On the Convex Hull of a Brownian Excursion with Parabolic Drift

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Abstract: The solutions of Burgers equation with white noise initial velocity are closely connected to the convex hull $\mathcal{H}_a$ of a Brownian excursion with parabolic drift $s \mapsto e_s + \frac{a}{2}\, s^2$. We derive from the law of the minimum σ and the location η of the minimum of $s \mapsto 2e_s/s(1-s)$ a complete description of the convex hull $\mathcal{H}_a$. As an application, we determine the statistical properties of the Burgers turbulence on the circle.
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1 Introduction

The study of the so-called entropic solutions of Burgers/Riemann equation

$$\partial_t u + u\, \partial_x u = 0\,, \tag{1}$$

with random initial conditions has raised a special interest during the last decade. It is mainly motivated by considerations on the phenomenon of turbulence (see Burgers [6]) and on the formation of the large scale structures of the universe (see Vergassola et *al.* [21]). When $u(\cdot, 0)$ denotes the velocity field at time $t = 0$, it is standard, according to the celebrated Hopf-Cole formula [8, 15], that the solution $u(\cdot, t)$ at time $t > 0$ of (1) can be explicitly expressed in terms of the convex hull of the path

$$s \mapsto \int_0^s u(x, 0)\, dx + \frac{1}{2t} s^2\,.$$

Suppose now that the so-called initial potential $W_s = \int_0^s u(x, 0)\, dx$ is 1-periodic and is distributed on a period as a Brownian bridge. It can be shown, with the help of a path transformation due to

Vervaat [22], that the study of the convex hull of $\left\{s \mapsto W_s + \frac{1}{2t}s^2;\ s \in \mathbb{R}\right\}$ amounts to the study of the convex hull of a Brownian excursion of duration 1 with parabolic drift $\left\{s \mapsto e_s + \frac{1}{2t}s^2;\ s \in [0,1]\right\}$. This observation motivates the present work.

Our analysis of the convex hull of a Brownian excursion with parabolic drift is related to several works on the convex hull of stochastic processes. Questions on empirical processes and queueing systems lead Groeneboom [13] (see also Pitman [18] and Çinlar [7]) to consider the convex hull of a one-sided Brownian motion. He showed that it is a.s. piecewise linear, and that the distribution of its derivative can be expressed in terms of a Poisson point process. Furthermore, he obtained a striking decomposition of the Brownian motion conditionally on its convex hull $\mathcal{H}$. Roughly speaking, the "excursions" of the Brownian motion above $\mathcal{H}$ are independent conditionally on $\mathcal{H}$ and are distributed as Brownian excursions. Few years later, Groeneboom [14] focused on the case of a two-sided Brownian motion with parabolic drift. His work was motivated this time by the analysis of the global behaviour of a wide class of estimators in statistics. He showed in this case that the convex hull is again a.s. piecewise linear and that its derivative can be characterized in terms of a Markov chain of known transitions. His work found in the late 90's a renewal of interest in the setting of the Burgers turbulence. Frachebourg & Martin [11] recovered independently the results of Groeneboom (see also Avellaneda & E [2]) and Giraud [12] studied the evolution in the variable t of the convex hull of $\left\{s \mapsto W_s + \frac{1}{2t}s^2;\ s \in \mathbb{R}\right\}$, when $\{W_s;\ s \in \mathbb{R}\}$ is a two-sided Brownian motion. He showed besides a decomposition of the Brownian motion with parabolic drift conditionally on its convex hull, similar to the case with no drift. Some other works have also been lead in a more general setting. We refer to Bass [3], Nagasawa & Tanaka [17] and Bertoin [4] for the Markov case, the Lévy case and the Cauchy case, respectively.

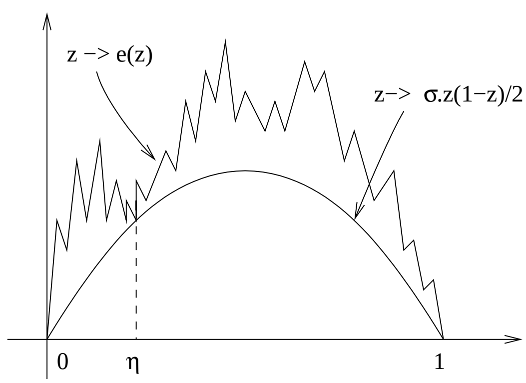

Figure 1: σ and η

The rest of the paper is organized as follows. We obtain in section 2 a simple formula for the joint probability density function of

$$\begin{cases} \sigma = \min\left\{\frac{2}{s(1-s)}\, e_s;\ s \in]0,1[\right\}, \\ \eta = \text{ right-most location of this minimum.} \end{cases} \tag{2}$$

In section 3, we first give a dynamical description of the convex hull of a Brownian excursion of duration 1 with parabolic drift in terms of a Markov process and then explicit its statistics. The last section is devoted to the description of the Burgers turbulence on the circle.

2 A key step

The variables σ and η defined above (see formula (2)), have been introduced in [12]. They play a major role in the description of the genealogical tree of a shock of the solution of Burgers equation (1) with white noise initial velocity. Their joint law is given in Lemma 4 (of [12]) by a complicated formula in terms of the derivative of an integral depending of a parameter. The formula we obtain in Theorem 1 is explicit and easily amenable to mathematical as well as to numerical analysis. It involves the Laplace transform $C(\lambda)$ of the integral of a Brownian excursion $\int_0^1 e_s\,ds$. According to Groeneboom's formula (see [14], Lemma 4.2.(iii)),

$$C(\lambda) := \mathbb{E}\left(-\lambda \int_0^1 e_s\,ds\right) = \lambda\sqrt{2\pi}\sum_{n=1}^{\infty} \exp\left(-2^{-1/3} w_n \lambda^{2/3}\right), \quad \text{for } \lambda > 0\,, \tag{3}$$

where $0 > -w_1 > -w_2 > \cdots$ denotes the zeros of the Airy function (see [1] on p. 446).

Theorem 1 (law of (σ, η))
For $a \geq 0$ and $0 < x < 1$, the probability density function of (σ, η) is given by

$$\mathbb{P}(\sigma \in da, \eta \in dx) = \frac{e^{-a^2/24}}{\sqrt{8\pi x(1-x)}}\, C\left(a\,x^{3/2}\right) C\left(a(1-x)^{3/2}\right)\, da\,dx\,.$$

Before starting the proof of Theorem 1, we make some few remarks.

Remarks

1- There is a symmetry $x \longleftrightarrow 1-x$. Indeed, $(\sigma,\eta) \overset{\text{law}}{=} (\sigma, 1-\eta)$.

2- For x close to 0, we have the asymptotic:

$$\mathbb{P}(\sigma \in da, \eta \in dx) \overset{x\to 0}{\sim} \frac{e^{-a^2/24}}{\sqrt{8\pi x}}\, C(a)\, da\,dx\,.$$

3- Recall from Lemma 4 in [12] that the probability distribution function of σ is

$$\mathbb{P}(\sigma \geq a) = e^{-a^2/24} C(a)\,. \tag{4}$$

It follows that the conditional probability density function of η given $\sigma = a$ is

$$\mathbb{P}(\eta \in dx \,|\, \sigma = a) = \frac{C\left(ax^{3/2}\right) C(a(1-x)^{3/2})}{C'(a) - \frac{a}{12}\,C(a)} \times \frac{da\,dx}{\sqrt{8\pi x(1-x)}}\,.$$

We may notice two facts. First, the variable η conditioned by $\sigma = 0$ is distributed as the celebrated arcsin law:

$$\mathbb{P}(\eta \in dx \,|\, \sigma = 0) = \frac{dx}{\pi\sqrt{x(1-x)}}\,.$$

Second, it is easy to check that, for any value of $a \geq 0$, the conditional probability density function $\mathbb{P}(\eta \in dx | \sigma = a)$ is convex in the variable x. Yet, one may observe a change of concavity when a goes to infinity, since $\lim_{a\to\infty} \mathbb{P}(\eta \in dx \,|\, \sigma = a) = 6\, x(1-x)\, dx$.

For the proof of Theorem 1, we need to compute the law of the first passage time Γ_a of a Brownian excursion e across the parabola $s \mapsto \frac{a}{2} s(1-s)$, viz

$$\Gamma_a = \inf\left\{ s > 0,\ e(s) = \frac{a}{2}\, s(1-s) \right\}.$$

Lemma 1 (Law of Γ_a)
For $a > 0$ and $0 < x < 1$ the probability density function of the first passage time Γ_a of a Brownian excursion across $s \mapsto \frac{a}{2}\, s(1-s)$ is given by

$$\mathbb{P}\left(\Gamma_a \in dx\right) = a.\exp\left(-\frac{a^2}{24}\left(x^3 + 3x(1-x)\right)\right) \frac{C\left(a\, x^{3/2}\right)}{\sqrt{8\pi x(1-x)}}\ dx\,.$$

