Program Article

Geometry and Dynamics for Discrete Subgroups of Higher Rank Lie Groups

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This semester, SLMath is hosting two related programs, one on *Geometry and Dynamics for Discrete Subgroups of Higher Rank Lie Groups* (the GDLG program) and the other on *Topological and Geometric Structures in Low Dimensions* (TGS). To vastly oversimplify, one could say that the two programs correspond to the legacies of G. Margulis and of W. Thurston, respectively. But this is far from suitably conveying the rich and intricate flow of ideas that has influenced each of the two topics since the end of the nineteenth century.

In this brief article we will give a few insights around the four notions appearing in the title of the GDLG program:

Geometry,
Higher rank Lie groups,
Discrete subgroups,
Dynamics.

Geometry

We would like to start by celebrating an anniversary: one hundred years ago, in 1926, the French mathematician Élie Cartan (1869–1951) published a pair of foundational articles in the *Bulletin de la Société Mathématique de France*, entitled "*Sur une classe remarquable d'espaces de Riemann*."

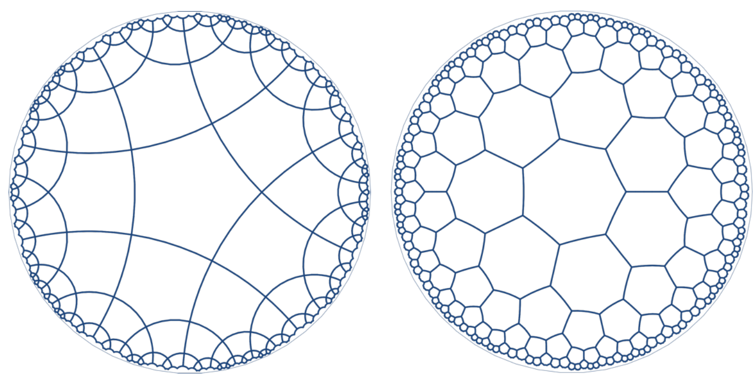
Cartan was looking for interesting examples of Riemannian manifolds beyond those of constant negative curvature — the famous hyperbolic manifolds. In the articles, he focused on simply connected complete Riemannian manifolds satisfying a natural geometric condition, namely that the curvature tensor is parallel. These spaces are now called *Riemannian symmetric spaces*.

Not only did Cartan discover many examples of Riemannian symmetric spaces, but he was able to classify them all by using *real semisimple Lie groups* — a nice family of Lie groups which he had previously classified in 1914. This family contains, for instance, the group $\mathrm{SL}(n,\mathbb{R})$ of real $n\times n$ matrices of determinant one, its subgroup $\mathrm{SO}(p,n-p)$ preserving a symmetric bilinear form of signature $(p,n-p)$ on $\mathbb{R}^n$, and the group $\mathrm{Sp}(2n,\mathbb{R})$ of real $2n\times 2n$ matrices preserving a nondegenerate skew-symmetric bilinear form on $\mathbb{R}^{2n}$ (where $n \geqslant p \geqslant 1$ are integers, and for $\mathrm{SO}(p,n-p)$ we take $n \geqslant 3$). More precisely, if X is a Riemannian symmetric space of *noncompact type* (that is, of nonpositive curvature, with no Euclidean factor), then its isometry group G is a noncompact real semisimple Lie group, and X identifies with the quotient G/K where K is a maximal compact subgroup of G; conversely, any such quotient G/K is naturally a Riemannian symmetric space.

Cartan was aware of the importance of his discovery. He wrote in the introduction of his article: *"Les résultats obtenus appellent un grand nombre de recherches nouvelles, ne serait-ce que l'étude individuelle des nouveaux espaces, qui semblent devoir jouer un rôle presque aussi important que celui des espaces à courbure constante, et qui sont du reste susceptible d'une définition géométrique directe."* ("The results we have obtained call for a great number of new avenues of research, even if only the individual study of each of these new spaces, which seem to play a role almost as important as that of the spaces of constant curvature, and which besides are susceptible to a direct geometric definition.")

Higher Rank Lie Groups

We now know, one century later, that the intuition of Élie Cartan was true beyond his own expectations.



