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Turbulence de Burgers et Agrégation de Particules lorsque L'état Initial est Aléatoire.

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Table des matières

partie

Introduction et synthèse

Ce mémoire rassemble quelques prospections sur la turbulence de Burgers et l'agrégation de particules collantes, éventuellement interagissant par gravitation, dans un espace unidimensionnel, lorsque les conditions initiales sont aléatoires. Il est constitué de quatre parties, chacune conçue comme un article autonome avec sa propre bibliographie. Nous procédons dans un premier temps à une introduction succincte à l'équation de Burgers et aux particules collantes ainsi qu'à une présentation synthétique des résultats mathématiques développés dans les quatre parties. Un appendice situé à la fin de l'introduction guidera le lecteur à travers les résultats déjà connus sur la turbulence de Burgers initialisée par un bruit blanc.

0 De l'équation de Burgers et des particules collantes.

L'équation de Burgers unidimensionnelle non visqueuse

$$\partial_t u + u\partial_x u = 0 \,, \tag{1}$$

aussi appelée équation de Riemann par la communauté mathématique, a été introduite en physique par Burgers [14, 15, 16] comme un modèle simplifié de turbulence hydrodynamique. Elle correspond à un gaz infiniment compressible, $u(\cdot, t)$ représentant alors le champ de vitesse au temps t. Ce n'est pas un modèle très précis de turbulence hydrodynamique ; nous nous référons à Kraichnan [35] pour une discussion sur les similitudes et les différences avec l'équation de Navier et Stokes. Son analyse possède cependant un véritable intérêt, à la fois mathématique, car il s'agit de l'EDP non linéaire la plus simple, que physique, car elle intervient dans des domaines aussi variés que l'astrophysique, l'étude de la sédimentation ou l'acoustique ; voir par exemple Woyczyǹski [54], Burgers [16] ou Vergassola et al. [53]. On s'intéresse aux solutions faibles de (1), satisfaisant à la condition dite entropique qu'en tout temps $t > 0$, la fonction $x \to u(x,t)$ ne présente que des discontinuités du premier ordre, constituées de sauts négatifs. Ces solutions peuvent être obtenues comme limites de l'unique solution (forte) de l'équation de Burgers avec viscosité, $\partial_t u + u\partial_x u = \mu\partial_{xx}^2 u$, lorsque l'on fait tendre la viscosité μ vers 0 (voir Hopf [31], Cole [17]). Dès que la condition

$$U(x) :\, = \int_0^x u(z,0)\, dz = o(x^2) \qquad \text{quand } |x| \to \infty \,,$$

est satisfaite, ces solutions sont données, en tout point où elles sont continues, par

$$u(x,t) = \frac{x - a(x,t)}{t},$$

où $a(x,t)$ représente l'abscisse où la fonction

$$a \to U(a) + \frac{1}{2t}(a-x)^2$$

atteint son maximum (celui-ci est atteint vue la condition imposée sur le potentiel initial U). On peut interpréter géométriquement $a(x,t)$ de la manière suivante. Considérons la parabole $a \to -(a-x)^2/2t + C$ où le paramètre C est choisi de sorte que le graphe de la parabole soit strictement en dessous de celui du potentiel initial U. Augmentons C, jusqu'à ce que la parabole entre en contact avec le graphe de U. Le plus à droite des points de contact entre la parabole et le potentiel a alors pour abscisse $a(x,t)$ (voir figure ci-dessous). L'application $x \to a(x,t)$ ainsi définie est croissante et présente éventuellement des sauts positifs qui constituent les discontinuités de $x \to u(x,t)$, communément appelés chocs.

L'équation de Burgers est intimement liée au modèle dit de l'agrégation balistique. Ce modèle fait intervenir les particules collantes (appelées *sticky particles* dans la littérature anglo-saxone) proposées par Zeldovich [55] pour étudier les superstructures de l'univers. Les particules collantes sont des particules massives et ponctuelles, qui ont la propriété d'entrer en collision de manière totalement inélastiques. Lors d'une collision, elles restent collées et la masse (respectivement le moment) de la nouvelle particule est donnée par la somme des masses (respectivement des moments) des particules entrées en collision. La dynamique est dite de l'agrégation balistique, lorsque les particules évoluent librement entre chaque collision. Elle est sujette à de nombreuses études physiques voir par exemple [18, 21, 22, 24, 37, 49, 55]. Il s'agit d'une solution (faible) particulière du système de conservation de masse et moment

$$\left\{ \begin{array}{l} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = 0 \, , \end{array} \right. \tag{2}$$

où $u(\cdot,t)$ et $\rho(\cdot,t)$ représentent les champs de vitesse et de densité massique au temps t ; voir [20], mais aussi [10, 11, 13, 27].

Le lien entre l'équation de Burgers et l'agrégation balistique est le suivant. Considérons au temps $t=0$ des particules collantes infinitésimales réparties sur $\mathbb{R}$ selon la mesure de Lebesgue et évoluant selon la dynamique de l'agrégation balistique. La particule située en x au temps $t=0$ est supposée avoir pour vitesse initiale $u(x,0)$. Alors la solution (faible) entropique de l'équation de Burgers (1) avec condition initiale $u(\cdot, 0)$ définie par

$$u(x,t) :\, = \frac{x - a(x,t)}{t} \qquad \text{si } a \text{ est continue en } x,$$

$$\text{et} \quad u(x,t) :\, = \frac{1}{a(x+,t) - a(x-,t)} \int_{a(x-,t)}^{a(x+,t)} u(z,0)\, dz = \frac{u(x+,t) + u(x-,t)}{2} \qquad \text{sinon,}$$

(avec la notation $f(x+) = \lim_{z \downarrow x} f(z)$ et $f(x-) = \lim_{z \uparrow x} f(z)$), décrit complètement l'évolution du système. En effet, la vitesse de la particule située en x au temps t est donnée par $u(x,t)$ et l'inverse continu à droite $a \to x(a,t)$ de la fonction $x \to a(x,t)$ donne, partout où il est continu, la position au temps t de la particule initialement en a. Détaillons-en les implications. Si la fonction lagrangienne $a \to x(a,t)$ présente une discontinuité en a, alors il n'y a pas de particule au temps t dans l'intervalle dit de raréfaction $]x(a-,t), x(a+,t)[$. Hors de ces intervalles, la fonction $x \to a(x,t)$ donne la position initiale de la particule située en x au temps t. En particulier, une discontinuité en x de $x \to a(x,t)$ (c'est-à-dire un choc de $u(\cdot,t)$ en x) correspond à la présence au temps t d'un amas en x composé des particules initialement dans l'intervalle $]a(x-,t), a(x+,t)[$ et donc de masse $a(x+,t) - a(x-,t)$. Géométriquement, cela se traduit par l'existence d'au moins deux points de contact entre la parabole $a \to -(a-x)^2/2t + C$ et le potentiel U. Le plus grand des points de contact donne $a(x+,t)$ et le plus petit $a(x-,t)$. La vitesse de l'amas vaut $u(x,t) = (2x - a(x-,t) - a(x+,t))/2t$; elle s'interprète comme étant la pente de la droite joignant les deux points de contact, voir le dessin ci-dessous. Parallèlement aux points de choc, un point x où $x \to a(x,t)$ est continue et strictement croissante (c'est-à-dire un point x de l'image de $x(\cdot,t)$ où $x(\cdot,t)$ est continue et strictement croissante) correspond à la position au temps t d'une particule restée isolée, c'est-à-dire ne s'étant pas agrégée à un amas. Il est appelé point (lagrangien) régulier et sa vitesse est donnée par $u(x,t)$. Signalons pour finir que si on définit la mesure de Stieltjes $\rho(dx,t)$ par $\rho([x,y],t) = a(y+,t) - a(x-,t)$, alors le couple (ρ, u) est solution (faible) du système (2) avec conditions initiales $\rho(dx,0) = dx$ et $u(\cdot,0)$; voir [20].

Nous nous intéresserons aussi dans ce mémoire au modèle où les particules collantes sont en interaction gravitationnelle. Les collisions entre les particules sont toujours supposées être totalement inélastiques, mais la dynamique entre les chocs est gouvernée ici par l'hamiltonien

$$H = \sum_i \frac{p_i^2}{2m_i} + \gamma \sum_{i \neq j} m_i m_j |x_i - x_j|,$$

où x_i, m_i, p_i représentent la position, la masse et le moment de la particule i au temps t et où γ est une constante gravitationnelle. L'accélération subie par une particule est alors proportionnelle à la différence des masses entre sa gauche et sa droite, elle est entre autre indépendante des distances interparticulaires. Cette dynamique est une solution du système

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = -\rho\, \partial_x \phi \\ \partial_{xx}^2 \phi = \gamma \rho \end{cases} \tag{3}$$

où ρ, u, ϕ représentent la densité massique, la vitesse et le potentiel gravitationnel en x au temps t, voir [20].

Les dynamiques considérées précédemment sont totalement déterministes. L'aléa va intervenir au niveau de la condition initiale, soit en donnant un champ de vitesse initial aléatoire (partie I, II et III de ce mémoire), soit en donnant une distribution spatiale initiale aléatoire aux particules (partie IV). Les motivations de ces études sont principalement issues de l'astrophysique. Les physiciens pensent que la distribution des superstructures de l'univers reflète les petites fluctuations de densité autour de la densité uniforme aux premiers instants de l'univers et plus précisement au moment du découplement du baryon et du photon. Les équations de Jeans-Vaslov-Poisson qui régissent la dynamique de la matière à ces âges, peuvent se ramener pour la matière noire (qui compose 90% de l'univers), soit à l'équation de Burgers multidimensionnelle $\partial_t u + (u, \nabla u) = 0$, moyennant quelques changements de variables et une hypothèse reliant le potentiel de vitesse initial au champ de gravitation, soit (en dimension 1) au système (3) en faisant abstraction de l'expansion de l'univers ; nous nous référons à [53] pour une discussion détaillée à ce sujet. Les astrophysiciens cherchent donc à comprendre comment une perturbation aléatoire de l'état homogène initial induite soit par un champ de vitesse initial aléatoire, soit par une densité aléatoire, se propage au cours du temps et à quelles structures elle donne lieu. Les trois premières parties de ce mémoire sont consacrées à l'étude de la turbulence de Burgers lorsque les profils de vitesse initiaux sont donnés par des processus stochastiques. La littérature à ce sujet abonde ; citons en autres [2, 3, 5, 8, 16, 23, 43, 46, 48, 51] ainsi que les synthèses [7, 53, 54]. Nous nous intéresserons pour notre part aux cas où le champ de vitesse initial est donné par un bruit blanc, un bruit stable, ou un mouvement brownien. Les cas du bruit blanc ou stable apparaissent naturellement comme limite infinitésimale d'un système discret où les particules sont uniformément réparties au temps initial avec des vitesses aléatoires i.i.d.(indépendantes et identiquement distribuées) de variance finie (bruit blanc) ou non (bruit stable). La quatrième partie est consacrée à l'étude du modèle d'agrégation gravitationnelle lorsque les particules sont indépendamment reparties au temps initial selon la loi uniforme avec moment nul. Cela correspond à l'évolution d'un gaz froid homogène sous sa propre gravitation. La littérature traitant de l'agrégation gravitationnelle est plus restreinte mais très récente ; on pourra se référer à [9, 38, 49]. Nous pouvons désormais procéder à la synthèse des résultats développés dans les quatre parties.

1 Étude des points réguliers de la turbulence de Burgers avec pour champ de vitesse initial un bruit stable *(à paraitre dans les annales de l'IHP)*.

La première partie est consacrée à l'étude de la turbulence de Burgers lorsque le profil de vitesse initial est donné par un bruit stable de Lévy. Un bruit stable est la dérivée (au sens des distributions) d'un processus de Lévy stable, à savoir d'un processus stochastique U continu à droite, à accroissements indépendants et stationnaires, vérifiant la propriété d'autosimilarité pour $\alpha \in]0, 2[$

$$U(\lambda \cdot) \sim \lambda^{1/\alpha} U(\cdot) \,.$$

Ils apparaissent comme processus limites dans des théorèmes du type théorème de Donsker pour des variables aléatoires à variance infinie. Dans cette première partie, nous nous focalisons sur l'étude des points (lagrangiens) réguliers de la turbulence. Rappelons qu'il s'agit des particules restées isolées jusqu'au temps t, c'est-à-dire n'ayant percuté aucune autre particule. Leur position est donnée par les abscisses x où $x \to a(x,t)$ est continue et strictement croissante. Lorsque la turbulence est initialisée par un bruit stable, il y a deux cas où il existe des points réguliers à un temps t donné, voir [6]. Le premier cas est celui où le profil de vitesse initial est donné par un bruit stable de Lévy d'indice de scaling $\alpha \in]1/2, 1[$ non complètement asymétrique, c'est-à-dire dérivant d'un processus de Lévy stable à variations bornées présentant à la fois des sauts positifs et négatifs. Le second cas est celui du bruit de Cauchy, il correspond à l'indice de scaling $\alpha = 1$. Nous donnons dans un premier théorème une description qualitative de l'ensemble des points réguliers.

Théorème 1 **i)** *Supposons que le profil de vitesse initial de la turbulence de Burgers soit donné par un bruit stable de Lévy d'indice* $\alpha \in]1/2, 1[$*, non complètement asymétrique. Alors en tout temps* $t > 0$*, l'ensemble des points lagrangiens réguliers est p.s. discret et régénératif. Il coïncide p.s. avec l'ensemble des zéros de* $u(\cdot, t)$.
ii) *Supposons que le profil de vitesse initial de la turbulence de Burgers soit donné par un bruit de Cauchy. Alors en tout temps* $t > 0$*, l'ensemble des points lagrangiens réguliers est p.s. non-dénombrable et de dimension de Minkowsky (et a fortiori de Hausdorff) nulle.*

Nous pouvons apporter dans le premier cas quelques précisions sur l'état de la turbulence au temps t, en terme de particules collantes.

Théorème 2 *Lorsque le profil de vitesse initial de la turbulence de Burgers est donné par un bruit stable de Lévy d'indice* $\alpha \in]1/2, 1[$*, non complètement asymétrique, la turbulence évolue indépendamment entre les points réguliers. De plus, entre deux points réguliers* *consécutifs les positions des amas forment une suite croissante qui peut être indexée par $\mathbb{Z}$ de sorte que la vitesse des amas soit négative (respectivement positive) lorsque l'indice est positif (respectivement négatif).*

Le fait que la turbulence évolue indépendamment entre les points réguliers est physiquement intuitif. Les points réguliers sont dans ce cas des particules de vitesse nulle, n'ayant percutées aucune autre particule jusqu'au temps t. Les particules présentes de part et d'autre d'un point régulier n'ont donc pas pu interagir. Les preuves de ces résultats résultent de la confrontation des propriétés trajectorielles des processus de Lévy aux conditions géométriques imposées par l'équation de Burgers. La présence d'un point régulier correspond, du moins dans le premier cas, à la présence d'un extremum local du potentiel initial U. L'idée essentielle est alors d'approximer les conditions géométriques imposées au potentiel à l'aide de subordinateurs (processus de Lévy croissants), afin de pouvoir exploiter la théorie des fluctuations pour les processus de Lévy.

2 Généalogie des chocs de la turbulence de Burgers initialisée par un bruit blanc *(Comm. Math. Phys. 223 (2001) No 1)*

L'état statistique à un temps fixe t de la turbulence de Burgers avec pour vitesse initiale un bruit blanc est connu ; on pourra se référer à l'appendice (situé à la fin de l'introduction) pour un rappel de ces différents résultats. En tout temps t positif, les particules sont agglomérées dans des amas dont les positions forment une suite discrète $(x_n(t)\,;\,n \in \mathbb{Z})$ de la droite réelle, avec la convention que $x_1(t)$ est la position du premier amas à droite de 0. On notera $(m_n(t), v_n(t))$ la masse et la vitesse de l'amas situé en $x_n(t)$. Nous souhaitons dans la deuxième partie de ce mémoire en savoir plus sur l'évolution temporelle de la turbulence. La dynamique d'agrégation des amas est totalement déterministe et engendre une perte d'information, dans la mesure où, pour deux temps $t_1 < t_2$, on ne peut pas retrouver l'état au temps t_1 à partir de celui au temps t_2. En terme probabiliste, cela se traduit par le fait que la filtration $\mathcal{F}_{t_2} = \sigma(x_n(t_2), m_n(t_2), v_n(t_2); n \in \mathbb{Z})$ est strictement incluse dans celle $\mathcal{F}_{t_1} = \sigma(x_n(t_1), m_n(t_1), v_n(t_1); n \in \mathbb{Z})$. Si on retourne le sens d'écoulement du temps, on obtient un processus stochastique de dislocation. Un amas M se fragmente après un certain temps ρ en deux amas M_1 et M_2, se fragmentant à leur tour respectivement après un temps ρ_1 et ρ_2 chacun en deux amas M_{11}, M_{12} et M_{21}, M_{22}, etc...

Le but de la deuxième partie est de décrire les lois de la dislocation.

On définit pour tout point de choc eulérien, c'est-à-dire tout point x correspondant à la position d'un amas au temps t, et tout temps $s < t$ la variable de dislocation

$$\mathcal{M}(x, s, t) = (m_1(x, s, t), \ldots, m_k(x, s, t)),$$

où $m_1(x, s, t), \ldots, m_k(x, s, t)$ représentent les masses des amas issus au temps s de la fragmentation de l'amas situé en x au temps t, rangées par l'ordre croissant de leur position. Les résultats de cette partie décrivent la loi du processus dit de fragmentation $(\mathcal{M}(x, t-r, t); 0 \leq r \leq t)$ conditionnellement à la donnée de l'état $\mathcal{F}_t$ de la turbulence au temps t.

L'analyse repose sur la constatation que les "excursions" du potentiel initial W (qui est ici un mouvement brownien) au dessus des paraboles "caratéristiques" du temps t

$$\varepsilon^{(x_n(t),t)}(z) = W(z + a_{n-1}(t)) - W(a_{n-1}(t)) - \frac{1}{2t} z(m_n(t) - z) - z v_n(t)\,, \quad 0 \leq z \leq m_n(t),$$

sont indépendantes sachant $\mathcal{F}_t$, et ont pour loi conditionnelle la loi de la différence entre une excursion brownienne de longueur $m_n(t)$ conditionnée à rester au dessus de la parabole

$$z \to \frac{1}{2t} z(m_n(t) - z)\,,$$

et ladite parabole. Nous pouvons ainsi nous ramener à l'étude de la loi d'une excursion brownienne de longueur m conditionnée à rester au dessus de la parabole $z \to z(m-z)/2t$. En particulier, les statistiques de dislocation peuvent s'obtenir à partir de la détermination des lois du couple (σ, η) défini par

$$\begin{cases} \sigma = \sup\left\{a \geq 0; e(z) \geq \frac{a}{2} z(1-z), \forall z \in [0,1]\right\} \\ \eta = \text{la plus grande abscisse } z_0 \text{ où } e(z_0) = \frac{\sigma}{2} z_0(1-z_0) \end{cases}$$

où e est une excursion brownienne normalisée de longueur unité. Le premier théorème nous montre que l'on peut se focaliser sur un seul amas.

Théorème 3 *Conditionnellement à l'état $\mathcal{F}_t$ de la turbulence au temps t, les processus de fragmentation $(\mathcal{M}(x_n, t-r, t);\ 0 \leq r \leq t)$, $n \in \mathbb{Z}$, sont indépendants, de loi conditionnelle $\mu(t, m_n(t))$, où $\mu(t,m)$ est la loi conditionnelle de $(\mathcal{M}(x_1, t-r, t);\ 0 \leq r \leq t)$ sachant que $m_1(t) = m$.*

Ce théorème affirme que, conditionnellement à l'état de la turbulence au temps t, lorsque l'on retourne le temps, les amas se disloquent indépendamment de leur vitesse et position, mais aussi indépendemment des autres amas. En ce sens on peut réellement parler de fragmentation des amas. Remarquons aussi que ce théorème affirme le caractère markovien du processus de fragmentation $(\mathcal{M}(x, t-r, t);\ 0 \leq r \leq t)$. Nous voyons enfin que pour décrire l'évolution du système en temps retourné, il nous suffit de connaître la loi $\mu(t,m)$. Appelons $\rho = \rho(t,m)$ le temps auquel l'amas situé en $x_1(t)$ au temps t se disloque. Nous allons voir que p.s. il se fragmente alors en exactement deux amas de masse m_1 et m_2. Nous noterons $R = R(t,m)$ le rapport $m_1/(m_1+m_2)$. Pour spécifier la loi $\mu(t,m)$, il est utile de définir l'opération de concaténation sur les suites finies. Si $M_1 = (m_1^1, \ldots, m_k^1)$ et $M_2 = (m_1^2, \ldots, m_l^2)$, la concaténation de M_1 et M_2 est donnée par

$$M_1 * M_2 = (m_1^1, \ldots, m_k^1, m_1^2, \ldots, m_l^2)\,.$$

Le résultat suivant explicite l'évolution d'un processus de fragmentation à un temps de dislocation.

Théorème 4 *Soit $t, m > 0$ et $M = (M(r); 0 \leq r \leq t)$ un processus de loi $\mu(t,m)$. Ce processus vérifie la propriété de Markov suivante au temps de dislocation ρ :*

$$M(\rho + r) = M_1(r) * M_2(r) \qquad \textit{pour} \;\; 0 \leq r \leq t - \rho,$$

où M_1 et M_2 sont deux processus indépendants conditionnellement à (ρ, R), de loi conditionnelle $\mu(t-\rho, Rm)$ et $\mu(t-\rho, (1-R)m)$.

Un amas de masse m au temps t se disloque donc après un temps $\rho_0 = \rho(t,m)$ en deux amas de masse $m_1 = R_0\, m$ et $m_2 = (1-R_0)m$ avec $R_0 = R(t,m)$. Les processus de dislocation des masses m_1 et m_2 ayant pour loi $\mu(t-\rho_0, m_1)$ et $\mu(t-\rho_0, m_2)$ on en déduit qu'elles se disloquent à leur tour après des temps $\rho(t-\rho_0, m_1)$ et $\rho(t-\rho_0, m_2)$ chacun en deux amas de masse

$$m_{11} = R(t-\rho_0, m_1)\, m_1, \quad m_{12} = (1 - R(t-\rho_0, m_1))\, m_1$$

et

$$m_{21} = R(t-\rho_0, m_2)\, m_2, \quad m_{22} = (1 - R(t-\rho_0, m_2))\, m_2.$$

Nous pouvons ainsi réitérer le théorème à volonté. Pour décrire complètement la fragmentation d'un amas, il ne nous reste plus qu'à donner la loi jointe de $(\rho(t,m), R(t,m))$. Nous introduisons dans cette optique la fonction

$$\mathcal{G}^{(x)}(a,b) = \sqrt{8\pi}\, e^{ab(1-x)^2(2x+1)/12}\, e^{\mathcal{O}(b^2)} \int_0^\infty e^{yb(x-1/2)} F^{(x,y)}(a) F^{(1-x,y+\beta)}(a-b)\, dy\,,$$

où $\mathcal{O}(b^2) = \frac{b^2}{24}(1-x)(8x^2-4x-1)$ et $\beta = \frac{b}{2}x(1-x)$. La fonction $F^{(x,y)}(\lambda)$ est donnée par

$$F^{(x,y)}(\lambda) = 2^{-1/3}\lambda^{2/3} \sum_{n=1}^{\infty} \frac{\mathtt{Ai}(2^{1/3}\lambda^{1/3}\mathtt{y} - \omega_\mathtt{n})}{\mathtt{Ai}'(-\omega_\mathtt{n})} \exp(-2^{-1/3}\lambda^{2/3} x\omega_n)\,,$$

où `Ai` est la fonction d'Airy (cf page 446 de [1]) et $0 > -\omega_1 > -\omega_2 > \cdots$ est la suite de ses zéros. Nous aurons aussi besoin de la transformée de Laplace de l'intégrale d'une excursion brownienne de durée 1. Elle a été calculée par Groeneboom [28] et vaut

$$C(\lambda) := \mathbb{E}\left(\exp\left(-\lambda \int_0^1 e(s)\, ds\right)\right) = \lambda\sqrt{2\pi} \sum_{n=1}^{\infty} \exp(-2^{-1/3}\omega_n \lambda^{2/3})\,.$$

Nous pouvons maintenant formuler la loi jointe du couple $(\rho(t,m), R(t,m))$.

Théorème 5 *Pour $0 < r < t$, $m > 0$ et $0 \le \alpha \le 1$, la loi du couple $(\rho(t,m), R(t,m))$ est donnée par la densité*

$$\mu(t,m)\,(\rho \in dr, R > \alpha) = \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2} - \frac{1}{(t-r)^2}\right)\right) \partial_2\, \mathcal{G}^{(\alpha)}\left(\frac{m^{3/2}}{t-r}, 0\right) \frac{m^3\, dr}{C(m^{3/2}/t)(t-r)^2}$$

avec les fonctions $\mathcal{G}^{(\alpha)}(a,b)$ et C définies précédemment.

En particulier, la loi du temps de dislocation $\rho(t,m)$ s'exprime sous la forme réduite

$$\mu(t,m)(\rho \ge r) = \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2} - \frac{1}{(t-r)^2}\right)\right) \frac{C(m^{3/2}/(t-r))}{C(m^{3/2}/t)}\,.$$

Tout ces résultats se basent sur l'étude de quelques propriétés des excursions browniennes conditionnées à rester au dessus d'une parabole. Cette étude fait principalement intervenir un résultat sur la décomposition d'un processus de Markov dû à Millar [41], la formule de Girsanov et diverses relations entre les ponts browniens et de Bessel.

3 Statistiques du flux de masse à travers l'origine induit par la turbulence de Burgers avec conditions initiales browniennes asymétriques. *(En collaboration avec J. Bertoin et Y. Isozaki, Comm. Math. Phys. 224 (2001) No 2)*

Nous nous intéressons dans la troisième partie à la turbulence de Burgers, lorsqu'au temps initial le champ de vitesse à gauche de l'origine est nul (particules au repos) alors qu'à droite il est donné par un bruit blanc ou un mouvement brownien. Nous étudions dans ce cas le flux de masse traversant l'origine. Il est entièrement décrit par la masse $a(0,t)$ des particules qui ont traversé l'origine de droite à gauche. En effet, la masse d'un amas traversant l'origine au temps t est donnée par $a(0,t) - a(0,t-)$ et sa vitesse par $-(a(0,t) + a(0,t-))/2t$. Nous allons spécifier le processus $t \to a(0,t)$. Les particules à gauche de l'origine étant au repos, le flux de matière à travers 0 s'effectue de la droite vers la gauche, c'est-à-dire que $t \to a(0,t)$ est une application croissante. Une conséquence importante en est la caractère markovien inhomogène du processus $t \to a(0,t)$. La fonction $a(x,t)$ vérifie, au sens faible, l'équation de transport

$$\partial_t a + u\partial_x a = 0\,,$$

ce qui nous suggère un lien entre le générateur infinitésimal de $t \to a(0,t)$ en t et celui de $x \to a(x,t)$ en 0. Ce dernier étant connu, cela nous permettrait de caractériser le processus $t \to a(0,t)$.

Considérons dans un premier temps, le cas où la vitesse initiale à droite de l'origine est donnée par un bruit blanc. Introduisons la fonction $g : \mathbb{R} \to \mathbb{R}^+$ définie par sa transformée de Fourier

$$\hat{g}(\xi) \;=\; \int_{-\infty}^{\infty} e^{\mathtt{i}\xi x} g(x)dx \;=\; \frac{2^{1/3}}{\mathtt{Ai}(\mathtt{i}2^{-1/3}\xi)}\,, \qquad \xi \in \mathbb{R}, \quad \mathtt{i}^2 = -1\,,$$

ainsi que la fonction $p : \mathbb{R}^+ \to \mathbb{R}^+$

$$p(x) \;=\; 2\sum_{k\geq 1} \exp\{-2^{1/3}\omega_k x\}\,,$$

qui peut être définie alternativement par la transformée de Laplace

$$\int_0^{\infty} e^{-\lambda x}\left(p(x) - (2\pi x^3)^{-1/2}\right)dx \;=\; 2^{2/3}\frac{\mathtt{Ai}'}{\mathtt{Ai}}\left(2^{-1/3}\lambda\right) + \sqrt{2\lambda}\,.$$

La structure des chocs à un temps s fixé est p.s. discrète à droite de l'origine, c'est-à-dire que toutes les particules à droite de 0 se sont agrégées en des amas dont les positions forment une suite discrète de la droite réelle. Il en découle que la fonction $t \to a(0,t)$

est p.s. en escalier. Comme la loi de $a(0,s)$ est donnée par le corollaire 3.1 de [28], et comme $t \to a(0,t)$ est un processus de Markov en escalier, les statistiques de l'évolution de $t \to a(0,t)$ sont entièrement déterminées par la loi conditionnelle du couple $(T(s), M(s))$ sachant $a(0,s)$, où

$$T(s) \;=\; \inf\{t > s : a(0,t) > a(0,s)\}$$

représente le premier instant de passage d'un amas en 0 après le temps s et où

$$M(s) \;=\; a(0, T(s)) - a(0,s)$$

la masse de cet amas. Nous donnons cette loi conditionnelle en terme de g et p dans le théorème suivant.