Proof of Lemma 1
Lemma 1 bears the same flavor as the formulaes of Groeneboom [14] on the first passage time of a Brownian motion with parabolic drift across a given level. Indeed, it is a consequence of Theorem 2-1 in [14]. We may state Groeneboom's result for our special case as follows. Consider a Brownian motion W^ϵ starting from $\epsilon > 0$ at time 0. For $a, x > 0$, the probability that this Brownian motion with parabolic drift $s \mapsto W_s^\epsilon - \frac{a}{2}s(1-s)$ crosses for the first time the level 0 in the time interval $[x, x+dx]$ equals

$$\exp\left(-\frac{a^2}{6}\left(\left(x-\frac{1}{2}\right)^3 + \frac{1}{2^3}\right) + \frac{a\epsilon}{2} - \frac{\epsilon^2}{2x}\right) \mathbb{E}\left(\exp\left(-a\int_0^x \beta^{[x]}_{0\to\epsilon}(s)\, ds\right)\right) \frac{\epsilon\, dx}{\sqrt{2\pi x^3}}, \tag{5}$$

where $\beta^{[x]}_{0\to\epsilon}$ is a three dimensional Bessel bridge of duration x from 0 to ϵ. The law of a normalized Brownian excursion is not absolutely continuous with respect to the law of a Brownian motion. Yet, it is the limit, when ϵ decreases to 0, of the law $\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}$ of a three dimensional Bessel bridge of duration 1 liking ϵ to ϵ, which is itself absolutely continuous on the time interval $[0, x]$ with respect to the law $\mathbb{P}^\epsilon$ of a Brownian motion starting from ϵ. Actually, the relation of absolute continuity on the canonical filtration $\mathcal{F}_x = \sigma(X_s;\, 0 \leq s \leq x)$ between the law $\mathbb{P}^{\mathrm{Bes}^3(\epsilon)}$ of a three dimensional Bessel process starting from ϵ and $\mathbb{P}^\epsilon$ is given by

$$\mathbb{P}^{\mathrm{Bes}^3(\epsilon)}\ |_{\mathcal{F}_x} = \frac{X_{x\wedge T_0}}{\epsilon}\ \mathbb{P}^\epsilon\ |_{\mathcal{F}_x}, \quad \text{for } 0 < x < 1,$$

where T_0 represents the first passage time of the canonical process across 0 (see [19] exercise 1.22 Chap. XI). If we write $p_t(x,y)$ for the transition densities of the three dimensional Bessel process

$$p_t(x,y) = \frac{y}{x\sqrt{2\pi t}}\left(\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right)\right), \quad \text{for } x, y > 0,$$

see [19] on p 446, we deduce of the above equality the relation of absolute continuity between $\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}$ and $\mathbb{P}^{\epsilon}$

$$\begin{aligned}\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}\,|_{\mathcal{F}_x} &= \frac{p_{1-x}(X_x,\epsilon)}{p_1(\epsilon,\epsilon)}\,\mathbb{P}^{\mathrm{Bes}^3(\epsilon)}\,|_{\mathcal{F}_x} \qquad \text{(see [19], p 463)}\\ &= \frac{p_{1-x}(X_x,\epsilon)}{p_1(\epsilon,\epsilon)}\times\frac{X_{x\wedge T_0}}{\epsilon}\,\mathbb{P}^{\epsilon}\,|_{\mathcal{F}_x}\,, \qquad \text{for } 0<x<1\,.\end{aligned}$$

Let $\Gamma_a(X)$ denote the first passage time of the canonical process across $s\mapsto\frac{a}{2}\,s(1-s)$. Since the event $\Gamma_a(X)\in dx$ is $\mathcal{F}_x$-measurable, the relation of absolutely continuity ensures that

$$\begin{aligned}\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}\left(\Gamma_a(X)\in dx\right) &= \mathbb{E}^{\epsilon}\left(\frac{p_{1-x}(X_x,\epsilon)}{p_1(\epsilon,\epsilon)}\times\frac{X_{x\wedge T_0}}{\epsilon}\,;\,\Gamma_a(X)\in dx\right)\\ &= \frac{p_{1-x}(a_x,\epsilon)}{p_1(\epsilon,\epsilon)}\times\frac{a_x}{\epsilon}\,\mathbb{P}^{\epsilon}\left(\Gamma_a(X)\in dx\right)\,,\end{aligned}$$

where $a_x=\frac{a}{2}\,x(1-x)$. The second equality stems from the fact that conditionally on $\Gamma_a(X)=x$, we have both $T_0>x$ and $X_x=a_x$. The probability $\mathbb{P}^{\epsilon}\left(\Gamma_a(X)\in dx\right)$ is given by Groeneboom's formula (5), so putting pieces together, we obtain

$$\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}\left(\Gamma_a(X)\in dx\right)=$$
$$\frac{p_{1-x}(a_x,\epsilon)}{p_1(\epsilon,\epsilon)}\exp\left(-\frac{a^2}{6}\left(\left(x-\frac{1}{2}\right)^3+\frac{1}{2^3}\right)+\frac{a\epsilon}{2}-\frac{\epsilon^2}{2x}\right)\mathbb{E}\left(\exp\left(-a\int_0^x\beta^{[x]}_{0\to\epsilon}(s)\,ds\right)\right)\frac{a_x\,dx}{\sqrt{2\pi x^3}}\,.$$

We now want to use the (weak) convergence of the law of a Bessel bridge $\beta^{[1]}_{\epsilon\to\epsilon}$ to the law of a Brownian excursion e. Even if the first passage time $\Gamma_a(X)$ of the canonical process across the parabola $s\mapsto\frac{a}{2}\,s(1-s)$, is not a continuous functional of X, standard arguments ensure the convergence

$$\mathbb{P}(\Gamma_a\in dx)=\lim_{\epsilon\downarrow 0}\mathbb{P}^{\beta^{[1]}_{\epsilon\to\epsilon}}\left(\Gamma_a(X)\in dx\right)\,.$$

At the limit $\epsilon\downarrow 0$, the three dimensional Bessel bridge $\beta^{[x]}_{0\to\epsilon}$ converges in law towards a Brownian excursion $e^{[x]}$ of duration x :

$$\mathbb{P}(\Gamma_a\in dx)=a\,\exp\left(-\frac{a^2}{24}\left(x^3+3x(1-x)\right)\right)\,\mathbb{E}\left(\exp\left(-a\int_0^x e^{[x]}(s)\,ds\right)\right)\frac{dx}{\sqrt{8\pi x(1-x)}}\,.$$

The scaling property of the Brownian excursion, and the very definition of the function C entail the equalities

$$\begin{aligned}\mathbb{E}\left(\exp\left(-a\int_0^x e^{[x]}(s)\,ds\right)\right) &= \mathbb{E}\left(\exp\left(-a x^{3/2}\int_0^1 e(s)\,ds\right)\right)\\ &= C\left(a\,x^{3/2}\right),\end{aligned}$$

from which the formula of Lemma 1 follows.

∎

Theorem 1 is at this point a simple consequence of the Markov property of the Brownian excursion at time x. It will be convenient in the following to use the notation

$$\mathcal{P}^{(a,m)} := \left\{ s \mapsto \frac{a}{2}\, s(m-s);\ s \in [0,m] \right\} .$$

The quantity we shall compute is $\mathbb{P}(\Gamma_a \in dx, \sigma \geq a-b)$, since

$$\mathbb{P}(\eta \in dx, \sigma \in da) = \lim_{b \downarrow 0} \frac{1}{b}\, \mathbb{P}(\Gamma_a \in dx, \sigma \geq a-b)\, da\, .$$

The Markov property at time x yields

$$\begin{aligned}
\mathbb{P}(\Gamma_a \in dx, \sigma \geq a-b) &= \mathbb{P}\left(\Gamma_a \in dx\, ;\ e_{x+\cdot} \geq \mathcal{P}^{(a-b,1)}(x+\cdot)\right) \\
&= \mathbb{P}\left(\Gamma_a \in dx\right)\ \mathbb{P}\left(e_{x+\cdot} \geq \mathcal{P}^{(a-b,1)}(x+\cdot)\,|\,\Gamma_a = x\right) \\
&= \mathbb{P}\left(\Gamma_a \in dx\right)\ \mathbb{P}\left(e_{x+\cdot} \geq \mathcal{P}^{(a-b,1)}(x+\cdot)\,|\,e_x = a_x\right) \\
&= \mathbb{P}\left(\Gamma_a \in dx\right)\ \mathbb{P}\left(\beta^{[1-x]}_{a_x \to 0} \geq \mathcal{P}^{(a-b,1)}(x+\cdot)\right) ,
\end{aligned}$$

where the last equality stems from the fact that $e(x+\cdot)$ conditioned by $e_x = a_x$ has the same law as a Brownian bridge $\beta^{[1-x]}_{a_x \to 0}$ of duration $1-x$ from a_x to 0. The calculus made p.81-82 in [12] justifies the equality

$$\mathbb{P}\left(\beta^{[1-x]}_{a_x \to 0} \geq \mathcal{P}^{(a-b,1)}(x+\cdot)\right) = \frac{b}{a}\, \mathbb{P}\left(\beta^{[1-x]}_{b_x \to 0} \geq \mathcal{P}^{(a-b,1-x)}\right) ,$$

with $b_x = \frac{b}{2}x(1-x)$. We thus obtain

$$\mathbb{P}(\Gamma_a \in dx, \sigma \geq a-b) = b.\exp\left(-\frac{a^2}{24}\left(x^3 + 3x(1-x)\right)\right) \frac{C\left(a\, x^{3/2}\right)}{\sqrt{8\pi x(1-x)}}\, \mathbb{P}\left(\beta^{[1-x]}_{b_x \to 0} \geq \mathcal{P}^{(a-b,1-x)}\right)\, dx\, .$$

The law of the three dimensional Bessel bridge $\beta^{[1-x]}_{b_x \to 0}$ converges to the law of a normalized Brownian excursion $e^{[1-x]}$ of duration $1-x$, when b tends to 0. So, using the scaling property of the Brownian excursion and formula (4), we get

$$\begin{aligned}
\lim_{b \downarrow 0} \mathbb{P}\left(\beta^{[1-x]}_{b_x \to 0} \geq \mathcal{P}^{(a-b,1-x)}\right) &= \mathbb{P}\left(e^{[1-x]} \geq \mathcal{P}^{(a,1-x)}\right) \\
&= \mathbb{P}\left(\sigma \geq a(1-x)^{3/2}\right) \\
&= \exp\left(-\frac{a^2}{24}(1-x)^3\right) C\left(a(1-x)^{3/2}\right) .
\end{aligned}$$

We can now conclude that

$$\begin{aligned}
\mathbb{P}(\eta \in dx, \sigma \in da) &= \lim_{b \downarrow 0} \frac{1}{b}\, \mathbb{P}(\Gamma_a \in dx, \sigma \geq a-b)\, da \\
&= \frac{e^{-a^2/24}}{\sqrt{8\pi x(1-x)}}\, C\left(a\, x^{3/2}\right) C\left(a(1-x)^{3/2}\right)\, dx\, da\, .
\end{aligned}$$

The proof of Theorem 1 is complete.