Tilings of the hyperbolic plane $\mathbb{H}^2$: of valence 4 by regular pentagons (left), and of valence 3 by regular heptagons (right).

The members of this Spring's TGS program would certainly agree with Cartan's humble phrase *"presque aussi important"* ("almost as important"), since the hyperbolic spaces of dimensions 2 and 3, namely $X=\mathbb{H}^2$ and $X=\mathbb{H}^3$, and their group of oriented isometries, namely $G=\mathrm{PSL}(2,\mathbb{R})$ and $G=\mathrm{PSL}(2,\mathbb{C})$, play a central, important role in low-dimensional geometry and topology.

On the other hand, the members of the GDLG program might be tempted to replace "almost as important" by "almost more important" since the theory of Riemannian symmetric spaces of noncompact type $X=G/K$ has had some striking applications (in particular to number theory) in the setting that G has *higher real rank*, which means that X contains Euclidean subspaces of dimension at least 2. Even the simplest examples like $G=\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$, $G=\mathrm{SL}(3,\mathbb{R})$, or $G=\mathrm{Sp}(4,\mathbb{R})$, whose symmetric spaces have respective dimensions 4, 5, and 6, are already very interesting.

Both in rank one (that is, when there are no Euclidean subspaces beyond geodesics) and in higher rank, one of the richest aspects of the theory involves *discrete subgroups* Γ of noncompact semisimple Lie groups G. These are the groups of symmetries of *periodic tilings* of the Riemannian symmetric space $X = G/K$: one can find an open tile T of X which is disjoint from all its translates γT with γ in Γ, and such that the closed tiles $\gamma\overline{T}$ cover X, as in the hyperbolic tiling figure on the previous page. Equivalently, any discrete subgroup Γ of G acts properly discontinuously on X, and the quotient is a manifold (or orbifold) $\Gamma\backslash X = \Gamma\backslash G/K$, called a *locally symmetric space*, which inherits the Riemannian metric of X and its local properties.

Discrete subgroups of semisimple Lie groups G have played a central role in several areas of mathematics since the end of the nineteenth century, especially in the rank-one cases of $G = \mathrm{PSL}(2,\mathbb{R})$ and $G = \mathrm{PSL}(2,\mathbb{C})$. Their study in higher real rank has gained momentum, especially since the 1960s, with striking rigidity results for lattices and flexibility results for “thinner” discrete subgroups, inspired in part by progress in low-dimensional geometry. The next two sections discuss these results.

Discrete Subgroups of Finite Covolume

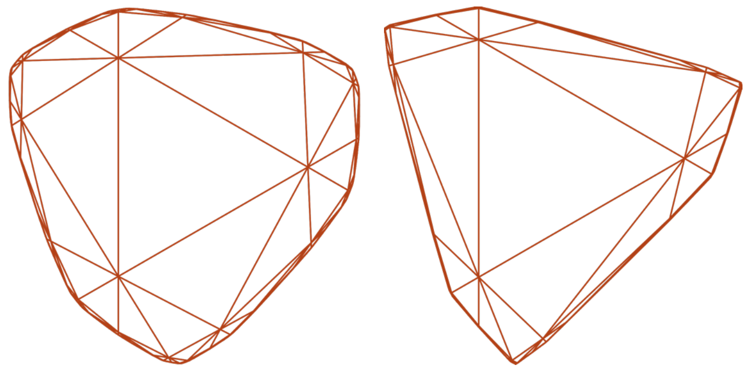
In a noncompact semisimple Lie group G, the discrete subgroups Γ defining tilings with finite-volume tiles are called *lattices*. Important examples include, by results of Siegel, Borel, and Harish-Chandra, all the *arithmetic subgroups* of G; these are essentially the subgroups obtained as the $\mathbb{Z}$-points of some $\mathbb{Q}$-algebraic structure on G, which naturally arise in number theory. For instance, $\Gamma = \mathrm{SL}(3,\mathbb{Z})$ and $\Gamma = \mathrm{Sp}(4,\mathbb{Z})$ are lattices in $G = \mathrm{SL}(3,\mathbb{R})$ and $G = \mathrm{Sp}(4,\mathbb{R})$, respectively, and $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ embeds naturally as a lattice in $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

