathfrak{h})$ and let ρ_H be the usual half sum of positive roots. Then the convex function $\rho_\mathfrak{h}$ coincides with the twice of the linear form ρ_H on the positive Weyl chamber $\mathfrak{a}_\mathfrak{h}^+$. If $W_H = W(\mathfrak{a}_\mathfrak{h}; \mathfrak{h})$ denotes the Weyl group, then for all $X \in \mathfrak{a}_\mathfrak{h}$,

$$\rho_\mathfrak{h}(X) = \max_{w \in W_H} 2\rho_H(wX).$$

In particular, $\rho_\mathfrak{h}$ is W_H-invariant.

Let Vol be the Lebesgue measure on the vector space $V \cong \mathbb{R}^d$. The following lemma shows that the function ρ_V reflects the volume decay of τ.

Lemma 4.6 ([BK22, Lem 2.8])**.** *For any compact neighborhood B of 0 in V, there exist constants $c, C > 0$ such that*

$$c \exp\{-\rho_V(X)\} \leq e^{-\operatorname{Tr}\tau(X)/2} \operatorname{Vol}\big(\tau(e^X)B \cap B\big) \leq C \exp\{-\rho_V(X)\}$$

uniformly for all $X \in \mathfrak{a}_{\mathfrak{h}}$. □

This result can be reframed in terms of unitary representations.

Corollary 4.7. *Let H be a real reductive group and $\tau : H \to \mathrm{GL}(V)$ be an algebraic representation. Consider the unitary representation $(T, \mathrm{L}^2(V))$ of the group G induced by*

$$T(g)f(v) = f(\tau(g)^{-1}v)\,(\det \tau(g))^{-\frac{1}{2}},$$

for $f \in \mathrm{L}^2(V)$ and $v \in V$. Then for any compact neighborhood B of 0 in V, there exist constants $c, C > 0$ such that

$$c \exp\{-\rho_V \kappa_H(h)\} \leq \langle T(h)\mathbb{1}_B, \mathbb{1}_B\rangle \leq C \exp\{-\rho_V \kappa_H(h)\}$$

uniformly for all $h \in H$, where $\kappa_H : H \to \mathfrak{a}_{\mathfrak{h}}^+$ is the Cartan projection.

Proof. Let $H = K_H A_H^+ K_H$ be the Cartan decomposition of H associated with κ_H. Let $D = \tau(K_H)B$ which is still a compact neighborhood of 0 in V. Then for any $k_1, k_2 \in K_H$ and $X \in \mathfrak{a}_{\mathfrak{h}}$, we have

$$\big\langle T(k_1 e^X k_2)\mathbb{1}_D, \mathbb{1}_D\big\rangle = e^{-\operatorname{Tr}\tau(X)/2} \operatorname{Vol}\big(\tau(e^X)D \cap D\big),$$

whence we conclude by applying Lemma 4.6. □

Now we can define the relative local volume decay exponent associated with a homogeneous space.

Definition 4.8. Let G be a real semisimple algebraic group and H be an algebraic subgroup of G. The *relative decay exponent* $\beta_{G/H}$ is defined by

$$\beta_{G/H} := \sup_{\mathfrak{h}} \frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}}},$$

where $\mathfrak{g}, \mathfrak{h}$ are viewed as $\mathfrak{h}$-module through the adjoint action. We take $0/0 = 0$ by convention.

Remark 4.9. (1) By definition, the number $\beta_{G/H}$ lies in $[0, 1]$.

(2) If H is an algebraic subgroup of G, then the Jordan decomposition implies that

$$\beta_{G/H} = \sup_{\mathfrak{a}_{\mathfrak{h}}} \frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}}}.$$

4.3 Volume growth in G

By $B \Subset G$ we denote that B be a compact subset of G of nonempty interior.

Proposition 4.10. *For any $B \Subset G$, there exist constants $c, C > 0$ such that*

$$c \exp\{2\rho\kappa(g)\} \leq \nu_G(BgB) \leq C \exp\{2\rho\kappa(g)\}$$

uniformly for all $g \in G$.

Proof. Let $X = \kappa(g)$. For the upper bound, by Lemma 2.11, there exists $r > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$, whence we have $BgB \subset Ke^{X+\mathfrak{a}(r)}K$. By Proposition 2.6,

$$\nu_G(BgB) \leq \int_K \int_{X+\mathfrak{a}(r)} \int_K e^{2\rho(Y)}\, \mathrm{d}k\, \mathrm{d}Y\, \mathrm{d}k' \leq C \exp\{2\rho(X)\}$$

uniformly for $g \in G$.

For the lower bound, first note that by Lemma 2.11, up to translation we can suppose that B contains a neighborhood of $e \in G$. Then we can find a small neighborhood B' of e with $kB'k^{-1} \subset B$ for all $k \in K$ (cf. Lemma 4.2). Write $g = k_1 e^X k_2$. By the unimodularity of G,

$$\nu_G(BgB) = \nu_G\big((k_1^{-1}Bk_1)e^X(k_2Bk_2^{-1})\big) \geq \nu_G\big(B'e^XB'\big).$$

Up to further shrinking B', we can assume $B' \subset \overline{N}MAN$. By the Bruhat decomposition (Theorem 2.7), there exist compact neighborhoods of e in the respective subgroups $B_{\overline{N}} \Subset \overline{N}$, $B_M \Subset M$, $B_A \Subset A$, and $B_N \Subset N$ such that $B_{\overline{N}}B_MB_AB_N \subset B'$. Hence by the unimodularity of G, we deduce

$$\begin{aligned}
\nu_G\big(B'e^XB'\big) &= \nu_G\big(B'e^XB'e^{-X}\big) \\
&\geq \nu_G\big(B_{\overline{N}}\left(e^XB_{\overline{N}}B_MB_AB_Ne^{-X}\right)\big) \\
&= \nu_G\big(\left(B_{\overline{N}}e^XB_{\overline{N}}e^{-X}\right)B_MB_A\left(e^XB_Ne^{-X}\right)\big).
\end{aligned}$$

By further applying Proposition 2.8, we obtain

$$\begin{aligned}
\nu_G\big(B'e^XB'\big) &\geq c\,\nu_{\overline{N}}\big(B_{\overline{N}}e^XB_{\overline{N}}e^{-X}\big)\,\nu_N\big(e^XB_Ne^{-X}\big) \\
&\geq c\,\nu_{\overline{N}}\big(B_{\overline{N}}\big)\,e^{2\rho(X)}\nu_N(B_N) = c' \exp\{2\rho(X)\}\,,
\end{aligned}$$

uniformly for $g \in G$. ∎

4.4 The volume growth exponent of closed subgroups

We define the volume growth exponent of closed subgroups of the real semisimple group G and relate it to certain series. Then we determine its value for reductive subgroups and discrete subgroups. Recall that the symmetric measure is given by $\mathrm{d}\nu_H(h) = \Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}h$ for a locally compact group H. If H is either reductive or discrete, then ν_H is just the Haar measure.

Definition 4.11. The *growth exponent* of H inside G is defined by

$$\delta_{G/H} := \max\left\{0,\ \sup_{B\Subset G}\limsup_{g\to\infty}\frac{\log\nu_H(H\cap BgB)}{\log\nu_G(BgB)}\right\},$$

where B ranges over compacta of G of nonempty interior.