Théorème 6 *Dans le cas du bruit blanc, $t \to a(0,t)$ est un processus de Markov (inhomogène) en escalier. Pour tout $m, s > 0$, la loi conditionnelle jointe de l'instant $T(s)$ et de la taille $M(s)$ du premier saut de $a(0,\cdot)$ après le temps s sachant que $a(0,s) = m$ est*

$$\begin{aligned}
&\mathbb{P}\left(T(s) \in dt, M(s) \in dy \mid a(0,s) = m\right) \\
&= \left(\frac{s}{t}\right)^{1/3} \exp\left(-\frac{m^3}{6}\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\right) \frac{y(2m+y)}{4t^3} p\left((2t)^{-2/3} y\right) \frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2s)^{-2/3} m\right)}\, dy\, dt\,,
\end{aligned}$$

pour $y > 0$ et $t > s$.

Le principal travail pour obtenir ce théorème est d'établir le lien

$$\begin{aligned}
&\frac{1}{h\eta}\mathbb{P}(a(0,t+h) - a(0,t) \in [y, y+\eta] | a(0,t) = m) \\
&\qquad \overset{h,\eta \to 0+}{\sim} \frac{1}{h\eta}\,\frac{2m+y}{2t}\mathbb{P}(a(h,t) - a(0,t) \in [y, y+\eta] | a(0,t) = m) \quad (4)
\end{aligned}$$

entre le taux de saut de $s \to a(0,s)$ en t et celui de $x \to a(x,t)$ en 0. Nous exploitons pour cela fortement le fait que la structure de choc soit p.s. discrète à droite de l'origine.

Considérons maintenant le cas où le profil de vitesse initial est un mouvement brownien à droite de l'origine. La principale différence avec le cas précédent tient au fait que les positions des amas à droite de l'origine sont p.s. partout denses dans $\mathbb{R}^+$. La loi de $a(0,s)$ étant donnée par le théorème 1 de [5], nous allons caractériser le processus de Markov $t \to a(0,t)$ en calculant la transformée de Laplace de l'accroissement $a(0,t) - a(0,s)$ sachant $a(0,s)$.

Théorème 7 *Dans le cas brownien, $t \to a(0,t)$ est un processus de Markov (inhomogène) de saut pur dont les transitions sont caractérisées par la transformée de Laplace*

$$\begin{aligned}
&\mathbb{E}\left(\exp\left\{-q\left(a(0,t) - a(0,s)\right)\right\} \mid a(0,s) = m\right) \\
&= \sqrt{\frac{s}{t} + \frac{t-s}{t\sqrt{2t^2q+1}}}\, \exp\left\{-\frac{m(t-s)}{st^2}\left(\sqrt{2t^2q+1} - 1\right)\right\}\,,
\end{aligned}$$

pour tout $q, m > 0$ et $0 < s < t$.

Il est difficile de prouver une relation analogue à (4) dans ce cas, à cause de la densité des chocs. L'idée est de décomposer $t \to a(0,t) - a(0,s)$ en deux processus indépendants, dont on sait calculer les transformées de Laplace des transitions grâce à la solution dite retardée $\widetilde{u}\,(x,r) = u(x, s+r)$ introduite par Bertoin dans [8].

4 Agrégation d'un gaz unidimensionnel auto-gravitant à basse température *(J. Stat. Phys. 105 (2001) No 3/4)*.

La quatrième et dernière partie est consacrée à l'analyse de l'évolution de n particules collantes de masse $1/n$ en interaction gravitationnelle, lorsqu'au temps initial elles sont réparties indépendamment sur $[0,1]$ selon la loi uniforme, avec moment nul. La dynamique est celle présentée en introduction. Quitte à effectuer le changement de temps $t' = \sqrt{\gamma}\, t$ nous pouvons supposer que $\gamma = 1$. Notre étude se porte sur les propriétés asymptotiques de l'évolution du système lorsque $n \to \infty$. Le système étant confiné, toutes les particules se sont agrégées après un temps fini. Nous explicitons dans un premier temps les lois limites des temps de première et dernière collisions.

Théorème 8 **i)** *La dernière collision a p.s. lieu entre deux amas macroscopiques, au temps $T_n^{l.c.}$ vérifiant la convergence en loi lorsque $n \to \infty$*

$$\sqrt{n}\left(T_n^{l.c.} - 1\right) \overset{loi}{\to} \sup_{x\in[0,1]} \left(\frac{1}{1-x} \int_x^1 b(t)\, dt - \frac{1}{x} \int_0^x b(t)\, dt \right) ,$$

où b représente un pont brownien.
ii) *Les temps $T_{n:k}$ de k-ième collision vérifient la convergence en loi lorsque $n \to \infty$*

$$(\sqrt{n}T_{n:1}, \dots , \sqrt{n}T_{n:k}) \overset{loi}{\to} (\sqrt{\mathsf{e}_1}, \cdots , \sqrt{\mathsf{e}_1 + \cdots + \mathsf{e}_k}) ,$$

où $\mathsf{e}_1, \cdots, \mathsf{e}_k$ sont des variables exponentielles indépendentes de paramètre 1.

Nous nous intéressons maintenant à la taille des amas présents dans le système. Un amas sera dit de taille k, s'il est issu de l'agrégation de k particules initiales, c'est-à-dire si sa masse vaut k/n.

Théorème 9 **i)** *Les amas de taille k apparaissent à un temps de l'ordre de $n^{-1/2(k-1)}$, dans le sens où, la probabilité qu'il existe un amas de taille au moins k au temps t_n tend vers 0 si $n\, t_n^{2(k-1)} \to 0$ et elle tend vers 1 si $n\, t_n^{2(k-1)} \to \infty$.*
ii) *En tout temps $0 < t < 1$, il existe des constantes $0 < C_t \le C'_t < \infty$ telles que la taille du plus gros amas soit p.s. comprise entre $C_t \log n$ et $C'_t \log n$, pour n suffisamment grand.*
iii) *La taille d'un amas caractéristique à un temps $t < 1$ est p.s. finie, dans le sens où, si l'on numérote les particules initiales selon leur position, l'amas contenant la particule $n/2$ est p.s. de taille finie en tout temps $t < 1$.*

Nous regardons la particule $n/2$ dans le **iii)** car, d'un point vue heuristique, c'est celle qui a le plus de chance d'être dans un gros amas, vu sa position centrale. Il est remarquable que le cas étudié ici génère beaucoup moins de collisions avant le temps 1, que le cas étudié par Martin et al. [9, 38]. C'est une conséquence de l'absence d'énergie cinétique initiale du système, c'est-à-dire de sa basse température. L'étude repose sur l'analyse de différents centres de gravité faisant intervenir quelques propriétés asymptotiques des fonctions de répartition de n variables uniformes i.i.d. ainsi que la représentation des positions initiales des particules à l'aide de variables exponentielles.

5 Appendice : De l'équation de Burgers avec condition initiale un bruit blanc. Synthèse des résultats connus.

Parmi les études concernant l'équation de Burgers avec condition initiale aléatoire, le cas du bruit blanc est sans doute celui qui a suscité la plus grande activité de recherche. Burgers lui-même s'y est intéressé [16], citons aussi les travaux de Kida [34], She et al. [46], Avellaneda et E [2, 3] et Ryan [43]. En fait la description statistique du système à un temps fixe t a été obtenue indirectement par Groeneboom dès le milieu des années 80, au cours d'une étude du mouvement brownien avec drift parabolique [28] reliée à un problème de statistique concernant les estimateurs isotoniques. Ces résultats ont récemment été retrouvés et complétés par Frachebourg et Martin [23]. Nous nous proposons dans cet appendice de les synthésiser pour la commodité du lecteur et d'expliquer rapidement l'apparition de la fonction d'Airy.

Nous considérons par la suite le cas où $u(\cdot, 0)$ est un bruit blanc, c'est-à-dire le cas où le potentiel initial, noté ici $(W(x)\,;\ x \in \mathbb{R})$, est donné par un mouvement brownien à deux branches. À un temps fixe $t > 0$, l'état du système est caractérisé d'un point de vue qualitatif par sa structure de chocs p.s. discrète, voir [2, 28]. Cela signifie que la fonction inverse du Lagrangien $x \to a(x,t)$ est p.s. une fonction (croissante) en escalier, où de manière équivalente que $x \to u(x,t)$ est p.s. une fonction en dents de scie. L'interprétation en terme d'agrégation balistique est que toutes les particules se sont p.s. agglomérées au temps t en amas macroscopiques dont les positions forment une suite discrète de $\mathbb{R}$. Pour décrire l'état statistique à un temps $t > 0$ donné, il nous suffit de décrire l'état statistique au temps $t = 1$, car la propriété de scaling du mouvement brownien se transmet à la turbulence, induisant l'égalité en loi

$$(u(x,t)\,;\ x \in \mathbb{R}) \overset{\text{loi}}{\sim} (t^{-1/3} u(xt^{-2/3}, 1)\,;\ x \in \mathbb{R})\,.$$

Nous nous focalisons désormais sur le temps $t = 1$ et nous écrirons $a(x)$ pour $a(x,1)$. Il est remarquable que l'état au temps 1 est entièrement décrit par la suite $((X_n, a(X_n))\,;\ n \in \mathbb{Z})$, où $(X_n\,;\ n \in \mathbb{Z})$ est la suite croissante des abscisses des sauts de $x \to a(x)$, avec la convention que X_1 représente le premier saut à droite de 0. En effet, la fonction $u(x,1)$ est totalement déterminée par $a(x)$ et, du point de vue de l'agrégation balistique, la position, la masse, et la vitesse du n-ième amas à droite de 0 sont données par

$$X_n\,,\quad M_n = a(X_n) - a(X_{n-1}),\ \text{ et } V_n = \frac{2X_n - a(X_n) - a(X_{n-1})}{2t}\,.$$

Nous allons voir que la suite $((X_n, a(X_n))\,;\ n \in \mathbb{Z})$ est une chaîne de Markov de transitions connues. Introduisons en reprenant les notations de Groeneboom les fonctions $g : \mathbb{R} \to \mathbb{R}^+$ et $p : \mathbb{R}^+ \to \mathbb{R}^+$ définies par les transformées de Fourier et Laplace

$$\hat{g}(\xi) \;=\; \int_{-\infty}^{\infty} e^{\mathrm{i}\xi x} g(x)dx \;=\; \frac{2^{1/3}}{\mathtt{Ai}(\mathrm{i}2^{-1/3}\xi)}\,, \qquad \xi \in \mathbb{R}$$

$$\text{et } \int_0^{\infty} e^{-\lambda x}\left(p(x) - (2\pi x^3)^{-1/2}\right) dx \;=\; 2^{2/3}\frac{\mathtt{Ai}'}{\mathtt{Ai}}\left(2^{-1/3}\lambda\right) + \sqrt{2\lambda}\,,$$

où $\mathtt{Ai}$ représente la fonction d'Airy (cf page 446 de [1]). Les fonctions g et p admettent une représentation sous forme de série

$$g(x) = 4^{1/3}\sum_{n=1}^{\infty}\frac{\exp\left(-2^{1/3}\omega_n|x|\right)}{\mathtt{Ai}(-\omega_n)}\,, \quad \text{pour } x < 0$$

$$\text{et } p(x) = \;2\sum_{k\geq 1}\exp\{-2^{1/3}\omega_k x\}\,, \quad \text{pour } x > 0,$$

où $0 > -\omega_1 > -\omega_2 > \ldots$ sont les zéros de la fonction d'Airy, rangés par ordre décroissant.

Résultat 1 (Groeneboom)
Le processus $((x, a(x))\,;\ x \in \mathbb{R})$ est un processus de Markov déterminé par la loi de $a(0)$

$$\mathbb{P}(a(0) \in da) = \frac{1}{2}\, g\left(-2^{-2/3}a\right) g\left(2^{-2/3}a\right)\, da\,, \qquad \textit{pour } a \in \mathbb{R}\,,$$

et les transitions de la chaîne de Markov $((X_n, a(X_n))\,;\ n \in \mathbb{Z})$

$$\mathbb{P}(X_n \in dx_n, a(X_n) - a(X_{n-1}) \in dm_n \,|\, X_{n-1} = x_{n-1}, a(X_{n-1}) = a_{n-1}) =$$
$$\exp\left(\frac{(a_{n-1} - x_n)^3 - (a_{n-1} - x_{n-1})^3}{6}\right)\frac{m_n p\left(2^{-2/3}m_n\right) g\left(2^{-2/3}(m_n + a_{n-1} - x_n)\right)}{2\,g\left(2^{-2/3}(a_{n-1} - x_{n-1})\right)}\, dx_n\, dm_n,$$

pour $m_n > 0$, $x_n > x_{n-1}$ et $a_{n-1} \in \mathbb{R}$.

Nous pouvons préciser de plus que le processus de Markov $(u(x,1)\,;\ x \in \mathbb{R})$ est stationnaire, la loi de $u(x,1)$ étant la même que celle de $a(0)$. Voyons comment ces résultats se dérivent de ceux de Groeneboom [28]. La loi de $a(0)$ est donnée par le corollaire 3.3 de [28]. Concernant les transitions de $((X_n, a(X_n))\,;\ n \in \mathbb{Z})$, la propriété de Markov appliquée en X_{n-1} justifie l'égalité

$$\begin{aligned}\mathbb{P}(X_n \in dx_n, a(X_n) - a(X_{n-1}) \in dm_n \,|\, X_{n-1} = x_{n-1}, a(X_{n-1}) = a_{n-1})\\ = \mathbb{P}(X_1 \in d(x_n - x_{n-1}), a(X_1) - a(0) \in dm_n \,|\, a(0) = a_{n-1} - x_{n-1})\,.\end{aligned}$$

Il suffit donc d'évaluer cette dernière quantité. Le theorème 4.1 de [28] donne le générateur infinitésimal du processus de Markov $((x, a(x, 1/2)\,;\; x \in \mathbb{R})$. On en déduit facilement que le taux de saut $\nu(x, a, m)$ de $a(\cdot)$ en x conditionnellement à $a(x) = a$ vaut

$$\nu(x,a,m) : \,= \lim_{h,\eta \downarrow 0} \frac{1}{h\eta} \mathbb{P}(a(x+h) - a(x) \in [m, m+\eta] \,|\, a(x) = a) = \frac{mp\left(2^{-2/3}m\right) g\left(2^{-2/3}(m+a-x)\right)}{2\, g\left(2^{-2/3}(a-x)\right)}\,.$$

La quantité

$$\mathbb{P}(X_1 \in [x, x+h], a(X_1) - a(0) \in [m, m+\eta] \,|\, a(0) = a)$$

peut s'exprimer sous la forme du produit de

$$A = \mathbb{P}(X_1 \geq x \,|\, a(0) = a)$$
$$\text{par } B = \mathbb{P}(X_1 \in [x, x+h], a(X_1) - a(0) \in [m, m+\eta] \,|\, a(0) = a, X_1 \geq x)\,.$$

Il découle du caractère markovien de $a(\cdot)$ que la quantité $B/h\eta$, tend lorsque h, η décroissent vers 0, vers le taux de saut $\nu(x, a, m)$. Il ne reste qu'à évaluer la quantité A, ce qui s'obtient facilement à partir du corollaire 3.1 de [28]. En effet, nous savons d'après le résultat de Millar [41], que le processus $(W_{a(0)+z} - W_{a(0)}\,;\; z \geq 0)$ conditionné par $a(0) = a$, a pour loi conditionnelle la loi limite, lorsque y décroît vers 0, d'un mouvement brownien issu de y conditionné à rester au dessus de $z \to -z(z+2a)/2$. En exprimant la condition géométrique imposée à $(W_{a(0)+z} - W_{a(0)}\,;\; z \geq 0)$ par la condition $X_1 \geq x$ (voir figure ci-dessous) et en exploitant le corollaire 3.1 de [28] pour la seconde égalité, on obtient que

$$\begin{aligned}\mathbb{P}(X_1 \geq x \mid a(0)=a) &= \lim_{y\downarrow 0} \frac{\mathbb{P}^y\left(W_z \geq -\frac{1}{2}z(z+2(a-x))\,;\ \text{ pour tout } z\geq 0\right)}{\mathbb{P}^y\left(W_z \geq -\frac{1}{2}z(z+2a)\,;\ \text{ pour tout } z\geq 0\right)} \\ &= \frac{e^{(a-x)^3/6} g\left(2^{-2/3}(a-x)\right)}{e^{a^3/6} g\left(2^{-2/3}a\right)}\,,\end{aligned}$$

où W a la loi d'un brownien issu de y sous $\mathbb{P}^y$; voir la preuve du Théorème 1 partie III, pour un raisonnement très similaire. En exprimant le produit $A\times B$, on aboutit au résultat recherché

$$\begin{aligned}&\mathbb{P}(X_1 \in dx, a(X_1)-a(0)\in dm \mid a(0)=a) = \\ &\qquad \exp\left(\frac{(a-x)^3-a^3}{6}\right) \frac{mp\left(2^{-2/3}m\right) g\left(2^{-2/3}(m+a-x)\right)}{g\left(2^{-2/3}a\right)}\, dx\, dm\,.\end{aligned}$$

Ces formules ont été retrouvés récemment par Frachebourg et Martin [23]. Il faut cependant faire attention car ils ne calculent pas exactement les mêmes quantités. Les correspondances entre les fonctions I et J de Frachebourg-Martin et celles p et g de Groeneboom, sont les suivantes :

$$\begin{aligned}I(\mu,\eta) &= 2\exp\left(-\frac{1}{2}\left(\frac{\eta^2}{\mu}+\frac{\mu^3}{12}\right)\right) p\left(2^{-2/3}\mu\right) \\ \text{et }\ J(\nu) &= 2^{-5/6}\exp\left(-\frac{\nu^3}{6}\right) g\left(-2^{-2/3}\nu\right).\end{aligned}$$

Cela nous permet de formuler le résultat complémentaire.

Résultat 2 (Frachebourg et Martin)
La densité de probabilité que la particule initialement en 0 ait une masse m et une vitesse v au temps 1 vaut $m\rho(m,v)$ avec la notation (48) de Frachebourg-Martin [23], c'est-à-dire en terme des fonctions p et g

$$4m\,p\left(2^{-2/3}m\right) g\left(2^{-2/3}\left(\frac{m}{2}-v\right)\right) g\left(2^{-2/3}\left(\frac{m}{2}+v\right)\right),$$

pour $m>0$ et $v\in\mathbb{R}$.

L'étape clef des calculs de Groeneboom et Frachebourg-Martin est la détermination du semi-groupe de transition du mouvement brownien avec drift parabolique $z\to cz^2$ tué lorsqu'il atteint 0. C'est à ce stade qu'apparaît la fonction d'Airy. Groeneboom se ramène grâce à la formule de Girsanov à la détermination de la résolvante

$$R^\partial_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q^\partial_t f(x)\, dt$$

associée au semi-groupe

$$Q_t^\partial f(x) = \mathbb{E}_\partial^{(0,x)} \left(f(W_t) e^{-2c\int_0^t W} \right),$$

où la loi de W sous $\mathbb{P}_\partial^{(0,x)}$ est celle d'un mouvement brownien tué en 0. La formule de Feynman-Kac permet de relier la résolvante $R_\lambda^\partial f$ à celle $\overline{R}_\lambda^\partial f$ associée au semi-groupe $\overline{Q}_t^\partial f(x) = \mathbb{E}_\partial^{(0,x)}(f(W_t))$ du mouvement brownien tué en 0 :

$$R_\lambda^\partial f(x) = \overline{R}_\lambda^\partial f(x) - \overline{R}_\lambda^\partial \left(v.R_\lambda^\partial f \right)(x), \qquad \text{avec } v(x) = 2cx\,.$$

En écrivant l'équation de la chaleur vérifiée par $\overline{R}_\lambda^\partial f$ on obtient que $R_\lambda^\partial f$ est l'unique solution de classe $\mathcal{C}^1$ de l'équation

$$\frac{1}{2} y''(x) - (\lambda + 2cx) y(x) = f(x)\,, \quad x > 0\,, \tag{5}$$

avec les conditions aux bords $y(x) \overset{x\to 0^+}{\to} 0$ et $y(x) \overset{x\to\infty}{\to} 0$. Un système fondamental de solutions de la version homogène associée est donnée par

$$t \to \mathtt{Ai}\left((2c^2)^{-1/3}\lambda + (4c)^{1/3} t \right) \; ; \; t \to \mathtt{Bi}\left((2c^2)^{-1/3}\lambda + (4c)^{1/3} t \right),$$

où $\mathtt{Ai}$ et $\mathtt{Bi}$ sont les fonctions d'Airy définies page 446 de [1]. La méthode de la variation de la constante permet alors d'obtenir $R_\lambda^\partial f$. Frachebourg et Martin calculent quant à eux directement le semi-groupe de transition, en résolvant l'équation de diffusion avec drift (formule (24) de [23]) dont il est solution. Cette résolution est obtenue en se ramenant à l'opérateur $(-\frac{1}{2}(\partial^2/\partial x^2) - x)$ à l'aide d'une renormalisation par un terme exponentiel (formule (25) à (29) de [23]).

Références

[1] M. Abramowitz, I.A. Stegun, *Handbook of mathematical functions.* Nat. Bur. Stand., Washington, 1964.

[2] M. Avellaneda, *Statistical Properties of Shocks in Burgers Turbulence II : Tail Probabilities for Velocities, Shock-strenghs and Rarefaction Intervals.* Comm. Math. Phys **169** (1995), pp 45-59.

[3] M. Avellaneda, Weinan E, *Statistical Properties of Shocks in Burgers Turbulence.* Comm. Math. Phys. **172** (1995), pp 13-38.

[4] J. Bertoin, *Large Deviation Estimates in Burgers Turbulence with Stable Noise Initial Data.* J. Stat. Phys. **91** no 3/4 (1998), pp 655-667.

[5] J. Bertoin, *The Inviscid Burgers Equation with Brownian Initial Velocity.* Comm. Math. Phys. **193** no 2 (1998), pp 397-406.

[6] J. Bertoin, *Structure of Shocks in Burgers Turbulence with Stable Noise Initial Data.* Comm. Math. Phys. **203** (1999), pp 729-741.

[7] J. Bertoin, *Some Properties of Burgers Turbulence with White or Stable Noise Initial Data.* In Lévy Processes : Theory and Applications. Eds Barndorff-Nielsen, Mikosh et Resnick. Birkhäuser (2001).

[8] J. Bertoin, *Clustering Statistics for Sticky Particles with Brownian Initial Velocity.* J. Math. Pures Appl. **79** no 2 (2000), pp 173-194.

[9] J.C. Bonvin, Ph.A. Martin, J. Piasecki, X. Zotos, *Statistics of Mass Aggregation in a Self-Gravitating One-Dimensional Gas.* J. Stat. Phys. **91** no 1/2 (1998), pp 177-197.

[10] F. Bouchut, *Advances in Kinetic Theory and Computing.* Ser. Adv. Math. Appl. Sci. **22**, World Scientific, River Edge, NJ, 1994

[11] F. Bouchut, F. James, *Equations de transport unidimensionnelles à coefficients discontinus.* C. R. Acad. Sci. Paris Sér. I Math., **320** (1995), pp 1097-1102.

[12] Y. Brenier, E. Grenier, *On the Model of Pressureless Gases with Sticky Particles.* CAM Report, Univ. of California, Los Angeles, 1994.

[13] Y. Brenier, E. Grenier, *Sticky Particles and Scalar Conservation Laws.* SIAM J. Numer. Anal. **35** no 6 (1998), pp 2317-2328.

[14] J.M. Burgers, Adv. Appl. Mech. **1** (1948), p 171.

[15] J.M. Burgers, Proc. Acad. Sci. Amst. **53** (1950), pp 247, 393, 718, 732.

[16] J.M. Burgers, *The Nonlinear Diffusion Equation.* Dordrecht, Reidel 1974.

[17] J.D. Cole, *On a Quasi Linear Parabolic Equation Occuring in Aerodynamics.* Quart. Appl. Math. **9** (1951), pp 225-236.

[18] G.F. Carnevale, Y. Pomeau, W.R. Young, *Statistics of Ballistic Agglomeration.* Phys. Rev. Lett. **64** no. 24 (1990), p 2913.

[19] J.D. Deuschel, D.W. Stroock, *Large Deviations.* Pure Appl. Math. **137**, Academic Press INC., 1989.

[20] Weinan E, Ya.G. Rykov, Ya.G. Sinai, *Generalized Variational Principles, Global Weak Soutions and Behavior with Random Initial Data for Systems of Conservation Laws Arising in Adhesion Particle Dynamics.* Comm. Math. Phys. **177** (1996), pp 349-380.

[21] L. Frachebourg, *Exact Solution of the One-Dimensional Ballistic Aggregation.* Phys. Rev. Lett. **82** no. 7 (1999), pp 1502-1505.

[22] L. Frachebourg, V. Jacquemet, Ph.A. Martin, *Inhomogeneous Ballistic Aggregation.* Preprint (2001).

[23] L. Frachebourg, Ph.A. Martin, *Exact Statistical Properties of the Burgers Equation.* J. Fluid. Mech. **417** (2000), pp 323-349.

[24] L. Frachebourg, Ph.A. Martin, J. Piasecki, *Ballistic Aggregation : a Solvable Model of Irreversible many Particles Dynamics.* Physica A **279** (2000), pp 69-99.

[25] B. E. Fristedt *Uniform local behavior of stable subordinators.* Ann. Proba. **7** (1979), pp 1003-1013.

[26] R. K. Getoor *Splitting times and shift functionals.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete **47** (1979), pp 69-81.

[27] E. Grenier, *Existence globale pour le système des gaz sans pression*, C. R. Acad. Sci. Paris Sér. I Math., **321** (1995), pp 171-174.

[28] P. Groeneboom, *Brownian Motion with a Parabolic Drift and Airy Functions.* Proba. Th. Rel. Fields **81** (1989), pp 79-109.

[29] H. Handa *A remark on shocks in inviscid Burgers turbulence.* Non linear waves and weak turbulence (1992) pp 339-345 Fitzmaurice et al., Eds, Birkhäuser-Boston.

[30] J. Hawkes *A lower Lipschitz condition for the stable subordinators.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete **17** (1971), pp 23-32.

[31] E. Hopf, *The Partial Differential Equation* $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. **3** (1950), pp 201-230.

[32] A. Janicki and W. A. Woyczynski, *Hausdorff dimension of regular points in stochastic flows with Lévy α-stable initial data.* J. Stat. Phys. **86** (1997), pp 277-299.

[33] N. Leonenko, *Limit Theorems for Random Fields with Singular Spectrum.* Kluwer, 1999.

[34] S. Kida, *Asymptotic Properties of Burgers Turbulence.* J. Fluid. Mech. **93** (1979), pp 337-377.

[35] R.H. Kraichnan, *Lagrangian History Statistical Theory for Burgers' Equation.* Phys. Fluids **11** (1968), pp 265-277.

[36] P.D. Lax, *Hyperbolic System of Conservation laws and the Mathematical Theory of Shock Waves.* Society for Industrial and Applied Mathematics, Philadelphia, 1973.

[37] Ph.A. Martin, J. Piasecki, *One Dimensional Ballistic Aggregation : Rigorous Long-time Estimates.* J. Stat. Phys. **76** (1994), pp 447-476.

[38] Ph.A. Martin, J. Piasecki, *Aggregation Dynamics in a Self-Gravitating One-Dimensional Gas.* J. Stat. Phys. **84** no 3/4 (1996), pp 837-857.

[39] D.M. Mason, *A Strong Limit Theorem for the Oscillation Modulus of the Empirical Quantile Process.* Stoch. Proc. and Appl. **17** (1984) pp 127-136.

[40] D.M. Mason, G.R. Shorack, J. A. Wellner, *Strong Limit Theorems for Oscillation Moduli of the Uniform Empirical Process.* Z. Wahrschein. verw. Gebiete **65** (1983) pp 127-136.

[41] P.W. Millar, *A Path Decompostion for Markov processes.* Ann. of Proba. **6** (1978), pp 345-348.

[42] J.S. Rao, M. Sobel, *Incomplete Dirichlet Integrals with Applications to Ordered Uniform Spacing.* J. Multivar. Anal **10** (1980) pp 603-610.

[15] D. Revuz, M. Yor *Continuous martingales and Brownian motion.* Grundlehren Math. Wiss. **293** Springer-Verlag (1991).

[43] R. Ryan, *Large-deviation Analysis of Burgers Turbulence with White-noise Initial Data.* Comm. Pure Appl. Math. **51** (1998), pp 47-75.

[44] R. Ryan, *The Statistics of Burgers Turbulence Initialized with White-noise Initial Data.* Comm. Math. Phys. **191** (1998), pp 71-86.

[45] S.F. Shandarin, Ya.B. Zeldovich, *The Large-scale Structures of the Universe : Turbulence, Intermittency, Structures in a Self-gravitating Medium.* Rev. Mod. Phys. **61** (1989), pp 185-220.

[46] Z.S. She, E. Aurell, U. Frisch, *The Inviscid Burgers Equation with Data of Brownian Type.* Comm. Math. Phys. **148** (1992), pp 632-641.

[47] G.R. Shorack, J.A. Wellner, *Empirical Processes with Applications to Statistics.* Wiley 1986.