■

3 The convex hull of a Brownian excursion with parabolic drift

We focus now on the convex hull $\mathcal{H}_a$ of a Brownian excursion with parabolic drift $s \mapsto e_s + \frac{a}{2}s^2$. It can be shown that the path $\mathcal{H}_a$ is a.s. piecewise linear. We call $\mathcal{M}_1^{(a)} < \cdots < \mathcal{M}_{N_a}^{(a)}$ the abscissaes of its N_a edges and $a\mathcal{X}_1^{(a)} < \cdots < a\mathcal{X}_{N_a+1}^{(a)}$ the slopes of its $N_a + 1$ pieces of line. It is convenient for the understanding to keep in mind the following geometrical interpretations of the convex hull $\mathcal{H}_a$. Consider for $0 < x < 1$, the locations of the minimum of $s \mapsto e_s + \frac{a}{2}s^2 - axs$. On the one hand, they coincide with the locations of the minimum of $s \mapsto e_s + \frac{a}{2}(s-x)^2$. On the other hand, the location of this minimum is unique and equals $\mathcal{M}_i^{(a)}$ when $x \in \,]\mathcal{X}_{i-1}^{(a)}, \mathcal{X}_i^{(a)}[$; whereas for $x = \mathcal{X}_i^{(a)}$ there are at least two locations of this minimum, the smallest of them is $\mathcal{M}_{i-1}^{(a)}$, and the largest $\mathcal{M}_i^{(a)}$. From a geometrical point of view, this means that if you bring up a line $s \mapsto ax\,s + K$ until it touches the graph of $s \mapsto e_s + \frac{a}{2}s^2$, or equivalently if you bring up a parabola $s \mapsto -\frac{a}{2}(s-x)^2 + K'$ until it touches the graph of $s \mapsto e_s$, then there is a unique contact point when $x \in \,]X_{i-1}^{(a)}, \mathcal{X}_i^{(a)}[$, the abscissa of which is $\mathcal{M}_i^{(a)}$. On the other hand, when $x = \mathcal{X}_i^{(a)}$ there are at least two contact points, the abscissa of the left-most of them is $\mathcal{M}_{i-1}^{(a)}$ and the abscissa of the right-most of them is $\mathcal{M}_i^{(a)}$, see Figure 2. We call *a-parabolic hull* of the excursion e, the set of parabolaes $\left\{s \mapsto -\frac{a}{2}(s - \mathcal{X}_i^{(a)})^2 + K_i;\ i = 1, N_a + 1\right\}$, with K_i chosen such that the i-th parabola touches $s \mapsto e_s$ at $s = \mathcal{M}_{i-1}^{(a)}$ and $s = \mathcal{M}_i^{(a)}$.

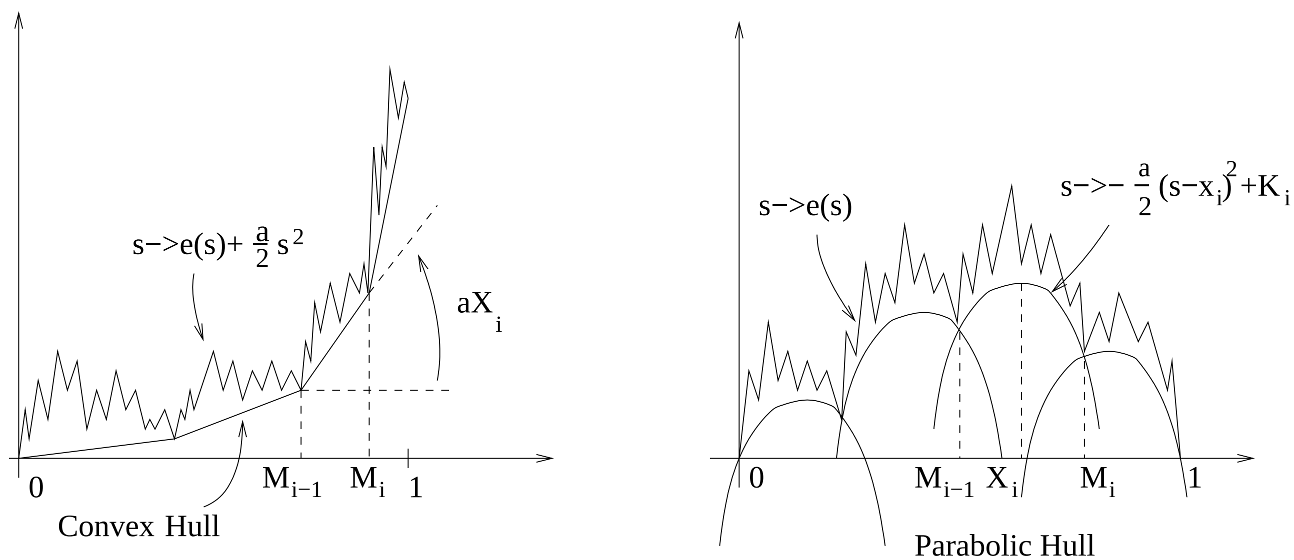

Figure 2: convex and parabolic hull

The set $\mathcal{M}^{(a)} := \left\{0, \mathcal{M}_1^{(a)}, \cdots, \mathcal{M}_{N_a}^{(a)}, 1\right\}$ corresponds to the set of the locations of the minimum of $s \mapsto e_s + \frac{a}{2}s^2 - axs$, when $x \in [0,1]$. For $0 < b < a$, the path $s \mapsto \left(e_s + \frac{a}{2}s^2 - axs\right) - \left(e_s + \frac{b}{2}s^2 - bxs\right) = \frac{a-b}{2}(s^2 - xs)$ is monotone increasing, which entails the embedding $\mathcal{M}^{(b)} \subset \mathcal{M}^{(a)}$. The set $\mathcal{M}^{(0)}$ a.s. equals $\{0,1\}$, and as time a runs, the edges appear one by one in the interval $[0,1]$.

Lemma 2 *The convex hull $\mathcal{H}_a$ can be recovered from the process $\left(\mathcal{M}^{(\alpha)}, 0 \le \alpha \le a\right)$, and conversely.*

Proof of Lemma 2

The convex hull $\mathcal{H}_a$ is completely determined by the coordinates of its N_a edges. The abscissaes $\mathcal{M}_1^{(a)}, \cdots, \mathcal{M}_{N_a}^{(a)}$ are given by $\mathcal{M}^{(a)}$. The ordinates can be computed in the following way. Let the time α increase from 0 to a. A first edge of abscissa $\mathcal{M}_{j_0}^{(a)}$ appears at a time α, we denote by $\alpha_{j_0}^{(a)}$. For $\alpha \geq \alpha_{j_0}^{(a)}$, its ordinate equals $e_{\mathcal{M}_{j_0}^{(a)}} + \frac{\alpha}{2}\left(\mathcal{M}_{j_0}^{(a)}\right)^2$, whereas at time $\alpha_{j_0}^{(a)}$, the edge j_0 belongs to the segment liking $(0,0)$ to $\left(1, \alpha_{j_0}^{(a)}/2\right)$, so that

$$e_{\mathcal{M}_{j_0}^{(a)}} + \frac{\alpha_{j_0}^{(a)}}{2}\left(\mathcal{M}_{j_0}^{(a)}\right)^2 = \frac{\alpha_{j_0}^{(a)}}{2}\mathcal{M}_{j_0}^{(a)}.$$

Putting pieces together, one obtains that the ordinate of the edge j_0 equals at time $\alpha \geq \alpha_{j_0}^{(a)}$

$$\frac{\alpha_{j_0}^{(a)}}{2}\mathcal{M}_{j_0}^{(a)} + \frac{\alpha - \alpha_{j_0}^{(a)}}{2}\left(\mathcal{M}_{j_0}^{(a)}\right)^2.$$

We compute in the same manner the ordinate of the next edge j_1 appearing when α increases. Suppose for example that $j_1 < j_0$. At time $\alpha_{j_1}^{(a)}$ the edge j_1 belongs to the segment liking $(0,0)$ to $\left(\mathcal{M}_{j_0}^{(a)}, \frac{\alpha_{j_0}^{(a)}}{2}\mathcal{M}_{j_0}^{(a)} + \frac{\alpha_{j_1}^{(a)}-\alpha_{j_0}^{(a)}}{2}\left(\mathcal{M}_{j_0}^{(a)}\right)^2\right)$, so that at time $\alpha \geq \alpha_{j_1}^{(a)}$ the ordinate of the edge j_1 equals

$$\frac{\alpha_{j_0}^{(a)}}{2}\mathcal{M}_{j_0}^{(a)} + \frac{\alpha_{j_1}^{(a)} - \alpha_{j_0}^{(a)}}{2}\left(\mathcal{M}_{j_0}^{(a)}\right)^2 + \frac{\alpha - \alpha_{j_1}^{(a)}}{2}\left(\mathcal{M}_{j_1}^{(a)}\right)^2.$$

Iterating the procedure, we calculate the ordinates of all edges of $\mathcal{H}_a$.

Conversely, we can recover $(\mathcal{H}_\alpha,\ 0 \leq \alpha \leq a)$ and thus $\left(\mathcal{M}^{(\alpha)},\ 0 \leq \alpha \leq a\right)$ from the convex hull $\mathcal{H}_a$, by reversing the procedure. Call $\mathcal{O}_j^{(a)}$ the ordinate of the edge j of the convex hull $\mathcal{H}_a$ and let α decrease from a to 0. At time $\alpha \leq a$, the ordinate of the edge j equals $\mathcal{O}_j^{(a)} + \frac{\alpha - a}{2}\left(\mathcal{M}_j^{(a)}\right)^2$. When three edges $j_1 < j_2 < j_3$ become aligned (while α is decreasing) erase the edge j_2. Then, the edges which remain at time α correspond to the edges of $\mathcal{H}_\alpha$.

■

The next corollary of Theorem 1 characterizes the process $\alpha \mapsto \mathcal{M}^{(\alpha)}$ in terms of the function C.