Remark 4.12. In the definition of $\delta_{G/H}$, we can restrict B to a cofinal subfamily of subsets, for example by additionally requiring B to be symmetric and K-bi-invariant.

Proposition 4.13. *For any closed subgroup H of the real semisimple group G, we have*

$$\delta_{G/H} = \inf\left\{t\in[0,\infty] : \int_H e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h) < \infty\right\}.$$

Proof. Denote the right hand side by τ. First show $\delta_{G/H}\le\tau$. Let $B\Subset G$. Then by Lemma 2.11, there exists $R>0$ such that $\kappa(BgB)\subset\kappa(g)+\mathfrak{a}(R)$ for all $g\in G$, whence for all $h\in H\cap BgB$ we have

$$2\rho\kappa(h)\le 2\rho\kappa(g)+C_1,$$

where the constant $C_1=\sup_{\mathfrak{a}(R)}2\rho$ is uniform. Since for any $t>\tau$,

$$\int_H e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h)<\infty,$$

we then have

$$\begin{aligned}\nu_H(H\cap BgB) &\le C_2\,e^{2t\rho\kappa(g)}\int_{H\cap BgB} e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h)\\ &\le C_2\,e^{2t\rho\kappa(g)}\int_H e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h)\\ &\le C_3\,e^{2t\rho\kappa(g)},\end{aligned}$$

uniformly for all $g\in G$. From the definition of the growth exponent $\delta_{G/H}$ we deduce that $t\ge\delta_{G/H}$, whence we have $\tau\ge\delta_{G/H}$.

Next we show $\delta_{G/H}\ge\tau$. Let $\mathcal{L}$ be a lattice of $\mathfrak{a}$ and $\mathcal{L}^+=\mathcal{L}\cap\mathfrak{a}^+$. Then there exists some $r>0$ such that $\mathfrak{a}^+$ is covered by the balls of radius r centered at elements in L^+. Fix any number $t>\delta_{G/H}$ and then fix a small number $\varepsilon>0$ so that $t-\varepsilon>\delta_{G/H}$.

Define the subset $B=Ke^{\mathfrak{a}(r)}K\Subset G$. Lemma 2.11 yields a constant $c>0$ such that whenever $h\in Be^XB$, we have $2\rho(X)\le 2\rho\kappa(h)+c$, whence

$$\int_{H\cap Be^XB} e^{-2t\rho\kappa(h)}\,\mathrm{d}\nu_H(h)\le C_4\,e^{-2t\rho(X)}\nu_H\big(H\cap Be^XB\big)$$

uniformly for all $X \in \mathfrak{a}^+$. But since $t - \varepsilon > \delta_{G/H}$, we deduce from Proposition 4.10 that

$$\nu_H\big(H \cap Be^X B\big) \le C_5\, e^{2(t-\varepsilon)\rho(X)}$$

uniformly for $X \in \mathfrak{a}^+$, whence for all $X \in \mathfrak{a}^+$ we have

$$\int_{H \cap Be^X B} e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h) \le C_6\, e^{-2\varepsilon\rho(X)}.$$

Since the construction implies that the subset $Be^X B$ contains $Ke^{X+\mathfrak{a}(r)}K$ for all $X \in \mathfrak{a}^+$, we obtain

$$\begin{aligned}\int_H e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h) &\le \sum_{X \in \mathcal{L}^+} \int_{H \cap Be^X B} e^{-2t\rho\kappa(h)}\, \mathrm{d}\nu_H(h)\\ &\le C_6 \sum_{X \in \mathcal{L}^+} e^{-2\varepsilon\rho(X)},\end{aligned}$$

which is finite as the lattice $\mathcal{L}$ grows polynomially, so $t > \tau$. We obtain $\delta_{G/H} \ge \tau$ and thus conclude the proof. □

4.4.1 The case of reductive subgroups

Proposition 4.14. *If H is a reductive subgroup of G, then*

$$\delta_{G/H} = \beta_{G/H}.$$

Before going into the proof, we make several preliminary remarks. Let A_H be a Cartan subgroup of H. Extending A_H to a Cartan subgroup A of G, we have $A_H = A \cap H$. By the reductiveness of H, there exists a Cartan decomposition $G = KAK$ of G such that the subgroup $K_H := K \cap H$ is a maximal compact subgroup of H, with the Cartan decomposition of H given by $H = K_H A_H K_H$.

Let $\mathfrak{a}, \mathfrak{a}_{\mathfrak{h}}$ denote respectively the Cartan subspaces and W_G, W_H denote respectively the associated Weyl groups. Since the Killing form is adjoint-invariant, the induced Euclidean norm $|\cdot|$ on $\mathfrak{a}$ is W_G-invariant and its restriction to $\mathfrak{a}_{\mathfrak{h}}$ is W_H-invariant.

By the Cartan decomposition of H, we have for all $h \in H$ that

$$h \in K_H\, e^{\{w\kappa(h) : w \in W_G\} \cap \mathfrak{a}_{\mathfrak{h}}} K_H. \tag{4.15}$$

Lemma 4.16. *For any $B \Subset G$, there exists a constant $C > 0$ such that*

$$\nu_H(H \cap BgB) \le C \exp\big\{2\beta_{G/H}\rho\kappa(g)\big\}$$

uniformly for all $g \in G$.

Proof. Write $X = \kappa(g)$. By Lemma 2.11, there exists a constant $r > 0$ such that $\kappa(BgB) \subset \kappa(g) + \mathfrak{a}(r)$ for all $g \in G$. Then (4.15) yields

$$H \cap BgB \subset \bigcup_{w \in W_G} K_H e^{(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_\mathfrak{h}} K_H. \tag{4.17}$$

Hence, by Proposition 2.6 and Example 4.5, we have

$$\nu_H(H \cap BgB) \le \sum_{w \in W_G} \int_{(wX+\mathfrak{a}(r)) \cap \mathfrak{a}_\mathfrak{h}} \exp\{\rho_\mathfrak{h}(Y)\} \,\mathrm{d}Y. \tag{4.18}$$

Since $\rho_\mathfrak{h} \le \beta_{G/H} \rho_\mathfrak{g}$ by the definition of $\beta_{G/H}$, and since $\rho_\mathfrak{g}$ is W_G-invariant and uniformly Lipschitz on $\mathfrak{a}_\mathfrak{h}$, we have

$$\begin{aligned} \exp\{\rho_\mathfrak{h}(Y)\} &\le \exp\bigl\{\beta_{G/H}\rho_\mathfrak{g}(Y)\bigr\} \le C_1 \exp\bigl\{\beta_{G/H}\rho_\mathfrak{g}(wX)\bigr\} \\ &= C_1 \exp\bigl\{\beta_{G/H}\rho_\mathfrak{g}(X)\bigr\} = C_1 \exp\bigl\{2\beta_{G/H}\rho(X)\bigr\} \end{aligned}$$

uniformly for $Y \in wX + \mathfrak{a}(r)$. By feeding back to (4.18), we conclude the proof. □

Proof of Proposition 4.14. For $B \Subset G$, Proposition 4.10 and Lemma 4.16 yield

$$\frac{\log \nu_H(H \cap BgB)}{\log \nu_G(BgB)} \le \frac{C_1 + 2\beta_{G/H}\rho\kappa(g)}{C_2 + 2\rho\kappa(g)}$$

uniformly for all $g \in G$. Since $\rho\kappa(g) \to \infty$ as $g \to \infty$, we get $\delta_{G/H} \le \beta_{G/H}$.