[48] Ya.G. Sinai, *Statistics of Shocks in Solution of Inviscid Burgers Equation.* Comm. Math. Phys. **148** (1992), pp 601-621.

[49] T. Suidan, *Adhesion Dynamics on the Line : Mass Aggregation Process* J. Stat. Phys. **101** no 3/4 (2000).

[50] T. Suidan, *Statistical Properties of Convex Minorants of Random Walks and Brownian Motions.*, preprint (2000), Princeton University. www.arXiv:math.PR

[51] T. Tatsumi, S. Kida, *The Burgers Model of Turbulence.* J. Fluid. Mech. **55** (1972), pp 659-675.

[52] R. Tribe, O. Zaboronski, *On the Large Time Asymptotics of Decaying Burgers Turbulence.* Comm. Math. Phys. **212** (2000), pp 415-436.

[53] M. Vergassola, B. Dubrulle, U. Frisch, A. Noullez, *Burgers' Equation, Devil's Staircases and the Mass Distribution Function for Large-scale Structures.* Astron. Astrophys. **289** (1994), pp 325-356.

[54] W.A. Woyczyǹski, *Göttingen Lectures on Burgers-KPZ turbulence.* Lecture Notes in Math. **1700**, Springer 1998.

[55] Ya.B. Zeldovich, *Gravitational Instability : an Approximate Theory for Large Density Perturbations.* Astron. Astrophys. **5** (1970), pp 84-89.

Première partie

On Regular Points in Burgers Turbulence with Stable Noise Initial Data

Abstract : We study the set of regular points (i.e. the points which have not been involved into shocks up to time t) for the inviscid Burgers equation in dimension 1 when initial velocity is a stable Lévy noise. We prove first that when the noise is not completely asymmetric and has index $\alpha \in (1/2, 1)$, the set of regular points is discrete a.s. and regenerative. Then, we show that in the case of the Cauchy noise, the set of regular points is uncountable, with Minkowsky dimension 0.

Résumé : Nous étudions l'ensemble des points réguliers (c'est-à-dire des points qui jusqu'au temps t n'ont pas été impliqués dans des chocs) relatifs à l'équation unidimensionnelle de Burgers lorsque la vitesse initiale est un bruit stable de Lévy. Nous prouvons en premier lieu que, pour un bruit non complètement asymétrique d'indice $\alpha \in (1/2, 1)$, l'ensemble des points réguliers est discret p.s. et régénératif. Nous établissons par la suite que, dans le cas du bruit de Cauchy, l'ensemble des points réguliers est non dénombrable et de dimension de Minkowsky nulle.

Key words : Burgers turbulence, stable Lévy noise, regular points.

A.M.S. Classification. 35Q53, 60H15, 60J65.

1 Introduction

Burgers equation

$$\partial_t u + u\ \partial_x u = \mu \partial^2_{xx} u$$

was first introduced as a simplification of the Navier Stokes equation, by neglecting the terms of pressure and force (see [7]). This is the simplest PDE that captures the interaction of nonlinear wave propagation and viscosity. Burgers equation also appears in the study of growth interface such as ballistic aggregation and is proposed as a model to describe the formation of the superstructures of the universe, see [21] and references therein.

There is a particular attention in the literature about the behavior of the Burgers turbulence when the viscosity $\mu \to 0$. It is known that the solution u_μ converges to $u_0 = u$, where u is the unique weak solution to the so-called inviscid Burgers equation

$$\partial_t u + u\ \partial_x u = 0 \tag{6}$$

satisfying the entropy condition. A physical interpretation of this solution is given by the model of sticky particles. At the initial time, infinitesimal particles are uniformly spread on the line, with initial velocity $u(.,0)$. They evolve with the dynamic of completely inelastic shocks. This means that the velocity of a particle only changes in case of collision, and when two (clusters of) particles collide, they stick and form a heavier cluster with conservation of masses and momenta.

There is an increasing interest in the inviscid Burgers equation (6) when the initial data $u(.,0)$ is a random process (see e.g. [7], [17] and [21]). Sinai [20] and Bertoin [4] have given a complete statistical description of the solution of (6) when $u(.,0)$ is a Brownian motion or a Lévy process with no positive jumps. Avellaneda and E [1][2], Ryan [18] and [9] have focused on the case of white noise initial velocity (i.e. when the initial velocity is the weak derivative of a Brownian motion). In fact, one could get a complete description of the system at a given time t from the work of Groeneboom [12]. Bertoin [6] specified the shock structure of the solution of (6), when the initial velocity is a stable noise, i.e. when $u(.,0)$ is the weak derivative of a stable Lévy process of index $\alpha \in (1/2, 2]$. Moreover, he has given large deviations estimates for the velocity (see [5]).

Regular points may be viewed as particles that have not been perturbed by the turbulence up to time t. She, Aurell and Frisch [19] as well as Janicki and Woyczynski [16] have investigated by numerical simulations the dimension of the set of regular points when the initial velocity is a fractional Brownian motion or a stable Lévy process. They have then conjectured that this dimension is related to the scaling exponent of these processes. Sinai [20] and Bertoin [4] have partially proved these results. In the same direction, Handa [13] has shown a link between the regularity of the initial data and the dimension of the set of regular points. In this paper we will turn our attention to the properties of the set of regular points (at a fixed time t) when the initial velocity $u(.,0)$ is a stable Lévy noise.

There are only two cases where the set $\mathcal{R}_t$ of regular points is not empty a.s. (see Bertoin [6]). The first is when the initial velocity is a non-completely asymmetric stable noise with index $\alpha \in (1/2, 1)$. We will prove here that the set $\mathcal{R}_t$ of regular points is then discrete a.s. and follows a regenerative property. Moreover, between two consecutive regular points, clusters form an increasing sequence indexed by $\mathbb{Z}$. The second case where the set $\mathcal{R}_t$ is not empty is when the initial velocity is a Cauchy noise. We will check that $\mathcal{R}_t$ is then a.s. uncountable but has Minkowsky dimension 0.

In the second section, we recall some basic features and definitions on the inviscid Burgers equation and stable noises. The third section will be devoted to the case $\alpha \in (1/2, 1)$, the next one to the Cauchy case, and we give in the last one numerical illustrations of both cases.

2 Preliminaries

2.1 Hopf-Cole solution of the inviscid Burgers equation.

Assume that the initial potential ψ_0, defined by $\psi_0(x) - \psi_0(y) = \int_x^y u(z,0)dz$ (the integral is purely formal), has discontinuities only of the first kind and satisfies $\psi_0(z) = o(z^2)$, when $|z| \to \infty$. Then, when $\mu \to 0$, the unique solution of Burgers equation with viscosity $\mu > 0$ converges (excepted on a countable set) to a weak solution of the inviscid Burgers equation, referred to as the Hopf-Cole solution, see [15], [8]. The right-continuous version of this solution is

$$u(x,t) = \frac{x - a(x,t)}{t}$$

where

$$a(x,t) = \min \ \underset{y}{\operatorname{argmin}} \left(\psi_0(z) - \frac{1}{2t}(z-x)^2 \right)$$

is the (largest) location of the supremum of $z \to \psi_0(z) - \frac{1}{2t}(z-x)^2$. One has the following geometrical interpretation : consider a realization of the initial potential ψ_0 and a parabola $z \mapsto (z-x)^2/2t + C$, where C is chosen such that the parabola is strictly above the path of ψ_0. Let C decrease until this parabola touches the graph of ψ_0. Then $a(x,t)$ is the largest abscissa of the contact points. One notices in particular that the solution is expressed in term of the initial potential, rather than the initial velocity.

The function $x \mapsto a(x,t)$ is non-decreasing and right continuous and its right continuous inverse $a \mapsto x(a,t)$ is known as the *Lagrangian function*, i.e. as the function that gives the position at time t of the particle initially located in a.

A discontinuity of $x \to u(x,t)$ is called a shock and occurs when $x \to a(x,t)$ jumps, i.e. when $a(x,t) \neq a(x-,t)$, where $a(x-,t)$ refers to the left limit of $a(.,t)$ in x. It happens when the function $\psi_0(z)-(z-x)^2/2t$ reaches its overall minimum at more than one location and then $a(x-,t)$ is the smallest of such locations. From the sticky particles point of view, the location of a shock corresponds to the location of a cluster at time t. This cluster results from the aggregation of the particles initially located in $[a(x-,t), a(x+,t)]$; its velocity is (according to the conservation of masses and momenta)

$$v(x) = -\frac{\psi_0(a(x)) - \psi_0(a(x-))}{a(x) - a(x-)} = \frac{u(a(x)) + u(a(x-))}{2}. \qquad (7)$$

The interval $[a(x-,t), a(x+,t)]$ is called a *shock interval* and x a *Eulerian shock point*.

Finally, a *Lagrangian regular point* is a right and left accumulation point of the closed range of $a(.,t)$. It represents the initial location of a particle that has not been involved into shocks up to time t. In the sequel, $\mathcal{R}_t$ will denote the set of Lagrangian regular points at time t.

2.2 Stable Lévy noises.

Stable Lévy noises are the weak derivatives of stable Lévy processes. A stable Lévy process of index $\alpha \in (0,2]$ is a right continuous stochastic process with independent and stationary increments that fulfills the scaling property

$$\psi_0(\lambda x) \sim \lambda^{1/\alpha}\psi_0(x) \qquad \forall \lambda > 0,$$

where $\sim$ means 'identity in law'. See Chapter VIII in [3] for much more on the topic. The requirement $\psi_0(z) = o(z^2)$ when $|z| \to \infty$ imposes $\alpha > 1/2$, so we will restrict our attention to this case.

First, let us recall some basic features on both kind of processes to which we will turn our interest. A non-completely asymmetric stable Lévy process with index $\alpha \in (1/2, 1)$ is a pure jump process with both positive and negative jumps. It has bounded variation a.s. A Cauchy process is the sum of a symmetric Cauchy process and a deterministic drift. Its index is $\alpha = 1$. It has unbounded variation a.s.

One deduces (see [19]) from the stationarity and the scaling property of ψ_0, that $u(., t)$ is a stationary process fulfilling the scaling property

$$u(., t) \sim t^{(1-\alpha)/(2\alpha-1)} u(., 1).$$

In particular, we can focus on time $t = 1$ and we will write $a(x) = a(x, 1)$, $u(x) = u(x, 1)$ and $\mathcal{R} = \mathcal{R}_1$ in the sequel.

3 The case $\alpha \in (1/2, 1)$.

3.1 Statement of the main results.

When $u(., 0)$ is a not completely asymmetric stable noise with index $\alpha \in (1/2, 1)$, Bertoin [6] has proved that with probability one, there exists regular points. These are exceptional, in the sense that the set $\mathcal{R}$ of Lagrangian regular points has a.s. Lebesgue measure 0.

Here, we shall first prove that regular points form a discrete set.

Theorem 1 *Suppose that the initial potential ψ_0 is a stable Lévy process with index $\alpha \in (1/2, 1)$ which is not completely asymmetric. Then a.s. the set $\mathcal{R}$ of Lagrangian regular points is discrete.*

Let us denote thenceforth $\mathcal{R} = \{r_n;\ n \in \mathbb{Z}\}$ where r_0 is the first Lagrangian regular point at the right of 0 and $r_i < r_{i+1}$ for any $i \in \mathbb{Z}$. The next proposition claims that between two consecutive regular points, clusters may also be indexed by $\mathbb{Z}$.

Proposition 1 *Suppose that the initial potential ψ_0 is a stable Lévy process with index $\alpha \in (1/2, 1)$ which is not completely asymmetric. Then, with probability one, there exists a unique sequence $(e_i;\ i \in \mathbb{Z})$ of (random) increasing maps*

$$e_i \ :\ \mathbb{Z} \to (r_i, r_{i+1})$$

such that

i) *the range of the sequence $(e_i(n);\ n \in \mathbb{Z})$ is exactly the set of Eulerian shock points in (r_i, r_{i+1})*

ii) $\begin{cases} u(e_i(n)) > 0 \text{ for any } n < 0 \\ u(e_i(n)) < 0 \text{ for any } n \geq 0. \end{cases}$

Moreover, $r_i = \lim_{n\downarrow -\infty} e_i(n)$ and $r_{i+1} = \lim_{n\uparrow\infty} e_i(n)$ a.s.

The following corollary ensures that the velocity of the clusters (given by (7) section 2-1) is positive (negative) at the right (left) of a regular point. Moreover, regular points are exactly the points where $u = 0$.

Corollary 1.1 *With probability one,*

$$\begin{cases} v(e_i(n)) > 0 \text{ for any } n < 0 \\ v(e_i(n)) < 0 \text{ for any } n > 0 \end{cases}$$

and $\mathcal{R} = \{r;\ u(r) = 0\}$.

In the next theorem we claim that $\mathcal{R}$ is a regenerative set and that the turbulence has evolved (up to time 1) independently at the right and the left of a regular point. Physically, this property is quite intuitive. Regular points are particles with initial velocity zero, which have not been perturbed up to time 1. So, there has been no interaction between the particles at the left of a regular point and those at its right. Mathematically, this property appears naturally as well. It is easily seen, mainly using splitting times (see [11] for a short introduction to splitting times) and the fact that ψ_0 has stationary increments, that u is a (homogeneous) simple Markov process. If we knew that u is a *strong* Markov process, the equality $\mathcal{R} = \{r \in \mathbb{R}\ ;\ u(r) = 0\}$ would ensure the regenerative property of $\mathcal{R}$. Yet, we cannot conclude directly that $\mathcal{R}$ is regenerative, because we do not know whether u is *strongly* Markovian or not.

By stationarity, we just state the regenerative property for r_0 which is the first regular point at the right of 0.

Theorem 2 *Suppose that the initial potential ψ_0 is a stable Lévy process with index $\alpha \in (1/2, 1)$, that is not completely asymmetric. Then, the processes $(\psi_0(r_0+x) - \psi_0(r_0);\ x \geq 0)$ and $(\psi_0(r_0 - x);\ x \geq 0)$ are independent.*

As a consequence, the processes $(u(r_0+x);\ x \geq 0)$ and $(u(r_0-x);\ x \geq 0)$ are independent and $\mathcal{R}$ is a regenerative set.

Let us now prove these results.

3.2 Regular points form a discrete set.

The proof of Theorem 1 is closely connected to the proof of Theorem 4 in [6]. In particular, we may refer to [6] for some technical arguments.

Thanks to the stationarity of the initial potential, we just have to prove that

$$\text{card}\{\mathcal{R} \cap [0, 1]\} < \infty \quad \text{a.s.}$$

Fix $\varepsilon > 0$. If we prove that there exists a constant $K > 0$ such that

$$\mathbb{P}(\mathcal{R} \cap [0, \varepsilon] \neq \emptyset) \leq K\varepsilon, \tag{8}$$

then by stationarity, $\mathbb{P}(\mathcal{R} \cap [a, a+\varepsilon] \neq \emptyset) \leq K\varepsilon$ for any $a \in \mathbb{R}$ and thanks to Fubini's Theorem

$$\mathbb{E}(\text{card}\{k = 0, .., [1/\varepsilon]; \mathcal{R} \cap [k\varepsilon, (k+1)\varepsilon] \neq \emptyset\}) \leq 2K,$$

which implies in particular that $\text{card}\{\mathcal{R} \cap [0,1]\} < \infty$ a.s.

Bertoin ([6], Lemma 5) has proved that, if $r \in \mathcal{R}$, then a.s. ψ_0 is continuous at r and $r = a(r)$ (which means that r had an initial velocity zero and has not been involved in the turbulence, so it stayed at the same location). In particular, we deduce from the definition of a that a.s.

$$\psi_0(r+h) - \frac{1}{2}h^2 \leq \psi_0(r) \quad \forall h \in \mathbb{R},$$

and hence that

$$\left\{\mathcal{R} \cap [0, \varepsilon] \neq \emptyset\right\} \subset \left\{\exists r \in [0, \varepsilon]/\forall h \in \mathbb{R} \; ; \; \psi_0(r \pm h) - \psi_0(r) \leq \frac{1}{2}h^2\right\}.$$

Evaluating the probability of the right-hand event involves the maximum of the inhomogeneous Markov process $\{\psi_0(r+h) - h^2/2, h \in \mathbb{R}\}$, which is not convenient. We will thus replace this inhomogeneous Markov process by an homogeneous one in replacing the quadratic term by a β-stable subordinator σ (remember it is a β-stable Lévy process with no negative jumps) fulfilling $\sigma(h) \geq h^2/2$ with a probability at least $1 - \delta$. We first state and prove a lemma that ensures the existence of such subordinators. We then see how to use them.

Lemma 1 *For any $\frac{1}{2} < \beta < 1$, consider a β-stable subordinator σ with Laplace transform*

$$\mathbb{E}(\exp(-\lambda\sigma_t)) = \exp(-tc\lambda^\beta).$$

Then, for any fixed $\delta > 0$, the parameter c can be chosen such that the event

$$\{\sigma(t+h) - \sigma(t) \geq h^2, \; \forall t \in [0,1], \quad h \in (0,2]\}$$

has a probability larger than $1 - \delta$.

Proof of the Lemma.
This proof is adapted from Theorem 1 in [14]. Denote, for any integer k, j, q such that $0 \leq j \leq 2^{q+1}$ and $1 \leq k \leq q$,

$$A_{jkq} = \left\{\sigma((j+k)/2^q) - \sigma(j/2^q) \leq \left(\frac{k+2}{2^q}\right)^2\right\}.$$

and

$$A = \bigcup_{q=1..\infty; j=0..2^{q+1}; k=1..q} A_{jkq}.$$

We want to chose the parameter c such that the event A has a probability less than δ. Thanks to the scaling property of σ and the stationarity of its increments we have

$$\mathbb{P}(A_{jkq}) = \mathbb{P}\left(\sigma(1) \le \left(\frac{2^q}{k}\right)^{1/\beta}\left(\frac{k+2}{2^q}\right)^2\right) \le \mathbb{P}\left(\sigma(1) \le 9k^{2-1/\beta}2^{-q(2-1/\beta)}\right).$$

Using the Markov inequality

$$\mathbb{P}(\sigma(1) \le x) \le e^{x.1/x}\mathbb{E}(e^{-\sigma(1)/x}) \le e.e^{-cx^{-\beta}},$$

we get setting $\rho = 2\beta - 1 \in (0,1)$

$$\begin{aligned}\mathbb{P}(A) &\le \sum_{q=1}^{\infty}\sum_{j=0}^{2^{q+1}}\sum_{k=1}^{q}\mathbb{P}(A_{jkq}) \\ &\le \sum_{q=1}^{\infty}\sum_{j=0}^{2^{q+1}}\sum_{k=1}^{q} e.\exp(-9^{-\beta}c2^{q\rho}k^{-\rho}) \\ &\le e\sum_{q=1}^{\infty} q2^{q+1}\exp(-9^{-\beta}c(2^q/q)^{\rho})\end{aligned}$$

So, if we choose c large enough, we obtain $\mathbb{P}(A) \le \delta$. We will check now that we have

$$\sigma(t+h) - \sigma(t) \ge h^2 \qquad \forall t \in [0,1], \quad h \in (0,2]$$

on the event A^c. Given $0 < h \le 2$, $t \in [0,1]$, let q, j, k be such that

$$\frac{1}{2^{q-1}} < h \le \frac{q}{2^q} \quad \text{and} \quad \frac{j-1}{2^q} < t \le \frac{j}{2^q} < \frac{j+k}{2^q} \le t+h < \frac{j+k+1}{2^q}.$$

We have on A^c

$$\sigma(t+h) - \sigma(t) \ge \sigma((j+k)/2^q) - \sigma(j/2^q) > \left(\frac{k+2}{2^q}\right)^2 > h^2,$$

which concludes the proof of the Lemma.
□

We are now ready to establish Theorem 1.

Proof of Theorem 1.

Fix $0 < \delta < 1$. Thanks to Lemma 1, we can choose two β-stable subordinators σ^+ and σ^- with index $\frac{1}{2} < \beta < \alpha$, which satisfy the following conditions: σ^+, σ^- and ψ_0 are independent and the event

$$A_\delta = \{\sigma^{+/-}(t+h) - \sigma^{+/-}(t) \ge h^2, \ \forall t \in [0,1], \quad h \in [0,2]\}$$

has probability larger than $1 - \delta$.

Recall that we want to estimate the probability of the event

$$B_\varepsilon = \{\exists r \in [0,\varepsilon]/\forall h \in \mathbb{R}\ ;\ \psi_0(r \pm h) - \psi_0(r) \le h^2/2\}.$$

We deduce from the independence of σ^+, σ^- and ψ_0 that

$$\mathbb{P}(B_\varepsilon) = \mathbb{P}(B_\varepsilon \cap A_\delta)/\mathbb{P}(A_\delta).$$

In particular, we just need to prove that

$$\mathbb{P}(B_\varepsilon \cap A_\delta) = O(\varepsilon). \tag{9}$$

Let us consider $\omega \in B_\varepsilon \cap A_\delta$ and a regular point $r = r(\omega) \in [0,\varepsilon]$. Recall that the initial potential ψ_0 then fulfills

$$\psi_0(r \pm h) - \psi_0(r) \le \frac{1}{2}h^2 \qquad \forall h \in \mathbb{R},$$

and as a consequence

$$\psi_0(x) - \frac{1}{2}(r-x)^2 \le \psi_0(r) \qquad \text{for } \ \varepsilon - 1 \le x \le r.$$

So, applying the condition on the growth of σ^- with $t = 1-\varepsilon+r$ and $h = r-x$, we get

$$\psi_0(x) + \sigma^-(1-\varepsilon+x) - \sigma^-(1-\varepsilon+r) \le \psi_0(r) \qquad \text{for } \ \varepsilon - 1 \le x \le r.$$

In particular, the Lévy process

$$Y^-(x) = \psi_0(x) + \sigma^-(1-\varepsilon+x) \qquad \text{for } x \ \ge \varepsilon - 1$$

reaches a new maximum at $r \in [0,\varepsilon]$. Let τ denote the first time after 0 where Y^- reaches a new maximum and write $\eta = r - \tau \in [0,\varepsilon]$. The inequality

$$\psi_0(r+h) - \psi_0(r) \le \frac{1}{2}h^2 \qquad \forall h > 0$$

implies

$$\psi_0(\tau+\eta+h) - \psi_0(\tau+\eta) \le \frac{1}{2}h^2 \le \sigma^+(\eta+h) - \sigma^+(\eta) \qquad \forall h \in [0, 2-\eta].$$

So the process

$$Y^+(x) = \psi_0(\tau+x) - \psi_0(\tau) - \sigma^+(x) \qquad \text{for } x \ge 0$$

does not reach a new maximum in $(\varepsilon, 1+\varepsilon)$. Moreover, it is a Lévy process independent of Y^- since τ is a stopping time. Putting the pieces together, one obtains that

$$\mathbb{P}(B_\varepsilon \cap A_\delta) \le \mathbb{P}_1(\varepsilon) \times \mathbb{P}_2(\varepsilon) \tag{10}$$

where

$$\begin{cases} \mathbb{P}_1 = \mathbb{P}(Y^- \text{ reaches a new max in } [0,\varepsilon]) \\ \mathbb{P}_2 = \mathbb{P}(Y^+ \text{ does not reach a new max in } [\varepsilon, 1+\varepsilon]). \end{cases}$$

Now, the proof follows the same lines as Lemma 7 in [6]. We first give an upper bound to $\mathbb{P}_2(\varepsilon)$. Like in [6] we have by the fluctuation theory for Lévy processes

$$\mathbb{P}_2(\varepsilon) = O(\phi(1/\varepsilon)^{-1})$$

where

$$\phi(1/\varepsilon) = \exp\left(\int_0^\infty \frac{e^{-s} - e^{-s/\varepsilon}}{s} \mathbb{P}(Y^+(s) \geq 0) ds\right).$$

We evaluate the latter quantity using the scaling property

$$\begin{aligned} \mathbb{P}(Y^+(s) \geq 0) =& \mathbb{P}(\psi_0(s) \geq \sigma^+(s)) \\ =& \mathbb{P}(\psi_0(1) \geq s^\gamma \sigma^+(1)) \\ =& \mathbb{P}(\psi_0(1) \geq 0) - \mathbb{P}(0 \leq \psi_0(1) < s^\gamma \sigma^+(1)) \end{aligned}$$

where $\gamma = 1/\beta - 1/\alpha > 0$. We get from the independence of ψ_0 and σ^+

$$\mathbb{P}(0 \leq \psi_0(1) < s^\gamma \sigma^+(1)) = \int_{y \geq 0} \mathbb{P}(\psi_0(1) \in dy) \mathbb{P}(\sigma^+(1) > y/s^\gamma).$$

For $s < 1$, Markov inequality yields

$$\mathbb{P}(\sigma^+(1) > y/s^\gamma) \leq \left(y^{-1/2} s^{\gamma/2} \mathbb{E}\left(\sqrt{\sigma^+(1)}\right)\right) \wedge 1,$$

so writing $k = \mathbb{E}\left(\sqrt{\sigma^+(1)}\right) < \infty$ (since σ^+ is β-stable with $\beta > 1/2$)

$$\begin{aligned} \mathbb{P}(0 \leq \psi_0(1) < s^\gamma \sigma^+(1)) \leq & \int_{y \geq 0} \mathbb{P}(\psi_0(1) \in dy) \left(\frac{k s^{\gamma/2}}{\sqrt{y}} \wedge 1\right) \\ \leq & \int_0^{k^2 s^\gamma} \mathbb{P}(\psi_0(1) \in dy) + \int_{y \geq k^2 s^\gamma} \frac{\mathbb{P}(\psi_0(1) \in dy)}{\sqrt{y}} k s^{\gamma/2} \\ \leq & K k^2 s^\gamma + K k s^{\gamma/2} \int_{k^2 s^\gamma}^1 \frac{dy}{\sqrt{y}} + k s^{\gamma/2} \int_1^\infty \mathbb{P}(\psi_0(1) \in dy) \\ \leq & \text{ constant}.s^{\gamma/2}, \end{aligned}$$

where $K = \sup_{y \in [0,1]} (\mathbb{P}(\psi_0(1) \in dy)/dy)$ is a finite constant, since stable laws have continuous densities. With the notation $\rho = \mathbb{P}(\psi_0(1) \geq 0)$ we deduce that

$$\phi(1/\varepsilon) \geq \varepsilon^{-\rho} \underbrace{\exp\left(-\int_0^\infty \frac{e^{-s}}{s} \mathbb{P}\left(0 \leq \psi_0(1) < s^\gamma \sigma^+(1)\right) ds\right)}_{\text{constant } > 0}$$

and we conclude that

$$\mathbb{P}_2(\varepsilon) = O(\varepsilon^\rho).$$

Let us now evaluate the probability that Y^- reaches a new maximum in $[0,\varepsilon]$. As in [6] we introduce the time-reversed process

$$Z(x) = Y^-(\varepsilon - x-) - Y^-(\varepsilon) \qquad \text{for } x \in [0,1].$$

This is a Lévy process, which has the same law as $-\psi_0(x) - \sigma^-(x)$. We thus have

$$\begin{aligned}&\mathbb{P}(Y^- \text{ reaches a new max in } [0,\varepsilon])\\ &\qquad = \mathbb{P}\left(Z \text{ does not reach a new max in } (\varepsilon, 1+\varepsilon]\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(Z(s) \geq 0\right) =& \mathbb{P}(\psi_0(1) \leq 0) - \mathbb{P}\left(-s^\gamma \sigma^-(1) < \psi_0(1) \leq 0\right)\\ =& 1 - \rho - \mathbb{P}\left(-s^\gamma \sigma^-(1) < \psi_0(1) \leq 0\right).\end{aligned}$$

We deduce from what has previously been done for Y^+, the inequality

$$\mathbb{P}_1(\varepsilon) = O(\varepsilon^{1-\rho}),$$

which allows us to conclude that (9) holds and this ends the proof of Theorem 1.

3.3 Precisions on the turbulence.

This subsection is broken into three lemmas. Putting the pieces together, one deduces the results stated in Proposition 1. The first lemma deals with the structure of Eulerian shock points.

Lemma 2 *With probability one, Eulerian shock points between two consecutive regular points form a (random) increasing sequence indexed by* $\mathbb{Z}$.