Corollary 1 (A dynamical description of the convex hull $\mathcal{H}_a$)

The process $\alpha \mapsto \mathcal{M}^{(\alpha)}$ is a pure jump (inhomogeneous) strong Markov process. For $a^* \geq a$ and $M_1 < \cdots < M_{i-1} < M^* < M_i < \cdots < M_N$, the conditional probability given $\mathcal{M}^{(a)} = \{0, M_1, \ldots, M_N, 1\}$ that the next edge appears during the time interval $[a^*, a^* + da^*]$ with an abscissa in $[M^*, M^* + dM^*]$, equals

$$\frac{(M_i - M_{i-1})^{3/2}\, da^*\, dM^*}{\sqrt{8\pi(M_i - M^*)(M^* - M_{i-1})}} \times \frac{C\left(a^*(M_i - M^*)^{3/2}\right) C\left(a^*(M^* - M_{i-1})^{3/2}\right)}{C\left(a^*(M_i - M_{i-1})^{3/2}\right)}$$
$$\prod_{j=1}^{N} \exp\left(-\frac{a^{*2} - a^2}{24}(M_j - M_{j-1})^3\right) \frac{C\left(a^*(M_j - M_{j-1})^{3/2}\right)}{C\left(a\,(M_j - M_{j-1})^{3/2}\right)}, \quad (6)$$

with the function C defined at the beginning of the previous section, see formula (3).

Specifying formula (6) for $a^*=a$, we obtain the rates of jump of $\alpha \mapsto \mathcal{M}^{(\alpha)}$:

$$\lim_{b\downarrow a}\frac{1}{b-a}\mathbb{P}\left(\begin{array}{l}\mathcal{M}^{(b)}=\{0,M_1,\ldots,M_{i-1},M^*,M_i,\ldots,M_N,1\};\\ M^*\in[M,M+dM]\end{array}\middle|\mathcal{M}^{(a)}=\{0,M_1,\ldots,M_N,1\}\right)$$
$$=\frac{(M_i-M_{i-1})^{3/2}\ dM}{\sqrt{8\pi(M_i-M)(M-M_{i-1})}}\times\frac{C\left(a\,(M_i-M)^{3/2}\right)C\left(a\,(M-M_{i-1})^{3/2}\right)}{C\left(a\,(M_i-M_{i-1})^{3/2}\right)}.$$

Proof of Corollary 1

Corollary 1 largely rephrases Theorem 1 and 2 in [12]. We sketch the main lines of the proof. The first step is to obtain a path decomposition of the excursion e conditionally on its convex hull $\mathcal{H}_a$. This decomposition is depicted in the next lemma. We omit its proof since it can be obtained by a slight variation of the proof of proposition 1 in [12], which mainly uses a path decomposition for Markov processes due to Millar [16]. We write as before $e^{[m]}$ for a Brownian excursion of duration m, $(\sigma(m),\eta(m))$ for the minimum and right-most location of the minimum of

$$s\mapsto\frac{2\,e^{[m]}(s)}{s(m-s)}$$

and $\nu(a,m)$ for the law of

$$s\mapsto e_s^{[m]}-\frac{a}{2}\,s(m-s),\quad \text{with } e^{[m]} \text{ conditioned by } \sigma(m)\geq a.$$

Lemma 3 (A path decomposition of the excursion e conditionally on convex hull $\mathcal{H}_a$)
The N_a+1 "excursions" of a Brownian excursion with parabolic drift $s\mapsto e_s+\frac{a}{2}\,s^2$ above its convex hull $\mathcal{H}_a$,

$$\varepsilon_i^{(a)}=\left(e\left(\mathcal{M}_{i-1}^{(a)}+s\right)+\frac{a}{2}\left(\mathcal{M}_{i-1}^{(a)}+s\right)^2-\mathcal{H}_a\left(\mathcal{M}_{i-1}^{(a)}+s\right)\ ;\ s\in\left[0,\mathcal{M}_i^{(a)}-\mathcal{M}_{i-1}^{(a)}\right]\right),\ i=1,\ldots,N_a+1$$

(with the convention $\mathcal{M}_0^{(a)}=0$ and $\mathcal{M}_{N_a+1}^{(a)}=1$) are independent conditionally on $\mathcal{H}_a$, with conditional law $\nu\left(a,\mathcal{M}_i^{(a)}-\mathcal{M}_{i-1}^{(a)}\right)$.

■

Let us now briefly explain the mechanism of growth of $\alpha\to\mathcal{M}^{(\alpha)}$. Keep in mind the geometrical interpretation in terms of the parabolic hull. The "excursions" $\varepsilon_i^{(a)}$, $i=1,\ldots,N_a+1$, exactly correspond to the "excursions" of $s\mapsto e_s$ above its a-parabolic hull. As time a runs, the parabolaes of the parabolic hull are drawn in the vertical direction. An edge of abscissa $M^*\in[\mathcal{M}_{i-1}^{(a)},\mathcal{M}_i^{(a)}]$, appears at time a^*, if the parabola $s\mapsto-\frac{a^*}{2}\left(s-\mathcal{X}_i^{(a^*)}\right)+K_i$ enters in contact with $s\mapsto e_s$ at $s=M^*$. The time a^* and the abscissa M^* then correspond to $\sigma\left(s\mapsto\varepsilon_i^{(a)}(s)+\frac{a}{2}\,s(\mathcal{M}_i^{(a)}-\mathcal{M}_{i-1}^{(a)}-s)\right)$ and $\eta\left(s\mapsto\varepsilon_i^{(a)}(s)+\frac{a}{2}\,s(\mathcal{M}_i^{(a)}-\mathcal{M}_{i-1}^{(a)}-s)\right)+\mathcal{M}_{i-1}^{(a)}$. After time a^*, we deal with $N_{a^*}+1=N_a+2$ "excursions" and the same procedure gives the next edge. And so on.

It follows from this mechanism (see Lemma 1 in [12] for a close argument) that there exists a function

$$F : \mathbb{R}^+ \times \bigcup_{m>0} \mathcal{C}([0,m],\mathbb{R}^+) \to \bigcup_{N\in\mathbb{N}} [0,1]^N$$

such that for any time $a > 0$ and index $i \in \{1,\dots,N_a+1\}$, the process $\left(\mathcal{M}^{(\beta)} \cap \left]\mathcal{M}_{i-1}^{(a)}, \mathcal{M}_i^{(a)}\right[;\, \beta \geq a\right)$ is given by $F\left(a, \varepsilon_i^{(a)}\right)$. The process $\left(\mathcal{M}^{(\beta)};\, \beta \geq a\right)$ considered on the whole interval $[0,1]$ is then

$$\left(\mathcal{M}^{(\beta)};\, \beta \geq a\right) = 0 * F\left(a, \varepsilon_1^{(a)}\right) * \cdots * F\left(a, \varepsilon_{N_a+1}^{(a)}\right),$$

where $*$ denotes the concatenation $(m_1,\dots,m_p) * (r_1,\dots,r_q) = (m_1,\dots,m_p,r_1,\dots,r_q)$. Since the conditional law of $\varepsilon_i^{(a)}$ given $\mathcal{H}_a$ (or $(\mathcal{M}^\alpha;\, 0 \leq \alpha \leq a)$) is $\nu\left(a, \mathcal{M}_i^{(a)} - \mathcal{M}_{i-1}^{(a)}\right)$, the conditional law of $\left(\mathcal{M}^{(\beta)};\, \beta \geq a\right)$ given $(\mathcal{M}^\alpha;\, 0 \leq \alpha \leq a)$ only depends of $\mathcal{M}^{(a)}$. The process $\alpha \mapsto \mathcal{M}^{(\alpha)}$ is thus Markovian.

Consider now what happens at a jump time. Call a^* the first time after a where an edge appears, and $M^* \in \left[\mathcal{M}_{i-1}^{(a)}, \mathcal{M}_i^{(a)}\right]$ the abscissa of this edge. Lemma 5 in [12] ensures that the "excursion" $\varepsilon_i^{(a)}$ splits at time a^* into two "excursions" $\varepsilon_i^{(a^*)}$ and $\varepsilon_{i+1}^{(a^*)}$, independent conditionally on $\left(\mathcal{M}^{(\alpha)};\, 0 \leq \alpha \leq a^*\right)$, with respective conditional laws $\nu\left(a^*, M^* - \mathcal{M}_{i-1}^{(a)}\right)$ and $\nu\left(a^*, \mathcal{M}_i^{(a)} - M^*\right)$. The evolution of $\mathcal{M}$ after time a^*, viz $\left(\mathcal{M}^{(\beta)};\, \beta \geq a^*\right)$ is given by $0 * F\left(a^*, \varepsilon_1^{(a^*)}\right) * \cdots * F\left(a^*, \varepsilon_{N_{a^*}+1}^{(a^*)}\right)$, where the processes $\varepsilon_j^{(a^*)};\, j = 1,\dots,N_{a^*}+1$, are independent conditionally on $\left(\mathcal{M}^{(\alpha)};\, 0 \leq \alpha \leq a^*\right)$, with conditional law $\nu\left(a^*, M_i^{(a^*)} - \mathcal{M}_{i-1}^{(a^*)}\right)$. The Markov property at time a^* follows. Since $\alpha \mapsto \mathcal{M}^{(\alpha)}$ is a pure jump Markov process fulfilling the Markov property at jump times, it is strong Markov process.