To show $\delta_{G/H} \ge \beta_{G/H}$, let $B \Subset G$ and $B_H := B \cap H \Subset H$. Then the intersection $H \cap BhB$ contains $B_H h B_H$ for all $h \in H$. Now Proposition 4.10 applies to the real reductive group H without modification, whence $\nu_H\bigl(H \cap Be^X B\bigr) \ge c\exp\{\rho_\mathfrak{h}(X)\}$ for all $X \in \mathfrak{a}_\mathfrak{h}$. By the continuity and homogeneity of $\rho_\mathfrak{h}, \rho_\mathfrak{g}$ on $\mathfrak{a}_\mathfrak{h}$, there exists $X \in \mathfrak{a}_\mathfrak{h} \setminus 0$ with

$$\beta_{G/H}\rho_\mathfrak{g}(X) = \rho_\mathfrak{h}(X).$$

Setting $g_n := e^{nX} \in H$ with $g_n \to \infty$ in G, we obtain

$$\delta_{G/H} \ge \limsup_{n\to\infty} \frac{\log \nu_H\bigl(H \cap Be^{nX}B\bigr)}{\log \nu_G(Be^{nX}B)} \ge \limsup_{n\to\infty} \frac{\log c\exp\{n\rho_\mathfrak{h}(X)\}}{\log C\exp\{n\rho_\mathfrak{g}(X)\}} = \beta_{G/H}$$

by applying the lower bound of Proposition 4.10 to H and the upper bound to G. □

4.4.2 The case of discrete subgroups For a discrete subgroup Γ, recall that the growth indicator function $\psi_\Gamma : \mathfrak{a}^+ \to \mathbb{R} \cup \{-\infty\}$ is defined by

$$\psi_\Gamma(X) := |X| \inf_{\mathfrak{C} \ni X} \inf \left\{ t \in \mathbb{R} : \sum_{\gamma \in \Gamma,\, \kappa(\gamma) \in \mathfrak{C}} e^{-t|\kappa(\gamma)|} < \infty \right\},$$

where $\mathcal{C}$ ranges over open cones in $\mathfrak{a}$ which contain X. It is easy to see that (cf. [Qui02, §I.1])

$$\sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho} = \inf \left\{ t \in \mathbb{R} : \sum_{\gamma \in \Gamma} e^{-2t\rho\kappa(\gamma)} < \infty \right\}.$$

As an immediate consequence of Proposition 4.13, we have the following.

Proposition 4.19. *If Γ is a discrete subgroup of G, then*

$$\delta_{G/\Gamma} = \max\left\{ \sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho},\, 0 \right\}. \qquad \square$$

Example 4.20. When Γ is a discrete subgroup of $G = \mathrm{SL}(2, \mathbb{R})$, the exponent $\delta_{G/\Gamma}$ coincides with the usual critical exponent δ_Γ. In general, when the semisimple group G is of real rank one, these two exponents coincide up to renormalization.

5 Decay of coefficients and volume growth

The goal of this section is to prove Theorem A on the equality between the exponents θ, δ, and p. The upper bounds on δ and p from the uniform decay exponent θ are rather straightforward. The main difficulty, which we will start with, is to establish uniform decay estimates from other data, but the method in [LWW24] has already paved the way.

Let G be a real semisimple algebraic group and H be a closed subgroup of G. Then by Lemma 2.1, the homogeneous space G/H admits a G-quasi-invariant Radon measure $\mathrm{d}(gH)$ and a continuous function $\delta : G \to \mathbb{R}^+$ which satisfy (2.2) and (2.3) in Lemma 2.1.

5.1 Matrix coefficients of induced representations

As a preliminary step, we transform the matrix coefficients of $\mathrm{L}^2(G/H)$ into more accessible terms. For later applications as well, we address more generally the coefficients of an induced representation $(\pi, \mathcal{H}) = \mathrm{Ind}_H^G(\sigma, \mathcal{V})$. Setting $\sigma = 1_H$ will recover $\mathrm{L}^2(G/H)$.

Given $\varphi_1, \varphi_2 \in C_\mathrm{c}(G)$ and $v_1, v_2 \in \mathcal{V}$, consider the equivariant functions $f_i := \mathrm{I}(\varphi_i, v_i) \in \mathcal{H}$ for $i = 1, 2$ (see Section 3.2). To study the matrix coefficient $\langle \pi(\cdot) f_1, f_2 \rangle$, we first expand it with the expressions of the vectors f_1, f_2. By using the unitarity of the representation σ, the σ-equivariance of the vectors f_1, f_2, the integration formula (2.3) on the homogeneous space G/H, Fubini's theorem, and the fact that $\delta(xh) = \delta(x)\Delta_H(h)$ for all $h \in H$, we can transform the matrix coefficient as follows:

$$
\begin{aligned}
&\langle \pi(g) f_1, f_2\rangle \\
&= \int_{G/H} \big\langle f_1(g^{-1}x), f_2(x)\big\rangle_{\mathcal{V}} \left(\frac{\delta(g^{-1}x)}{\delta(x)}\right)^{\frac{1}{2}} \mathrm{d}\mu(xH) \\
&= \int_{G/H}\int_H \big\langle \varphi_1(g^{-1}xh)\sigma(h)v_1, f_2(x)\big\rangle_{\mathcal{V}} \left(\frac{\delta(g^{-1}x)}{\delta(x)}\right)^{\frac{1}{2}} \mathrm{d}h\,\mathrm{d}\mu(xH) \\
&= \int_{G/H}\int_H \langle v_1, f_2(xh)\rangle_{\mathcal{V}}\, \varphi_1(g^{-1}xh) \left(\frac{\delta(g^{-1}xh)}{\delta(xh)}\right)^{\frac{1}{2}} \mathrm{d}h\,\mathrm{d}\mu(xH) \\
&= \int_G \langle v_1, f_2(x)\rangle_{\mathcal{V}}\, \varphi_1(g^{-1}x) \big(\delta(g^{-1}x)\delta(x)\big)^{\frac{1}{2}}\,\mathrm{d}x \\
&= \int_G\int_H \langle v_1, \sigma(h)v_2\rangle_{\mathcal{V}}\, \varphi_1(g^{-1}x)\overline{\varphi_2(xh)} \big(\delta(g^{-1}x)\delta(x)\big)^{\frac{1}{2}}\,\mathrm{d}h\,\mathrm{d}x \\
&= \int_H \langle v_1, \sigma(h)v_2\rangle_{\mathcal{V}} \int_G \varphi_1(g^{-1}x)\overline{\varphi_2(xh)} \big(\delta(g^{-1}x)\delta(x)\big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}h \\
&= \int_H \langle v_1, \sigma(h)v_2\rangle_{\mathcal{V}} \int_G \varphi_1(g^{-1}x)\overline{\varphi_2(xh)} \big(\delta(g^{-1}x)\delta(xh)\big)^{\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}\nu_H(h),
\end{aligned}
$$

where $\mathrm{d}\nu_H(h) = \Delta_H(h)^{-1/2}\,\mathrm{d}h$ is the symmetric measure on H. By changing h to h^{-1}, we obtain

$$
\langle \pi(g) f_1, f_2\rangle = \int_H \langle \sigma(h)v_1, v_2\rangle_{\mathcal{V}}\, \Phi(h,g)\,\mathrm{d}\nu_H(h), \tag{5.1}
$$

where we define

$$
\Phi(h,g) := \int_G \varphi_1(g^{-1}x)\overline{\varphi_2(xh^{-1})} \big(\delta(g^{-1}x)\delta(xh^{-1})\big)^{\frac{1}{2}}\,\mathrm{d}x. \tag{5.2}
$$

Using the volume decay in G, we can obtain the first estimates for decay of coefficients.