While lattices in $G = \mathrm{PSL}(2,\mathbb{R})$ have a rich deformation theory (*Teichmüller theory*), lattices in other semisimple Lie groups G tend to be much more rigid, especially when G has higher real rank. Beyond local rigidity and Mostow rigidity, Margulis proved that lattices in higher-rank G are *superrigid*, which means that they can be realized in essentially only one way. He used this to show that all lattices in higher-rank G are arithmetic; this contrasts with the case of some rank-one groups such as $G = \mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SO}(n,1)$.

When Γ is a lattice in G, there is a natural G-invariant probability measure on the quotient $\Gamma\backslash G$, and one can use results in ergodic theory to study the dynamics of the elements of G on this quotient. This has numerous applications; let us note just two of them.

The first one is a purely algebraic statement due to Margulis, which has no analogue in rank one: *every normal subgroup of a lattice Γ in a higher-rank simple Lie group is either finite or of finite index*. A key point in the proof is a deep understanding of the dynamics of Γ on the *full flag variety* $\mathcal{F}$ of G, which is the quotient G/P of G by a maximal cocompact amenable subgroup P. For instance, if $G = \mathrm{SL}(n,\mathbb{R})$, then $\mathcal{F}$ is the space of full flags of $\mathbb{R}^n$.

The second one is an asymptotic counting result (due to Duke–Rudnick–Sarnak and Eskin–McMullen), which is already striking for $\Gamma = \mathrm{SL}(3,\mathbb{Z})$: *given any lattice Γ and any point x in the symmetric space X of G, the number of points of the Γ-orbit of x inside the ball of radius R centered at x is equivalent, as R tends to infinity, to the volume of the ball.* Here the volume is normalized so that $\mathrm{vol}(\Gamma\backslash G) = 1$. This statement is a generalization of a much older counting result of Hedlund for lattices of $G = \mathrm{SL}(2,\mathbb{R})$. A key point in the proof is an equidistribution property for the projection to $\Gamma\backslash X$ of large spheres of X.



Two properly convex domains Ω in the projective plane $\mathbb{P}(\mathbb{R}^3)$, with tilings by triangles. The symmetry groups Γ of these tilings are discrete subgroups of $G = \mathrm{SL}(3,\mathbb{R})$ of infinite covolume, which are Anosov in the sense described in the “Infinite Covolume” section below.