For $a^* \geq a$ and $M_1 < \cdots < M_{i-1} < M^* < M_i < \cdots < M_N$, we compute the conditional probability given $\mathcal{M}^{(a)} = \{0, M_1, \dots, M_N, 1\}$ that the next edge appears in the time interval $[a^*, a^* + da^*]$ with an abscissa in $[M^*, M^* + dM^*]$. It corresponds to the event $\sigma\left(s \mapsto \varepsilon_i^{(a)}(s) + \frac{a}{2}\, s(M_i - M_{i-1} - s)\right) \in [a^*, a^* + da^*]$, $\eta\left(s \mapsto \varepsilon_i^{(a)}(s) + \frac{a}{2}\, s(M_i - M_{i-1} - s)\right) \in [M^* - M_{i-1}, dM^* + M^* - M_{i-1}]$ and also $\sigma\left(s \mapsto \varepsilon_j^{(a)}(s) + \frac{a}{2}\, s(M_j - M_{j-1} - s)\right) \geq a^*$, for $j \neq i$. Since for $j = 1,\dots,N_a+1$, the processes $s \mapsto \varepsilon_j^{(a)}(s) + \frac{a}{2}\, s(M_j - M_{j-1} - s)$ are distributed conditionally on $\left(\mathcal{M}^{(\alpha)};\, 0 \leq \alpha \leq a\right)$ as independent Brownian excursions of duration $M_j - M_{j-1}$ conditioned by $\sigma(M_j - M_{j-1}) \geq a$, the probability of this event equals

$$\frac{\mathbb{P}\left(\sigma(M_i - M_{i-1}) \in da^*, \eta(M_i - M_{i-1}) \in d(M^* - M_{i-1})\right)}{\mathbb{P}\left(\sigma(M_i - M_{i-1}) \geq a\right)} \prod_{j\neq i} \frac{\mathbb{P}\left(\sigma(M_j - M_{j-1}) \geq a^*\right)}{\mathbb{P}\left(\sigma(M_j - M_{j-1}) \geq a\right)}.$$

Formula (6) follows now from the scaling property $(\sigma(m), \eta(m)) \overset{\text{law}}{=} \left(m^{-3/2}\sigma, m\eta\right)$, Theorem 1 and formula (4).

■

It is interesting for the physical applications of this work to compute the probability that $\mathcal{H}_a$ has

N edges, satisfying

$$\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N} \in \prod_{i=1}^{N} dM_i \times dX_i.$$

It is to be noticed that $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N_a}$ completely determines the coordinates of the edges, which means that the variable $\mathcal{X}_{N_a+1}^{(a)}$ is redundant. Indeed, since $a\mathcal{X}_{N_a+1}^{(a)}$ is the slope of the segment liking the N_a^{th} edge and $(1, a/2)$, it can be expressed in terms of $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N_a}$:

$$\mathcal{X}_{N_a+1}^{(a)} = \frac{1}{1-\mathcal{M}_{N_a}^{(a)}} \left(\frac{1}{2} - \sum_{i=1}^{N_a} \mathcal{X}_i^{(a)} \left(\mathcal{M}_i^{(a)} - \mathcal{M}_{i-1}^{(a)}\right)\right).$$

We must first determine the values $(M_i, X_i)_{1,N}$ which are acceptable for $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{1,N_a}$. It is convenient to call $(Y_i;\ i = 1, \ldots, N)$, the ordinates of the intersection points of the parabolaes of the a-parabolic hull of e, when $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{1,N_a} = (M_i, X_i)_{1,N}$, viz

$$Y_i = a\left(X_1 M_1 + X_2\left(M_2 - M_1\right) + \cdots + X_i\left(M_i - M_{i-1}\right) - \frac{1}{2} M_i^2\right), \text{ for } i = 1, \ldots, N. \tag{7}$$

The values $(M_i, X_i)_{i=1,N}$ are acceptable for $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N_a}$ if and only if they belong to the set

$$\mathcal{A} := \left\{\begin{array}{l} 0 < M_1 < \cdots < M_N < 1, \\ 0 < X_1 < \cdots < X_N < 1, \\ X_1 M_1 + X_2(M_2 - M_1) + \cdots + X_i(M_i - M_{i-1}) \geq \frac{1}{2} M_i^2, \text{ for i} = 1, \ldots, \text{N} \\ X_N < \frac{1}{1-M_N}\left(\frac{1}{2} - X_1 M_1 - X_2(M_2 - M_1) - \cdots - X_N(M_N - M_{N-1})\right) \end{array}\right\}.$$

The two first conditions are obvious. The third one expresses that Y_i is non-negative for $i = 1, \ldots, N$, and the last one ensures that $\mathcal{X}_N^{(a)} < \mathcal{X}_{N+1}^{(a)}$.

In order to state our result, we need to associate to a vector $(M_i, X_i)_{i=1,N} \in \mathcal{A}$ the genealogical tree $\mathcal{T}$ induced by $\left(\mathcal{M}^{(\alpha)};\ 0 \leq \alpha \leq a\right)$ when $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{1,N_a} = (M_i, X_i)_{1,N}$. The root i_0 corresponds to the index of the first edge appearing when time α goes from 0 to a. The index i_1 (respectively i_2) is the index of the first edge appearing at the left (resp. right) of $\mathcal{M}_{i_0}^{(a)}$, when α runs, and so on. The genealogical tree $\mathcal{T}$ can be obtained from the geometrical configuration of $(M_i, X_i)_{1,N}$ as follows. Recall that $(M_i, Y_i)_{i=1,N}$ represents the coordinates of the intersection points of the parabolaes $s \mapsto \frac{a}{2}(s - X_i)^2 + K_i$, see formula (7). Let α increase from 0 until the parabola $s \to \frac{\alpha}{2} s(1-s)$ touches one of the points (M_i, Y_i), we call (M_{i_0}, Y_{i_0}). Consider now two parabolaes with leading coefficient $-\frac{\alpha}{2}$, the first one which runs trough $(0,0)$ and (M_{i_0}, Y_{i_0}), and the second one which runs trough (M_{i_0}, Y_{i_0}) and $(0,1)$. Let α increase in the first one, until it touches a third point (or $\alpha = a$), we denote by (M_{i_1}, Y_{i_1}). Do the same in the second one, and call (M_{i_2}, Y_{i_2}) the third point. We deals at this point with four parabolaes with leading coefficient $-\frac{\alpha}{2}$. The first one runs trough $(0,0)$ and (M_{i_1}, Y_{i_1}), the second one runs trough (M_{i_1}, Y_{i_1}) and (M_{i_0}, Y_{i_0}), the third one runs trough (M_{i_0}, Y_{i_0}) and (M_{i_2}, Y_{i_2})

and the last one runs trough (M_{i_2}, Y_{i_2}) and $(0,1)$. As before, let α increase in each of them, until it touches a third point (or $\alpha = a$), the index of which is called i_{11} for the first parabola, i_{12} for the second one, i_{21} for the third one and i_{22} for the last one; and so on, see Figure 3. We have excluded the cases where the parabolaes touche two points at the same time, since the set of values of $(M_i, X_i)_{i=1,N}$ for which this phenomenon occurs has Lebesgue measure 0. We say that the point i_0 is the father of i_1 and i_2, or equivalently that i_1 and i_2 are the sons of i_0. In the same manners, we say that i_0 is of the first generation, i_1 and i_2 of the second generation, and so on. The quantities we are interested with are the functions l and r defined as

$$\begin{aligned} l(i) &= \sup\{0; j < i, \text{ such that } j \text{ ancestor of } i\} \\ \text{and} \quad r(i) &= \inf\{1; j > i, \text{ such that } j \text{ ancestor of } i\}, \end{aligned}$$

where j is said to be an ancestor of i, if it is of a smaller generation than i. We can now give the explicit statistics of $\mathcal{H}_a$.

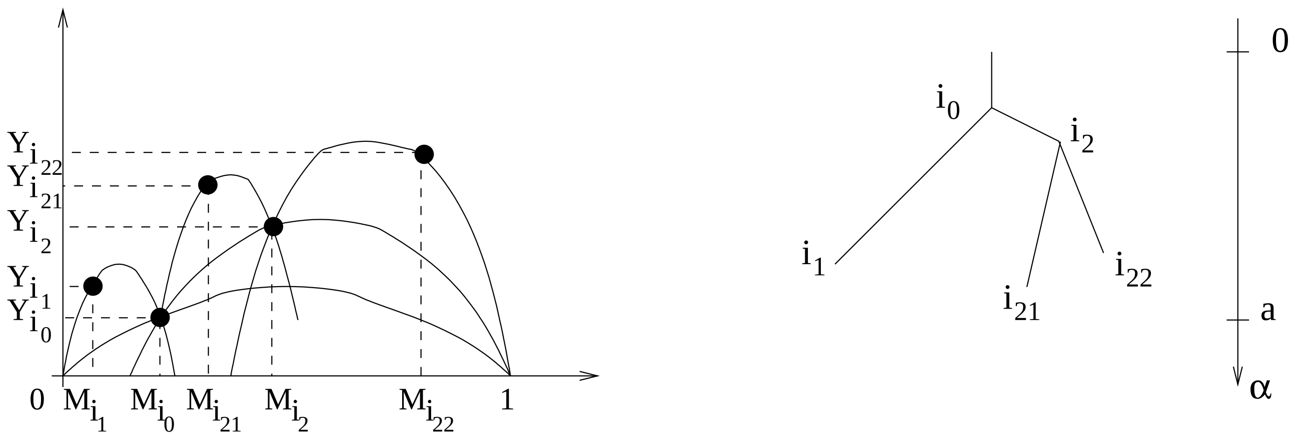

Figure 3: Genealogical tree associated to $(M_i, Y_i)_{1,N}$

Corollary 2 (Explicit statistics of $\mathcal{H}_a$)
For any $a > 0$, $N \in \mathbb{N}$ and $(M_i, X_i)_{i=1,N} \in \mathcal{A}$, the probability that $\mathcal{H}_a$ has N edges and $\left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N}$ belongs to $\prod_{i=1}^N dM_i \times dX_i$ equals

$$\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges; } \left(\mathcal{M}_i^{(a)}, \mathcal{X}_i^{(a)}\right)_{i=1,N} \in \prod_{i=1}^N dM_i \times dX_i\right) =$$

$$\frac{\sqrt{2\pi}}{a\, m_{N+1}} \exp\left(-a^2 \Phi\left((M_i, X_i)_{i=1,N}\right)\right) \prod_{j=1}^{N+1} \exp\left(-\frac{a^2}{24} m_j^3\right) \frac{a\, C\left(a\, m_j^{3/2}\right)}{\sqrt{2\pi m_j}} \quad dM_1 \ldots dM_N\, dX_1 \ldots dX_N,$$

where $m_j = M_j - M_{j-1}$ *and*

$$\Phi\left((M_i, X_i)_{i=1,N}\right) = \sum_{j=1}^{N} \frac{(M_{r(j)} - M_j)(M_j - M_{l(j)})}{2(M_{r(j)} - M_{l(j)})} \left(\sum_{k=l(j)+1}^{j} \frac{m_k\, X_k}{M_j - M_{l(j)}} - \sum_{k=j+1}^{r(j)} \frac{m_k\, X_k}{M_{r(j)} - M_j} + \frac{M_{r(j)} - M_{l(j)}}{2} \right)^2 .$$