Lemma 5.3. *Let B_G be the open neighborhood of $e \in G$ given in Lemma 4.2. For any $v_1, v_2 \in \mathcal{V}$ and any $\varphi_1, \varphi_2 \in C_{\mathrm{c}}(G)$ whose supports $B_i := \operatorname{supp}\varphi_i$ satisfy $B_i B_i^{-1} \subset B_G$, let $f_i := \mathrm{I}(\varphi_i, v_i) \in \mathcal{H}$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$
|\langle \pi(g) f_1, f_2\rangle| \le C \exp\{-2\rho\kappa(g)\} \int_{H\cap(B_2^{-1} g B_1)} |\langle \sigma(h)v_1, v_2\rangle_{\mathcal{V}}|\,\mathrm{d}\nu_H(h),
$$

uniformly for all $g \in G$.

Proof. If $\Phi(h,g) \neq 0$, then there exists $x \in G$ such that $g^{-1}x =: b_1 \in B_1$ and $xh^{-1} =: b_2 \in B_2$, so $h = b_2^{-1} g b_1 \in B_2^{-1} g B_1$. By using the unimodularity of G, we deduce

$$\begin{aligned}\Phi(h,g) &= \int_G \varphi_1(g^{-1}x)\overline{\varphi_2(xb_1^{-1}g^{-1}b_2)}\,\big(\delta(g^{-1}x)\delta(xh^{-1})\big)^{\frac{1}{2}}\,\mathrm{d}x \\ &= \int_G \varphi_1(g^{-1}ygb_1)\overline{\varphi_2(yb_2)}\,\big(\delta(g^{-1}ygb_1)\delta(yb_2)\big)^{\frac{1}{2}}\,\mathrm{d}y \end{aligned} \tag{5.4}$$

by setting $y = xb_1^{-1}g^{-1} = xhb_2^{-1}$. For $i = 1, 2$, define

$$\tilde{\varphi}_i(x) := \sup_{b\in B_i} |\varphi_i(xb)|\,\delta(xb)^{\frac{1}{2}}.$$

Then $\tilde{\varphi}_i \in C_c(B_G)$ by the hypothesis on B_i. Applied to $\tilde{\varphi}_1, \tilde{\varphi}_2$, Proposition 4.3 yields

$$\int_G \tilde{\varphi}_1(g^{-1}yg)\tilde{\varphi}_2(y)\,\mathrm{d}y \leq C\exp\{-2\rho\kappa(g)\}$$

uniformly for all $g \in G$. Feeding back to (5.4), we deduce

$$|\Phi(h,g)| \leq C\exp\{-2\rho\kappa(g)\}\,\mathbb{1}_{B_2^{-1}gB_1}(h)$$

uniformly for $g \in G$ and $h \in H$. We conclude by feeding back to (5.1). $\square$

5.2 From volume growth to uniform decay

Proposition 5.5. $\theta_{G/H} \leq \delta_{G/H}$.

Proof. Apply Lemma 5.3 to the induced representation $\lambda_{G/H} = \operatorname{Ind}_H^G 1_H$ and we obtain

$$\big|\langle \lambda_{G/H}(g)f_1, f_2\rangle\big| \leq C\exp\{-2\rho\kappa(g)\}\,\nu_H\big(H\cap B_2^{-1}gB_1\big).$$

Let $B \Subset G$ contain $B_2^{-1}\cup B_1$. By the definition of $\delta_{G/H}$ and by Proposition 4.10, for any $\delta > \delta_{G/H}$, there exists $C_1 = C_1(\delta) > 0$ such that

$$\nu_H\big(H\cap B_2^{-1}gB_1\big) \leq \nu_H(H\cap BgB) \leq C_1\exp\{2\delta\rho\kappa(g)\}$$

for all $g \in G$, whence

$$\big|\langle \lambda_{G/H}(g)f_1, f_2\rangle\big| \leq C_2\exp\{-2(1-\delta)\rho\kappa(g)\} \tag{5.6}$$

uniformly for $g \in G$. By Lemma 5.3, this uniform decay holds for all functions f_1, f_2 in

$$\big\{\mathrm{I}(\varphi,1) : \varphi \in C_c(G) \text{ with } (\operatorname{supp}\varphi)(\operatorname{supp}\varphi)^{-1}\subset B_G\big\},$$

which spans $C_c(G)$ by Corollary 3.4. Hence, (5.6) is valid for any functions $f_1, f_2 \in C_c(G)$ and any number $\delta > \delta_{G/H}$, which yields $\delta_{G/H} \geq \theta_{G/H}$. $\square$

5.3 From uniform decay to integrability

Recall that the integrability exponent $p_{G/H}$ optimizes the condition that for all $f_1, f_2 \in C_c(G/H)$ (or equivalently in $\mathrm{L}^\infty_c(G/H)$), we have

$$\langle \lambda_{G/H}(\cdot) f_1, f_2 \rangle \in \bigcap_{p > p_{G/H}} \mathrm{L}^p(G).$$

Proposition 5.7. $p_{G/H} \leq 1/(1-\theta_{G/H})$.

Proof. Suppose $\theta_{G/H} < 1$. For any $\theta > \theta_{G/H}$ and any $f_1, f_2 \in C_c(G/H)$, we obtain from Theorem A that

$$\left| \langle \lambda_{G/H}(g) f_1, f_2 \rangle \right| \leq C \exp\{(2\theta - 2)\rho\kappa(g)\}$$

uniformly for $g \in G$. Then by applying Proposition 2.6, we have

$$\int_G \left| \langle \lambda_{G/H}(g) f_1, f_2 \rangle \right|^p \mathrm{d}g \leq C \int_{\mathfrak{a}^+} \exp\{(2\theta p - 2p + 2)\rho(X)\} \, \mathrm{d}X,$$

which is finite as long as $p > 1/(1-\theta)$. By the arbitrariness of $\theta > \theta_{G/H}$, we conclude that $p_{G/H} \leq 1/(1-\theta_{G/H})$. $\square$

5.4 From integrability to volume growth

Lemma 5.8. *Let B be a symmetric compact neighborhood of e in G. For*

$$f(xH) = \int_H \mathbb{1}_B(xh)\delta(xh)^{-\frac{1}{2}} \, \mathrm{d}h \in \mathrm{L}^\infty_c(G/H),$$

we have for any $g \in G$ that

$$\int_{BBgBB} \langle \lambda_{G/H}(x) f, f \rangle \, \mathrm{d}x = \nu_G(B)^2 \, \nu_H(H \cap BgB), \tag{5.9}$$

where $\mathrm{d}x$ denotes the Haar measure of G.