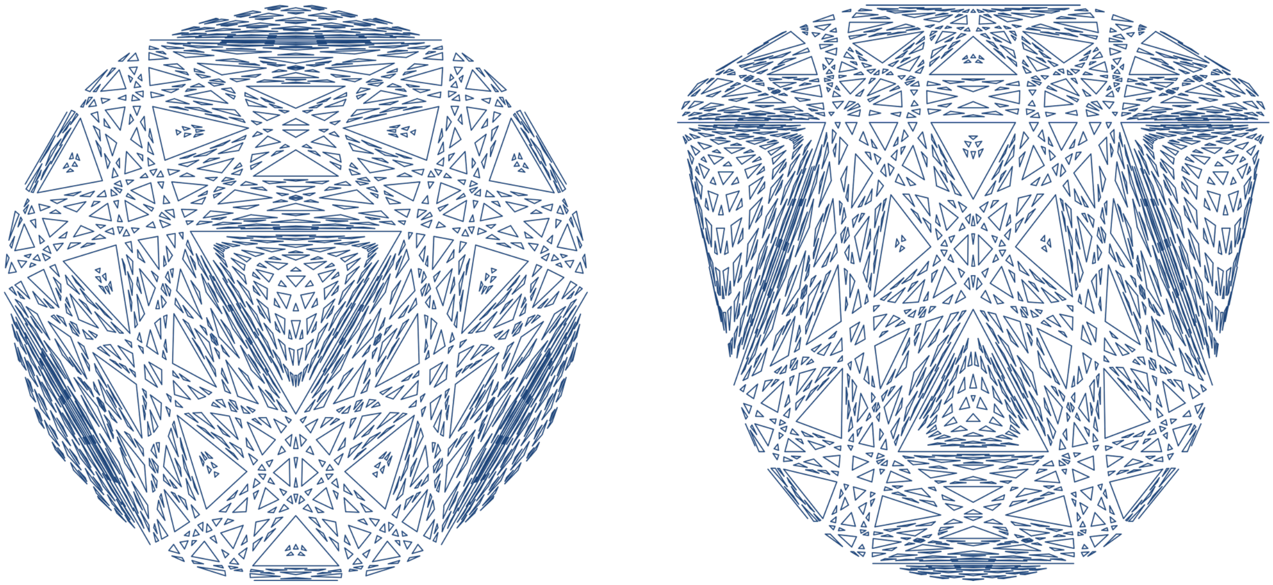
Discrete Subgroups of Infinite Covolume

In a noncompact semisimple Lie group G, discrete subgroups Γ of infinite covolume tend to be much more flexible than lattices.

In real rank one, examples of such subgroups were constructed early on by Schottky, Poincaré, Klein, and others. A full conjectural description of all finitely generated discrete subgroups of $G = \mathrm{SL}(2,\mathbb{C})$ was proposed by Bers, Marden, Sullivan, and Thurston in the 1970s and 80s, which was finally established fifteen years ago as a major achievement. Among these discrete subgroups, the simplest to understand are the *convex cocompact* subgroups, which act cocompactly on a convex subset of the hyperbolic space.

In higher real rank, discrete subgroups of G remain much more mysterious, and we are very far from a full classification. For instance, even for $G = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, we do not know whether there exists a discrete subgroup of G which is not a lattice and whose projections to both $\mathrm{SL}(2,\mathbb{R})$ factors are dense. However, some exciting progress has been made in the past 25 years:

- Many new interesting discrete subgroups have been found. For $G = \mathrm{SL}(n,\mathbb{R})$, this includes the discrete subgroups acting cocompactly on a *properly convex* (that is, convex and bounded in some affine chart) domain in the projective space $\mathbb{P}(\mathbb{R}^n)$ as in the figure above and the one at the top of the next page. For $G = \mathrm{SO}(p,q)$, this includes the discrete subgroups acting *convex cocompactly* on a properly convex domain in $\mathbb{H}^{p,q-1}$ — the



The boundary $\partial\Omega$ (top and front view) of a properly convex domain Ω in the projective space $\mathbb{P}(\mathbb{R}^4)$, where Ω admits a cocompact action by a discrete subgroup of $G = SL(4, \mathbb{R})$. This discrete subgroup has infinite covolume. It is not Anosov in the sense decsribed in the "Infinite Covolume" section because it contains infinitely many copies of $\mathbb{Z}^2$, each of which tiles a triangle of Ω whose edges lie in $\partial\Omega$.

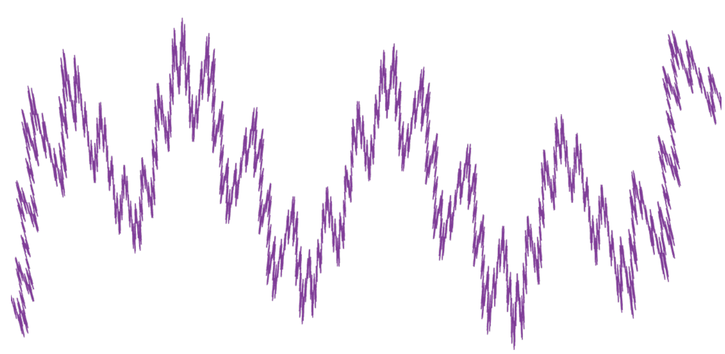
pseudo-Riemannian analogue of hyperbolic space in signature $(p, q-1)$. For $G = Sp(2n, \mathbb{R})$, this includes the discrete subgroups isomorphic to the fundamental group of a closed hyperbolic surface and which maximize a topological invariant, the *Toledo invariant*.