Proof of the Lemma.
The lemma exactly means that a.s. Lagrangian regular points are the only accumulation points of the range of the inverse of the Lagrangian function a. So, let us consider a right accumulation point d of the range of a (similar arguments apply to left accumulation points). We will prove that d is a regular point. The line of the proof is the following: we first check that ψ_0 is a.s. continuous at d, we then prove that a.s. $a(d) = d$ and a is continuous at d, and finally we conclude.
i) Potential ψ_0 is continuous at d. Suppose that d is a jump time for ψ_0. We will check that we then have $a(d) = d$, and that this cannot agree with our assumption. Let $d_n = a(x_n)$ be a sequence decreasing to d and set $c = \lim_{n\to\infty} x_n$. The right continuity of a implies that $a(c) = d$. The identity $d_n = a(x_n)$ entails that

$$(\psi_0(d_n-) \vee \psi_0(d_n)) - \frac{1}{2}(x_n - d_n)^2 \geq \psi_0(d) - \frac{1}{2}(x_n - d)^2$$

so

$$\frac{(\psi_0(d_n-) \vee \psi_0(d_n)) - \psi_0(d)}{d_n - d} \geq \frac{d_n + d - 2x_n}{2}$$

and finally

$$\limsup_{h \downarrow 0} \frac{\psi_0(d+h) - \psi_0(d)}{h} \geq d - c. \tag{11}$$

Consider now $d < d_n$ and $c < x_n$. The identity $d_n = a(x_n)$ yields

$$\begin{aligned} \psi_0(d_n + h) - (\psi_0(d_n-) \vee \psi_0(d_n)) &\leq \frac{1}{2}(d_n + h - x_n)^2 - \frac{1}{2}(d_n - x_n)^2 \\ &\leq h(d_n - x_n + h/2). \end{aligned}$$

Using the right continuity of ψ_0, we get

$$\psi_0(d+h) - \psi_0(d) \leq h(d-c) + \frac{1}{2}h^2$$

which implies that

$$\limsup_{h \downarrow 0} \frac{\psi_0(d+h) - \psi_0(d)}{h} \leq d - c. \tag{12}$$

We deduce from (11) and (12) that

$$\limsup_{h \downarrow 0} \frac{\psi_0(d+h) - \psi_0(d)}{h} = d - c.$$

A result of Khintchine (see Theorem VIII.5 in [3] for an accessible reference) give the following information on the local behavior of ψ_0 :

$$\text{for any } \beta > 0, \text{ a.s. } \limsup_{h \downarrow 0} \frac{\psi_0(h)}{h^\beta} = \begin{cases} 0 & \text{if } \alpha < 1/\beta \\ \infty & \text{if } \alpha \geq 1/\beta. \end{cases} \tag{13}$$

Since the set of discontinuities of ψ_0 is a countable set of stopping times ; one deduces from (13) that a.s.

$$\limsup_{h \downarrow 0} \frac{\psi_0(d+h) - \psi_0(d)}{h} = d - c = 0,$$

i.e. $a(d) = d$.

Now, (13) ensures that a.s. for any time d of discontinuity of ψ_0

$$\limsup_{h \downarrow 0} \frac{\psi_0(d+h) - \psi_0(d)}{h^2} = \infty,$$

which makes the equality $d = a(d)$ impossible. We have proved that ψ_0 is a.s. continuous at any right-accumulation point d.

ii) A.s., for any right accumulation point d, we have a(d)=d. The argument of Lemma 5 in [6] shows that with probability 1 at any point c of the range of a where ψ_0 is continuous, we have

$$\begin{cases} c = a(c) \\ x \neq c \Longrightarrow a(x) \neq a(c). \end{cases} \tag{14}$$

In particular $a(d) = d$ and a is strictly increasing in d.
iii) The inverse of the Lagrangian function a is continuous at d. We first check that if $a(d-) \neq a(d) = d$, then $a(d-)$ is a time of negative jump for ψ_0. It is known that $\psi_0 = S^{(1)} - S^{(2)}$, where $S^{(1)}$ and $S^{(2)}$ are two independent α-stable subordinators (recall that a subordinator is a Lévy process with no negative jump). Suppose that $t = a(d-)$ is not the time of a negative jump for ψ_0. Then

$$S^{(1)}(t+h) - S^{(1)}(t) \leq S^{(2)}(t+h) - S^{(2)}(t) - h(\underbrace{t-d}_{<0} - h/2).$$

A result of Fristedt [10] claims that for any α-stable subordinator S we have a.s.

$$\text{for any } t \in \mathbb{R},\ \liminf_{h\to 0} \frac{S(t+h) - S(t)}{h^{1/\alpha}} < \infty,$$

from which follows that a.s.

$$\liminf_{h\to 0} \frac{S^{(2)}(t+h) - S^{(2)}(t)}{h} = 0.$$

As a consequence, there exists some small h such that $S^{(1)}(t+h) - S^{(1)}(t) < 0$, which is absurd. So, $t = a(d-)$ is a time of negative jump for ψ_0.

Since $a(d) = d$ we have the equality

$$\psi_0(t-) = \psi_0(d) + \frac{1}{2}(d-t)^2,$$

which leads us to consider the set $\mathcal{T}$ of time τ such that there exists a negative jump of ψ_0, say $T < \tau$, checking

$$\psi_0(T-) = \psi_0(\tau) + \frac{1}{2}(T-\tau)^2.$$

Since negative jumps of ψ_0 form a countable set of stopping times, one deduces that the set $\mathcal{T}$ is a countable set of stopping times τ. In particular (13) ensures that a.s. for any $\tau \in \mathcal{T}$

$$\limsup_{h\downarrow 0} \frac{\psi_0(\tau+h) - \psi_0(\tau)}{h^2} = \infty,$$

which impedes τ to be a right accumulation point of the range of a. In particular, $d \in \mathcal{T}$ is impossible a.s. and $a(d-) = a(d) = d$.

iv) *Conclusion.* Recall that (14) ensures that a is strictly increasing at d, and since a is continuous at d, one deduces that d is a left and right accumulation point in the range of a, i.e. d is regular.

□

We now turn our attention to the local behavior of ψ_0 at a point of the range of a.

Lemma 3 *With probability one we have:*
i) *Regular points are exactly the points of the range of $x \to a(x)$ where ψ_0 is continuous.*
ii) *Suppose that x is not a regular point. If $a(x) > x$, then $a(x)$ is a time of positive jump for ψ_0, whereas it is a time of negative jump, if $a(x) < x$.*

proof of the Lemma
i) We have seen in the proof of the previous Lemma that ψ_0 is continuous at any right accumulation point of the range of a. In particular, ψ_0 is continuous at any regular point. Conversely, if ψ_0 is continuous at a point c of the range of a, then (14) ensures that a.s. $c = a(c)$ and that a is strictly increasing at c. Since a is right continuous, one deduces that c is a.s. a right accumulation point of the range of a, which means using the previous lemma that a.s. c is Lagrangian regular.
ii) The first part of the lemma ensures that a point $a(x)$ that is not a regular point, is a time of jump for ψ_0. Since the times of jump of ψ_0 form a countable set of stopping times, one see from (13) that with probability one for any time T of jump of ψ_0

$$\limsup_{h \downarrow 0} \frac{\psi_0(T+h) - \psi_0(T)}{h} = 0 \quad \text{and} \quad \limsup_{h \downarrow 0} \frac{\psi_0(T-h) - \psi_0(T-)}{h} = 0.$$

If $a(x)$ is a time of positive jump of ψ_0 then

$$\psi_0(a(x)+h) - \psi_0(a(x)) \le \frac{1}{2}(a(x)+h-x)^2 - \frac{1}{2}(a(x)-x)^2,$$

i.e.

$$\frac{\psi_0(T+h) - \psi_0(T)}{h} \le a(x) - x + h/2$$

and

$$\limsup_{h \downarrow 0} \frac{\psi_0(T+h) - \psi_0(T)}{h} = 0 \le a(x) - x.$$

We deduce in the same way that if $a(x)$ is a time of a negative jump of ψ_0 then $a(x) \le x$. The proof of the Lemma is complete.

□

In the final lemma, we focus on the behavior of u.

Lemma 4 *With probability one, the set $\mathcal{R}$ of regular points is exactly the set of points where u has velocity 0. Moreover, when x increases from a given regular point to the next, the velocity $u(x)$ is a.s. first positive and then negative.*

As a consequence, when x increases the velocity of the clusters between two consecutive regular points is a.s. first positive and then negative.

proof of the Lemma.
Bertoin (Lemma 5 in [6]) has proved that with probability one, any regular point has velocity 0. Conversely, if $a(r) = r$, r cannot be a time of jump for ψ_0 since (13) ensures that a.s. for any time T of jump of ψ_0

$$\limsup_{h\downarrow 0} \frac{\psi_0(T+h) - \psi_0(T)}{h^2} = \infty \ .$$

We deduce from the previous lemma that r is regular.

Between two regular points, u is a tooth path, made of pieces of line of slope 1, split by negative jumps (shocks). In particular, u is continuous at any point of increase, and u cannot go from negative value to positive value without crossing a regular point. We shall prove now that u cannot stay always positive or negative between two regular points. Let us prove for example that u cannot stay always positive between r_{-1} and r_0 (recall that r_0 is the first regular point at the right of 0, whereas r_{-1} is the first one at the left of 0). Define

$$T = \inf \left\{ t > 0; \psi_0(z) - \psi_0(t) \leq \frac{1}{2}(z-t)^2, \ \forall z \leq t \right\} .$$

Obviously, T is a stopping time, such that $T \leq r_0$. We first prove that if $u > 0$ on (r_{-1}, r_0), then $T = r_0$. Suppose that $T < r_0$. There exists $y \in (r_{-1}, r_0)$ such that $T \in (a(y-), a(y)]$. Call $\mathcal{C}_y$ the parabola defined by $\{\frac{1}{2}(z-y)^2 + C; z \in \mathbb{R}\}$, where C is chosen such that $\mathcal{C}_y$ is above the graph of ψ_0 but touches it at $a(y-)$ and $a(y)$. Then, geometrically, $\mathcal{C}_y$ is strictly above the half-parabola $\{\psi_0(T) + \frac{1}{2}(z-T)^2; z < T\}$, because $T \leq a(y) < y$ (remember that $u(y) > 0$) and $\mathcal{C}_y$ is above $\psi_0(T)$. In particular, $a(y-)$ is strictly above this half-parabola, what contradicts the definition of T. So $T = r_0$. Yet, T is a stopping time, so $\psi_0(T + .) - \psi_0(T)$ has the same law as ψ. We deduce that a.s. T is not a regular point, so u cannot be positive between r_{-1} and r_0. We finally deduce that after a regular point, the velocity u is first positive and then negative, with one and only one change of sign.

Consider an Eulerian shock point x. The identity $v(x) = (u(a(x)) + u(a(x-)))/2$ ensures that the velocity of the clusters between two consecutive regular points is a.s. first positive and then negative (when x increases).

□

Remark

Bertoin (Theorem 3 in [6]) has proved that a local maximum of ψ_0 has a positive probability to be a regular point. Conversely, one may easily see from the previous lemma, that a regular point is a.s. a local maximum of ψ_0.

3.4 Regenerative property of regular points.

The idea of the proof of Theorem 2 is the following. We consider an approximation $T_{\delta,\varepsilon}$ of the first regular point r_0 at the right of 0 that satisfies a Markov type property and we then obtain the regeneration property for r_0 by taking the limits.

Let us first introduce some notations. Let $\mathcal{T} = \{t \in \mathbb{R}^+ : \psi_0(t) \neq \psi_0(t-)\}$ denote the set of positive jump points of the initial potential ψ_0, and write

$$\psi_0^a = \{\psi_0(a+t) - \psi_0(a), t \in \mathbb{R}\}, \psi_{0-}^a \quad = \{\psi_0(t), t \leq a\}, \psi_{0+}^a = \{\psi_0(a+t) - \psi_0(a), t \geq 0\},$$

and

$$\mathcal{A}_{(a,b)}^\varepsilon = \{g \in \mathbb{D} : g(s) \leq f^\varepsilon(s) \ \forall s \in (a,b)\}$$

where

$$f^\varepsilon(s) = \mathbf{1}_{s<0}\left(\frac{1}{2}(s-\varepsilon)^2 - \frac{1}{2}\varepsilon^2\right) + \mathbf{1}_{s\geq 0}\left(\frac{1}{2}s^2\right)$$

and $\mathbb{D}$ is the space of right continuous with left limits functions $g : \mathbb{R} \to \mathbb{R}$. We will consider as an approximation of the first positive regular point, the time

$$T_{\delta,\varepsilon} = \inf\{\tau \in \mathcal{T} : \psi_0(\tau) - \psi_0(\tau-) < -\delta \text{ and } \psi_0^\tau \in \mathcal{A}_{\mathbb{R}}^\varepsilon\}.$$

The following lemma ensures that $T_{\delta,\varepsilon}$ tends to r_0 when we first let $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$.

Lemma 5 *With probability one*
i) *When δ decreases to 0, $T_{\delta,\varepsilon}$ decreases to a time T_ε fulfilling $T_\varepsilon \leq r_0$.*
ii) *When ε decreases to 0, T_ε increases to r_0.*

proof of the Lemma.
i) In the notation of proposition 1, consider the sequence of Eulerian shock points $(e_0(n);\ n < 0)$ that decreases to r_0 when n tends to $-\infty$. Remember that a.s. $r_0 = a(r_0)$, and $u(e_0(n)) > 0$ for $n < 0$. So, there exists $N \in \mathbb{N}$ such that $0 < e_0(n) - a(e_0(n)) \leq \varepsilon$ for $n \leq -N$. In particular, $\psi_0^{e_0(-N)} \in \mathcal{A}_{\mathbb{R}}^\varepsilon$. Moreover, Lemma 4 ensures that $e_0(-N)$ is a time of negative jump and if we choose δ small enough $\psi_0(e_0(-N)) - \psi_0(e_0(-N-)) < -\delta$. This implies that $T_\varepsilon \leq T_{\delta,\varepsilon} \leq e_0(-N)$. Finally, we let $\delta \downarrow 0$, and we deduce that a.s. $T_\varepsilon \leq r_0$.
ii) The function f^ε is increasing with ε, which makes the condition $\mathcal{A}_{\mathbb{R}}^\varepsilon$ more and more restrictive when ε decreases to 0. So T^ε increases to, say, $T \leq r_0$ when ε decreases to 0. The process ψ_0^T fulfills the condition $\mathcal{A}_{\mathbb{R}}^0$. In particular, we see from (13) that T is not a time of jump for ψ. So Lemma 3i) ensures that T must be regular and finally $T = r_0$.
□
The following lemma ensures that $T_{\delta,\varepsilon}$ is finite a.s..

Lemma 6 *If one chooses δ small enough, then $\mathbb{P}(T_{\delta,\varepsilon} < \infty) = 1$.*

proof of the Lemma.
The lemma relies mainly on the 0-1 law of Kolmogorov. The previous lemma ensures that $T_\varepsilon \leq r_0$ a.s., so if one chooses δ small enough then $\mathbb{P}(T_{\delta,\varepsilon} < \infty) > 0$. Call a δ,ε-regular point (resp. local-δ,ε-regular point), a point τ fulfilling the conditions

$$\tau \in \mathcal{T}, \psi_0(\tau) - \psi_0(\tau-) < -\delta \text{ and } \psi_0^\tau \in \mathcal{A}_{\mathbb{R}}^\varepsilon \quad (\text{resp. } \psi_0^\tau \in \mathcal{A}_{(-1,1)}^\varepsilon).$$

The first δ,ε-regular point at the right of 0 is $T_{\delta,\varepsilon}$ and let us denote for any $M>0$, T_{loc}^M the first local-δ,ε-regular point at the right of M.

Let X_i denotes the pieces of path $(\psi_0(i+x)-\psi_0(i)\ ;\ x\in(0,1))$. The sequence $(X_i\ ;\ i\in\mathbb{N})$ is a sequence of i.i.d. variables. The existence of arbitrarily large local-δ,ε-regular points is an event of the tail σ-field

$$\bigcap_{n\in\mathbb{N}}\sigma(X_i\ ;\ i>n).$$

In particular, the 0-1 law of Kolmogorov ensures that the existence of arbitrarily large local-δ,ε-regular points has probability 0 or 1. The identity in law $\psi_0(M+.)-\psi_0(M)\sim\psi_0$ leads us to

$$\mathbb{P}(T_{loc}^M<\infty)=\mathbb{P}(T_{loc}^0<\infty)\geq\mathbb{P}(T_{\delta,\varepsilon}<\infty)>0,$$

which implies in particular that there exists arbitrarily large local-δ,ε-regular points with probability >0. This probability is then 1. Moreover, a point T_{loc}^M has a positive probability (independent of M) to be a δ,ε-regular point, so one deduces that there exists δ,ε-regular point with probability one.

□

We now turn our attention to the Markov property type of $T_{\delta,\varepsilon}$.

Lemma 7 *In the above notation, $\psi_{0+}^{T_{\delta,\varepsilon}}$ and $\psi_{0-}^{T_{\delta,\varepsilon}}$ are independent.*

proof of the Lemma.

We want to prove that for any (f,g) $\mathcal{B}orel(\mathbb{D})$-measurable bounded functions the following equality holds

$$\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right)g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right)=\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right)\right)\mathbb{E}\left(g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right).$$

Let $\{\tau_1,..,\tau_i,..\}$ denote the increasing sequence of positive times where the initial potential makes a jump of size less than $-\delta$. We have

$$\begin{aligned}\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right)g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right)&=\sum_{i=1}^{\infty}\mathbb{E}(f(\psi_{0-}^{\tau_i})g(\psi_{0+}^{\tau_i});T_{\delta,\varepsilon}=\tau_i)\\&=\sum_{i=1}^{\infty}\mathbb{E}(f(\psi_{0-}^{\tau_i})g(\psi_{0+}^{\tau_i});\psi_0^{\tau_i}\in\mathcal{A}_{\mathbb{R}}^{\varepsilon};\psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}}\notin\mathcal{A}_{\mathbb{R}}^{\varepsilon}).\end{aligned}$$

We deduce from the convexity of f^{ε} the identity

$$\{\psi_0^{\tau_i}\in\mathcal{A}_{\mathbb{R}}^{\varepsilon};\psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}}\notin\mathcal{A}_{\mathbb{R}}^{\varepsilon}\}=\{\psi_0^{\tau_i}\in\mathcal{A}_{\mathbb{R}}^{\varepsilon};\psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}}\notin\mathcal{A}_{(-\infty,\tau_i)}^{\varepsilon}\}.$$

Moreover, we know that each τ_i is a stopping time, so the processes $\psi_{0-}^{\tau_i}$ and $\psi_{0+}^{\tau_i}$ are independent, which ensures the next identities:

$$\begin{aligned}
&\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right) g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right)\\
&=\sum_{i=1}^{\infty} \mathbb{E}\left(f(\psi_{0-}^{\tau_i}) g(\psi_{0+}^{\tau_i}); \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}; \psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}} \notin \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}; \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(\tau_i,\infty)}\right)\\
&=\sum_{i=1}^{\infty} \mathbb{E}\left(f(\psi_{0-}^{\tau_i}); \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}; \psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}} \notin \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}\right) \mathbb{E}\left(g(\psi_{0+}^{\tau_i}); \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(\tau_i,\infty)}\right).
\end{aligned}$$

The stationarity of ψ_0 yields

$$\begin{aligned}
\mathbb{E}(g(\psi_{0+}^{\tau_i}); \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(\tau_i,\infty)}) &= \mathbb{E}(g(\psi_{0+}^{\tau_j}); \psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(\tau_j,\infty)})\\
\text{and } \mathbb{P}(\psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(\tau_i,\infty)}) &= \mathbb{P}(\psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(\tau_j,\infty)}) \text{ for any integer } i,j.
\end{aligned}$$

Remember that

$$\begin{aligned}
1 =&\mathbb{P}(T_{\delta,\varepsilon} < \infty)\\
=&\sum_{j=1}^{\infty} \mathbb{P}\left(\psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(-\infty,\tau_j)}; \psi_0^{\tau_1},..,\psi_0^{\tau_{j-1}} \notin \mathcal{A}^{\varepsilon}_{(-\infty,\tau_j)}\right) \mathbb{P}\left(\psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(\tau_j,\infty)}\right),
\end{aligned}$$

so we get with the preceding the following identities

$$\begin{aligned}
&\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right) g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right)\\
=&\sum_{i=1}^{\infty} \mathbb{E}\left(f(\psi_{0-}^{\tau_i}); \psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}; \psi_0^{\tau_1},..,\psi_0^{\tau_{i-1}} \notin \mathcal{A}^{\varepsilon}_{(-\infty,\tau_i)}\right) \mathbb{P}\left(\psi_0^{\tau_i} \in \mathcal{A}^{\varepsilon}_{(\tau_i,\infty)}\right)\\
\times&\sum_{j=1}^{\infty} \mathbb{E}\left(g(\psi_{0+}^{\tau_j}); \psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(\tau_j,\infty)}\right) \mathbb{P}\left(\psi_0^{\tau_j} \in \mathcal{A}^{\varepsilon}_{(-\infty,\tau_j)}; \psi_0^{\tau_1},..,\psi_0^{\tau_{j-1}} \notin \mathcal{A}^{\varepsilon}_{(-\infty,\tau_j)}\right)\\
=&\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right)\right) \mathbb{E}\left(g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right)
\end{aligned}$$

which complete the proof of the lemma.
□

Let us now conclude. For any bounded continuous function $f, g\colon \mathcal{B}(\mathbb{D}) \to \mathbb{R}$, let first $\delta \downarrow 0$, and then $\varepsilon \downarrow 0$ in the equality

$$\mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right) g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right) = \mathbb{E}\left(f\left(\psi_{0-}^{T_{\delta,\varepsilon}}\right)\right) \mathbb{E}\left(g\left(\psi_{0+}^{T_{\delta,\varepsilon}}\right)\right).$$

This yields

$$\mathbb{E}\left(f\left(\psi_{0-}^{r_0}\right) g\left(\psi_{0+}^{r_0}\right)\right) = \mathbb{E}\left(f\left(\psi_{0-}^{r_0}\right)\right) \mathbb{E}\left(g\left(\psi_{0+}^{r_0}\right)\right),$$

which ensures the independence of $\psi_{0+}^{r_0}$ and $\psi_{0-}^{r_0}$.

Moreover, $(u(r_0+x);\ x \geq 0)$ is $\sigma(\psi_{0+}^{r_0})-$measurable, whereas $(u(r_0-x);\ x \geq 0)$ is $\sigma(\psi_{0-}^{r_0})-$measurable. We obtain then, the second part of Theorem 2.

4 The Cauchy case.

Assume that the initial potential ψ_0 is a Cauchy process. Bertoin ([6], Lemma 6 and Theorem 5) has proved that a.s. the range of a is contained into the set of continuity points of ψ_0 and that a is a.s. strictly increasing. In this section we will evaluate the size of the set of Lagrangian regular points.

Theorem 3 *When the initial potential ψ_0 is a Cauchy process, the set $\mathcal{R}$ of Lagrangian regular points is a.s. uncountable and has Minkowsky dimension 0.*

In particular, $\mathcal{R}$ has Hausdorff dimension 0.
Theorem 3 results from the following technical lemma.

Lemma 8 *Assume that the initial potential ψ_0 is a Cauchy process. Then for any $\delta > 0$ there exists a finite constant $c(\delta)$ such that for any $0 < \varepsilon < \delta/2$ and $b \in \mathbb{R}$*

$$\mathbb{P}(\exists y = a(x) \in [0, \varepsilon] \text{ with } x \in [b, b+\delta]) \leq c(\delta).\varepsilon^{1-\delta}.$$

The proof of this lemma is postponed to the end of the section.
Proof of Theorem 3.

One deduces from the strict monotonicity of a that $\mathcal{R}$ is uncountable. Indeed, a induces a bijection between the set of point of continuity of a and $\mathcal{R}$.

Let us prove now that $\mathcal{R}$ has Minkowsky dimension 0. Again, the proof is closely connected to the proof of Theorem 4 in [6]. Our intention is to prove that for any $\delta > 0$, $\mathcal{R}$ has Minkowsky dimension at most 2δ. In this aim, we will split $[0,1]$ into small intervals of size ε, and prove that the number of intervals containing a regular point is at most $\varepsilon^{2\delta}$.

For any fixed $n, \delta > 0$ and $\varepsilon < \delta/2$ define

$$N_\varepsilon^{(n)} = \text{ card } \{k = 0..[1/\varepsilon]; \exists a(x) \in [k\varepsilon, (k+1)\varepsilon], \text{ with } |x| \leq n\}.$$

One obtains with Fubini's theorem and the previous lemma the following upper bound

$$\begin{aligned}
\mathbb{E}\left(\varepsilon^{2\delta} N_\varepsilon^{(n)}\right) &= \sum_{k=0}^{[1/\varepsilon]} \sum_{p=0}^{2n/\delta} \varepsilon^{2\delta} \mathbb{P}(\exists a(x) \in [k\varepsilon, k\varepsilon + \varepsilon]; x \in [-n + p\delta, -n + p\delta + \delta]) \\
&\leq \sum_{k=0}^{[1/\varepsilon]} \sum_{p=0}^{2n/\delta} c(\delta).\varepsilon^{1+\delta} \\
&\leq \text{cst}(n, \delta).\varepsilon^{\delta}.
\end{aligned}$$

Then take $\varepsilon = 2^{-p}$. So

$$\mathbb{E}\left(\sum_{p \geq 0} 2^{-2p\delta} N_{2^{-p}}^{(n)}\right) < \infty,$$

which ensures that a.s.

$$\limsup_{p \to \infty} 2^{-2p\delta} N_{2^{-p}}^{(n)} = 0.$$

If we choose now $2^{-p} < \varepsilon \le 2^{-p+1}$, the monotonicity of $N_\varepsilon^{(n)}$ in the variable ε implies that

$$\varepsilon^{2\delta} N_\varepsilon^{(n)} \le 2^{2\delta} 2^{-2p\delta} N_{2^{-p}}^{(n)}$$

and in particular

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^{2\delta} N_\varepsilon^{(n)} = 0.$$

This exactly means that the Minkowsky dimension of $\{a(x) \in [0,1]; |x| \le n\}$ is at most 2δ. We can choose δ and n at our will, so the Minkowsky dimension of $\{a(x) \in [0,1]\}$ is zero, and due to the stationarity of $u(x)$ the Minkowsky dimension of $\mathcal{R}$ is zero.

□

It only remains now to prove lemma 8.

Proof of lemma 8

Lemma 8 bears the same flavor as the result (8) in section 2.1. The proof is then similar. In the proof of Theorem 1 we replaced a parabola by the graph of two β-stable subordinators, in order to deal with homogeneous Markov process. In the same way, we will replace here a parabola by an appropriated vertical cone. Let us explain this.

A Cauchy process ψ_0 is the sum of a symmetric Cauchy process C and a drift: $\psi_0(y) = C(y) + dy$. Adding a drift to the potential ψ_0 (i.e. adding a constant to the initial velocity) has no effect on the set of regular points. In particular, we may assume that $d = 0$. Remember that Bertoin [6] has proved that C is continuous at any point of the range of a. We deduce in particular that for every regular point $r = a(x)$

$$\begin{aligned} C(r \pm h) - C(r) \le& \frac{1}{2}(r \pm h - x)^2 - \frac{1}{2}(r-x)^2 \\ \le& h\left(h/2 \pm (r-x)\right). \end{aligned}$$

Suppose that $r = a(x) \in [0, \varepsilon]$ with $x \in [b, b+\delta]$ and $h \in (0, \varepsilon+\delta)$. Then

$$\begin{cases} C(r+h) - C(r) \le h\left((\varepsilon+\delta)/2 + \varepsilon - b\right) \\ C(r-h) - C(r) \le h\left((\varepsilon+\delta)/2 + b + \delta\right). \end{cases} \tag{15}$$

If we define

$$Y^-(y) = C(y) + y\left(b + \frac{\varepsilon + 3\delta}{2}\right) \qquad \text{for } y \ge -\delta,$$

then (15) implies that Y^- reaches a new maximum at r. Call now τ the first time after 0 where Y^- reaches a new maximum (notice that $0 \le \tau \le r \le \varepsilon$). If Y^+ denotes the process

$$Y^+(y) = C(\tau + y) - C(\tau) + y\left(b - \frac{3\varepsilon + \delta}{2}\right) \qquad \text{for } y \ge 0,$$

then the first condition in (15) implies that Y^+ does not reach a new maximum in $(\varepsilon, \varepsilon+\delta)$. Notice that τ is a stopping time for C, so Y^+ and Y^- are two independent Cauchy process (with different drift). Putting the pieces together, one deduces that

$$\mathbb{P}(\exists y = a(x) \in [0, \varepsilon] \text{ with } x \in [b, b+\delta]) \le \mathbb{P}_1 \mathbb{P}_2,$$

where

$$\begin{cases} \mathbb{P}_1 = \mathbb{P}\left(Y^{-}\text{reaches a new max on } [0,\varepsilon]\right) \\ \mathbb{P}_2 = \mathbb{P}\left(Y^{+}\text{reaches no new max on } [\varepsilon,\varepsilon+\delta]\right). \end{cases}$$

Again, like in lemma 7 of [6], we have

$$\mathbb{P}_2 \leq \ \text{cst}(\delta) \times \phi(1/\varepsilon)^{-1}$$

where

$$\phi(1/\varepsilon) = \exp \int_0^\infty \frac{e^{-s} - e^{-s/\varepsilon}}{s} \mathbb{P}(Y^+(s) \geq 0) ds.$$

Using the scaling property of Y^+, one deduces that $\mathbb{P}\left(Y^+(s) \geq 0\right)$ does not depend on $s > 0$ and equals

$$\frac{1}{2} + \frac{1}{\pi} \arctan\left(b - \frac{3\varepsilon + \delta}{2}\right).$$

Thus, we have

$$\mathbb{P}_2 \leq \ \text{cst}(\delta) \times \exp\left[\left(\frac{1}{2} + \frac{1}{\pi}\arctan\left(b - \frac{3\varepsilon+\delta}{2}\right)\right)(\log\varepsilon)\right].$$

Once again, like in lemma 7 of [6], we evaluate $\mathbb{P}_1$ by using time reversal. It is easily seen that then

$$\begin{aligned} \mathbb{P}_1 &\leq \text{cst}(\delta) \times \exp\left[(1 - \mathbb{P}(Y_s^-))(\log\varepsilon)\right] \\ &\leq \text{cst}(\delta) \times \exp\left[\left(\frac{1}{2} - \frac{1}{\pi}\arctan\left(b + \frac{\varepsilon + 3\delta}{2}\right)\right)(\log\varepsilon)\right]. \end{aligned}$$

We finally deduce that

$$\begin{aligned} \mathbb{P}_1\mathbb{P}_2 &\leq \text{cst}(\delta) \\ &\times \exp\left[\left(1 + \frac{1}{\pi}\arctan\left(b - \frac{3\varepsilon+\delta}{2}\right) - \frac{1}{\pi}\arctan\left(b - \frac{\varepsilon+3\delta}{2}\right)\right)(\log\varepsilon)\right] \\ &\leq \text{cst}(\delta).\varepsilon^{1-2\pi^{-1}(\delta+\varepsilon)} \\ &\leq \text{cst}(\delta).\varepsilon^{1-\delta} \end{aligned}$$

in using the inequality $\arctan(x) - \arctan(x') \leq x - x'$ for any $x' \leq x$. Lemma 8 has been proved.
□

5 Numerical illustration.

We give here a numerical illustration of the both cases studied in this paper. The simulation of u has been made in breaking the line into 1600 points, and replacing a Lévy process by a random walk.