Proof. Let us apply the preliminary computations in Section 5.1 to the induced representation $\lambda_{G/H} = \mathrm{Ind}_H^G 1_H$ and the functions

$$\varphi_1(x) = \varphi_2(x) = \mathbb{1}_B(x)\delta(x)^{-\frac{1}{2}} \in \mathrm{L}^\infty_c(G),$$

so that the functions $f_1 = f_2 = f$. Then (5.2) gives $\Phi(h,g) = \nu_G(gB \cap Bh)$ and (5.1) gives

$$\langle \lambda_{G/H}(g) f_1, f_2 \rangle = \int_{H \cap BgB} \nu_G(gB \cap Bh) \, \mathrm{d}\nu_H(h).$$

Let $B_g := BBgBB$ for $g \in G$. Then by Fubini's theorem,

$$\int_{B_g} \langle \lambda_{G/H}(x) f_1, f_2 \rangle \, \mathrm{d}x = \int_{H \cap BgB} \int_{B_g} \nu_G(xB \cap Bh) \, \mathrm{d}x \, \mathrm{d}\nu_H(h).$$

Then Fubini's theorem further yields, for any $h \in H \cap BgB$ that

$$\int_{B_g} \nu_G(xB \cap Bh) \, \mathrm{d}x = \int_G \int_G \mathbb{1}_{B_g}(x) \mathbb{1}_{xB}(y) \mathbb{1}_{Bh}(y) \, \mathrm{d}x \, \mathrm{d}y$$
$$= \int_G \int_G \mathbb{1}_{B_g}(x) \mathbb{1}_{xB}(zh) \mathbb{1}_B(z) \, \mathrm{d}x \, \mathrm{d}z = \int_B \nu_G(B_g \cap zhB) \, \mathrm{d}z = \nu_G(B)^2,$$

where $z := yh^{-1}$ and for the last equality we note that $B_g \supset zhB$ for any $z \in B$ and $h \in BgB$. We conclude by combining the equations above. $\square$

Proposition 5.10. $p_{G/H} \geq 1/(1 - \delta_{G/H})$.

Proof. By the definition of $\delta_{G/H}$ and by Remark 4.12, for any number $\delta < \delta_{G/H}$ there exists a symmetric, K-bi-invariant, compact neighborhood B of e in G, and a sequence (X_n) in $\mathfrak{a}^+$ going to infinity, such that

$$\log \nu_H\big(H \cap Be^{X_n}B\big) \geq \delta \log \nu_G\big(Be^{X_n}B\big) + c_1$$

uniformly for $n \in \mathbb{N}$, whence we can deduce from Proposition 4.10 that

$$\nu_H\big(H \cap Be^{X_n}B\big) \geq c_2 \exp\{2\delta\rho(X_n)\} \tag{5.11}$$

uniformly for $n \in \mathbb{N}$. By passing to a subsequence, we can assume that the subsets $BBe^{X_n}BB$ with $n \in \mathbb{N}$ are pairwise disjoint.

Let $f \in \mathrm{L}^2(G/H)$ be given as in Lemma 5.8. We proceed to study the integrability of the matrix coefficient $\langle \lambda_{G/H}(\cdot) f, f \rangle$. By applying the Hölder inequality to (5.9), we obtain for $p > 1$ that

$$\int_{BBgBB} \big|\langle \lambda_{G/H}(x) f, f \rangle\big|^p \, \mathrm{d}x \geq \nu_G(BBgBB)^{1-p} \left(\int_{BBgBB} \langle \lambda_{G/H}(x) f, f \rangle \, \mathrm{d}x \right)^p$$
$$\geq c_3 \exp\{2(1-p)\rho\kappa(g)\} \; \nu_H(H \cap BgB)^p,$$

where the last inequality follows from Proposition 4.10. By substituting $g = e^{X_n}$ and applying (5.11), we obtain

$$\int_{BBe^{X_n}BB} \big|\langle \lambda_{G/H}(x) f, f \rangle\big|^p \, \mathrm{d}x \geq c_4 \exp\{(2 - 2p + 2p\delta)\rho(X_n)\}$$

uniformly for $n \in \mathbb{N}$. Hence, for any $p > p_{G/H}$, from $\langle \lambda_{G/H}(\cdot) f, f \rangle \in \mathrm{L}^p(G)$ we deduce

$$\sum_{n \in \mathbb{N}} \exp\{2(1 - p + p\delta)\rho(X_n)\} < \infty,$$

which implies that $p > 1/(1-\delta)$. By the arbitrariness of $p > p_{G/H}$ and $\delta < \delta_{G/H}$, we conclude that $p_{G/H} \geq 1/(1 - \delta_{G/H})$. $\square$

5.5 Conclusion of proofs

Proof of Theorem A. The concatenation of Proposition 5.5, Proposition 5.7, and Proposition 5.10 yields

$$\delta_{G/H} \geq \theta_{G/H} \geq 1 - \frac{1}{p_{G/H}} \geq \delta_{G/H}.$$

Hence, they are all equal. $\square$

Applications to special classes of subgroups are immediate.

Proof of Corollary C. When the subgroup H is reductive algebraic, it follows from Theorem A and Proposition 4.14 that

$$\theta_{G/H} = \delta_{G/H} = \beta_{G/H}. \quad \square$$

Proof of Corollary D. When the subgroup Γ is discrete, it follows from Theorem A and Proposition 4.19 that

$$\theta_{G/\Gamma} = \delta_{G/\Gamma} = \sup_{\mathfrak{a}^+} \frac{\psi_\Gamma}{2\rho}. \quad \square$$

6 Uniform decay of induced representations

The goal of this section is to prove Proposition E. We will prove the equivalence between the exponents θ and β above $1/2$, and then the equivalence between the four exponents will follow from Theorem A.

Let G be a real semisimple algebraic group and H be an algebraic subgroup. For a preliminary reduction, we can assume the algebraic subgroup H to be Zariski connected. Indeed, the values of these exponents remain unchanged if we pass to an open subgroup of finite index.

Following the strategy of Benoist-Kobayashi in [BK22], we will examine the uniform decay property along a sequence of induced representations. Let us begin with two ingredients therein.

6.1 Key ingredients from Benoist-Kobayashi

The first ingredient is the existence of nice intermediate subgroups.

Lemma 6.1 ([BK22, Lem 4.1])**.** *There exist two intermediate algebraic subgroups $H \subset F \subset Q \subset G$ with the following properties:*

(1) Q is a parabolic subgroup of G of minimal dimension containing H.

(2) Let U be the unipotent radical of Q. There exist a Levi decomposition $Q = LU$ such that L is a maximal reductive subgroup of Q and that $H = (L \cap H)(U \cap H)$.

(3) $S = L \cap H$ *is a maximal reductive subgroup of* H *and* $W = U \cap H$ *is the unipotent radical of* H.

(4) $F = SU$.

That is to say, we have a chain of algebraic subgroups with compatible Levi decompositions

$$H = SW \subset F = SU \subset Q = LU \subset G. \qquad \square$$

The notation of these groups will be standing throughout this section. We can suppose that $Q \neq G$, for otherwise the algebraic subgroup H is already reductive and we can conclude by Corollary C.