Proof of Corollary 2

We will not iterate, as expected, formula (6) at the successive times of jump of $\alpha \mapsto \mathcal{M}^{(\alpha)}$, but rather follow the genealogical tree $\mathcal{T}$. We must first compute for $(a_1, \ldots, a_N) \in [0,a]^N$ and $0 < M_1 < \ldots < M_N < 1$ the probability density that $\left(\mathcal{M}_j^{(a)}, \alpha_j^{(a)}\right)_{1,N_a} = (M_j, a_j)_{1,N}$, where as before $\alpha_j^{(a)}$ denotes the time at which the edge of abscissa $\mathcal{M}_j^{(a)}$ appears. Let us first describe the genealogical tree $\mathcal{T}$ induced by $\left(\mathcal{M}^{(\alpha)};\, 0 \le \alpha \le a\right)$ when $\left(\mathcal{M}_j^{(a)}, \alpha_j^{(a)}\right)_{1,N_a} = (M_j, a_j)_{1,N}$. The root i_0 corresponds to the index for which a_j is minimal, viz $a_{i_0} = \min\{a_j;\, 1 \le j \le N\}$; i_1 and i_2 are the indices satisfying $a_{i_1} = \min\{a_j;\, 1 \le j < i_0\}$ and $a_{i_2} = \min\{a_j;\, i_0 < j \le N\}$, and so on.

When we follow the genealogical tree $\mathcal{T}$, we can take advantage at any time of jump α of the independence of the "excursions" $\varepsilon_i^{(\alpha)}$, $i = 1, \ldots, N_\alpha + 1$ conditionally on $\mathcal{H}_a$. The probability that $\left(\mathcal{M}_{i_0}^{(a)}, \alpha_{i_0}^{(a)}\right) \in dM_{i_0} \times da_{i_0}$ equals $\mathbb{P}(\sigma \in da_{i_0}, \eta \in dM_{i_0})$. Conditionally on $\left(\mathcal{M}_{i_0}^{(a)} = M_{i_0}, \alpha_{i_0}^{(a)} = a_{i_0}\right)$, $\varepsilon_1^{(a_{i_0})}$ and $\varepsilon_2^{(a_{i_0})}$ are independent with conditional law $\nu(a_{i_0}, M_{i_0})$ and $\nu(a_{i_0}, 1 - M_{i_0})$. The variables $\alpha_{i_1}^{(a)}$ and $\mathcal{M}_{i_1}^{(a)}$ equal $\sigma\left(s \mapsto \varepsilon_s^{(a_{i_0})} + \frac{a_{i_0}}{2}\, s(M_{i_0} - s)\right)$ and $\eta\left(s \mapsto \varepsilon_s^{(a_{i_0})} + \frac{a_{i_0}}{2}\, s(M_{i_0} - s)\right)$, with $s \mapsto \varepsilon_s^{(a_{i_0})} + \frac{a_{i_0}}{2}\, s(M_{i_0} - s)$ distributed conditionally on $\left(\mathcal{M}_{i_0}^{(a)} = M_{i_0}, \alpha_{i_0}^{(a)} = a_{i_0}\right)$ as a Brownian excursion of duration M_{i_0} conditioned by $\sigma(M_{i_0}) \ge a_{i_0}$. Similar formulaes also hold for $\alpha_{i_2}^{(a)}$ and $\mathcal{M}_{i_2}^{(a)}$, so that the probability of the event $\left(\mathcal{M}_{i_1}^{(a)}, \alpha_{i_1}^{(a)}\right) \in dM_{i_1} \times da_{i_1}$ $\left(\text{respectively } \left(\mathcal{M}_{i_2}^{(a)}, \alpha_{i_2}^{(a)}\right) \in dM_{i_2} \times da_{i_2}\right)$ conditionally on $\left(\mathcal{M}_{i_0}^{(a)} = M_{i_0}, \alpha_{i_0}^{(a)} = a_{i_0}\right)$ equals

$$\begin{aligned} &\mathbb{P}\left(\sigma(M_{i_0}) \in da_{i_1}, \eta(M_{i_0}) \in dM_{i_1}\right) \,/\, \mathbb{P}\left(\sigma(M_{i_0}) \ge a_{i_0}\right), \\ \text{respectively}\quad &\mathbb{P}\left(\sigma(1 - M_{i_0}) \in da_{i_2}, \eta(1 - M_{i_0}) \in d(M_{i_2} - M_{i_0})\right) \,/\, \mathbb{P}\left(\sigma(1 - M_{i_0}) \ge a_{i_0}\right). \end{aligned}$$

More generally, if we write $f(j)$ for the father of j and $a_{f(i_0)} = 0$, the probability that $\left(\alpha_j^{(a)}, \mathcal{M}_j^{(a)}\right) \in da_j \times dM_j$, given $\left(\alpha_{f(j)}^{(a)} = a_{f(j)}\right)$ and $\left(\mathcal{M}_{l(j)}^{(a)} = M_{l(j)}, \mathcal{M}_{r(j)}^{(a)} = M_{r(j)}\right)$, is

$$\begin{aligned} &\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \in da_j, \eta\left(M_{r(j)} - M_{l(j)}\right) \in d(M_j - M_{l(j)}) \,|\, \sigma\left(M_{r(j)} - M_{l(j)}\right) \ge a_{f(j)}\right) \\ &\qquad = \frac{\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \in da_j, \eta\left(M_{r(j)} - M_{l(j)}\right) \in d(M_j - M_{l(j)})\right)}{\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \ge a_{f(j)}\right)}. \end{aligned}$$

An index j is said to be a left-leaf (respectively a right-leaf) of $\mathcal{T}$, if it has no left son (resp. right son) in the tree $\mathcal{T}$. We shall compute the probability that left-leaves (resp. right-leaves) of $\mathcal{T}$ have no left sons (resp. right sons). When j is a left-leaf (resp. a right-leaf) of $\mathcal{T}$, the probability that, conditionally on $\left(\mathcal{M}_j^{(a)} = M_j, \mathcal{M}_{j-1}^{(a)} = M_{j-1}, \alpha_j^{(a)} = a_j\right)$, the interval $\left[\mathcal{M}_{j-1}^{(a)}, \mathcal{M}_j^{(a)}\right]$ $\left(\text{respectively } \left[\mathcal{M}_j^{(a)}, \mathcal{M}_{j+1}^{(a)}\right]\right)$ does not split before $\alpha = a$ equals

$$\begin{aligned}
\mathbb{P}\left(\sigma(M_j - M_{j-1}) \geq a \,|\, \sigma(M_j - M_{j-1}) \geq a_j\right) &= \frac{\mathbb{P}\left(\sigma(M_j - M_{j-1}) \geq a\right)}{\mathbb{P}\left(\sigma(M_j - M_{j-1}) \geq a_j\right)} \\
\text{resp.} \quad \mathbb{P}\left(\sigma(M_{j+1} - M_j) \geq a \,|\, \sigma(M_{j+1} - M_j) \geq a_j\right) &= \frac{\mathbb{P}\left(\sigma(M_{j+1} - M_j) \geq a\right)}{\mathbb{P}\left(\sigma(M_{j+1} - M_j) \geq a_j\right)}.
\end{aligned}$$

Putting pieces together, we obtain by conditioning at the successive times $(a_j)_{1,N}$ taken in their increasing order

$$\begin{aligned}
&\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges}, \left(\mathcal{M}_j^{(a)}, \alpha_j^{(a)}\right)_{1,N} \in \prod_{j=1}^{N} dM_j \times da_j\right) \\
&\quad = \prod_{j=1}^{N} \frac{\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \in da_j, \eta\left(M_{r(j)} - M_{l(j)}\right) \in d(M_j - M_{l(j)})\right)}{\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \geq a_{f(j)}\right)} \\
&\qquad \prod_{\lambda \text{ left-leaf of } \mathcal{T}} \frac{\mathbb{P}\left(\sigma(M_\lambda - M_{\lambda-1}) \geq a\right)}{\mathbb{P}\left(\sigma(M_\lambda - M_{\lambda-1}) \geq a_\lambda\right)} \prod_{\lambda \text{ right-leaf of } \mathcal{T}} \frac{\mathbb{P}\left(\sigma(M_{\lambda+1} - M_\lambda) \geq a\right)}{\mathbb{P}\left(\sigma(M_{\lambda+1} - M_\lambda) \geq a_\lambda\right)}. \qquad (8)
\end{aligned}$$

Formula of Theorem 1, combined with the scaling property $(\sigma(m), \eta(m)) \stackrel{\text{law}}{=} \left(m^{-3/2}\sigma, m\,\eta\right)$ and formula (4) yields

$$\begin{aligned}
&\mathbb{P}\left(\sigma\left(M_{r(j)} - M_{l(j)}\right) \in da_j, \eta\left(M_{r(j)} - M_{l(j)}\right) \in d(M_j - M_{l(j)})\right) \\
&\quad = \frac{\left(M_{r(j)} - M_{l(j)}\right)^{3/2}}{2\sqrt{2\pi\left(M_{r(j)} - M_j\right)\left(M_j - M_{l(j)}\right)}} \exp\left(-\frac{a_j^2}{8}\left(M_{r(j)} - M_{l(j)}\right)\left(M_{r(j)} - M_j\right)\left(M_j - M_{l(j)}\right)\right) \\
&\qquad \times \mathbb{P}\left(\sigma\left(M_{r(j)} - M_j\right) \geq a_j\right) \mathbb{P}\left(\sigma\left(M_j - M_{l(j)}\right) \geq a_j\right) da_j\, dM_j\,.
\end{aligned}$$