5.1 $\alpha = 0.85$.

The following simulation of u has been made for $\alpha = 0.85$ and $t = 1$. One may notice a regular point around $x = 260$.

5.2 $\alpha = 1$.

We have here a simulation of u in the Cauchy case at time $t = 1$. One may notice the proliferation of small shocks.

Références

[1] M. AVELLANEDA and W. E *Statistical properties of shocks in Burgers Turbulence.* Comm. Math. Phys. 172 (1995) p. 13-38.

[2] M. AVELLANEDA *Statistical properties of shocks in Burgers Turbulence II.* Comm. Math. Phys. 169 (1995) p. 45-59.

[3] J. BERTOIN *Lévy processes.* Cambridge University Press, Cambridge (1996).

[4] J. BERTOIN *The inviscid Burgers equation with brownian initial velocity.* Comm. Math. Phys. 193 (1998) p. 397-406.

[5] J. BERTOIN *Large deviation estimate in Burgers turbulence with stable noise initial data.* J.Stat.Phys. 91 (1998) p. 655-667.

[6] J. BERTOIN *Structure of shocks in Burgers turbulence with stable noise initial data.* Comm. Math. Phys. 203 (1999) p. 729-741.

[7] J. M. BURGERS *The nonlinear diffusion equation.* Dordrecht, Reidel (1974).

[8] J. D. COLE *On a quasi linear parabolic equation occuring in aerodynamics.* Quart. Appl. Math. 9 (1951) p. 225-236.

[9] L. FRACHEBOURG, Ph. A. MARTIN *Exact Statistical Properties of the Burgers Equation.* J. Fluid. Mech. 417 (2000), p. 69-99.

[10] B. E. FRISTEDT *Uniform local behavior of stable subordinators.* Ann. Proba. 7 (1979) p. 1003-1013.

[11] R. K. GETOOR *Splitting times and shift functionals.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete 47 (1979) p. 69-81.

[12] P. GROENEBOOM *Brownian motion with a parabolic drift and Airy functions.* Probability Theory and Rel. Fields 81 (1989) p.79-109.

[13] H. HANDA *A remark on shocks in inviscid Burgers turbulence.* Non linear waves and weak turbulence (1992) p. 339-345 Fitzmaurice et al., Eds, Birkhäuser-Boston.

[14] J. HAWKES *A lower Lipschitz condition for the stable subordinators.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete 17 (1971), p. 23-32.

[15] E. HOPF, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl.Math. 3 (1950) p. 201-230.

[16] A. JANICKI and W. A. WOYCZYNSKI, *Hausdorff dimension of regular points in stochastic flows with Lévy* α*-stable initial data.* J. Stat. Phys. 86 (1997) p. 277-299.

[17] N. LEONENKO *Limit theorems for random fields with singular spectrum.* Math. and Appl. (1999) Kluwers Academic Publishers.

[18] R. RYAN, *Large-deviation analysis of Burgers turbulence with white-noise initial data.* Comm. Pure Applied Math. 51 (1998) p. 47-75.

[19] Z. SHE, E. AURELL and U. FRISCH, *The inviscid Burgers equation with initial data of Brownian type.* Comm. Math. Phys. 148 (1992) p. 623-641.

[20] Y. SINAI, *Statistics of shocks in solutions of inviscid Burgers Equation.* Comm. Math. Phys. 148 (1992) p. 601-621.

[21] W. A. WOYCZYNSKI, *Burgers-KPZ Turbulence, Göttingen lectures* Lectures Notes in Math. 1700, Springer 1998.

Deuxième partie

Genealogy of Shocks in Burgers Turbulence with White Noise Initial Velocity

Abstract : As time passes, the shocks of the solution of the inviscid Burgers equation aggregate. We characterize, in the case of white noise initial velocity, the stochastic fragmentation process obtained when time runs backwards. In other words, we specify the law of the genealogy of the shocks of the Burgers turbulence with white noise initial velocity.
Key words : Burgers turbulence, sticky particles, white noise initial velocity, Brownian excursion.
A.M.S. classification : 35Q53, 60J65.

1 Introduction.

Burgers equation

$$\partial_t u + u.\nabla u = \mu \Delta u \tag{16}$$

is a simplified version of the Navier-Stokes equation, in which the terms of pressure and force are neglected. First introduced by Burgers (see [6]) as a simple model for hydrodynamic turbulence, Burgers equation also appears in different fields such as the theory of the interfaces for ballistic aggregation or in cosmology for the formation of superstructures in the universe (see [19] and references therein).

We focus thenceforth on dimension one when the viscosity μ tends to 0. It is known that the solution u_μ of (16) converges when $\mu \to 0$ to $u_0 = u$, which is the (weak) entropy solution of the so-called inviscid Burgers equation

$$\partial_t u + u \partial_x u = 0. \tag{17}$$

Several studies concern statistics of the solution of (17) when the initial velocity $u(.,0)$ is a random process, see [6, 13, 19]. The solution $u(.,t)$ at fixed time t of (17) is now well known in many cases, such as for example when $u(.,0)$ is a Brownian motion (see [18, 4])or a white noise (see [2, 3, 6, 11, 16, 17]). Recently, Bertoin has studied the evolution in time of the solution u when the initial velocity $u(.,0)$ is a Brownian motion, and proved a striking connection with the additive coalescent, see [5].

The aim of this paper is to describe the evolution in time of u when $u(.,0)$ is a white noise. The solution $u(.,t)$ is then a toothpath with a discrete sequence of discontinuities and segments of slope $1/t$. In particular, the solution $u(.,t)$ is determined by the location and amplitude of its discontinuities, which are usually called shocks. As time passes, when two shocks "collide", they form a single shock with amplitude the sum of the amplitudes of the two previous shocks. The evolution in time of u is thus governed by the deterministic dynamic of clustering of the shocks. This dynamic induces a loss of information, in the sense that if $t_1 < t_2$, the solution $u(.,t_1)$ cannot be recovered from $u(.,t_2)$. When time runs backwards, we thus obtain a random process of fragmentation : a shock S splits after a certain time ρ into two shocks S_1 and S_2, which in turn split after a time ρ_1 and ρ_2, respectively into S_{11}, S_{12} and S_{21}, S_{22}, etc...

The purpose of the present work is to describe the law of this process of fragmentation, that is the genealogy of the shocks.

In this direction it is convenient to use the sticky particles interpretation of the inviscid Burgers equation. Consider at time $t = 0$, infinitesimal particles uniformly distributed on the line, with initial velocity $u(.,0)$. We suppose that they evolve with the dynamics of completely inelastic shocks. This means that the velocity of a particle only changes in case of collision, and when two (clusters) of particles collide, they form a heavier cluster with conservation of masses and momenta. The evolution of the system is completely described by the entropy solution of (17), and $u(x,t)$ then represents the velocity of the particle located at x at time t. The case of white noise initial velocity arises for example at the infinitesimal limit of a discrete sticky particles system, when particles are evenly spread with i.i.d. velocities (with zero mean and finite variance). A shock of $u(.,t)$ represents a cluster at time t with mass given by $t\times$the amplitude of the shock. The fragmentation of the shocks then corresponds to the fragmentation of the clusters obtained by reversing the dynamics of the sticky particles.

We prove here that, in the case of white noise initial data, conditionally on the state of the system at time t, masses split independently of their location, velocity and also of their environment. We compute the law of the parameters which describe the splitting of a mass, and which also characterize the law of the fragmentation process.

We present in the second section some material for the study of the system. We specified the fragmentation in the third section. The fourth section is devoted to the proofs of the preliminary results. The Appendix contains the somewhat technical proof of Lemma 6.

2 Preliminaries.

2.1 Inviscid Burgers equation and sticky particles.

Let us consider the entropy solution u of the inviscid Burgers equation. This is the weak solution of (17) such that $x \mapsto u(x,t)$ has only discontinuities of the first kind and no positive jumps. We shall work here with the version satisfying

$$u(x,t) = \frac{u(x+,t) + u(x-,t)}{2},$$

where $u(x+,t)$ and $u(x-,t)$ refers to the right and left limit of $u(.,t)$ at x. We call initial potential the process

$$W(z) = \int_0^z u(x,0)dx, \ \ z \in \mathbb{R}.$$

When $W(z) = o(|z|^2)$ for $z \to \pm\infty$, we denote by $a(x,t)$ the largest location of the minimum of the function $z \to W(z) + \frac{1}{2t}(z-x)^2$. We have the following two equivalent geometrical interpretations of $a(.,t)$.

First, consider a parabola $z \mapsto -\frac{1}{2t}(z-x)^2 + C$, with C chosen such that this parabola is strictly below the path of W. Let C increase until this parabola touches the graph of W. The largest abscissa of the contact points is then $a(x,t)$. Alternatively, bring up a line of slope x/t, until it touches the graph of $z \mapsto W(z) + \frac{1}{2t}z^2$. The largest abscissa of contact is again $a(x,t)$. The function $a(.,t)$ may be thus described in terms of the convex minorant of the initial potential with a $\frac{1}{2t}$-parabolic drift, $z \mapsto W(z) + \frac{1}{2t}z^2$, or in terms of the "$\frac{1}{2t}$-parabolic" minorant of the initial potential W. One notices in particular that $a(.,t)$ is right continuous and non decreasing.

We can express the entropy solution of the inviscid Burgers equation in terms of $a(.,t)$. Yet, it is known (see Hopf [12] or Cole [7]) that

$$u(x,t) = \frac{x - a(x,t)}{t}$$

at every x where $a(.,t)$ is continuous, and

$$u(x,t) = \frac{u(x+,t) + u(x-,t)}{2},$$

elsewhere.

We have seen that the function $a(.,t)$ is right continuous and non decreasing. It possesses therefore a right continuous inverse $x(a,t) = \inf\{y \in \mathbb{R} : a(y,t) > a\}$, which is known as the *Lagrangian function*. In the sticky particles interpretation, the latter gives the location at time t of the particle started from $a \in \mathbb{R}$. In particular, the *Eulerian shock points* at time t, which are the abscissaes x of discontinuity of $a(.,t)$, are the locations of clusters in the system at time t. They have mass $m(x,t) = a(x+,t) - a(x-,t)$ and velocity $u(x,t)$, since (conservation of momenta)

$$u(x,t) = \frac{u(x+,t) + u(x-,t)}{2} = \frac{1}{a(x+,t) - a(x-,t)} \int_{a(x-,t)}^{a(x+,t)} u(z,0)dz.$$

When the initial potential W is a Brownian motion (i.e. $u(.,0)$ is a white noise), it was shown (see [2, 11]) that $a(.,t)$ is a.s. a step function, which means that $u(.,t)$ is a toothpath. The particles are thus located at time $t > 0$ on a discrete set of clusters. One says that the shock structure is discrete.

We are interested in the fragmentation of a cluster located at x at time t, when time runs backwards. We write $m_1(x,s,t),..,m_k(x,s,t)$ for the masses of the clusters at time s which form this cluster at time t. One may notice that $m_1(x,s,t),..,m_k(x,s,t)$ are exactly the length of the intervals of the partition of $[a(x-,t),a(x,t)]$ induced by the range of $a(.,s)$. We give therefore the following definition.

Definition 1 *For any Eulerian shock point x at time t and $0<s<t$, we set*

$$\mathcal{M}(x,s,t) = (m_1(x,s,t),..,m_k(x,s,t)),$$

where $m_1(x,s,t),..,m_k(x,s,t)$ are the length of the intervals appearing in the partition of $[a(x-,t),a(x,t)]$ by the range of $a(.,s)$ ranked according to the increasing order of their location. $\mathcal{M}(x,s,t)$ takes values in the space of finite positive numerical sequence $\mathbb{S} = \cup_{n\in\mathbb{N}^*}]0,\infty[^n$.

The process $(\mathcal{M}(x,t-r,t); 0\le r\le t)$ will be called the fragmentation process.

We notice in the following lemma that the fragmentation of a cluster only depends of the "excursions" of the initial potential above the "parabolic minorant". We write in the sequel $\mathcal{E} = \cup_{m>0}\{m\}\times \mathrm{C}([0,m],\mathbb{R}^+)$ for the space of positive excursions, where m is meant to represent the duration of the excursion.

Lemma 1 *There exists a function $F:\mathbb{R}^+\times\mathbb{R}^+\times\mathcal{E}\to\mathbb{S}$ such that for any initial potential W which induces a discrete shock structure, and any Eulerian shock point x at time t, we have*

$$\mathcal{M}(x,s,t) = F(s,t,m(x,t),\varepsilon^{(x,t)}), \qquad \textit{for} \quad 0<s<t,$$

where

$$\varepsilon^{(x,t)}(z) = W(a(x-,t)+z) - W(a(x-,t)) - \frac{1}{2t}z(m(x,t)-z) - zu(x,t), \ \textit{for } 0\le z\le m(x,t).$$

Proof of Lemma 1.

We set

$$\overline{\varepsilon^{(x,t)}}(z) = W(a(x-,t)+z) - W(a(x-,t)) - \frac{1}{2t}z(m(x,t)-z) - zu(x,t), \text{ for } z\in\mathbb{R}.$$

For any $y\in\mathbb{R}$ and $0<s<t$, we have

$$\begin{aligned} a(y,s) &= a(x-,t) + \underset{z\in R}{\operatorname{argmin}}\left(W(a(x-,t)+z) - W(a(x-,t)) + \frac{1}{2s}(z-y)^2\right) \\ &= a(x-,t) + \underset{z\in R}{\operatorname{argmin}}\left(\overline{\varepsilon^{(x,t)}}(z) + \frac{1}{2t}z(m(x,t)-z) + \frac{1}{2s}(z-y-su(x,t))^2\right), \end{aligned}$$

where $\text{argmin}_{z\in\mathbb{R}}(g(z))$ denotes the largest z that minimizes g. We deduce so that the range of $a(.,s)-a(x-,t)$ only depends on $s,t,m(x,t)$ and $\overline{\varepsilon^{(x,t)}}$. Now the intersection of the range of $a(.,s)-a(x-,t)$ with $[0,m(x,t)]$ depends on $s,t,m(x,t)$ and the restriction of $\overline{\varepsilon^{(x,t)}}$ to $[0,m(x,t)]$, which is $\varepsilon^{(x,t)}$. Since $M(x,s,t)$ is the finite sequence of the length of the intervals of the partition of $[0,m(x,t)]$ by the range of $a(.,s)-a(x-,t)$, it is therefore a function of $s,t,m(x,t)$ and $\varepsilon^{(x,t)}$. The proof is complete.
□

2.2 Laplace transform of the integral of a 3-d Bessel bridge (after Groeneboom).

We recall in this subsection the value of the Laplace transform of the integral of a 3-d Bessel bridge, which shall appear in further calculations. Let $0>-\omega_1>-\omega_2>\cdots$ denotes the zeros of the Airy function $\mathtt{Ai}$ as defined on p. 446 of [1]. We introduce

$$\begin{aligned} C(\lambda) =& \sqrt{2\pi}\lambda\sum_{n=1}^{\infty}\exp(-2^{-1/3}\omega_n\lambda^{2/3}),\\ \text{and } F^{(x,y)}(\lambda) =& 2^{-1/3}\lambda^{2/3}\sum_{n=1}^{\infty}\frac{\mathtt{Ai}(2^{1/3}\lambda^{1/3}\mathtt{y}-\omega_{\mathtt{n}})}{\mathtt{Ai}'(-\omega_{\mathtt{n}})}\exp(-2^{-1/3}\lambda^{2/3}x\omega_n). \end{aligned}$$

Groeneboom has computed the Laplace transform of the integral of a three dimensional Bessel bridge, see [11] Theorem 2-1 and formulaes (4-9), (4-13).

Lemma 2 *For $x,y\geq 0$, let $\beta^{[x]}_{0\to y}$ be a three dimensional Bessel bridge with duration x starting at 0 and ending at y. We set*

$$L^{(x,y)}(\lambda)=\mathbb{E}\left(\exp\left(\lambda\int_0^x u d\beta^{[x]}_{0\to y}(u)\right)\right).$$

We have

$$L^{(x,y)}(\lambda)=\begin{cases}\sqrt{2\pi x^3}\exp\left(\lambda xy+y^2/2x\right)F^{(x,y)}(\lambda)/y & \text{for } y>0\\ \mathbb{E}\left(\exp\left(-\lambda\int_0^x e^{[x]}(s)\,ds\right)\right)=C\left(x^{3/2}\lambda\right) & \text{for } y=0\ ,\end{cases}$$

where $e^{[x]}$ denotes a Brownian excursion with duration x.

2.3 Excursions above parabolaes.

We state here some useful results on excursions conditioned to stay above a parabola. The proof of these results are given in section 4. For $m>0$, we set $\mathbb{P}^{[m]}$ for the law of a Brownian excursion (or of a 3D-Bessel bridge starting and ending at 0) with duration m. For any $a,m>0$ we introduce the probability measure

$$\nu(a,m)=\frac{\exp\left(-a\int_0^m X_s ds\right)}{C(am^{3/2})}\,\mathbb{P}^{[m]}$$

which is absolutely continuous with respect to $\mathbb{P}^{[m]}$, where X_s denotes the canonical process and the normalizing factor C has been defined in the previous subsection. In the sequel, we will often write $e^{[m]}$ for a Brownian excursion with duration m. We first connect $\nu(a,m)$ to the law of an excursion conditioned to stay above a parabola.

Lemma 3 *For any $a, m > 0$, we set for $0 \le z \le m$*

$$p^{(a,m)}(z) = \frac{a}{2} z(m-z).$$

If $e^{[m]}$ is a Brownian excursion with duration m, then the law of $e^{[m]} - p^{(a,m)}$ conditionally on $e^{[m]} \ge p^{(a,m)}$ is $\nu(a,m)$.

We now associate to a Brownian excursion with duration m, $e^{[m]}$, the two variables

$$\begin{cases} \sigma(m) = \sup\{a \ge 0; e^{[m]} \ge p^{(a,m)}\} \\ \eta(m) = \text{ the largest abscissa } z_0 \in (0,1) \text{ such that } e^{[m]}(z_0) \ge p^{(\sigma(m),m)}(z_0). \end{cases}$$

We shall write in the sequel $e = e^{[1]}$, $\sigma = \sigma(1)$ and $\eta = \eta(1)$. Let us specify the law of $(\sigma(m), \eta(m))$.

Lemma 4 *Law of $(\sigma(m), \eta(m))$.*
i) *For any $m > 0$ we have the scaling identity*

$$(\sigma(m), \eta(m)) \overset{law}{=} (m^{-3/2}\sigma, m\,\eta).$$

ii) *The law of (σ, η) is given by*

$$\mathbb{P}\left(\eta > x, \sigma \in da\right) = e^{-\frac{a^2}{24}}\, \partial_2\, G^{(x)}(a,0)$$

where

$$G^{(x)}(a,b) = \sqrt{8\pi}\, e^{ab(1-x)^2(2x+1)/12}\, e^{\mathcal{O}(b^2)} \int_0^\infty e^{yb(x-1/2)} F^{(x,y)}(a) F^{(1-x,y+\beta)}(a-b)\, dy\,,$$

with $\mathcal{O}(b^2) = \frac{b^2}{24}(1-x)(8x^2-4x-1)$, $\beta = p^{(b,1)}(x) = \frac{b}{2}x(1-x)$ and $F^{(x,y)}$ defined in the previous subsection.

In particular we have

$$\mathbb{P}(\sigma \ge a) = e^{-a^2/24} C(a)\,,$$

with C defined in the previous section.

We give in the last lemma the law of an excursion conditionally on $(\sigma = a, \eta = x)$. In this direction, it is convenient to define the concatenation of two processes $(\varepsilon_1(z); 0 \le z \le m_1)$ and $(\varepsilon_2(z); 0 \le z \le m_2)$ as the process

$$(\varepsilon_1(z)\mathbf{1}_{0\le z\le m_1} + \varepsilon_2(z)\mathbf{1}_{m_1\le z\le m_1+m_2};\ 0 \le z \le m_1+m_2)\,.$$

Lemma 5 *Conditionally on $(\sigma = a, \eta = x)$, $e - p^{(\sigma,1)}$ has the law of the concatenation of two independent processes of law $\nu(a,x)$ and $\nu(a, 1-x)$.*

As a consequence, the law of $e^{[m]} - p^{(\sigma,m)}$ under $\nu(a,m)$ conditionally on $(\sigma(m) = b-a, \eta(m) = xm)$ is the law of the concatenation of two independent processes of law $\nu(xm, b)$ and $\nu((1-x)m, b)$.

Remark
The previous lemma ensures in particular that there exists a.s. a unique abscissa $z_0 \in (0,1)$ such that $e^{[m]}(z_0) = p^{(a,m)}(z_0)$. Indeed, an excursion is a.s. positive on (0,1), and this property still holds under $\nu(a,m)$. We can start now our investigations.

2.4 Conditional distribution of the initial data.

Clusters are ranked according to the increasing order of their location, with the convention that $x_1(t)$ is the location of the first cluster at the right of 0. From a physical point of view, the state of the system at time t is described by the sequence $((x_n(t), m_n(t), v_n(t)); n \in \mathbb{Z})$, where $m_n(t)$ and $v_n(t)$ are the mass and velocity of the n-th cluster. Yet, one may notice that the useful datum is the sequence $((x_n(t), a_n(t)); n \in \mathbb{Z})$, where $a_n(t) = a(x_n(t), t)$. Indeed, we have

$$m_n(t) = a_n(t) - a_{n-1}(t)$$

$$\text{and} \qquad v_n(t) = \frac{2x_n(t) - (a_n(t) + a_{n-1}(t))}{2t}.$$

We thus introduce $\mathcal{F}_t = \sigma((x_n(t), a_n(t)); n \in \mathbb{Z})$ which is the datum given by the state of the system at time t. The law of $((x_n(t), m_n(t), v_n(t)); n \in \mathbb{Z})$ is known see formulaes (101) and the following remark, (50), (54), (55), (66), (67) and (70) in [8].

We specify now the law of the initial potential W conditionally on the state of the turbulence at time t. We prove that the pieces of Brownian motion between the Lagrangian points $a_n(t)$ are independent conditionally on $\mathcal{F}_t$ and we connect their law to $\nu(a,m)$ (recall that this distribution has been introduced in section 2-3).

Lemma 6 *The "excursions" of the Brownian motion above the "$\frac{1}{2t}$-parabolic minorant"*

$$\varepsilon^{(x_n(t),t)}(z) = W(z + a_{n-1}(t)) - W(a_{n-1}(t)) - p^{(m_n(t),1/t)}(z) - zv_n(t), \ 0 \le z \le m_n(t),$$

are independent conditionally on $\mathcal{F}_t$ and their conditional law is $\nu(m_n(t), 1/t)$.

This lemma is the key of our analysis, since it connects the study of the fragmentation to the law $\nu(a, m)$. Let us sketch an explanation for the origin of this result.

Groeneboom [10] has studied the convex minorant of the Brownian motion (he actually focused on the concave majorant, which is its symmetric about the abscissaes line). He has shown it is a piecewise linear path, and conditionally on the edge points of this path, the Brownian motion realizes independent Brownian excursions above each segment of the convex minorant. By Girsanov Theorem, adding a parabolic drift $z \mapsto \frac{1}{2t}z^2$ to the Brownian motion amonts to work under the probability measure

$$\exp\left(\frac{1}{t}\int_0^T z dW_z - \frac{T^3}{6t^2}\right)\mathbb{P})_{\mathcal{G}_T},$$

where $\mathcal{G}_T = \sigma(W_s; 0 \leq s \leq T)$. In other words conditionally on the "$\frac{1}{2t}$−parabolic minorant" of the Brownian motion, the excursions $\varepsilon^{(x_n(t),t)}$ of the Brownian motion above the "$\frac{1}{2t}$−parabolic minorant" are independent and have the law $\nu(m_n(t), 1/t)$.

Nevertheless, we need to investigate the convex minorant of a Brownian motion on a finite interval in order to apply Girsanov Theorem, which cannot be deduced easily from the convex minorant of a Brownian motion on $[0, \infty)$. However, we can prove Lemma 6 from results in [11], and this somewhat technical proof is given in the Appendix.

3 Fragmentation statistics.

3.1 Statement of the main results.

We thenceforth turn our attention to the process of fragmentation generated by the dynamic of sticky particles as time runs backwards. Since the clustering dynamic of sticky particles is deterministic and induces a loss of information, the fragmentation process obtained by time reversal is a stochastic Markovian process. Our aim is to describe this process.

Recall that $(x_n(t), m_n(t), v_n(t))$ denotes the (location, mass, velocity) of the n-th cluster at time t and $\mathcal{F}_t = \sigma((x_n(t), m_n(t), v_n(t)); n \in \mathbb{Z})$. We point out that $(\mathcal{F}_t, t \geq 0)$ is a backwards filtration, since the evolution of the system is deterministic and induces a loss a information. The variable $\mathcal{M}(x, s, t)$ is defined in section 2-1 (definition 1). We first specify the dependence of the fragmentation of a cluster and its environment. In this direction, we write $\mu(t, m)$ for the law of $(\mathcal{M}(x_1(t), t - r, t); 0 \leq r \leq t)$ conditionally on $(m_1(t) = m)$.

Theorem 1 *Fix $t > 0$. Conditionally on the state $\mathcal{F}_t$ of the turbulence at time t, the fragmentation processes $(\mathcal{M}(x_n(t), t - r, t); 0 \leq r \leq t)$, $n \in \mathbb{Z}$, are independent, and their conditional laws only depend on $(t, m_n(t))$. More precisely, they are given by $\mu(t, m_n(t))$.*

Roughly, Theorem 1 claims that conditionally on the state of the system at time t, the masses of the clusters at time $t - r$ are obtained by breaking into pieces each cluster at time t independently of its location and velocity and of the other clusters. In particular, we deduce from Theorem 1 that for any Eulerian shock point x, the fragmentation process $(\mathcal{M}(x,t-r,t); 0 \le r \le t)$ is an inhomogeneous Markov process.

We want now to describe the law $\mu(t,m)$. We denote by $\rho(t,m)$ (or simply ρ) the time at which the cluster located at $x_1(t)$ at time t splits conditionally on $(m(x_1(t),t) = m)$. We will check that we have a binary splitting $\mathcal{M}(x_1(t), t-\rho, t) = (m_1, m_2)$ and we denote by $R(t,m)$ (or simply R) the ratio $R = m_1/(m_1+m_2)$. The next theorem characterizes the law $\mu(t,m)$, in terms of the law of the variable (ρ, R). We introduce in this aim the operation $*$ on the space of finite numerical sequences: for any $M_1 = (m_1^1, .., m_{k_1}^1)$ and $M_2 = (m_1^2, .., m_{k_2}^2)$, we write

$$M_1 * M_2 = (m_1^1, .., m_{k_1}^1, m_1^2, .., m_{k_2}^2).$$

We can state now our result which is a splitting property at time ρ.

Theorem 2 *For any $t, m > 0$, let $M = (M(r); 0 \le r \le t)$ be a process of law $\mu(t,m)$. We have*

$$M(\rho + r) = M_1(r) * M_2(r) \qquad \textit{for } \ 0 \le r \le t - \rho,$$

where M_1 and M_2 are independent conditionally on (ρ, R) with conditional law $\mu(t-\rho, Rm)$ and $\mu(t-\rho, (1-R)m)$.