The second ingredient is the domination of group actions. On the homogeneous space U/W, the reductive group S acts by conjugation and the unipotent group U acts by left translation.

Lemma 6.2 ([BK22, Prop 4.4]). *Let* U/W *be equipped with a* U*-invariant Radon measure* Vol. *Then for every compact subset* $D \subset U/W$*, there exists a compact subset* $D_0 \subset U/W$ *such that for all* $s \in S$ *and* $u \in U$*, we have*

$$\mathrm{Vol}(suD \cap D) \leq \mathrm{Vol}(sD_0 \cap D_0). \qquad \square$$

Since $F/H = U/W$, this lemma says that the coefficients of the unitary representation $(\lambda_{F/H}, \mathrm{L}^2(F/H))$ are majorated by those of the unitary representation $(\sigma_0, \mathrm{L}^2(U/W))$ whose action is given by $\sigma_0(su) = \lambda_{F/H}(s)$ for all $s \in S$ and $u \in U$. This corresponds to the concept of domination in [BK23, Def 4.2].

Now this majoration carries to induced representations (cf. [BK23, §4.2]), which implies that the coefficients of $\lambda_{Q/H} = \mathrm{Ind}_F^Q \lambda_{F/H}$ are majorated by those of the induced representation $\pi_0 := \mathrm{Ind}_F^Q \sigma_0$. Since the unitary representation σ_0 is trivial on the unipotent radical U, the induced representation π_0 is also trivial on U, whence we have $\pi_0|_L = \mathrm{Ind}_S^L(\sigma_0|_S)$. What we have obtained can be precised as the following proposition, which is also implied by the proof of [BK22, Prop 4.9].

Proposition 6.3. *For any* $f_1, f_2 \in C_c(Q/H)$*, there exist* $\varphi_1, \varphi_2 \in C_c(L)$ *and* $D_1, D_2 \Subset U/W$ *such that*

$$\left|\left\langle \lambda_{Q/H}(q) f_1, f_2 \right\rangle\right| \leq \left\langle \pi_0(l_q) \mathrm{I}_S^L(\varphi_1, \mathbb{1}_{D_1}), \mathrm{I}_S^L(\varphi_2, \mathbb{1}_{D_2}) \right\rangle$$

uniformly for all $q \in Q$*, where* l_q *is the* L*-component of* q. $\square$

6.2 Uniform decay of reductive induction

For the next step, we inspect the uniform decay of $\pi_0|_L = \mathrm{Ind}_S^L(\sigma_0|_S)$. The real reductive group L admits the maximal compact subgroup K_M which is contained in K and the Cartan subgroup A which is also a Cartan subgroup of G (cf. Section 2.3). The Cartan projection of L is denoted by

$$\kappa_L : L \to \mathfrak{a}/W_L.$$

The Weyl group W_L can be identified with a subgroup of W_G. The following decay estimates bring the local decay exponent $\beta_{G/H}$ into play. The proof relies on a refined control of the preliminary decay estimates in Section 5.1.

Proposition 6.4. *Given any $\varphi_1, \varphi_2 \in C_c(L)$ and $D_1, D_2 \Subset V$, we form $f_i = \mathrm{I}_S^L(\varphi_i, \mathbb{1}_{D_i})$ for $i = 1, 2$. Then there exists a constant $C > 0$ such that*

$$|\langle \pi_0(l) f_1, f_2 \rangle| \le C \exp\left\{ \frac{1}{2} \left(-\rho_{\mathfrak{l}} + (2\beta_{G/H} - 1)\rho_{\mathfrak{g}} \right)(\kappa_L(l)) \right\}$$

uniformly for all $l \in L$.

Proof. By unipotency, the exponential map identifies the homogeneous space U/W with the vector space $\mathfrak{u}/\mathfrak{w}$ equivariantly with respect to the adjoint action of S. By identifying $(\sigma_0|_S, \mathrm{L}^2(U/W))$ with $(\sigma_0|_S, \mathrm{L}^2(\mathfrak{u}/\mathfrak{w}))$, we can apply Lemma 5.3 the induced representation $\pi_0|_L = \mathrm{Ind}_S^L(\sigma_0|_S)$. Together with Lemma 3.3 applied to φ_1, φ_2, we have

$$|\langle \pi_0(l) f_1, f_2 \rangle| \le C_1 \exp\{-\rho_{\mathfrak{l}} \kappa_L(g)\} \int_{S \cap BlB} |\langle \sigma_0(s) \mathbb{1}_{D_1}, \mathbb{1}_{D_2} \rangle| \, \mathrm{d}s \tag{6.5}$$

uniformly for all $l \in L$, for some given $B \Subset L$.

For the reductive group S, fix a Cartan decomposition $S = K_S A_S K_S$ with the Cartan projection $\kappa_S : S \to \mathfrak{a}_{\mathfrak{s}}/W_S$. By applying Corollary 4.7 to the algebraic S-module $\mathfrak{u}/\mathfrak{w}$, we deduce that

$$|\langle \sigma_0(s) \mathbb{1}_{D_1}, \mathbb{1}_{D_2} \rangle| \le C_2 \exp\left\{ -\rho_{\mathfrak{u}/\mathfrak{w}} \kappa_S(s) \right\} \tag{6.6}$$

uniformly for all $s \in S$.

Since L conjugates the subspace $\mathfrak{a}_{\mathfrak{s}}$ into its Cartan subspace $\mathfrak{a}$, we can identify $\mathfrak{a}_{\mathfrak{s}}$ with a subspace of $\mathfrak{a}$. Write $X = \kappa_L(l)$. As in (4.17) in the proof of Lemma 4.16, we have for some uniform constant $r > 0$ that

$$S \cap BlB \subset \bigcup_{w \in W_L} K_S \exp\{(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_{\mathfrak{s}}\} \, K_S.$$

Combining this with (6.6) yields the following uniform estimates

$$\begin{aligned}
&\int_{S \cap BlB} |\langle \sigma_0(s) \mathbb{1}_{D_1}, \mathbb{1}_{D_2} \rangle| \, \mathrm{d}s \le C_2 \int_{S \cap BlB} \exp\left\{ -\rho_{\mathfrak{u}/\mathfrak{w}} \kappa_S(s) \right\} \mathrm{d}s \\
&\le C_2 \sum_{w \in W_L} \int_{K_S \exp\{(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_{\mathfrak{s}}\} K_S} \exp\left\{ -\rho_{\mathfrak{u}/\mathfrak{w}} \kappa_S(s) \right\} \mathrm{d}s \\
&\le C_3 \sum_{w \in W_L} \int_{(wX + \mathfrak{a}(r)) \cap \mathfrak{a}_{\mathfrak{s}}} \exp\left\{ \rho_{\mathfrak{s}}(Y) - \rho_{\mathfrak{u}/\mathfrak{w}}(Y) \right\} \mathrm{d}Y,
\end{aligned}$$

where the last inequality follows from Proposition 2.6. But since

$$\rho_{\mathfrak{s}} - \rho_{\mathfrak{u}/\mathfrak{w}} = \rho_{\mathfrak{s}} + \rho_{\mathfrak{w}} - \rho_{\mathfrak{u}} = \rho_{\mathfrak{h}} - (\rho_{\mathfrak{g}} - \rho_{\mathfrak{l}})/2 \leq \big(\beta_{G/H} - 1/2\big)\rho_{\mathfrak{g}} + \rho_{\mathfrak{l}}/2,$$

we deduce, by the uniform Lipschitz property of $\rho_{\mathfrak{g}}, \rho_{\mathfrak{l}}$ on $\mathfrak{a}$, that

$$\sum_{w\in W_L} \int_{(wX+\mathfrak{a}(r))\cap\mathfrak{a}_{\mathfrak{s}}} \exp\big\{\rho_{\mathfrak{s}}(Y) - \rho_{\mathfrak{u}/\mathfrak{w}}(Y)\big\}\,\mathrm{d}Y \\ \leq C_4 \exp\left\{\frac{1}{2}\left(\rho_{\mathfrak{l}} + (2\beta_{G/H} - 1)\rho_{\mathfrak{g}}\right)(X)\right\}$$

uniformly for all $l \in L$. We conclude the proof by feeding back to (6.5). $\square$

