MAP 553 PC4

$\varphi_j'' = -\pi^2 a_j \varphi_j$

3 Liens splines $\leftrightarrow$ moyaux

1. $W_2 = \{ f : [0,1] \to \mathbb{R},\ f' \text{ abs. continue et } \int_0^1 (f'')^2 < +\infty,\quad f(0)=f(1),\ f'(0)=f'(1) \}$

$= \{ f = \sum_{j=1}^{\infty} b_j \varphi_j \text{ où } \sum_{j\geq 1} a_j^2 b_j^2 < +\infty \}$ où $\begin{cases} \varphi_j(x) = \sqrt{2}\cos(\pi j x) & j \text{ pair} \\ = \sqrt{2}\sin(\pi(j-1)x) & j \text{ impair} \\ = 1 & j=1. \end{cases}$

• $\int (f'')^2 = \int \left(\sum_j b_j \varphi_j''\right)^2 = \int \left(\sum_j b_j \pi^2 a_j \varphi_j\right)^2$

$= \pi^4 \sum_{j,k} b_j b_k a_j a_k \underbrace{\int \varphi_j \varphi_k}_{\delta_{jk}} = \pi^4 \sum_{j\geq 1} b_j^2 a_j^2$

donc $\frac{1}{m} \sum_{i=1}^m (Y_i - f(x_i))^2 + \kappa \int_0^1 (f'')^2 = \frac{1}{m}\|Y\|^2 - \frac{2}{m} \sum_i Y_i \sum_{j\geq 1} b_j \varphi_j(x_i) + \frac{1}{m}\sum_i \left(\sum_{j\geq 1} b_j \varphi_j(x_i)\right)^2 + \kappa \pi^4 \sum_{j\geq 1} b_j^2 a_j^2$

$= \frac{1}{n}\|Y\|^2 + \sum_{j\geq 1} \left(-2 b_j \times \underbrace{\frac{1}{m}\sum_i Y_i \varphi_j(x_i)}_{\hat\theta_j} + \kappa\pi^4 b_j^2 a_j^2\right) + \underbrace{\sum_{j,k} b_j b_k \frac{1}{m}\sum_{i=1}^m \varphi_j(x_i)\varphi_k(x_i)}_{(I)}$

$(I) = \sum_{\substack{j,k \leq m \\ j\neq k}} + \sum_{j=k} + 2\sum_{\substack{j \geq m \\ j\neq k}}$

$= 0 + \sum_{j\geq 1} b_j^2 + R_n$ où $|R_m| \leq 2 \sum_{j\geq m}\sum_{k\geq 1} |b_j||b_k|$ car $\left|\frac{1}{m}\sum_i \underbrace{\varphi_j(x_i)\varphi_k(x_i)}_{|\ |\leq 1}\right| \leq 1$

$\leq 2\sqrt{\sum_{k\geq 1} b_k^2 a_k^2 \sum_{k\geq 1} \frac{1}{a_k^2}}\ \sqrt{\sum_{j\geq m} b_j^2 a_j^2 \underbrace{\sum_{j\geq m} \frac{1}{a_j^2}}_{\sim \frac{1}{m^3}}}$

$\leq C\left(\sum_{k\geq 1} a_k^2 b_k^2\right) \times \frac{1}{m^{3/2}}$

donc

(1) $\Leftrightarrow$ $\min_{b_j} \sum_{j\geq 1} \left(-2 b_j \hat\theta_j + b_j^2 (\kappa\pi^4 a_j^2 + 1)\right)(1 + O(1/m))$

2. si $O(1/m) = 0$: $b_j = \frac{\hat\theta_j}{1+\kappa\pi^4 a_j^2}$ et $\hat f(x) = \sum_{j\geq 1} d_j \hat\theta_j \varphi_j(x)$ où $d_j = \frac{1}{1+\kappa\pi^4 a_j^2}$

3. $\tilde f(x) = \sum_{j\geq 1} d_j \hat\theta_j \varphi_j(x) = \frac{1}{m}\sum_{i=1}^m Y_i \underbrace{\left(\sum_{j\geq 1} d_j \varphi_j(x_i)\varphi_j(x)\right)}_{\approx \frac{1}{h} K\left(\frac{x_i - x}{h}\right)}$

$\mathcal{F}[K](\omega) = \frac{1}{1+\omega^4}$

donc $\int_0^1 \frac{1}{h} K\left(\frac{x_i - x}{h}\right) e^{i 2\pi k x}\, dx \overset{y = \frac{x}{h}}{=} \int_0^{1/h} K\left(\frac{x_i}{h} - y\right) e^{i 2\pi k h y}\, dy \approx \mathcal{F}\left[K\left[\frac{x_i}{h} - y\right]\right](2\pi kh)$

$\simeq \frac{e^{i(2\pi kh) x_i/h}}{1 + (2\pi kh)^4}$

Re + Im : $\left\langle \frac{1}{h} K\left(\frac{x_i - \cdot}{h}\right), \varphi_j \right\rangle \approx \sqrt{2} \times \frac{\varphi_j(x_i)}{\sqrt{2}} \times \frac{1}{1 + (\pi j h)^4} \simeq \frac{1}{1 + \pi^4 a_j^2 \kappa}$