Many terms at the numerator and denominator of formula (8) will cancel. If s_1 is a left son of j then $l(s_1) = l(j)$ and $r(s_1) = j$, and in the same way if s_2 is a right son of j then $l(s_2) = j$ and $r(s_2) = r(j)$. This implies that if j has a left son s_1 then

$$\frac{\mathbb{P}\left(\sigma\left(M_j - M_{l(j)}\right) \geq a_j\right)}{\sqrt{M_j - M_{l(j)}}} \quad \text{cancels with} \quad \frac{\sqrt{M_{r(s_1)} - M_{l(s_1)}}}{\mathbb{P}\left(\sigma\left(M_{r(s_1)} - M_{l(s_1)}\right) \geq a_{f(s_1)}\right)}$$

and if j has a right son s_2 then

$$\frac{\mathbb{P}\left(\sigma\left(M_{r(j)} - M_j\right) \geq a_j\right)}{\sqrt{M_{r(j)} - M_j}} \quad \text{cancels with} \quad \frac{\sqrt{M_{r(s_2)} - M_{l(s_2)}}}{\mathbb{P}\left(\sigma\left(M_{r(s_2)} - M_{l(s_2)}\right) \geq a_{f(s_2)}\right)}.$$

After cancelling also every terms involving right or left leaves, it remains

$$\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges}, \left(\mathcal{M}_j^{(a)}, \alpha_j^{(a)}\right)_{1,N} \in \prod_{j=1}^N dM_j \times da_j\right)$$
$$= \prod_{j=1}^N \frac{M_{r(j)} - M_{l(j)}}{2\sqrt{2\pi}} \exp\left(-\frac{a_j^2}{8}\left(M_{r(j)} - M_{l(j)}\right)\left(M_{r(j)} - M_j\right)\left(M_j - M_{l(j)}\right)\right) dM_j\, da_j$$
$$\prod_{\lambda \text{ left leaf of } \mathcal{T}} \frac{\mathbb{P}\left(\sigma(M_\lambda - M_{\lambda-1}) \geq a\right)}{\sqrt{M_\lambda - M_{\lambda-1}}} \prod_{\lambda \text{ right leaf of } \mathcal{T}} \frac{\mathbb{P}\left(\sigma(M_{\lambda+1} - M_\lambda) \geq a\right)}{\sqrt{M_{\lambda+1} - M_\lambda}}.$$

Since for any $i \in \{0, \dots, N\}$, either i is a right leaf or $i+1$ is a left leaf, we finally obtain with the notation $m_i = M_i - M_{i-1}$

$$\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges}, \left(\mathcal{M}_j^{(a)}, \alpha_j^{(a)}\right)_{1,N} \in \prod_{j=1}^N dM_j \times da_j\right)$$
$$= \prod_{j=1}^N \frac{M_{r(j)} - M_{l(j)}}{2\sqrt{2\pi}} \exp\left(-\frac{a_j^2}{8}\left(M_{r(j)} - M_{l(j)}\right)\left(M_{r(j)} - M_j\right)\left(M_j - M_{l(j)}\right)\right) dM_j\, da_j$$
$$\prod_{i=1}^{N+1} m_i^{-1/2} \exp\left(-\frac{a^2}{24} m_i^3\right) C\left(a\, m_i^{3/2}\right).$$

We now want to come back to the variable $\left(\mathcal{M}_j^{(a)}, \mathcal{X}_j^{(a)}\right)_{1,N_a}$. Geometrical considerations show that

$$\alpha_j^{(a)} = \frac{2\left(\mathcal{Y}_j^{(a)}\left(\mathcal{M}_{r(j)}^{(a)} - \mathcal{M}_{l(j)}^{(a)}\right) - \mathcal{Y}_{l(j)}^{(a)}\left(\mathcal{M}_{r(j)}^{(a)} - \mathcal{M}_j^{(a)}\right) - \mathcal{Y}_{r(j)}^{(a)}\left(\mathcal{M}_j^{(a)} - \mathcal{M}_{l(j)}^{(a)}\right)\right)}{\left(\mathcal{M}_{r(j)}^{(a)} - \mathcal{M}_{l(j)}^{(a)}\right)\left(\mathcal{M}_j^{(a)} - \mathcal{M}_{l(j)}^{(a)}\right)\left(\mathcal{M}_{r(j)}^{(a)} - \mathcal{M}_j^{(a)}\right)},$$

where

$$\mathcal{Y}_j^{(a)} = a\left(\mathcal{X}_1^{(a)}\mathcal{M}_1^{(a)} + \mathcal{X}_2^{(a)}\left(\mathcal{M}_2^{(a)} - \mathcal{M}_1^{(a)}\right) + \dots + \mathcal{X}_j^{(a)}\left(\mathcal{M}_j^{(a)} - \mathcal{M}_{j-1}^{(a)}\right) - \frac{1}{2}\left(\mathcal{M}_j^{(a)}\right)^2\right).$$

A change of variable thus leads to

$$\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges}, \left(\mathcal{M}_j^{(a)}, \mathcal{X}_j^{(a)}\right)_{1,N} \in \prod_{j=1}^N dM_j \times dX_j\right) = \exp\left(-a^2 \Phi\left((M_i, X_i)_{i=1,N}\right)\right)$$
$$\prod_{j=1}^N \frac{a\, m_j\left(M_{r(j)} - M_{l(j)}\right) dM_j\, dX_j}{\sqrt{2\pi}\left(M_j - M_{l(j)}\right)\left(M_{r(j)} - M_j\right)} \prod_{i=1}^{N+1} m_i^{-1/2} \exp\left(-\frac{a^2}{24} m_i^3\right) C\left(a\, m_i^{3/2}\right),$$

with Φ defined in the statement of Corollary 2. Cancellations occur again, excepted for left or right-leaves, and after mixing the two products, there remains the expected formula

$$\mathbb{P}\left(\mathcal{H}_a \text{ has } N \text{ edges}, \left(\mathcal{M}_j^{(a)}, \mathcal{X}_j^{(a)}\right)_{1,N} \in \prod_{j=1}^N dM_j \times dX_j\right) =$$
$$\frac{\sqrt{2\pi}}{a\, m_{N+1}} \exp\left(-a^2 \Phi\left((M_i, X_i)_{i=1,N}\right)\right) \prod_{j=1}^{N+1} \exp\left(-\frac{a^2}{24} m_j^3\right) \frac{a\, C\left(a\, m_j^{3/2}\right)}{\sqrt{2\pi\, m_j}} \quad dX_1 \dots dX_N\, dM_1 \dots dM_N.$$

4 Application to Burgers Turbulence on the circle

We apply in this section the previous results to the analysis of the (weak) solutions of Burgers/Riemann equation (1), when the initial condition $u(\cdot, 0)$ is a periodic white noise. The solutions of Burgers equation are not unique in general. We focus henceforth to the so-called entropic solution, which may be view as the limit of the (unique) solution of the viscous Burgers equation $\partial_t u + u\,\partial_x u = \epsilon\,\partial^2_{xx} u$, when the viscosity ϵ tends to 0. This solution develops shocks which can be interpreted in terms of a ballistic model of aggregation. Consider at time $t = 0$ infinitesimal particles uniformly spread on the real axis, with velocity field $u(\cdot, 0)$. Suppose that they evolve according to the dynamic of free sticky particles, which means that between the collisions the motion of the particles is free, and when particles meet, they merge into a new heavier particle, with mass (respectively momentum), the sum of the masses (respectively momenta) of the particles involved into the collision. The macroscopic clusters present at time t are isomorphic to the shocks of the solution of $u(\cdot, t)$ of Burgers equation (1) at time t. The locations of the clusters coincide with the locations of the shocks, and their masses with the amplitude of the shocks.

When the so-called initial potential

$$W(s) = \int_0^s u(x, 0)\,dx$$

satisfies the condition of growth, $W(s) \ll s^2$ as $|s| \to \infty$, it is standard, according to the celebrated Hopf-Cole formula (see [8, 15]), that the solution at time t of Burgers equation may be determined in terms of the convex hull of $s \mapsto W(s) + \frac{1}{2t}s^2$. When this convex hull is piecewise linear, viz of the form $s \mapsto \sum_i t^{-1}(X_i s + k_i)\mathbf{1}_{[M_{i-1}, M_i]}(s)$, every particle (of the ballistic model) has merged into macroscopic cluster of mass $m_i = M_i - M_{i-1}$, located at X_i and one says that the shock structure is discrete. The vertex of a parabola of the $\frac{1}{2t}$-parabolic hull of W thus represents the location of a cluster, whose mass corresponds to the difference between the abscissaes of the two contact points of the parabola with its neighbours. See Figure 4 for the geometrical interpretation of the physical quantities.