Recall from Section 2.5 that the spherical functions of the real reductive group $L = K_M A K_M$ are given for $\chi \in \mathfrak{a}^*$ by

$$\Xi^L_\chi(l) = \int_{K_M} e^{-(\chi+\rho_L)\eta(l^{-1}k_M)}\,\mathrm{d}k_M.$$

Corollary 6.7. *Let $\chi = (2\beta_{G/H}-1)^+\rho := \max\big\{2\beta_{G/H} - 1, 0\big\}\rho \in \mathfrak{a}^*$. With the same assumptions as Proposition 6.4, we have uniformly for $l \in L$,*

$$|\langle \pi_0(l)f_1, f_2\rangle| \leq C \sum_{w\in W_G} \Xi^L_{w\chi}(l).$$

Proof. If $\beta_{G/H} \leq 1/2$, then Proposition 6.4 implies that $\pi_0|_L$ is tempered. We have $\chi = 0$ and we conclude by Theorem 3.10.

From now on assume $\beta_{G/H} > 1/2$. Since $\rho_{\mathfrak{g}} = 2\max_{w\in W_G} w\rho$ on $\mathfrak{a}$ by Example 4.5, the uniform estimates of Proposition 6.4 gives

$$|\langle \pi_0(l)f_1, f_2\rangle| \leq C \max_{w\in W_G} \exp\big\{\big(-\rho_L + (2\beta_{G/H} - 1)w\rho\big)(\kappa^+_L(l))\big\}\,,$$

where $\kappa^+_L : L \to \mathfrak{a}$ is the Cartan projection for the positive system Σ^+_M and $\rho_L = \rho_M$ as in Section 2.3. We have also used the fact that $\rho_{\mathfrak{l}}\kappa_L = 2\rho_L\kappa^+_L$. Then Lemma 2.13 provides the uniform majoration by spherical functions

$$|\langle \pi_0(l)f_1, f_2\rangle| \leq C \max_{w\in W_G} \Xi^L_{w\chi}(l) \leq C \sum_{w\in W_G} \Xi^L_{w\chi}(l). \qquad \square$$

Now the domination of $\lambda_{Q/H}$ by π_0 (Proposition 6.3) and the uniform majoration of π_0 (Corollary 6.7) yields the following majoration of $\lambda_{Q/H}$.

Corollary 6.8. *Let $\chi = (2\beta_{G/H} - 1)^+\rho$. Then for any $\xi_1, \xi_2 \in C_c(Q/H)$, there exists a constant $C > 0$ such that*

$$\big|\big\langle \lambda_{Q/H}(q)\xi_1, \xi_2\big\rangle\big| \leq C \sum_{w\in W_G} \Xi^L_{w\chi}(l_q)$$

uniformly for all $q \in Q$, where l_q is the L-component of q. $\square$

6.3 Uniform decay of parabolic induction

With the premise of Corollary 6.8, the last piece of the proof is to establish uniform decay estimates for a unitary representation of G which is induced from the parabolic subgroup Q. We will establish a variant of [Kna86, Prop 7.14], highlighting the compatibility of spherical functions with parabolic inductions.

Theorem 6.9. *Let $\chi = (2\beta_{G/H} - 1)^+\rho$. Then for any $f_1, f_2 \in C_c(G/H)$, there exists a constant $C > 0$ such that uniformly for all $g \in G$, we have*

$$\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \le C\, \Xi^G_\chi(g).$$

Proof. Since $\lambda_{G/H} = \mathrm{Ind}_Q^G \lambda_{Q/H}$, we can dominate the coefficient on the left hand side by passing to some positive functions $f_i = \mathrm{I}_Q^G(\varphi_i, \xi_i)$ for some $\varphi_1, \varphi_2 \in C_c(G)$ and $\xi_1, \xi_2 \in C_c(Q/H)$. Lemma 5.3 and Lemma 3.3 yield

$$\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \le C_1 e^{-2\rho\kappa(g)} \int_{Q\cap BgB} \left|\left\langle \xi_1, \lambda_{Q/H}(q^{-1})\xi_2 \right\rangle\right| \mathrm{d}\nu_Q(q)$$

uniformly for $g \in G$, for some given $B \Subset G$. Next Corollary 6.8 yields

$$\left|\left\langle \xi_1, \lambda_{Q/H}(q^{-1})\xi_2 \right\rangle\right| \le C_2 \sum_{w\in W_G} \Xi^L_{w\chi}(l_q^{-1})$$

uniformly for $q \in Q$, whence

$$\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \le C_3 e^{-2\rho\kappa(g)} \sum_{w\in W_G} \int_{Q\cap BgB} \Xi^L_{w\chi}(l_q^{-1})\, \mathrm{d}\nu_Q(q)$$

uniformly for $g \in G$. Since K_M normalizes N_Q and since $\eta(\cdot)$ is N_Q-right-invariant, for $q \in Q$ we always have

$$\Xi^L_{w\chi}(l_q^{-1}) = \int_{K_M} e^{-(w\chi+\rho_L)\eta(l_q k_M)}\, \mathrm{d}k_M = \int_{K_M} e^{-(w\chi+\rho_L)\eta(q k_M)}\, \mathrm{d}k_M,$$

whence from $\mathrm{d}\nu_Q(q) = e^{\rho_Q \eta(q)}\, \mathrm{d}q$ (cf. Section 2.3) we obtain

$$\begin{aligned} &\left|\left\langle \lambda_{G/H}(g) f_1, f_2 \right\rangle\right| \\ &\qquad \le C_3 e^{-2\rho\kappa(g)} \sum_{w\in W_G} \int_{Q\cap BgB} \int_{K_M} e^{-(w\chi+\rho_L)\eta(qk_M)+\rho_Q\eta(q)}\, \mathrm{d}k_M\, \mathrm{d}q. \end{aligned} \tag{6.10}$$

Up to enlarging B, we can assume B to be K-bi-invariant. Note that

- $K(Q\cap BgB) = BgB$, and
- $x \mapsto \int_{K_M} e^{-(w\chi+\rho_L)\eta(xk_M)+\rho_Q\eta(x)}\, \mathrm{d}k_M$ is a K-left-invariant function.