In other words, conditionally on $m(x_1(t),t)$ the cluster located at $x_1(t)$ at time t splits after a time $\rho_0 = \rho(t, m(x_1(t),t))$ into two clusters of mass $m_1 = R_0\, m(x_1(t),t)$ and $m_2 = (1 - R_0)\, m(x_1(t),t)$, where $R_0 = R(t, m(x_1(t),t))$. Moreover the fragmentation processes of these two clusters are independent conditionally on $(m(x_1(t),t), R_0, \rho_0)$ and their conditional law are $\mu(t-\rho_0, m_1)$ and $\mu(t-\rho_0, m_2)$. We can therefore iterate Theorem 2 in order to obtain that in turn each of these two clusters splits into two clusters of mass

$$m_{11} = R(t-\rho_0, m_1)\, m_1, \quad m_{12} = (1 - R(t-\rho_0, m_1))\, m_1$$

and

$$m_{21} = R(t-\rho_0, m_2)\, m_2, \quad m_{22} = (1 - R(t-\rho_0, m_2))\, m_2$$

at time $\rho_1 = \rho(t-\rho_0, m_1)$ and $\rho_2 = \rho(t-\rho_0, m_2)$, and so on.

The last theorem gives the joint law of $\rho(t,m)$ and $R(t,m)$ which completes the characterisation of the law $\mu(t,m)$.

Theorem 3 *For any $0 < r < t$, $m > 0$ and $0 \le \alpha \le 1$, the law of $(\rho(t,m), R(t,m))$ is given by*

$$\mu(t,m)\,(\rho \in dr, R > \alpha) = \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2} - \frac{1}{(t-r)^2}\right)\right) \partial_2\, G^{(m\alpha)}\left(\frac{m^{3/2}}{t-r}, 0\right) \frac{m^3\, dr}{C(m^{3/2}/t)(t-r)^2}$$

where $G^{(\alpha)}(a,b)$ is defined in Lemma 4 and C in section 2-2.

In particular, we have

$$\mu(t,m)(\rho \ge r) = \exp\left(\frac{m^3}{24}\left(\frac{1}{t^2} - \frac{1}{(t-r)^2}\right)\right) \frac{C(m^{3/2}/(t-r))}{C(m^{3/2}/t)}\,.$$

We deduce therefore the following asymptotics for the distribution of the splitting time ρ.

Corollary 3.1 *For any $t, m > 0$, we have*

$$\mu(t,m)(\rho \geq r) \sim_{r \to t} \frac{\exp\left(m^3/24t^2\right)}{C(m^{3/2}/t)} \frac{\sqrt{2\pi} m^{3/2}}{t-r} \exp\left(-\frac{m^3}{24(t-r)^2} - 2^{-1/3}\omega_1 \frac{m}{(t-r)^{2/3}}\right),$$

with $\omega_1 \approx 2.3381$, and

$$\mu(t,m)(\rho \leq r) \sim_{r \to 0} \left(\frac{m^{3/2}}{12t} - \frac{C'(m^{3/2}/t)}{C(m^{3/2}/t)}\right) \times \frac{r}{t^2},$$

where C is defined in section 2-2.

Let us prove now these results.

3.2 Numerical illustrations.

The joint density of the time $\rho(1,1)$ and the position $R(1,1)$ computed in Theorem 3 is plotted in Figure 1. The joint density is not plotted at the extremal values $\alpha = 0$ and $\alpha = 1$ of the position $R(1,1)$, since the serie $F^{(x,y)}(\lambda)$ does not converge when $x = 0$.

FIG. 1: *Joint density of time $\rho(1,1)$ and the position $R(1,1)$*

FIG. 2: *Theoretical and simulated density of the time* $\rho(1,1)$

In Figures 2 and 3, the plots of the theoretical and the simulated marginal densities of $\rho(1,1)$ and $R(1,1)$ are given. The simulated curves are obtained in drawing a Brownian excursion on 1000 steps. The paths which are not everywhere above the parabola $z \to z(1-z)/2$ are rejected. We then compute σ and η to deduce $(\rho(1,1), R(1,1))$. The plots result from 3.10^6 iterations. Theoretical and simulated curves do not fit exactely. The difference is the consequence of the discretisation of the Brownian excursion. Increasing the number of step of the simulated Brownian excursion leads to the convergence of the simulated curves towards the theoretical ones. The density of $R(1,1)$ is only plotted on $[0.5, 0.95]$, since the serie $F^{(x,y)}(\lambda)$ converges slowly for small values of x.

FIG. 3: *Theoretical and simulated density of the position* $R(1,1)$

3.3 Proofs.

Proof of Theorem 1.
Lemma 1 ensures that the fragmentation processes may be written

$$\mathcal{M}(x_n(t), t - ., t) = F(t - ., t, m_n(t), \varepsilon^{(x_n(t),t)}).$$

We thus deduce from Lemma 6 that they are independent conditionally on $\mathcal{F}_t$. Since the $\mathcal{F}_t$-conditional law of the processes $\varepsilon^{(x_n(t),t)}$, $n \in \mathbb{Z}$ is $\nu(t, m_n(t))$, the $\mathcal{F}_t$-conditional law of $\mathcal{M}(x_n(t), t - ., t)$, $n \in \mathbb{Z}$ only depends on $(t, m_n(t))$ and is $\mu(t, m_n(t))$.

Proof of Theorem 2.
The law of the excursion $\varepsilon^{(x_1(t),t)}$ of the Brownian motion above the "$\frac{1}{2t}$-parabolic minorant" is $\nu(m(x_1(t), t), 1/t)$. We write $\tau = t - \rho$,

$$\varepsilon_1(z) = \varepsilon^{(x_1(t),t)}(z) - p^{(m(x_1(t),t),1/\tau-1/t)}(z), \qquad \text{for} \quad 0 \leq z \leq m_1 = Rm(x_1(t), t)$$

and

$$\varepsilon_2(z)=\varepsilon^{(x_1(t),t)}(m_1+z)-p^{(m(x_1(t),t),1/\tau-1/t)}(m_1+z),\text{ for }\ 0\le z\le m_2=m(x_1(t),t)-m_1.$$

Since

$$\frac{1}{\tau}=\sup\left\{a\ge 0;\varepsilon^{(x_1(t),t)}(z)\ge p^{(m(x_1(t),t),a-1/t)},\ \text{for }0\le z\le m(x_1(t),t)\right\},$$

Lemma 5 ensures that conditionally on $(m(x_1(t),t)=m,1/\tau=1/t_1-1/t,R=x)$ the law of ε_1 is $\nu(xm,1/t_1)$ and the law of ε_2 is $\nu((1-x)m,1/t_1)$.

Moreover, since the fragmentation of $[a(x_1(t)-,t),a(x_1(t),t)]$ after time ρ is obtained from the fragmentation of $[a(x_1(t)-,t),a(x_1(t)-,t)+Rm(x_1(t),t)]$ and $[a(x_1(t)-,t)+Rm(x_1(t),t),a(x_1(t),t)]$ we deduced from Lemma 1 that

$$\begin{aligned}\mathcal{M}(x_1(t),\tau-r,t)=&F(\tau-r,\tau,m(x_1(t),t),\varepsilon^{(x_1(t),\tau)})\\=&F(\tau-r,\tau,m_1,\varepsilon_1)*F(\tau-r,\tau,m_2,\varepsilon_2).\end{aligned}$$

Putting pieces together, one obtains Theorem 2.

Proof of Theorem 3.
Conditionally on $m(x_1(t),t)=m$, we have :

$$\frac{1}{\tau}=\frac{1}{t-\rho}=\sup\left\{a\ge 0;\varepsilon^{(x_1(t),t)}(z)\ge p^{(m,a-1/t)},\ \text{for }0\le z\le m\right\}.$$

We thus have with Lemma 6 and the scaling property of Brownian excursions

$$\mu(t,m)(\rho\in dr,R>\alpha)=\mathbb{E}\left(\frac{\exp\left(-m^{3/2}t^{-1}\int_0^1 e_s ds\right)}{C\left(m^{3/2}/t\right)};\sigma\in d\left(\frac{m^{3/2}}{t-r}-\frac{m^{3/2}}{t}\right);\eta>m\alpha\right),$$

where σ and η are defined in section 2-3. The Lemma 3 ensures now that

$$\begin{aligned}\mu(t,m)(\rho\in dr,R>\alpha)&=\mathbb{P}\left(\sigma\in d\left(\frac{m^{3/2}}{t-r}\right);\eta>m\alpha|\sigma\ge\frac{m^{3/2}}{t}\right)\\&=\mathbb{P}\left(\sigma\in d\left(\frac{m^{3/2}}{t-r}\right);\eta>m\alpha\right)/\mathbb{P}\left(\sigma\ge\frac{m^{3/2}}{t}\right).\end{aligned}$$

We deduce so the joint law of (τ,R) from the joint law of (σ,η), which has been computed in Lemma 4.

4 Proof of the preliminary results.

4.1 Proof of Lemma 3.

Lemma 3 is mainly an application of Girsanov formula. Let W be a Brownian motion starting from $x>0$ under $\mathbb{P}^x$. For any Borel bounded functional f

$$\mathbb{E}^x\left(f\left(W_s-p_s^{(a,1)};0\le s\le 1\right)\right)=e^{-a^2/24}\mathbb{E}^x\left(f(W_s;0\le s\le 1)\exp\left(-\frac{a}{2}\int_0^1(1-2s)dW_s\right)\right).$$

For $y > 0$, it follows from the equality $p^{(a,1)}(1) = 0$ that

$$\mathbb{E}^x \left(f \left(W_s - p_s^{(a,1)}; 0 \le s \le 1 \right) | W_1 = y \right)$$
$$= \frac{\mathbb{E}^x \left(f \left(W_s - p_s^{(a,1)}; 0 \le s \le 1 \right) \; ; \; W_1 - p^{(a,1)}(1) \in \, dy \right)}{\mathbb{P}^x \left(W_1 \in \, dy \right)}$$
$$= e^{-a^2/24} \mathbb{E}^x \left(f(W_s; 0 \le s \le 1) \exp \left(-\frac{a}{2} \int_0^1 (1-2s) dW_s \right) | W_1 = y \right).$$

After an integration by parts, we obtain

$$\mathbb{E} \left(f \left(b_{x \to y}^{[1]}(s) - p^{(a,1)}(s); 0 \le s \le 1 \right) \right) =$$
$$e^{a(x+y)/2 - a^2/24} \mathbb{E} \left(f \left(b_{x \to y}^{[1]}(s); 0 \le s \le 1 \right) \exp \left(-a \int_0^1 b_{x \to y}^{[1]}(s) ds \right) \right),$$

where $b_{x \to y}^{[1]}$ is a Brownian bridge of duration 1 from x to y. We have in particular

$$\mathbb{E} \left(f \left(b_{x \to y}^{[1]}(s) - p^{(a,1)}(s); 0 \le s \le 1 \right); b_{x \to y}^{[1]} - p^{(a,1)} \ge 0 \right) =$$
$$e^{a(x+y)/2 - a^2/24} \mathbb{E} \left(f \left(b_{x \to y}^{[1]}(s); 0 \le s \le 1 \right) \exp \left(-a \int_0^1 b_{x \to y}^{[1]}(s) ds \right); b_{x \to y}^{[1]} \ge 0 \right).$$

We shall use now the fact that a Brownian bridge $b_{x \to y}^{[1]}$ conditioned to stay positive has the law of a Bessel(3) bridge $\beta_{x \to y}^{[1]}$. We divide the previous equality first by $\mathbb{P} \left(b_{x \to y}^{[1]} \ge 0 \right)$

$$\mathbb{E} \left(f \left(\beta_{x \to y}^{[1]}(s) - p^{(a,1)}(s); 0 \le s \le 1 \right); \beta_{x \to y}^{[1]} \ge p^{(a,1)} \right) =$$
$$e^{a(x+y)/2 - a^2/24} \mathbb{E} \left(f \left(\beta_{x \to y}^{[1]}(s); 0 \le s \le 1 \right) \exp \left(-a \int_0^1 \beta_{x \to y}^{[1]}(s) ds \right) \right),$$

and then by $\mathbb{P} \left(\beta_{x \to y}^{[1]} \ge p^{(a,1)} \right)$:

$$\mathbb{E} \left(f \left(\beta_{x \to y}^{[1]}(s) - p^{(a,1)}(s); 0 \le s \le 1 \right) \mid \beta_{x \to y}^{[1]} \ge p^{(a,1)} \right) =$$
$$\mathbb{E} \left(\exp \left(-a \int_0^1 \beta_{x \to y}^{[1]}(s) ds \right) \right)^{-1} \times \mathbb{E} \left(f \left(\beta_{x \to y}^{[1]}(s); 0 \le s \le 1 \right) \exp \left(-a \int_0^1 \beta_{x \to y}^{[1]}(s) ds \right) \right).$$

This means that for a Bessel(3) bridge $\beta_{x \to y}^{[1]}$ of duration 1 under $\mathbb{P}$, the law of $\beta_{x \to y}^{[1]} - p^{(a,1)}$ conditionally on $\left(\beta_{x \to y}^{[1]} \ge p^{(a,1)} \right)$ is

$$\mathbb{E} \left(\exp \left(-a \int_0^1 \beta_{x \to y}^{[1]}(s) ds \right) \right)^{-1} \times \exp \left(-a \int_0^1 \beta_{x \to y}^{[1]}(s) ds \right) \mathbb{P}.$$

Now, since the law of a Bessel(3) bridge $\beta_{x \to y}^{[1]}$ converges to the law of an excursion e when $x, y \to 0$, we claim that the law of $e - p^{(a,1)}$ conditionally on $(e \ge p^{(a,1)})$ is $\nu(a,1)$. Using the scaling property of Brownian excursions, one obtains Lemma 3.

4.2 Proof of Lemma 4.

We give first some relations between Bessel bridges.

Lemma 7 *Let $\beta^{[m]}_{x\to y}$ denotes a Bessel(3) bridge from x to y of duration m. For any $y,z,\alpha\geq 0$ and $x>0$, the process*

$$\left(\beta^{[x]}_{z\to y+\alpha}(t)-\frac{\alpha}{x}t;\ 0\leq t\leq x\right)$$

conditioned to stay positive has the same law as $\beta^{[x]}_{z\to y}$.

As a consequence, for $a,y\geq 0$, $x>0$ and $\alpha=p^{(a,1)}(x)=\frac{a}{2}x(1-x)$ the law of the process

$$\beta^{[x]}_{0\to y+\alpha}-p^{(a,1)}$$

conditioned to stay positive is the same as the law of $\beta^{[x]}_{0\to y}-p^{(a,x)}$ conditioned to stay positive.

Proof of Lemma 7.

For any $y,z>0$, let W^{z-y} be a Brownian motion starting from $z-y$. It is well known that the process

$$y+W^{z-y}(t)-\frac{t}{x}W^{z-y}(x),\qquad 0\leq t\leq x,$$

is a Brownian bridge $b^{[x]}_{z\to y}$ which is independent of W^{z-y}_x. We can therefore conditioned W^{z-y} by $(W^{z-y}_x=\alpha)$ which gives

$$\left(y+b^{[x]}_{z-y\to\alpha}(t)-\frac{t}{x}\alpha;\ 0\leq t\leq x\right)\overset{law}{=}\left(b^{[x]}_{z\to y}(t);\ 0\leq t\leq x\right),$$

which may be written

$$\left(b^{[x]}_{z\to y+\alpha}(t)-\frac{\alpha}{x}t;\ 0\leq t\leq x\right)\overset{law}{=}\left(b^{[x]}_{z\to y}(t);\ 0\leq t\leq x\right).$$

Since a Brownian bridge conditioned to stay positive has the law of a Bessel(3) bridge, the process $\left(\beta^{[x]}_{z\to y+\alpha}(t)-\frac{\alpha}{x}t;\ 0\leq t\leq x\right)$ conditioned to stay positive has same law as $\beta^{[x]}_{z\to y}$. The result still holds for $y=0$ or $z=0$ by taking the limits $y\to 0$, $z\to 0$.

The second part of the lemma follows from the first part and the equality

$$p^{(a,1)}(t)=p^{(a,x)}(t)+\frac{\alpha}{x}t. \tag{18}$$

The proof is complete.

□

We prove now Lemma 4. We set

$$\tau_a=\inf\left\{u>0,e(u)\leq p^{(a,1)}(u)\right\}.$$

We want to compute

$$\begin{aligned}\mathbb{P}(\eta > x, \sigma \in da) &= \mathbb{P}\left(\tau_a > x; \sigma \in da\right)\\ &= \lim_{b \downarrow 0} \frac{\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) - \mathbb{P}\left(\tau_a > x; \sigma \geq a\right)}{b}.\end{aligned}$$

Conditionally on $(e_x = z)$, a Brownian excursion is the concatenation of two independent Bessel bridges $\beta^{[x]}_{0\to z}$ and $\beta^{[1-x]}_{z\to 0}$. This decomposition leads us to

$$\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) = \int_0^\infty \mathbb{P}(e_x - \alpha \in dy)\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right)\mathbb{P}\left(\beta^{[1-x]}_{y+\alpha\to 0} \geq p^{(a-b,1)}(x+\cdot)\right),$$

where $\alpha = p^{(a,1)}(x)$. We deduce from Lemma 7 and (18) that

$$\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right) = \mathbb{P}\left(\beta^{[x]}_{0\to y} \geq p^{(a,x)}\right)\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha}(s) \geq \frac{\alpha}{x}s;\, 0 \leq s \leq x\right).$$

If one follows the same way as in proof of Lemma 3, one obtains

$$\begin{aligned}\mathbb{P}\left(\beta^{[x]}_{0\to y} \geq p^{(a,x)}\right) =& e^{-a^2x^3/24}\mathbb{E}\left(\exp\left(-\frac{a}{2}\int_0^x (x-2s)\, d\beta^{[x]}_{0\to y}(s)\right)\right)\\ =& e^{-a^2x^3/24}e^{-axy/2}L^{(x,y)}(a),\end{aligned}$$

where the second equality stems from Lemma 2. Let us compute $\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha}(s) \geq \frac{\alpha}{x}s;\, 0 \leq s \leq x\right)$. Let B denotes a three dimensional Bessel process starting from z under $\mathbb{P}^z$. Under $\mathbb{P}^0$, we have the identity in law

$$(sB_{1/s}; s \geq 0) \overset{law}{=} (B_s; s \geq 0),$$

which implies that

$$\begin{aligned}\mathbb{P}^0\left(B_s \geq \frac{\alpha}{x}s; 0 \leq s \leq x | B_x = y+\alpha\right) &= \mathbb{P}^0\left(B_{1/s} \geq \frac{\alpha}{x}; 0 \leq s \leq x | B_{1/x} = \frac{y+\alpha}{x}\right)\\ &= \mathbb{P}^{(y+\alpha)/x}\left(B_s \geq \frac{\alpha}{x}; s \geq 0\right),\end{aligned}$$

where the last equality follows from the Markov property of B. Moreover, it is known (see chap.VI Corollary (3-4) in [15]) that this last quantity is equal to $y/(y+\alpha)$. Putting the pieces together, we obtain

$$\mathbb{P}\left(\beta^{[x]}_{0\to y+\alpha} \geq p^{(a,1)}\right) = \frac{y}{y+\alpha}e^{-a^2x^3/24}e^{-axy/2}L^{(x,y)}(a).$$

Now, since (with $\beta = p^{(b,1)}(x)$)

$$\begin{aligned}\mathbb{P}\left(\beta^{[1-x]}_{y+\alpha\to 0} \geq p^{(a-b,1)}(x+\cdot)\right) &= \mathbb{P}\left(\beta^{[1-x]}_{0\to y+\alpha} \geq p^{(a-b,1)}\right)\\ &= \frac{y+\beta}{y+\alpha}e^{-(a-b)^2(1-x)^3/24}e^{-(a-b)(1-x)(y+\beta)/2}L^{(1-x,y+\beta)}(a-b),\end{aligned}$$

and

$$\mathbb{P}(e_x \in d(y+\alpha)) = \frac{2(y+\alpha)^2}{\sqrt{2\pi x^3(1-x)^3}} \exp\left(-\frac{(y+\alpha)^2}{2x(1-x)}\right) dy$$

we obtain

$$\mathbb{P}\left(\tau_a > x; \sigma \geq a-b\right) = e^{-a^2/24}\, G^{(x)}(a,b)\,, \tag{19}$$

and at last

$$\mathbb{P}\left(\tau_a > x; \sigma \in da\right) = -e^{-a^2/24}\;\partial_2\, G^{(x)}(a,0)\,.$$

With Lemma 2 and the decomposition of an excursion into two Bessel bridges, one may check that $G^{(x)}(a,0) = C(a)$ for any $x \in [0,1]$. Finally, taking $x=0$ and $b=0$ in formula (19) gives

$$\mathbb{P}(\sigma \geq a) = e^{-a^2/24} C(a).$$

The proof of Lemma 4 is complete.

4.3 Proof of Lemma 5.

We shall mainly use here a path decomposition for Markov processes due to Millar [14]. For any $x>0$ and $0 \leq z \leq 1$, let $\mathbb{P}^{(x,z)}$ be the law of $\{(\beta^{[1-z]}_{x\to 0}(s), z+s); 0 \leq s \leq 1-z\}$ where $\beta^{[m]}_{x\to y}$ is a Bessel(3) bridge of duration m from x to y. The canonical process X is Markovian under $\mathbb{P}^{(x,z)}$ in its natural filtration $\{\mathcal{G}_s, s>0\}$. We set

$$f(x,y) = \begin{cases} 2y(1-y)/x & \text{if } x>0 \\ +\infty & \text{else.} \end{cases}$$

Under $\mathbb{P}^{(0,0)}$, we write $X(s) = (e(s), s)$ where e is a Brownian excursion. We thus have

$$\sigma = \inf_{s>0} f(X_s)$$

and η is the right most time where $f(X_s)$ reaches its overall minimum. The theorem of Millar in [14] allows us to decompose e at time η. Indeed, we apply this theorem to X under $\mathbb{P}^{(0,0)}$, and obtain that $(e(\eta+s); 0 \leq s \leq 1-\eta)$ is a Markov process independent of $(e(s); 0 \leq s \leq \eta)$ conditionally on (η, σ), with transitions

$$\mathbb{E}\left(g(e_{\eta+t}) \mid e_u; u \leq \eta+s\right) = \int g(y) H_{t-s}(e_{\eta+s}, \eta+s, \sigma; dy),$$

where

$$H_t(x,u,a;dy) = \mathbb{P}\left(\beta^{[1-u]}_{x\to 0}(t) \in dy \mid T_a = \infty\right)$$

and

$$T_a = \inf\left\{t>0, \beta^{[1-u]}_{x\to 0}(t) < p^{(a,1)}(u+t)\right\}.$$

This means that conditionally on $(\sigma = a, \eta = x)$, the process $e_{\eta+\cdot}$ has the law of a Bessel(3) bridge $\beta^{[1-x]}_{p^{(a,1)}(x)\to 0}$ conditioned by $(T_a = \infty)$.

We have as in Lemma 7, that the process

$$\left(\beta^{[1-x]}_{\alpha\to 0}(t) - p^{(a,1)}(x+t);\ 0 \le t \le 1-x\right)$$

conditioned to stay positive has the law of

$$\left(e^{[1-x]}(t) - p^{(a,1-x)}(t);\ 0 \le t \le 1-x\right)$$

conditioned to stay positive, which is $\nu(a, 1-x)$. We have proved that the law of

$$\left(e(\eta+s) - p^{(\sigma,1)}(\eta+s); 0 \le s \le 1-\eta\right)$$

conditionally on $(\sigma = a, \eta = x)$ is $\nu(a, 1-x)$. Using the symmetry

$$(e(s); 0 \le s \le 1) \overset{law}{=} (e(1-s); 0 \le s \le 1)$$

of Brownian excursion, we obtain that the law of $\left(e(s) - p^{(\sigma,1)}(s); 0 \le s \le \eta\right)$ conditionally on $(\sigma = a, \eta = x)$ is $\nu(a, x)$. The first part of Lemma 5 is proved.

For any $0 \le a \le b$ and $0 \le x \le 1$, we write

$$e = e_2 + p^{(a,1)}$$

and

$$\sigma_2 = \sigma - a = \sup\left\{\alpha \ge -a, e_2(s) \ge p^{(\alpha,1)}(s),\ \text{for } 0 \le s \le 1\right\}.$$

Conditionally on $(\sigma = b, \eta = x)$, $e - p^{(\sigma,1)}$ has the law of the concatenation of two independent processes with law $\nu(b, x)$ and $\nu(b, 1-x)$ (first part of the lemma). Since

$$e - p^{(\sigma,1)} = e_2 - p^{(\sigma_2,1)}$$

we have that conditionally on $(\sigma = b, \eta = x)$, the law of e_2 is $\nu(a, 1)$ conditioned on $(\sigma_2 = b - a, \eta = x)$. The second part of Lemma 5 follows now from the scaling property of Brownian excursion.

5 Appendix.

The key of the proof of Lemma 6 is the following result.

Lemma 8 *We write $Y(z) = W(z) + z^2/2t$ for the Brownian motion with parabolic drift and $Y^* = \inf_{z\in\mathbb{R}} Y(z)$ for its minimum. For any $a \in \mathbb{R}$ and $m \ge 0$ the law of the process $(Y(a+z) - Y(a); 0 \le z \le m)$ conditionally on $Y^* = Y(a) = Y(a+m)$ is $\nu(m, 1/t)$.*

Proof of Lemma 8.
It follows from the work of Millar [14], that the law of $(W(a+z)-W(a), 0\leq z\leq m)$ conditionally on $Y^* = Y(a)$ is the weak limit when x decreases to 0 of the conditional law

$$\mathbb{P}^x\left(\cdot \,|\bar{W}\,(z)\geq -\frac{1}{2t}z(z+2a),\ \text{for } 0\leq z\leq m\right),$$

where $\bar{W}$ is a Brownian motion starting from x under $\mathbb{P}^x$. As a consequence the law of $(W(a+z)-W(a), 0\leq z\leq m)$ conditionally on $Y(a)=Y^*=Y(a+m)$ is the weak limit when x decreases to 0 of the conditional law

$$\mathbb{P}^x\left(\cdot \,|\bar{W}\,(z)\geq -\frac{1}{2t}z(z+2a),\ \text{for } 0\leq z\leq m;\ \bar{W}\,(m)=x-\frac{1}{2t}m(m+2a)\right),$$

which is actually the weak limit when x decreases to 0 of the law of a Brownian bridge $b^{[m]}_{x\to x-\alpha}$ of duration m relying x to $x-\alpha$ (with $\alpha = m(m+2a)/2t$), conditioned to stay above the path $z\to -z(z+2a)/2t$. The same argument as in the proof of Lemma 7 ensures that the law of $b^{[m]}_{x\to x-\alpha}$ conditioned to stay above the path $z\to -z(z+2a)/2t$ is the same as the law of $\left(b^{[m]}_{x\to x}(z)-\alpha z/m; 0\leq z\leq m\right)$ with $b^{[m]}_{x\to x}$ conditioned to stay above $z\to z(m-z)/2t$. Now, in the one hand the law of a Brownian bridge $b^{[m]}_{x\to x}$ conditioned to stay above $z\to z(m-z)/2t$ is the same as the law of a Bessel Bridge $\beta^{[m]}_{x\to x}$ of duration m relying x to x conditioned to stay above $z\to z(m-z)/2t$ (see the proof of Lemma 3 for very close arguments) and in the other hand the law of $\beta^{[m]}_{x\to x}$ converges when x decreases to 0 towards the law of a Brownian excursion $e^{[m]}$ of duration m. Putting pieces together, we deduce that the pieces of Brownian motion $(W(a+z)-W(a), 0\leq z\leq m)$ conditioned by $Y(a) = Y^* = Y(a+m)$ has the law of $\left(e^{[m]}(z)-\alpha z/m; 0\leq z\leq m\right)$ where $e^{[m]}$ is conditioned to stay above $z\to z(m-z)/2t$. Lemma 8 follows now from Lemma 3 and the equality

$$Y(a+z)-Y(a)=W(a+z)-W(a)+\frac{1}{2t}z(z+2a).$$

□

Proof of Lemma 6.
Let us define for $n\in\mathbb{Z}$ the processes

$$W_-^{a_n(t)} = \left(W(z);\ -\infty < z\leq a_n\,(t)\right),$$
$$W_+^{a_n(t)} = \left(W(z+a_n\,(t))-W(a_n\,(t)); z\geq 0\right).$$

We know from the theory of splitting times (see [9] for a short introduction to splitting times) that $W_-^{a(0,t)}$ is independent of $W_+^{a(0,t)}$ conditionally on $a\,(0,t)$. We want to prove now that $W_-^{a_n(t)}$ is independent of $W_+^{a_n(t)}$ conditionally on $\mathcal{F}_t$.