The statistics of the system are well known, when the initial velocity is a white noise, namely when $s \mapsto W_s$ is a one or two-sided Brownian motion, see [5, 6, 11, 12, 14]. We shall focus thenceforth on the case where the initial velocity is a periodic white noise, and more precisely on the case where the initial potential $s \mapsto W_s$ is 1-periodic, and coincides on $[0, 1]$ with a Brownian bridge from 0 to 0. This case may be thought as a circular system of free sticky particles, which is a system of sticky particles evolving on a circle, with constant angular velocity between collisions. Indeed, it corresponds to the hydrodynamic limit of such a system starting from a regular setting, with random angular velocities $(\omega_i)_{1,N}$ i.i.d., of finite variance and satisfying $\sum_{i=1}^N \omega_i = 0$. We will describe what happens on $\mathbb{R}/\mathbb{Z}$. If we think to the system as a circular system, it can be shown that for t large enough all particles have merged into a single cluster. As the total momentum of the circular system is conserved and equals 0, the velocity of the final cluster is 0. Since the parabola $s \mapsto \frac{1}{2t}s^2$ becomes flat when t tends to infinity,

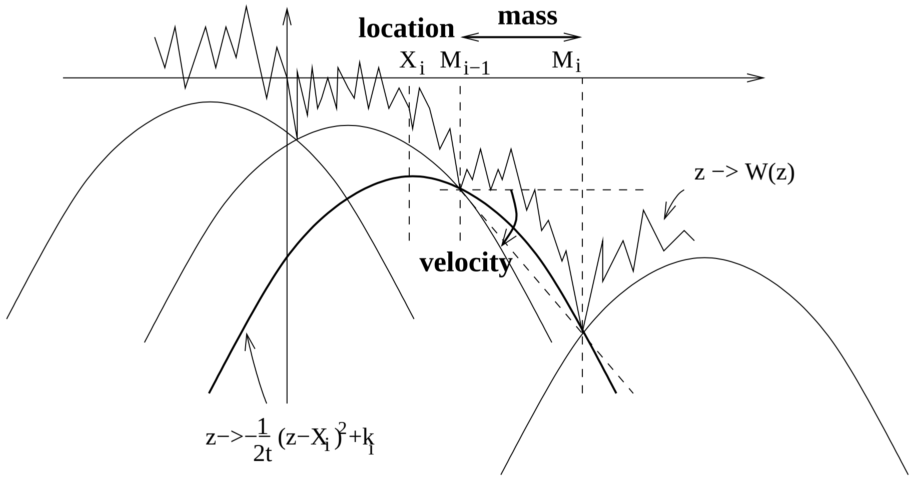

Figure 4: Physical interpretation of the parabolic hull

the abscissaes of the edges of the convex hull of $s \mapsto W_s + \frac{1}{2t}s^2$ corresponds, for t large enough, to the location L of the minimum of the Brownian bridge $(W_s,\, 0 \leq s \leq 1)$ modulo 1. In this case, the vertices of the parabolaes of the parabolic hull coincide with $L + \frac{1}{2}$ modulo 1. From a circular point of view, it means that the final cluster is located at $\lfloor L + \frac{1}{2} \rfloor_F$, where $\lfloor x \rfloor_F$ denotes the fractional part of x, defined as the difference between x and its entire part. According to Vervaat's transformation (see [22]), the process $\mathcal{E} = \big(W_{\lfloor L+s \rfloor_F},\, 0 \leq s \leq 1\big)$ is distributed as a Brownian excursion, independent of L. Moreover, the state of the system can be obtained from the convex hull $\mathcal{H}_{1/t}$ of $s \mapsto \mathcal{E}_s + \frac{1}{2t}s^2$. At time t, all particles have merged into N macroscopic clusters of mass $\mathcal{M}_i^{(1/t)} - \mathcal{M}_{i-1}^{(1/t)}$, located at $\lfloor L + \mathcal{X}_i^{(1/t)} \rfloor_F$, $i = 1, \ldots, N$. The statistics of the system thus simply rephrases those of $t \mapsto \mathcal{H}_{1/t}$.

The genealogical tree of the final cluster is given by $t \mapsto \mathcal{M}^{(1/t)}$. Indeed, at time t there are $N_{1/t} + 1$ clusters in the system, of mass $\mathcal{M}_i^{(1/t)} - \mathcal{M}_{i-1}^{(1/t)}$, $i = 1, \ldots, N_{1/t} + 1$. It is also interesting to calculate the statistics of the system at a given time t. We write $p_N(m_1, \ldots, m_N, \theta_1, \ldots, \theta_N)$ for the probability density that there exists N clusters of mass $m_1, \ldots, m_N$, located at $0 \leq \theta_1 < \cdots < \theta_N < 1$. Let us compute in this case, the value L of the location of the minimum of the initial potential $s \mapsto W_s$ on $[0, 1]$. The locations $\theta_1, \ldots, \theta_N$ correspond to the abscissaes in $[0, 1]$ of the vertices of the parabolic hull of $s \mapsto W_s$. We denote by $M_0 < \cdots < M_N$ the abscissaes of the intersection points of the parabolic hull, viz $M_i = M_0 + m_1 + \cdots + m_k$ is the abscissa of the intersection of $s \mapsto \frac{-1}{2t}(s - \theta_i)^2 + K_i$ and $s \mapsto \frac{-1}{2t}(s - \theta_{i+1})^2 + K_{i+1}$. It is standard that the velocity of the cluster number j then equals

$$u_j = \frac{2\,\theta_j - M_j - M_{j-1}}{2t}\,.$$

We write $Y_j = W_{M_j}$, and have the relation $Y_j = Y_0 + \sum_{i=1}^{j} m_i\, u_i$. The 1-periodicity implies the equality $Y_0 = Y_N = Y_0 + \sum_{i=1}^N m_i\, u_i$, from which follows

$$M_0 = \sum_{i=1}^N m_i \left(\theta_i - \sum_{k=1}^i m_k + \frac{m_i}{2} \right).$$

In this notation L coincides with $M_{j_0} = M_0 + \sum_{i=1}^{j_0} m_i$, where j_0 is the index which minimizes $\sum_{i=1}^j m_i\, u_i$. In terms of $(m_i,\theta_i)_{1,N}$, the value of L is given by

$$L = \sum_{i=1}^{j_0} m_i + \sum_{i=1}^N m_i \left(\theta_i - \sum_{k=1}^i m_k + \frac{m_i}{2} \right),$$

where j_0 is the index j which minimizes

$$\sum_{k=1}^j m_k \left(\theta_k + \frac{m_k}{2} - \sum_{p=1}^k m_p - \sum_{i=1}^N m_i \left(\theta_i - \sum_{p=1}^i m_p + \frac{m_i}{2} \right) \right) \quad \text{for} \quad j = 1, \ldots, N.$$

We can also specify the characteristics of the convex hull $\mathcal{H}_{1/t}$ of $\left(W_{\lfloor s+L \rfloor_F} + \frac{1}{2t}s^2,\, 0 \le s \le 1\right)$ in terms of $(m_i,\theta_i)_{1,N}$. Actually, $\mathcal{H}_{1/t}$ has $N-1$ edges, of abscissa

$$\mathcal{M}_j^{(1/t)} = M_{j_0,j} := \begin{cases} m_{j_0+1} + \ldots + m_{j_0+j} & \text{when } 1 \le j \le N - j_0 \\ m_{j_0+1} + \ldots + m_N + m_1 + \ldots + m_{j+j_0-N} & \text{else} \end{cases},$$

connected by segments of slope $t^{-1}\mathcal{X}_j^{(1/t)} = t^{-1}\theta_{j_0,j} := t^{-1}\lfloor \theta_{j+j_0} - (m_1 + \ldots + m_{j_0}) \rfloor_F$. Putting pieces together, we obtain, using the independence of L and $\mathcal{E}$

$$\begin{aligned} &p_N(m_1,\ldots,m_N,\theta_1,\ldots,\theta_N)\, dm_1 \ldots dm_{N-1}\, d\theta_1 \ldots d\theta_N = \\ &\qquad \delta(m_1 + \ldots + m_N = 1)\, \mathbb{P}\left(L \in d\left(\sum_{i=1}^{j_0} m_i + \sum_{i=1}^N m_i \left(\theta_i - \sum_{p=1}^i m_p + \frac{m_i}{2} \right) \right) \right) \\ &\qquad\qquad \times \mathbb{P}\left(\mathcal{H}_{1/t} \text{ has } N-1 \text{ edges}, \left(\mathcal{M}_j^{(1/t)}, \mathcal{X}_j^{(1/t)} \right)_{1,N-1} \in \prod_{j=1}^{N-1} dM_{j_0,j} \times d\theta_{j_0,j} \right). \end{aligned}$$

The formula of Corollary 2 ensures that the probability density function equals (recall that L is uniformly distributed on $[0,1]$)

$$\begin{aligned} &p_N(m_1,\ldots,m_N,\theta_1,\ldots,\theta_N) = \delta(m_1 + \ldots + m_N = 1) \times m_{j_0} \\ &\qquad \times \frac{t\sqrt{2\pi}}{m_{j_0}} \exp\left(-\frac{1}{t^2}\Phi\left((M_{j_0,i},\theta_{j_0,i})_{i=1,N-1} \right) \right) \prod_{j=1}^{N-1} \frac{1}{t} \exp\left(-\frac{m_{j_0+j}^3}{24\, t^2} \right) \frac{C\left(m_{j_0+j}^{3/2}/t \right)}{\sqrt{2\pi m_{j_0+j}}}, \end{aligned}$$

when $(M_{j_0,j},\theta_{j_0,j})_{j=1,N-1} \in \mathcal{A}$ and 0 else. We can simplify the previous expression as follows.

Proposition 1 *In the above notations, the probability density that there exists N clusters at time t of mass $m_1, \dots, m_N$ located at $0 < \theta_1 < \cdots < \theta_N < 1$ is given by*

$$p_N(m_1, \dots, m_N, \theta_1, \dots, \theta_N) =$$

$$t\sqrt{2\pi}\ \delta(m_1 + \ldots + m_N = 1)\ \exp\left(-\frac{1}{t^2}\Phi\left((M_{j_0,i}, \theta_{j_0,i})_{i=1,N-1}\right)\right) \prod_{j=1}^{N} \exp\left(-\frac{m_j^3}{24\,t^2}\right) \frac{C\left(m_j^{3/2}/t\right)}{t\sqrt{2\pi m_j}},$$

when $(M_{j_0,j}, \theta_{j_0,j})_{j=1,N-1} \in \mathcal{A}$ and 0 else.

We conclude with the remark that the law of the genealogical tree of a given cluster of mass m at time t is exactly the law $\mu(t,m)$ described in [12]. We can simplify the formula of Theorem 3 in [12], thanks to Theorem 1. Indeed, the joint probability density function of $(\rho(t,m), R(t,m))$ (see [12] for the definition of ρ and R) also equals

$$\mu(t,m)(\rho \in dr, R \in d\alpha) =$$

$$\exp\left(\frac{m^3}{24}\left(\frac{1}{t^2} - \frac{1}{(t-r)^2}\right)\right) \frac{C\left((\alpha m)^{3/2}/t - r\right)\ C\left(((1-\alpha)m)^{3/2}/t - r\right)}{C\left(m^{3/2}/t\right)} \times \frac{m^{3/2}\ dr\, d\alpha}{(t-r)^2\sqrt{8\pi\alpha(1-\alpha)}}.$$

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