Then Proposition 2.10 yields

$$\int_{Q\cap BgB}\int_{K_M} e^{-(w\chi+\rho_L)\eta(qk_M)+\rho_Q\eta(q)}\,\mathrm{d}k_M\,\mathrm{d}q \\ = \int_{BgB}\int_{K_M} e^{-(w\chi+\rho_L)\eta(xk_M)-\rho_Q\eta(x)}\,\mathrm{d}k_M\,\mathrm{d}x,$$

and then with $BgB \subset Ke^{\kappa(g)+\mathfrak{a}(r)}K$ (Lemma 2.11), Proposition 2.6 gives

$$\begin{aligned}
&\int_{BgB}\int_{K_M} e^{-(w\chi+\rho_L)\eta(xk_M)-\rho_Q\eta(x)}\,\mathrm{d}k_M\,\mathrm{d}x \\
&\leq \int_K\int_{\kappa(g)+\mathfrak{a}(r)}\int_K\int_{K_M} e^{-(w\chi+\rho_L)\eta(k_1e^Yk_2k_M)-\rho_Q\eta(k_1e^Yk_2)+2\rho(Y)}\,\mathrm{d}k_M\,\mathrm{d}k_2\,\mathrm{d}Y\,\mathrm{d}k_1 \\
&= \int_{\kappa(g)+\mathfrak{a}(r)}\int_K\int_{K_M} e^{-(w\chi+\rho_L)\eta(e^Ykk_M)-\rho_Q\eta(e^Yk)+2\rho(Y)}\,\mathrm{d}k_M\,\mathrm{d}k\,\mathrm{d}Y \\
&\leq C_4e^{2\rho\kappa(g)}\int_{\kappa(g)+\mathfrak{a}(r)}\int_K\int_{K_M} e^{-(w\chi+\rho_L)\eta(e^Ykk_M)-\rho_Q\eta(e^Yk)}\,\mathrm{d}k_M\,\mathrm{d}k\,\mathrm{d}Y,
\end{aligned}$$

uniformly for all $g \in G$. But since K_M normalizes both A_Q and N_Q, we have $\rho_Q\eta(xk_M) = \rho_Q\eta(x)$ for any $k_M \in K_M$ and $x \in G$, whence by Fubini,

$$\begin{aligned}
&\int_K\int_{K_M} e^{-(w\chi+\rho_L)\eta(e^Ykk_M)-\rho_Q\eta(e^Yk)}\,\mathrm{d}k_M\,\mathrm{d}k \\
&= \int_{K_M}\int_K e^{-(w\chi+\rho_L)\eta(e^Yk)-\rho_Q\eta(e^Ykk_M^{-1})}\,\mathrm{d}k\,\mathrm{d}k_M \\
&= \int_K e^{-(w\chi+\rho_L+\rho_Q)\eta(e^Yk)}\,\mathrm{d}k = \int_K e^{-(w\chi+\rho)\eta(e^Yk)}\,\mathrm{d}k \\
&= \Xi^G_{w\chi}(e^Y) = \Xi^G_\chi(e^Y),
\end{aligned}$$

where the last equality follows from Lemma 2.12. But any $Y \in \kappa(g)+\mathfrak{a}(r)$ satisfies $\eta(e^Yk) = \eta(e^Zk')+\eta(e^{\kappa(g)}k)$ for some $Z \in \mathfrak{a}(r)$ and $k' \in K$, whence

$$\Xi^G_\chi(e^Y) \leq C_5\,\Xi^G_\chi(e^{\kappa(g)}) = C_5\,\Xi^G_\chi(g)$$

uniformly for all $Y \in \kappa(g)+\mathfrak{a}(r)$. Chasing back to (6.10), we have proven

$$\bigl|\bigl\langle\lambda_{G/H}(g)f_1, f_2\bigr\rangle\bigr| \leq C_6\,\Xi^G_\chi(g)$$

uniformly for all $g \in G$. □

6.4 Putting all together

Proof of Proposition E. First by Theorem 6.9 and Lemma 2.13, we have for any $\varepsilon > 0$ and any $f_1, f_2 \in C_c(G/H)$ the uniform decay

$$\bigl|\bigl\langle\lambda_{G/H}(g)f_1, f_2\bigr\rangle\bigr| \leq C\exp\bigl\{-2\bigl(1-\varepsilon-\max\bigl\{\beta_{G/H}, 1/2\bigr\}\bigr)\rho\kappa(g)\bigr\}$$

for all $g \in G$. By the definition of $\theta_{G/H}$, we get $\theta_{G/H} \le \max\{\beta_{G/H}, 1/2\}$.

For the other direction, we follow the proof of [BK22, Prop 3.7]. Choose an A-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{v}$ with $\mathfrak{v} \cong \mathfrak{g}/\mathfrak{h}$. There exists a small neighborhood D of 0 in $\mathfrak{v}$ such that the exponential map from D to G/H, which maps Y to $e^Y H$, is a homeomorphism onto its image $B \Subset G/H$. By taking into account the Radon-Nikodym derivative and the A-invariance of $\mathfrak{v}$, we deduce that

$$\left\langle \lambda_{G/H}(e^X)\mathbb{1}_B, \mathbb{1}_B \right\rangle \ge c_1\, e^{-\operatorname{Tr}_{\mathfrak{v}}(Y)/2} \operatorname{Vol}\big(\operatorname{Ad}(e^X)D \cap D\big)$$

uniformly for $X \in \mathfrak{a}_{\mathfrak{h}}$. Applying Lemma 4.6 to $\operatorname{ad} : \mathfrak{h} \to \operatorname{GL}(\mathfrak{g}/\mathfrak{h})$ yields

$$e^{-\operatorname{Tr}_{\mathfrak{v}}(Y)/2} \operatorname{Vol}\big(\operatorname{Ad}(e^X)D \cap D\big) \ge c_2 \exp\big\{-\rho_{\mathfrak{g}/\mathfrak{h}}(X)\big\}\,,$$

whence uniformly for all $X \in \mathfrak{a}_{\mathfrak{h}}$,

$$\left\langle \lambda_{G/H}(e^X)\mathbb{1}_B, \mathbb{1}_B \right\rangle \ge c_3 \exp\big\{-\rho_{\mathfrak{g}/\mathfrak{h}}(X)\big\}\,.$$

But the definition of $\theta_{G/H}$ implies the uniform decay of coefficients

$$\left\langle \lambda_{G/H}(e^X)\mathbb{1}_B, \mathbb{1}_B \right\rangle \le C_4 \exp\{-(1-\theta)\rho_{\mathfrak{g}}(X)\}\,.$$

Thus by the homogeneity of the rho-functions, we must have

$$-(1-\theta)\rho_{\mathfrak{g}} \ge -\rho_{\mathfrak{g}/\mathfrak{h}} \text{ on } \mathfrak{a}_{\mathfrak{h}}.$$

Since $\rho_{\mathfrak{g}/\mathfrak{h}} = \rho_{\mathfrak{g}} - \rho_{\mathfrak{h}}$, we have $\rho_{\mathfrak{h}} \le \theta\rho_{\mathfrak{g}}$, whence $\beta_{G/H} \le \theta_{G/H}$. We obtain $\max\{\beta_{G/H}, 1/2\} = \max\{\theta_{G/H}, 1/2\}$. Then we conclude by Theorem A. $\square$

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