- An important class of discrete subgroups Γ analogous to the convex cocompact subgroups of $SL(2, \mathbb{C})$, the so-called *Anosov subgroups*, was introduced by Labourie in 2006. Since then, many equivalent definitions of these subgroups have been found. Their dynamics on certain partial flag varieties has been investigated, in particular the shapes and dimensions of *limit sets* (closures of attracting fixed points of elements of Γ) as in the figure below right. Limit sets have been used to construct interesting *domains of discontinuity* (open sets on which Γ acts properly discontinuously) in partial flag varieties. Asymptotic counting (as introduced in the previous section) for Anosov subgroups has also given rise to a lot of active research.
- Other nice classes of subgroups are now emerging: whereas Anosov subgroups do not contain copies of $\mathbb{Z}^2$, generalizations are currently being developed that may contain $\mathbb{Z}^2$ subgroups and have interesting dynamical and geometric behavior generalizing that of Anosov subgroups.
- Whereas lattices in higher-rank G are rigid (as described in the previous section), certain discrete subgroups of infinite covolume have been shown to have large deformation spaces. This is the case for certain discrete subgroups isomorphic to fundamental groups of closed hyperbolic surfaces (*higher Teichmüller theory*), but also for others isomorphic to fundamental groups of higher-dimensional manifolds, and which may contain copies of $\mathbb{Z}^2$.

Homogeneous Dynamics

In a broad sense, homogeneous dynamics is the study of the action of subgroups of a Lie group G (for example, discrete subgroups, or continuous abelian subgroups corresponding to flows, or larger continuous subgroups) on homogeneous spaces of G, using deep results from ergodic theory. In the "Finite Covolume" section, we already saw two remarkable achievements in this field.

In the early 1990s, Ratner introduced a new dynamical tool called *polynomial drift*. She proved that when Γ is a lattice in a semisimple Lie group G, the closure $\overline{Hx}$ of the orbit Hx of an element x in $\Gamma\backslash G$ under a subgroup H of G generated by unipotent elements is always an orbit for a larger subgroup of G. The key point is a classification of probability measures on $\Gamma\backslash G$ that are invariant under unipotent subgroups. Her results also hold over the p-adic numbers and have



Part of the limit set, in the projective plane $\mathbb{P}(\mathbb{R}^3)$, of an Anosov subgroup of $G = SL(3, \mathbb{R})$. This limit set is a fractal-looking topological circle, with Hölder regularity.

had important applications in number theory, for instance to the theory of quadratic forms.

Extensions of Ratner's theorem to certain classes of discrete subgroups Γ of G of infinite covolume now exist, for instance Anosov subgroups (as described in the previous section). There are also extensions of Ratner's theorem where H is not generated by unipotent elements anymore. In these extensions, the *polynomial drift* argument has been replaced by an *exponential drift* argument. There are even extensions of Ratner's theorem where the orbit Hx is replaced by certain submanifolds of the Riemannian symmetric space $X = G/K$ (see the "Geometry"section), which are transverse to some boundary at infinity.

In infinite covolume, a major goal is to understand quantitatively how the Γ-orbits in X accumulate at infinity. This information is controlled by certain measures on flag varieties, called *Patterson-Sullivan measures*, which are no longer absolutely continuous with respect to the Haar measure. They are valuable tools for understanding Hausdorff dimensions of limit sets as in the lower figure on the previous page, and their fine ergodic properties (as well as those of certain related measures on $\Gamma\backslash G$) are crucial for obtaining counting results.

Not only is homogeneous dynamics important in its own right, but it has also led to key progress in low-dimensional geometry through powerful analogies. We all know the fundamental role that analogies play in mathematics, indicating what new goals one should aim for and what technical steps should be achieved on the way. One of these fruitful analogies, developed for instance in Mirzakhani's thesis, is between lattices Γ of semisimple Lie groups G, acting on the corresponding Riemannian symmetric space X and on the full flag variety $\mathcal{F}$ of G, and the *mapping class group* of a surface S, acting on the *Teichmüller space* and on the space of *projective measured laminations* of S. This is yet another example of the nice interplay between the topics of the GDLG and TGS programs at SLMath this Spring.