For any real y, let us call $N_t\,(y)$ the index of the first cluster located at the right of y. By stationarity of W, the processes $(W\,(a_0\,(t)+z)\,; z\in\mathbb{R})$ and $\left(W\left(a_{N_t(y)}\,(t)+z\right); z\in\mathbb{R}\right)$

have the same law. In particular, $W_-^{a_{N_t(y)}(t)}$ is independent of $W_+^{a_{N_t(y)}(t)}$ conditionally on $a_{N_t(y)}(t)$. Let $(q_k; k \in \mathbb{N})$ be an enumeration of the rational numbers, and define the event

$$\left\{q_k \in A_n^t\right\} = \left\{N_t(q_k) = n; N_t(q_l) \neq n, \ \forall l < k\right\}$$

which is $\mathcal{F}_t-$measurable. For any f, g Borel functional on path, we have

$$\begin{aligned}
&\mathbb{E}\left(f\left(W_-^{a_n(t)}\right) g\left(W_+^{a_n(t)}\right) | \mathcal{F}_t\right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left(f\left(W_-^{a_{N_t(q_k)}(t)}\right) g\left(W_+^{a_{N_t(q_k)}(t)}\right); q_k \in A_n^t | \mathcal{F}_t\right) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left(\mathbb{E}\left(f\left(W_-^{a_{N_t(q_k)}(t)}\right) | \mathcal{F}_t\right) \mathbb{E}\left(g\left(W_+^{a_{N_t(q_k)}(t)}\right) | \mathcal{F}_t\right); q_k \in A_n^t\right) \\
&= \mathbb{E}\left(f\left(W_-^{a_n(t)}\right) | \mathcal{F}_t\right) \mathbb{E}\left(g\left(W_+^{a_n(t)}\right) | \mathcal{F}_t\right).
\end{aligned}$$

This ensures the independence of $W_-^{a_n(t)}$ and $W_+^{a_n(t)}$ conditionally on $\mathcal{F}_t$ and by the way the independence of the processes $\left(\varepsilon^{(x_n(t),t)} \ ; \ n \in \mathbb{Z}\right)$, conditionally on $\mathcal{F}_t$.

Let us give now the law of $\varepsilon^{(x_n(t),t)}$ conditionally on $\mathcal{F}_t$. As a consequence of the stationarity of the Brownian motion W, the process $\varepsilon^{(x_n(t),t)}$ conditioned by $a_{n-1}(t) - x_n(t) = a$ and $m_n(t) = m$ has the same law as the process $(Y(a+z) - Y(a); 0 \leq z \leq m)$ conditioned by $Y^* = Y(a) = Y(a+m)$. The conditional law of $\varepsilon^{(x_n(t),t)}$ given $\mathcal{F}_t$ is then $\nu(m_n(t), 1/t)$. □

Références

[1] M. ABRAMOWITZ and I. A. STEGUN *Handbook of mathematical functions.* Nat. Bur. Stand., Washington, 1964.

[2] M. AVELLANEDA and W. E *Statistical properties of shocks in Burgers Turbulence.* Comm. Math. Phys. 172 (1995) p. 13-38.

[3] M. AVELLANEDA *Statistical properties of shocks in Burgers Turbulence II.* Comm. Math. Phys. 169 (1995) p. 45-59.

[4] J. BERTOIN *The inviscid Burgers equation with Brownian initial velocity.* Comm. Math. Phys. 193 (1998) p. 397-406.

[5] J. BERTOIN *Clustering statistics for sticky particles with Brownian initial velocity.* J. Math. Pures Appl. 79-2 (2000) p. 173-194.

[6] J. M. BURGERS *The nonlinear diffusion equation.* Dordrecht, Reidel (1974).

[7] J. D. COLE *On a quasi linear parabolic equation occuring in aerodynamics.* Quart. Appl. Math. 9 (1951) p. 225-236.

[8] L. FRACHEBOURG and P. A. MARTIN *Exact statistical properties of the Burgers equation.* To appear in J. Fluids Mech.

[9] R. K. GETOOR *Splitting times an shift functionals.* Z. Wahrscheinlichkeitstheorie Verw. Gebiete 47 (1979) p. 69-81.

[10] P. GROENEBOOM *The concave majorant of Brownian motion.* Ann. of Proba. 11-4 (1983) p. 1016-1027.

[11] P. GROENEBOOM *Brownian motion with a parabolic drift and Airy functions.* Probability Theory and Rel. Fields 81 (1989) p.79-109.

[12] E. HOPF, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. 3 (1950) p. 201-230.

[13] N. LEONENKO *Limit theorems for random fields with singular spectrum.* Math. and Appl. (1999) Kluwers Academic Publishers.

[14] P. W. MILLAR, *A path decomposition for Markov processes.* Ann. of Proba. 6 (1978) p. 345-348.

[15] D. REVUZ, M. YOR *Continuous martingales and Brownian motion.* Grundlehren Math. Wiss. 293 Springer-Verlag (1991).

[16] R. RYAN, *Large-deviation analysis of Burgers turbulence with white-noise initial data.* Comm. Pure Applied Math. 51 (1998) p. 47-75.

[17] Z. SHE, E. AURELL and U. FRISCH, *The inviscid Burgers equation with initial data of Brownian type.* Comm. Math. Phys. 148 (1992) p. 623-641.

[18] Y. SINAI, *Statistics of shocks in solutions of inviscid Burgers Equation.* Comm. Math. Phys. 148 (1992) p. 601-621.

[19] W. A. WOYCZYNSKI, *Burgers-KPZ Turbulence, Göttingen lectures* Lectures Notes in Math. 1700, Springer 1998.

Troisième partie

Statistics of a Flux in Burgers Turbulence with One-sided Brownian Initial Data

Summary. We study the statistics of the flux of particles crossing the origin, which is induced by the dynamics of ballistic aggregation in dimension 1, under certain random initial conditions for the system. More precisely, we consider the cases when particles are uniformly distributed on $\mathbb{R}$ at the initial time, and if $u(x,t)$ denotes the velocity of the particle located at x at time t, then $u(x,0) = 0$ for $x < 0$ and $(u(x,0), x \geq 0)$ is either a white noise or a Brownian motion.

Key words. Burgers turbulence, ballistic aggregation, sticky particles, shocks, Brownian data.

A.M.S. Classification. 35 Q 53, 60 H 15, 60 J 65.

1 Introduction

This work is motivated by a question in the so-called Burgers turbulence that is perhaps more easily formulated in terms of the model of ballistic aggregation (i.e. system of free sticky particles). Specifically, consider infinitesimal particles that are uniformly spread in the one-dimensional space $\mathbb{R}$, and suppose that at the initial time, particles receive some impulse, inducing a velocity field. Then let the system evolve according to the dynamics of completely inelastic shocks. As time passes, this induces the formation of clusters (i.e. point masses); more precisely, particles clump in case of collision to form a single cluster whose mass and velocity are determined by the law of conservation of mass and momentum. Of course, the velocity of particles is unchanged as long as they are not involved in shocks.

Suppose now that the particles in $(-\infty, 0]$ are at rest at the initial time and that sufficiently many particles in $[0, \infty)$ have a negative velocity. We may observe as time passes a flux of clusters that cross the point 0 from the right to the left. Typically, if we denote by $a(0, t)$ the total mass of particles that crossed 0 up to time t, then t is a jump time of $a(0, \cdot)$ if and only if there is a cluster crossing 0 at time t, and the mass of this cluster is just the size of the jump, $a(0, t) - a(0, t-)$. Moreover, it can be seen that the velocity of this cluster is then a function of t, $a(0, t-)$ and $a(0, t)$, so that the flux is completely determined by the process $(a(0, t), t \geq 0)$.

The purpose of this work is to investigate the statistics of this flux when the initial velocities on $[0, \infty)$ are random. Specifically, we will be interested in the cases when they are given by either a white noise or a Brownian motion, which have been considered first by Burgers [5], and by Sinai [15] and She et al. [14]. We also refer to Tribe and Zaboronski [16] and Frachebourg et al. [7] for recent works on Burgers turbulence where the initial impulse is given by a Gaussian white noise supported by a finite or semi-infinite interval. Roughly, in both cases $a(0, \cdot)$ is pure jump and satisfies a time-inhomogeneous Markov property [1]; more precisely jumps occur on a discrete set of times in the case of white noise initial velocity, whereas the jump times are everywhere dense for Brownian initial velocity. Our main results specify the statistics of the jumps of $a(0, \cdot)$ in the white noise case, and its transition probabilities in the Brownian case.

The rest of this note is organized as follows. In the next section we recall some standard features on the evolution of sticky particles and its connection with the inviscid Burgers equation. Section 3 and 4 are devoted to the statements and proofs of our results in the case of white noise and Brownian initial velocities, respectively.

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1. The Markov property relies crucially on the hypothesis that at the initial time, the particles in $(-\infty, 0)$ have velocity zero, and would fail if we considered instead e.g. a two-sided Brownian motion as initial data. Indeed, it may happen in that case that at time $t > 0$, there is a cluster with mass m that passes across 0 from the left to the right. The dynamics of ballistic aggregation impose that, if the next cluster that will cross 0 arrives from the right, then it mass will be greater than m, which clearly impedes the Markov property.

2 Some background on Burgers equation

Provided that at the initial time the mass distribution of particles is given by the Lebesgue measure on $\mathbb{R}$, it is well-known (see [5], [11], [10], [6], [12], ...) that the state of the system at time $t > 0$ for the dynamics of ballistic aggregation can be completely described in terms of the initial velocity field $u(\cdot, t = 0)$ as follows.

First, we introduce the so-called initial potential $U(\cdot) = \int_0^{\cdot} u(y, 0)dy$, so for $x \geq 0$, $U(x)$ represents the initial momentum of particles located in $[0, x]$. We assume that the initial data ensure that $U(x) = o(x^2)$ as $x \to \pm\infty$. For every $x \in \mathbb{R}$ and $t > 0$, we denote by $a(x, t)$ the right-most location of the overall minimum of the function

$$z \to U(z) + \frac{(z - x)^2}{2t} ;$$

see the first figure below. Alternatively, this quantity can be viewed as the right-most location of the overall minimum of

$$z \to \int_0^z (tu(y, 0) + y - x)\, dy , \qquad z \in \mathbb{R} .$$

We shall refer to $a(x, t)$ as the *inverse Lagrangian function* evaluated at location x and time t. It corresponds to the right-most initial location of the particles that lie in $(-\infty, x]$ at time t.

For every fixed $t > 0$, $x \to a(x, t)$ is a right-continuous and increasing function, and its jumps are related to the clusters in the system. More precisely, there is a cluster located at $x \in \mathbb{R}$ at time t if and only if $a(x, t) > a(x-, t)$. By the conservation of mass and momentum, the mass of this cluster is given by the size of the jump $a(x, t) - a(x-, t)$ and its velocity by

$$u(x, t) \;=\; \frac{1}{a(x, t) - a(x-, t)} \int_{a(x-,t)}^{a(x,t)} u(y, 0)dy \;=\; \frac{1}{2t}(2x - a(x, t) - a(x-, t)) . \qquad (20)$$

See the second figure above.

When the initial velocities are identically zero on $(-\infty, 0]$ (which is assumed in this work), $a(0,t)$ represents the total mass of the particles that crossed 0 up to time t. In particular the map $t \to a(0,t)$ is also right-continuous and increasing. On the other hand, it is easily seen that its left-limit $a(0,t-)$ at $t > 0$ coincides with $a(0-,t)$. So the velocity of the cluster crossing 0 at time t (if any) can also be expressed as

$$u(0,t) \;=\; -\frac{a(0,t)+a(0-,t)}{2t} \;=\; -\frac{a(0,t)+a(0,t-)}{2t}\,, \tag{21}$$

which shows that the process $(a(0,t), t \geq 0)$ completely describes the flux of clusters crossing 0.

Let us turn our attention to the connection with the inviscid Burgers equation. If we define

$$u(x,t) \;=\; \frac{x - a(x,t)}{t} \tag{22}$$

for every point $x \in \mathbb{R}$ at which $a(\cdot,t)$ is continuous and by (20) otherwise, then u is the well-known entropy solution to the inviscid Burgers equation

$$\partial_t u + u\partial_x u \;=\; 0\,. \tag{23}$$

In particular, describing the solution at the fixed location 0 as time varies, $(u(0,t), t \geq 0)$, is the same as describing the flux of particles crossing 0 in the model of ballistic aggregation. An interesting consequence of (22) and (23) for the study of the flux is that it solves

$$\partial_t a \;+\; u\partial_x a \;=\; 0 \tag{24}$$

in the weak sense (in (23) and (24), it is of course crucial to define properly the value of $u(x,t)$ at its discontinuity points, which is the purpose of (20)). The physical interpretation of this PDE is clear: $\partial_x a$ gives the mass field and u the velocity field of the particles, so (24) is just the transport equation. Observe also that taking derivatives with respect to the spatial variable x in (24) yields the equation of conservation of masses, see e.g. [6].

3 White noise initial velocity

3.1 Main results

In this section, we focus on white noise initial velocity, i.e.

$$u(x,0) \;=\; 0 \text{ for } x \leq 0 \text{ and } u(x,0) \;=\; \frac{dW_x}{dx} \text{ for } x > 0$$

where $(W_x, x \geq 0)$ is a standard Brownian motion which thus coincides with the initial potential U. So, strictly speaking, the initial velocities $u(\cdot,0)$ are not given by a classical function, but rather by a generalized function which is the derivative of a random continuous path. However it is well known that the analysis of the inviscid Burgers equation can be extended to such singular initial data. More precisely the singularity immediately disappears, in the sense that the solution $u(\cdot, s)$ to Burgers equation is a classical function for any $s > 0$.

An important guide for our analysis is the fact that the shock structure is discrete (see [2]). This means that at time $s > 0$, all the particles are concentrated on clusters whose locations form a discrete set in space. As a consequence, $(a(0,s), s \geq 0)$ is a.s. a step process, in the sense that it increases only by jumps and the number of the jumps (i.e. of clusters of particles that cross 0) during the time interval $[\varepsilon, 1/\varepsilon]$ is finite for every $\varepsilon > 0$.

We shall show that $a(0, \cdot)$ is a time-inhomogeneous Markov process. Its one-dimensional distributions have been determined by Groeneboom [9], see also Section 3 in Frachebourg et al. [7]. To give an explicit expression, it is convenient to first introduce the following notation. We write $\mathtt{Ai}$ for the Airy function (see [1] on page 446), and $-\omega_1 > -\omega_2 > \dots$ for its zeros ranked in the decreasing order. Following Groeneboom [9], we introduce the function $g : \mathbb{R} \to \mathbb{R}_+$ which has Fourier transform

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} \varepsilon^{\mathrm{i}\xi x} g(x) dx = \frac{2^{1/3}}{\mathtt{Ai}(\mathrm{i}2^{-1/3}\xi)}, \qquad \xi \in \mathbb{R},$$

and the function $h(w, .) : \mathbb{R}_+ \to \mathbb{R}_+$ that is determined via its Laplace transform

$$\int_0^{\infty} e^{-\lambda x} h(w,x)\, dx = \frac{\mathtt{Ai}(2^{2/3}w + 2^{-1/3}\lambda)}{\mathtt{Ai}(2^{-1/3}\lambda)}.$$

There is also a representation of $h(w,m)$ as a series:

$$h(w,m) = 2^{1/3} \sum_{n=1}^{\infty} \frac{\mathtt{Ai}(2^{2/3}w - \omega_n)}{\mathtt{Ai}'(-\omega_n)} \exp\left(-2^{1/3} m \omega_n\right).$$

Then according to Corollary 3.1 in [9], we have

$$\mathbb{P}(a(0,s) \in dm) = (2s)^{-2/3} g\left((2s)^{-2/3} m\right) \left(\int_0^{\infty} h\left(w, (2s)^{-2/3} m\right) dw \right) dm.$$

By the Markov property, since the one-dimensional distributions of the process $a(0,\cdot)$ are known, the statistics of its evolution are completely determined by the family of conditional distribution of the pair $(T(s), M(s))$ given $a(0,s)$, where

$$T(s) = \inf\{t > s : a(0,t) > a(0,s)\} \quad , \quad M(s) = a(0, T(s)) - a(0,s)$$

denote the first instant after time s at which a cluster crosses 0, and the mass of this cluster. The calculations of these conditional laws rely again crucially on the work of Groeneboom [9]; more precisely, the fact that the shock structure is discrete will enable us to relate the rates of jump of the map $s \to a(0,s)$ to that of $x \to a(x,s)$ at $x = 0$ (which has been computed by Groeneboom). In order to state the result, we need to introduce the function $p : \mathbb{R}_+ \to \mathbb{R}_+$

$$p(x) \;=\; 2\sum_{k\geq 1} \exp\{-2^{1/3}\omega_k x\}\,.$$

Alternatively, p is determined by the Laplace transform

$$\int_0^\infty \varepsilon^{-\lambda x}\left(p(x) - (2\pi x^3)^{-1/2}\right) dx \;=\; 2^{2/3}\mathtt{Ai}'(2^{-1/3}\lambda)/\mathtt{Ai}(2^{-1/3}\lambda) + \sqrt{2\lambda}\,.$$

Theorem 1 *For white noise initial velocity, $a(0,\cdot)$ is a time-inhomogeneous Markov step process. For any $m, s > 0$, the conditional distribution of the instant and the size of the first jump of $a(0,\cdot)$ after time s given $a(0,s) = m$ is*

$$\begin{aligned}&\mathbb{P}\left(T(s) \in dt, M(s) \in dy \;\middle|\; a(0,s) = m\right)\\ &= \left(\frac{s}{t}\right)^{1/3} \exp\left(-\frac{m^3}{6}\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\right) \frac{y(2m+y)}{4t^3} p\left((2t)^{-2/3}y\right) \frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2s)^{-2/3}m\right)}\, dy\, dt\,,\end{aligned}$$

where $y > 0$ and $t > s$.

Specifying the above formula for $t = s$, we obtain the rates of jump of $s \to a(0,s)$:

$$\begin{aligned}&\lim_{t\to s} \frac{1}{t-s}\mathbb{P}\left(a(0,t) - a(0,s) \in dy \;\middle|\; a(0,s) = m\right)\\ &= \frac{y(2m+y)}{4s^3}\, p\left((2s)^{-2/3}y\right) \frac{g\left((2s)^{-2/3}(m+y)\right)}{g\left((2s)^{-2/3}m\right)}\, dy\,.\end{aligned}$$

Before starting to prove Theorem 1, recall from Corollary 3.4(ii) in [9] that

$$g(x) \;\sim\; 4x\exp\left\{-\frac{2}{3}x^3\right\} \qquad \text{as } x \to \infty\,,$$

which enables us to estimate the decay of the rates of jump e.g. when the size $y \to \infty$. For instance one gets for $s = 1/2$ and $a(0, 1/2) = m$ that the rate of jump of size y is of order

$$\asymp\; y^3 \exp\left(-\frac{2}{3}(y+m)^3 - 2^{1/3}\omega_1 y\right)$$

when y is large (the first root of the Airy function is $-\omega_1 \approx -2.3381$).

3.2 Proof of Theorem 1

We have already explained why $a(0,\cdot)$ must be a step process. An argument closed to that in Avellaneda and E [2] shows that the (time-inhomogeneous) Markov property follows from a general path decomposition for Markov processes due to Millar [13]. More precisely, Millar's result entails that conditionally on $a(0,s) = m$, the processes $(W_x, x \leq a(0,s))$ and

$\left(W_{a(0,s)+x} - W_{a(0,s)}, x \geq 0\right)$ are independent. For $r < s < t$, $a(0,r)$ can be viewed as the location of the minimum of $x \to W_x + x^2/2r$ for $x \leq m$, whereas $a(0,t) - a(0,s)$ coincides with the location of the minimum of $x \to W_{m+x} - W_m + (x+m)^2/2t$ for $x \geq 0$; see the figure below. The Markov property follows.

We now turn our attention to the conditional distribution of the pair $(T(s), M(s))$ given $a(0,s) = m$, that is we have to calculate the asymptotic of

$$\mathbb{P}\left(T(s) \in [t, t+h], M(s) \in [y, y+\eta] \mid a(0,s) = m\right)$$

as $h, \eta \to 0+$. As a first step, we express the latter quantity in the form $A \times B$ where

$$\begin{aligned} A &= \mathbb{P}\left(T(s) \geq t \mid a(0,s) = m\right) \\ B &= \mathbb{P}\left(T(s) \in [t, t+h], M(s) \in [y, y+\eta] \mid a(0,s) = m, T(s) \geq t\right) . \end{aligned}$$

We start with the computation of the conditional probability A. For every $z > 0$, let $\mathbb{P}^z$ denote the law of a Brownian motion $(W_x, x \geq 0)$ starting from z. On the one-hand, it is known from the work of Millar [13] that the conditional law

$$\mathbb{P}^z(\cdot \mid W_x \geq -x(x+2m)/2s \text{ for all } x \geq 0)$$

has a weak limit when z decreases to 0, which serves as the conditional distribution of the process $(W_{a(0,s)+x} - W_{a(0,s)}, x \geq 0)$ given $a(0,s) = m$. On the other hand, the analysis of the dynamics of ballistic aggregation sketched in Section 2 shows that the first instant $T(s)$ after time s at which a cluster crosses 0, occurs after time t if and only if $W_{a(0,s)+x} - W_{a(0,s)} \geq -\frac{1}{2t}x(x + 2a(0,s))$ for all $x \geq 0$. As a consequence

$$A = \mathbb{P}\left(W_{a(0,s)+x} - W_{a(0,s)} \geq -\frac{1}{2t}x(x+2a(0,s)) \text{ for all } x \geq 0 \mid a(0,s) = m\right)$$
$$= \lim_{z \downarrow 0} \frac{\mathbb{P}^z\left(W_x \geq -\frac{1}{2t}x(x+2m) \text{ for all } x \geq 0\right)}{\mathbb{P}^z\left(W_x \geq -\frac{1}{2s}x(x+2m) \text{ for all } x \geq 0\right)}$$

According to Corollary 3.1 in [9], we have for every $r > 0$ that

$$\lim_{z \downarrow 0} \frac{\partial}{\partial z}\mathbb{P}^z\left(W_x \geq -\frac{1}{2r}x(x+2m) \text{ for all } x \geq 0\right) = (2r)^{-1/3} \exp\left(\frac{m^3}{6r^2}\right) g\left((2r)^{-2/3}m\right),$$

where g has been defined before Theorem 1. By l'Hospital's rule, we conclude that the first conditional probability is

$$A = \left(\frac{s}{t}\right)^{1/3} \exp\left(-\frac{m^3}{6}\left(\frac{1}{s^2} - \frac{1}{t^2}\right)\right) \frac{g\left((2t)^{-2/3}m\right)}{g\left((2s)^{-2/3}m\right)}. \tag{25}$$

We next turn our attention to the conditional probability B. The Markov property of $s \to a(0,s)$ applied at time t shows that

$$B = \mathbb{P}\left(T(t) \in [t, t+h], M(t) \in [y, y+\eta] \mid a(0,t) = m\right).$$

We shall first estimate this quantity when $h \to 0+$ and $\eta \to 0+$, in terms of the jump rate of $x \to a(0,x)$.

Lemma 1 *We have the following estimate when $h, \eta \to 0+$:*

$$\begin{aligned} &\mathbb{P}\left(T(t) \in [t, t+h], M(t) \in [y, y+\eta] \mid a(0,t) = m\right) \\ \sim\ &\mathbb{P}\left(a((m+y/2)h/t, t) - a(0,t) \in [y, y+\eta] \mid a(0,t) = m\right). \end{aligned}$$

Let us sketch a heuristic argument for Lemma 1 (the rigorous proof is postponed to the end of this section). It is convenient to consider the left-most cluster located in $[0,\infty)$ at time t, so the location of this cluster is $\ell = \inf\{x \geq 0 : a(x,t) > a(0,t)\}$ and its mass $\mu = a(\ell,t) - a(0,t)$. On the one hand, as the shock structure is discrete, we may neglect the effects of collisions during small time-intervals. Therefore, a cluster of mass $\approx y$ and velocity $\approx v$, crosses 0 during the time-interval $[t, t+h]$ "if and only if" the left-most cluster in $[0,\infty)$ at time t is located at $\ell \in [0, -vh]$ and has mass $\mu \approx y$ and velocity $\approx v$ (roughly, this is a loose interpretation of the transport equation (24)). Recall from (21) that given $a(0,t) = m$, we must have $v \approx -(m+y/2)/t$. On the other hand, because the process $x \to a(x,t)$ is an inhomogeneous Markov step process (see [9] or [2]), we have as $h \to 0+$

$$\begin{aligned} &\mathbb{P}\left(\ell \in [0, (m+y/2)h/t] \text{ and } \mu \in [y, y+\eta] \mid a(0,t) = m\right) \\ \sim\ &\mathbb{P}\left(a((m+y/2)h/t, t) - a(0,t) \in [y, y+\eta] \mid a(0,t) = m\right). \end{aligned} \tag{26}$$

We thus arrive at the stated estimate.

In order to evaluate the quantities appearing in Lemma 1, we can invoke Theorem 4.1 of Groeneboom[2]. If we denote by $\nu_t(m,y)$ the rate of jump of size y for $x \to a(x,t)$ at $x=0$, conditionally on $a(0,t)=m$, then we have for $t=1/2$

$$\nu_{1/2}(m,y) \;=\; 2yp(y)\,\frac{g\,(m+y)}{g\,(m)}\,.$$

This means that for every $y' > y$, we have the following estimate as $h \to 0+$

$$\mathbb{P}\left(a(h,1/2)-a(0,1/2)\in[y,y'] \mid a(0,1/2)=m\right) \;\sim\; h\int_y^{y'}\nu_{1/2}(m,z)\,dz\,.$$

It is an easy matter to deduce from this the expression of $\nu_t(m,y)$ for an arbitrary $t>0$. Indeed, the scaling property of Brownian motion propagates to the inverse Lagrangian function, in the sense that for every $t>0$

$$(a(x,t), x\geq 0) \;\overset{\text{law}}{=}\; \left((2t)^{2/3}a(x(2t)^{2/3},1/2), x\geq 0\right)\,,$$

see e.g. Equation (21) in She et al. [14]. It follows readily that the conditional rates of jump at time t are given by

$$\begin{aligned}\nu_t(m,y) \;&=\; (2t)^{-4/3}\nu_{1/2}\left((2t)^{-2/3}m,(2t)^{-2/3}y\right)\\ &=\; \frac{y}{2t^2}\,p((2t)^{-2/3}y)\,\frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2t)^{-2/3}m\right)}\,.\end{aligned}$$

By Lemma 1, we get as $h,\eta \to 0+$ that

$$\begin{aligned}&\mathbb{P}\left(T(t)\in[t,t+h], M(t)\in[y,y+\eta] \mid a(0,t)=m\right)\\ \sim\;\; & h\eta\,\frac{y(2m+y)}{4t^3}\,p((2t)^{-2/3}y)\,\frac{g\left((2t)^{-2/3}(m+y)\right)}{g\left((2t)^{-2/3}m\right)}\,.\end{aligned}$$

Combining this with (25) yields the formula stated in Theorem 1.

So all that we need to complete the proof of Theorem 1 is to establish Lemma 1.

Proof of Lemma 1: We shall follow the heuristic approach sketched above; the main technical problem is to check that the effects of shocks during a small time interval can be neglected. Let us work conditionally on $a(0,t)=m$ and fix $\varepsilon>0$. Denote by $\rho(\ell,\mu,m,\varepsilon)$ the conditional probability given the location ℓ and the mass μ of the first cluster at the right of 0 at time t, that during the time interval $(t,t+\varepsilon]$ there is at least one cluster that crosses the location ℓ. By the same argument based on Corollary 3.1 in [9] as that we used to obtain (25), we see that

$$\rho(\ell,\mu,m,\varepsilon) = 1-\frac{k(t+\varepsilon,m+\mu-\ell)}{k(t,m+\mu-\ell)}\,,$$

2. In fact, Groeneboom considered a two-sided white noise as initial velocity, which induces spatial stationarity for the inviscid Burgers turbulence. Replacing the two-sided white noise by a one-sided white noise modifies the one-dimensional distributions of the process $a(\cdot,1/2)$, but not its transition probabilities nor its infinitesimal generator (this is easily seen from a perusal of Groeneboom's argument).

where

$$k(r,m) \;=\; (2r)^{-1/3}\exp\left(\frac{m^3}{6r^2}\right) g\left((2r)^{-2/3}m\right) \,.$$

Hence, for every $c>0$, if we set

$$\rho^\star(\varepsilon,m,c) : \;=\; \sup_{0\le \ell,\mu\le c} \rho(\ell,\mu,m,\varepsilon)\,,$$

then

$$\rho^\star(\varepsilon,m,c) \le \varepsilon \times \sup_{\substack{t\le\tau\le t+\varepsilon\\ 0\le\mu,\ell\le c}} \frac{|\partial_1 k(\tau,m+\mu-\ell)|}{k(t,m+\mu-\ell)}\,, \tag{27}$$

and this uniform upper bound tends to 0 as ε decreases to 0.

Next, we write for simplicity T for $T(t)$ and M for $M(t)$, i.e. T is the first instant after time t at which a cluster crosses 0, and M is the mass of this cluster. We thus have $a(0,T-) = m$ and $a(0,T) = m+M$, and by (21), the velocity of this cluster is $v = -(m+M/2)/T$. By the conservation of masses and momenta, the center of the mass of the particles that constitute this cluster is located at $-v(T-t)$ at time t. As a consequence, the location ℓ of the left-most cluster in $[0,\infty)$ at time t must fulfill

$$\ell \,\le\, (m+M/2)(T-t)/T \,\sim\, (m+M/2)(T-t)/t \qquad \text{as } T\to t+ \,.$$

Combining these two observations, we deduce the following for every fixed $0<dt\ll\varepsilon$. On the one hand, the event, say Λ, that $T\in[t,t+dt]$ and $M\in[y,y']$, is implied by the event that $\ell\in[0,(m+y/2)dt/t]$, $\mu\in[y,y']$, and that no cluster crosses ℓ during the time interval $(t,t+\varepsilon]$. This yields the lower-bound

$$\begin{aligned} & \mathbb{P}\left(T\in[t,t+dt], M\in[y,y'] \mid a(0,t)=m\right) \\ \ge\;\; & \mathbb{P}\left(\ell\in[0,(m+y/2)dt/t]\,, \mu\in[y,y'] \mid a(0,t)=m\right)\left(1-\rho^\star(\varepsilon,m,y')\right)\,. \end{aligned}$$

On the other hand, this same event Λ forces that $\ell\in[0,(m+y'/2)dt/t]$, and either that $\mu\in[y,y']$, or that $\mu<y$ and there is at least one cluster that crosses the location ℓ during the time interval $(t,t+\varepsilon]$. This yields the upper-bound

$$\begin{aligned} & \mathbb{P}\left(T\in[t,t+dt], M\in[y,y'] \mid a(0,t)=m\right) \\ \le\;\; & \mathbb{P}\left(\ell\in[0,(m+y'/2)dt/t]\,, \mu\in[y,y'] \mid a(0,t)=m)\right) \\ & +\mathbb{P}\left(\ell\in[0,(m+y'/2)dt/t]\,, \mu<y \mid a(0,t)=m)\right)\rho^\star(\varepsilon,m,y)\,; \end{aligned}$$

Finally, letting $\varepsilon\to 0+$ and then $y'\to y$, we obtain from (27) that

$$\begin{aligned} & \frac{\mathbb{P}\left(T\in[t,t+dt], M\in[y,y+dy] \;\mid a(0,t)=m\right)}{dt\,dy} \\ = & \frac{\mathbb{P}\left(\ell\in[0,(m+y/2)dt/t], \mu\in[y,y+dy] \;\mid a(0,t)=m\right)}{dt\,dy}\,. \end{aligned}$$

By (26), the proof of Lemma 1 is complete. ∎

4 Brownian initial velocity

4.1 Main results

Throughout this section, we assume Brownian initial velocities, i.e.

$$u(x,0) \;=\; 0 \text{ for } x<0 \text{ and } (u(x,0), x\geq 0) \text{ is a standard Brownian motion.}$$

One major difference with the white noise case is that now there are no rarefaction intervals, i.e. at any time $s>0$, the locations of clusters form an everywhere dense subset on $[0,\infty)$ with probability 1 (cf. Sinai [15]). We now state the main result of this work in this setting.

Theorem 2 *For Brownian initial velocities, $a(0,\cdot)$ is a time-inhomogeneous Markov process that increases only by jumps. The transition probabilities can be described as follows. For every $q,m>0$ and $0<s<t$, we have*

$$\begin{aligned}&\mathbb{E}\left(\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right)\\ =\;& \sqrt{\frac{s}{t}+\frac{t-s}{t\sqrt{2t^2q+1}}}\,\exp\left\{-\frac{m(t-s)}{st^2}\left(\sqrt{2t^2q+1}-1\right)\right\}.\end{aligned}$$

It is interesting to observe that

$$\lim_{q\to\infty}\mathbb{E}\left(\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right) \;=\; 0\,,$$

which entails that on any non-void time-interval $(s,t]$, there are clusters of particles that cross 0. This is certainly not surprising due to the absence of rarefaction intervals. We also point out that it is easy to derive from the formula in Theorem 2 the jump rates of $a(0,\cdot)$ (that is the rate of clusters with a given mass crossing zero) conditional on its current value. Specifically, we see that when t decreases to s,

$$\begin{aligned}&\frac{1}{t-s}\mathbb{E}\left(1-\exp\left\{-q\left(a(0,t)-a(0,s)\right)\right\} \mid a(0,s)=m\right)\\ \sim\;& \frac{m}{s^3}\left(\sqrt{2s^2q+1}-1\right)+\frac{1}{2s}\left(1-1/\sqrt{2s^2q+1}\right).\end{aligned}$$

Inverting the Laplace transform, we obtain the following formula for the rate of jump of size y for $a(0,\cdot)$ at time s conditionally on $a(0,s)=m$

$$\begin{aligned}&\frac{1}{t-s}\mathbb{P}\left(a(0,t)-a(0,s)\in[y,y+dy] \mid a(0,s)=m\right)\\ \sim\;& \left(\frac{2m+y}{2s}\right)\frac{1}{s\sqrt{2\pi y^3}}\exp\left(-\frac{y}{2s^2}\right)dy\,. \qquad (28)\end{aligned}$$

As a check, it may be interesting to recover (28) by the following informal argument, inspired by our approach via Lemma 1 in the white noise case. If we neglect the effects of collisions during a small time-interval, then we should have

$$\begin{aligned} & \mathbb{P}\left(a(0,t)-a(0,t-)\in[y,y+dy] \text{ for some } t\in[s,s+ds] \mid a(0,s)=m\right) \\ \sim\ & \mathbb{P}\left(a(x,s)-a(x-,s)\in[y,y+dy] \text{ for some } x\in[0,\frac{2m+y}{2s}ds] \mid a(0,s)=m\right) \end{aligned} \quad (29)$$

On the other hand, we know from Theorem 1 in [3] that the point process of the jumps of the inverse Lagrangian function, $(a(x,s)-a(x-,s), x\geq 0)$ (that is the process of the masses of clusters at time s as a function of their location) is independent of $a(0,s)$ and has the distribution of a Poisson point process on $(0,\infty)$ with characteristic measure

$$\nu_s(dy) \ = \ \frac{1}{s\sqrt{2\pi y^3}}\exp\left(-\frac{y}{2s^2}\right)\, dy\,.$$

So the probability of the right-hand side in (29) equals

$$\left(\frac{2m+y}{2s}\right) ds\, \nu_s(dy)\,,$$

which is the formula found in (28). Unfortunately, it does not seem easy to make this heuristic argument rigorous, because it does not take into account the collisions during the time interval $[s,s+ds]$ (recall that there are no rarefaction intervals a.s.).

The rest of this section is devoted to the proof of Theorem 2.

4.2 Proof of Theorem 2

Let us first establish the easiest parts of the statement. To start with, recall that with probability one, at any positive time almost every particle belongs to some clusters. See Sinai [15] or E et al. [6]. We may therefore identify $a(0,s)$ with the total mass of the clusters that crossed 0 by time s, and the latter, viewed as a process depending on s, is obviously pure jump. Next, we lift from Lemma 1 in [3] the key fact that $a(0,s)$ is a splitting time for the Brownian motion $u(\cdot,0)$. Specifically, write $\mathcal{F}_s$ for the sigma-field generated by $(u(x,0), x\leq a(0,s))$. Because $a(0,s)$ increases with s, $(\mathcal{F}_s)_{s\geq 0}$ is a filtration and the process $a(0,\cdot)$ is clearly $(\mathcal{F}_s)$-adapted. Then for each $s>0$

$$(u(x+a(0,s),0), x\geq 0) \text{ is independent of } \mathcal{F}_s. \quad (30)$$

On the other hand, the increment $a(0,t)-a(0,s)$ can be identified as the (right-most) location of the minimum of the function

$$z\to\int_0^z (tu(y+a(0,s),0)+y+a(0,s))\,dy\,, \qquad z\geq 0\,.$$

We thus see from (30) that the conditional distribution of $a(0,t)$ given $\mathcal{F}_s$ only depends on $a(0,s)$, which establishes the Markov property. So all that we need now is to calculate the (time-inhomogeneous) transition probabilities.

In this direction, we consider the solution v to the inviscid Burgers equation with initial velocity $v(\cdot,0)$ given for $x \leq 0$ by $v(x,0)=0$ and for $x \geq 0$ by

$$v(x,0) \;=\; u(x,s) - u(0,s) \;=\; u(x,s) + a(0,s)/s\,.$$

We write

$$(x,r) \;\to\; \alpha(x,r) \;=\; x - rv(x,r)$$

for the associated inverse Lagrangian function.

Lemma 2 *We have for every* $x \geq 0$

$$\frac{t}{t-s}\alpha(x+x_0, t-s) \;=\; \frac{sx}{t-s} + a(x,t)\,,$$

where

$$x_0 \,:\, = \frac{t-s}{s}a(0,s)\,.$$

Proof : It is convenient to consider first the so-called delayed solution (cf. Section 4.3 in [4]), that is for every $x \in \mathbb{R}$ and $r \geq 0$, we set $\tilde{u}(x,r) = u(x,s+r)$. Then $\tilde{u}$ is the entropy solution to the inviscid Burgers equation $\partial_t\tilde{u} + \tilde{u}\partial_x\tilde{u} = 0$ with initial velocity $\tilde{u}(\cdot,0) = u(\cdot,s)$.

Let us compare the two sticky particle systems with respective initial velocities $v(\cdot,0)$ and $\tilde{u}(\cdot,0)$. Because the initial relative velocities $v(x,0) - v(y,0) = \tilde{u}(x,0) - \tilde{u}(y,0)$ are the same for $x,y \geq 0$ and the initial velocities $\tilde{u}(\cdot,0)$ and $v(\cdot,0)$ both are non-positive on $(-\infty,0]$, it should be plain from the dynamics of the system and the interpretation of v and $\tilde{u}$ as a velocity fields that at any time $r>0$, we have the identity

$$v(x - r\tilde{u}(0,0), r) \;=\; \tilde{u}(x,r) - \tilde{u}(0,0) \;=\; u(x,s+r) - u(0,s)\,.$$

In particular, for $r = t-s$, if we set

$$x_0 \;:\; = -r\tilde{u}(0,0) \;=\; \frac{t-s}{s}a(0,s)\,,$$

then we have

$$\begin{aligned}\frac{x+x_0-\alpha(x+x_0,t-s)}{t-s} &= v(x+x_0,t-s)\\ &= u(x,t)-u(0,s)\\ &= \frac{x-a(x,t)}{t}+\frac{a(0,s)}{s}\,;\end{aligned}$$

and we arrive at the stated identity. ■

We may now use results in [3] to analyze the distribution of variables arising in Lemma 2. In this direction, we re-write the identity there for $x=0$ in the form

$$a(0,t) \;=\; a(0,s) + \tau_{s,t}(a(0,s)) + \frac{t}{t-s}\,\alpha(0,t-s)\,, \tag{31}$$

where

$$\tau_{s,t}(y) \;=\; \frac{t}{t-s}\left(\alpha\left(\frac{t-s}{s}y, t-s\right) - \alpha(0,t-s)\right) - y\,, \qquad y \geq 0\,.$$

Theorem 2 now follows from (31) and an appeal to the following lemma.

Lemma 3 (i) *The process* $(\tau_{s,t}(x), x \geq 0)$ *is independent of the variables* $a(0,s)$ *and* $\alpha(0,t-s)$. *It is a subordinator (increasing process with stationary and independent increments) with Laplace transform*

$$\mathbb{E}\left(\exp\left\{-q\tau_{s,t}(x)\right\}\right) \;=\; \exp\left\{-\frac{x(t-s)}{st^2}\left(\sqrt{2t^2q+1}-1\right)\right\}.$$

(ii) *The Laplace transform of the random variable* $t\alpha(0,t-s)/(t-s)$ *is given by*

$$\mathbb{E}\left(\exp\left\{-\frac{qt}{t-s}\alpha(0,t-s)\right\}\right) \;=\; \sqrt{\frac{s}{t}+\frac{t-s}{t\sqrt{2t^2q+1}}}\,.$$

Proof : (i) According to Theorem 1 in [3], the process $v(\cdot,0)$ is a Lévy process with no positive jumps which is independent of $a(0,s)$. Because the inverse Lagrangian function $\alpha(\cdot,t-s)$ is measurable with respect to $v(\cdot,0)$, it is also independent of $a(0,s)$. Then Theorem 2 in [3] applies to the initial velocity $v(\cdot,0)$, and we get that the the process $\alpha(\cdot,t-s)$ is a subordinator started from $\alpha(0,t-s)$. We thus see that the process $\tau_{s,t}(\cdot)$ is independent of $a(0,s)$ and $\alpha(0,t-s)$, and has stationary and independent increments.

Then it follows from Lemma 2 that increments of $\tau_{s,t}$ have the same distribution as those of $x \to a(sx/(t-s),t)$. According to Theorem 1 in [3] (and the scaling property), its Laplace transform is given by2

$$\mathbb{E}\left(\exp\left\{-q\left(a(x,t)-a(0,t)\right)\right\}\right) \;=\; \exp\left\{-\frac{x}{t^2}\left(\sqrt{2t^2q+1}-1\right)\right\},$$

and the formula for the Laplace transform of $\tau_{s,t}(x)$ follows.

(ii) We can use the identity (31) and the independence stated in part (i) to determine the Laplace transform of $t\alpha(0,t-s)/(t-s)$ in terms of the other variables. Specifically, we get

$$\mathbb{E}\left(\exp\left\{-qt\alpha(0,t-s)/(t-s)\right\}\right) \;=\; \frac{\mathbb{E}\left(\exp\left\{-qa(0,t)\right\}\right)}{\mathbb{E}\left(\exp\left\{-qa(0,s)-q\tau_{s,t}(a(0,s))\right\}\right)}\,.$$

On the one hand, the Laplace exponent of the subordinator $\tau_{s,t}$ has been given in part (i). On the other hand, we know from Theorem 1 in [3] (and the scaling property of Brownian motion) that for every $r>0$, $a(0,r)$ has a Gamma$(1/4, 1/2r^2)$ distribution, i.e. its Laplace transform is given by

$$\mathbb{E}\left(\exp\left\{-qa(0,r)\right\}\right) \;=\; \left(2r^2q+1\right)^{-1/4}\,.$$

Straightforward (but lengthy) calculations then yield the stated formula.

We mention that there is also an alternative proof which we now sketch. Because $v(\cdot, 0)$ is a Lévy process with no positive jumps, we can also determine the distribution of $\alpha(0, t-r)$ by adapting the argument of [3] on pages 402-403, which deals with Brownian initial velocities. Specifically, one gets that if $\tilde{\alpha}(0, t-r)$ is an independent copy of $\alpha(0, t-r)$, then the sum $\alpha(0, t-r) + \tilde{\alpha}(0, t-r)$ has the same distribution as the last passage time

$$\ell_{s,t} \ := \ \sup \{x \geq 0 : (t-s)v(x,0) + x = 0\} \ .$$

So all that is needed is to calculate the Laplace transform of $\ell_{s,t}$, which is an easy task since the distribution of the Lévy process $v(\cdot, 0)$ is known explicitly. One finds

$$\mathbb{E}(\exp\{-q\ell_{s,t}\}) \ = \ \frac{s}{t} + \frac{t-s}{t\sqrt{2qt(t-s)+1}} \, ,$$

so

$$\mathbb{E}(\exp\{-q\alpha(0,t-r)\}) \ = \ \sqrt{\frac{s}{t} + \frac{t-s}{t\sqrt{2qt(t-s)+1}}} \, ,$$

and we recover the formula (ii). ∎

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Références

[1] M. Abramowitz and I. A. Stegun : *Handbook of Mathematical functions.* Nat. Bur. Stand, Washington, 1964.

[2] M. Avellaneda and W. E : Statistical properties of shocks in Burgers turbulence. *Commun. Math. Phys.* **172** (1995), 13-38.

[3] J. Bertoin : The inviscid Burgers equation with Brownian initial velocity, *Comm. Math. Phys.* **193** (1998), 397-406.

[4] J. Bertoin : Clustering statistics for sticky particles with Brownian initial velocity. *J. Math. Pures Appl.* **79** (2000), 173-194.

[5] J. M. Burgers : *The nonlinear diffusion equation.* Dordrecht, Reidel, 1974.

[6] W. E, Yu. G. Rykov and Ya. G. Sinai : Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics. *Commun. Math. Phys.* **177** (1996), 349-380.

[7] L. Frachebourg, V. Jacquemet and Ph. A. Martin : Inhomogenenous ballistic aggregation. Preprint (2001).

[8] L. Frachebourg and Ph. A. Martin : Exact statistical properties of the Burgers equation. *J. Fluid Mech.* **417** (2000), 323-349.

[9] P. Groeneboom : Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Relat. Fields* **81** (1989), 79-109.

[10] S. Kida : Asymptotic properties of Burgers turbulence. *J. Fluid Mech.* **93** (1979), 337-377.

[11] P. D. Lax : *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.* Society for Industrial and Applied Mathematics, Philadelphia, 1973.

[12] Ph. A. Martin and J. Piasecki : One dimensional ballistic aggregation : Rigorous long-time estimates. *J. Stat. Phys.* **76** (1994), 447-476.

[13] P. W. Millar : A path decomposition for Markov processes. *Ann. Probab.* **6** (1978), 345-348.

[14] Z. S. She, E. Aurell and U. Frisch : The inviscid Burgers equation with initial data of Brownian type. *Commun. Math. Phys.* **148** (1992), 623-641.

[15] Ya. Sinai : Statistics of shocks in solution of inviscid Burgers equation. *Commun. Math. Phys.* **148** (1992), 601-621.

[16] R. Tribe and O. Zaboronski : On the large time asymptotics of decaying Burgers turbulence. *Commun. Math. Phys.* **212** (2000), 415-436.

[17] W. A. Woyczyński : *Göttingen Lectures on Burgers-KPZ turbulence.* Lecture Notes in Maths, Springer, 1998.

Quatrième partie

Clustering in a Self-Gravitating One-Dimensional Gas at Zero Temperature

Abstract : We study a system of gravitationally interacting sticky particles. At the initial time, we have n particles, each with mass $1/n$ and momentum 0, independently spread on $[0, 1]$ according to the uniform law. Due to the confining of the system, all particles merge into a single cluster after a finite time. We give the asymptotic laws of the time of the last collision and of the time of the k-th collision, when $n \to \infty$. We prove also that clusters of size k appear at time $\sim n^{-1/2(k-1)}$. We then investigate the system at a fixed time $t < 1$. We show that the biggest cluster has size of order $\log n$, whereas a typical cluster is of finite size.
Key words : sticky particles, gravitational interacting, uniform law, Brownian bridge.
A.M.S. classification : 70F10, 70F45, 60G50.

1 Introduction

The dynamics of gravitationally interacting sticky particles are a model that has been suggested by Zeldovich [11] and other authors to investigate the formation of large scale structure in the universe, see [10] for a survey article. We focus in this paper on the one-dimensional case. Sticky particles are particles which collide in a completely inelastic way. When particles meet, they form a new massive particle with conservation of mass and momentum. More precisely, when particles with mass m_i and m_j collide, they merge into a single particle with mass $m_i + m_j$, which follows the trajectory of their center of mass. It must be noticed that in the one dimensional case, only the nearest neighboring particles can collide. Following [1, 4], we consider the case when these sticky particles are attracting each other with forces proportional to the product of their masses, independently of the interparticle distance. Rigorously, the dynamic between collisions is governed by the Hamiltonian

$$H = \sum_i \frac{p_i^2}{2m_i} + \gamma \sum_{i \neq j} m_i m_j |x_i - x_j|,$$

where x_i, m_i, p_i denote the location, mass and momentum of the particle i and γ is the gravitational constant. The acceleration of a particle is then proportional to the difference between the total masses at its right and at its left. It is to be mentioned that these dynamics give rise to global weak solutions to the system of conservation laws

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2) = -\gamma\, \rho\, \partial_x \Phi \\ \partial_{xx}\Phi = \rho \end{cases}$$

where $\rho(x,t), u(x,t), \Phi(x,t)$ are meant to represent the density, velocity and gravitational potential at x at time t. This connection occurs when the initial density $\rho(.,0)$ is a purely singular measure supported on a finite or countable set, see [3] for further explanations.

The subject that is of interest in the study of gravitationally interacting sticky particles is the mass distributions induced by small perturbations of an initial homogeneous state. When a finite number of identical particles are initially spread on the regular lattice sites $x_j = ja,\ j = 1 \dots n$, with mass m and momentum 0, they all merge simultaneously into a single cluster at the characteristic time $t^* = 1/\sqrt{\gamma\rho}$, where $\rho = m/a$, see [4] section 3. Martin, Piasecki, Bonvin and Zotos [1, 4] and also Suidan [9] have focused on the evolution of the system, when the initial homogeneous state is perturbed by introducing random uncorrelated velocities to the initial particles. More specifically, they mainly work with a system of n initial particles located on the lattice sites $x_j = ja,\ j = 1 \dots n$, with mass m and uncorrelated velocities distributed according to the Gaussian law. They study the statistics of the continuum limit $a \to 0$, while keeping $\rho = m/a$ constant. In the concluding remarks of [4], they raise the question of the evolution of a system, in which the perturbation of the initial homogeneous state should result from the randomization of the initial locations of the particles. This question mainly motivates the present work.

We focus thenceforth on a system of initially n gravitationally interacting sticky particles with mass $1/n$, momentum 0 and which are independently spread on $[0, 1]$ according to the uniform law. Up to the change of time $t' = \sqrt{\gamma} t$, we can fix $\gamma = 1$. It should be mentioned that the evolution of the system is isomorphic to the evolution of n gravitationally interacting sticky particles with unit mass and momentum 0, independently spread at the initial time on $[0, n]$ according to the uniform law. We shall mainly investigate here the asymptotics of the statistics when n tends to infinity.

We start with giving in section 2 some material needed to study the system. In section 3 and 4, we specify the asymptotic laws of the first and last collisions, as well as the time scale of appearance of a cluster of finite size k. In section 5, we determine the size of scale of the biggest cluster at a fixed time t. Some elements on the evolution of a marked particle are presented in the last section.

2 Preliminaries

2.1 Analyzing the System

We shall give in this section some key results for our analysis. We first properly define the system. For any $n \geq 1$, $(X_{n,i}, i = 1, \ldots, n)$ shall denote n independent random variables with uniform law on $[0, 1]$. We write $0 \leq X_{n:1} \leq \cdots \leq X_{n:n} \leq 1$ for the ordered statistics. We consider henceforth a system of n particles of mass $1/n$ spread at the initial time on the sites $X_{n,i}$ with momentum 0.The particle initially located at $X_{n:i}$ should be called the i-th particle. These particles are assumed to evolve as time runs according to the dynamics of gravitationally interacting sticky particles described previously.

Our investigations are mainly based on an analysis made independently by Martin and Piasecki in [4] (see also [1]) and E, Rykov, Sinai in [3]. Let us consider the k particles $(i + 1, \ldots, i + k)$. Recall that masses and momenta are conserved during collisions and that the acceleration acting on these k particles is equal to the difference of the masses at their right and at their left. As a consequence, if these k particles have not collide with surrounding particles before time t, their center of mass $G(i + 1, \ldots, i + k)$ follows the trajectory (see for close formulaes (11) in [1] and (1-19) in [3])

$$G(i+1, \ldots, i+k)(s) = G(i+1, \ldots, i+k)(0) + \frac{(n - (k+2i))s^2}{2n} \quad \text{for any } s \leq t.$$

We shall state now the key of the analysis of the system : a necessary and sufficient condition for the k particles $(i + 1, \ldots, i + k)$ to merge into a single cluster of size k at time t, is that these particles did not collide with surrounding particles before time t and that for any partition into two subclusters $(i+1, \ldots, i+r)$ and $(i+r+1, \ldots, i+k)$, the centers of mass of these subclusters cross before time t (see formula (6) in [4] and also formula (1-12) in [3]). In particular, the condition

$$G(i+1, \ldots, i+r)(t) \geq G(i+r+1, \ldots, i+k)(t), \quad \text{for } r = 1, \ldots, k-1 . \tag{32}$$

is a necessary condition for the merging of $(i + 1, \ldots, i + k)$ into a single cluster of size k before time t. It is also a sufficient condition for the merging of $(i + 1, \ldots, i + k)$ into a cluster of size at least k. Indeed, this results from the fact that when the cluster $(i-l,\dots,i)$ collide with the cluster $(i+1,\dots,1+r)$ at time s, their trajectories cross and for $t \ge s$

$$G(i-l,\dots,i)(t) \ge G(i-l,\dots,i+r)(t) \ge G(i+1,\dots,i+r)(t),$$

see lemma 6 in [3] (and also lemmas 2 and 3 there) for very close arguments. Expressing condition (32) in terms of the initial locations of the particles gives

$$\frac{1}{r}\sum_{j=1}^{r} X_{n:i+j} - \frac{1}{k-r}\sum_{j=r+1}^{k} X_{n:i+j} + \frac{kt^2}{2n} \ge 0, \qquad \text{for } r = 1,\dots,k-1\,,$$

which is not easily amenable to mathematical analysis. We thus look for (weaker) necessary and sufficient conditions which only involve $X_{n:i+k} - X_{n:i+1}$. First, it follows from the inequality

$$X_{n:i+1} - X_{n:i+k} \le \frac{1}{r}\sum_{j=1}^{r} X_{n:i+j} - \frac{1}{k-r}\sum_{j=r+1}^{k} X_{n:i+j}\,,$$

that a sufficient condition for the merging of $(i+1,\dots,i+k)$ into a cluster of size at least k before time t is

$$X_{n:i+1} - X_{n:i+k} + \frac{kt^2}{2n} \ge 0\,. \tag{33}$$

Second, when $(i+1,\dots,i+k)$ merge into a cluster of size k before time t, the trajectory $s \to G(i+1)(s)$ of the particle $i+1$ crosses the trajectory $s \to G(i+1,\dots,i+k)(s)$ before time t, which implies that $G(i+1)(t) \ge G(i+1,\dots,i+k)(t)$. For the same reasons, the inequality $G(i+k)(t) \le G(i+1,\dots,i+k)(t)$ holds, which enables us to formulate a simplier necessary condition for the merging of $(i+1,\dots,i+k)$ into a single cluster of size k before time t as

$$X_{n:i+1} - X_{n:i+k} + \frac{kt^2}{n} \ge 0\,. \tag{34}$$

We next want to give necessary and sufficient conditions for the existence of a cluster of size at least k at time t. It follows from the previous analysis that a sufficient condition is

$$\begin{aligned} &\exists\, i \in \{0,\dots,n-k\} \text{ such that for } r = 1,\dots,k-1, \\ &\qquad \frac{1}{r}\sum_{j=1}^{r} X_{n:i+j} - \frac{1}{k-r}\sum_{j=r+1}^{k} X_{n:i+j} + \frac{kt^2}{2n} \ge 0\,, \end{aligned} \tag{35}$$

and thus it suffices that

$$\exists\, i \in \{0,\dots,n-k\} \text{ such that } X_{n:i+1} - X_{n:i+k} + \frac{kt^2}{2n} \ge 0\,. \tag{36}$$

Let us give now a necessary condition. There exists a cluster of size at least k at time t if and only if, for some time $s \le t$ there exists a cluster of size between k and $2k$, which only occurs if there exists $s \le t$, $k \le p \le 2k$, and $i \in \{0, \dots, n-p\}$ such that for any $r \in \{1, \dots, n-p\}$, $G(i+1, \dots, i+r)(s) \ge G(i+r+1, \dots, i+p)(s)$. A necessary condition is thus

$$\exists\, k \le p \le 2k \text{ and } i \in \{0, \dots, n-p\} \text{ such that for } r = 1, \dots, p-1,$$
$$\frac{1}{r}\sum_{j=1}^{r} X_{n:i+j} - \frac{1}{p-r}\sum_{j=r+1}^{p} X_{n:i+j} + \frac{pt^2}{2n} \ge 0\,, \tag{37}$$

and a fortiori

$$\exists\, k \le p \le 2k \text{ and } i \in \{0, \dots, n-p\} \text{ such that } X_{n:i+1} - X_{n:i+p} + \frac{pt^2}{n} \ge 0\,. \tag{38}$$

Armed of this collection of necessary or sufficient conditions, we are ready to start our investigations, after recalling some basic features on empirical and quantile processes.

2.2 Uniform Empirical and Quantile Processes

Our study involves some basic and some more advanced results on the uniform quantile process. For the convenience of the reader, we recall a few basic facts in this field. We refer to [8] for a classical text-book. We associate to the n i.i.d. random variables $(X_{n,i}, i = 1 \dots n)$ with uniform law on $[0,1]$, the uniform empirical distribution function

$$G_n(t) :\, = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}_{\{X_{n,i} \le t\}}$$

and its left-continuous inverse $G_n^{-1}(t) :\, = \inf\{x : G_n(x) \ge t\}$. It is easily seen that

$$G_n^{-1}(t) = X_{n:i} \qquad \text{for } \frac{i-1}{n} < t \le \frac{i}{n},$$

and $G_n^{-1}(0) = 0$, where $X_{n:i}$ denotes the ordered statistics. The law of large numbers implies that $G_n(t) \stackrel{n\to\infty}{\to} t$ a.s. and as a consequence $G_n^{-1}(t) \stackrel{n\to\infty}{\to} t$ a.s. Actually, if I is the identity function on $[0,1]$, it follows from the central limit theorem, that the so-called uniform empirical process $u_n :\, = \sqrt{n}\,(G_n^{-1} - I)$ converges in law towards a Brownian bridge, when n tends to infinity. In analogy with the uniform empirical process, we define the uniform quantile process as $v_n = \sqrt{n}\,(G_n^{-1} - I)$, i.e.

$$v_n(t) = \sqrt{n}\,(X_{n:i} - t), \qquad \text{for } \frac{i-1}{n} < t \le \frac{i}{n},$$

and $v_n(0) = 0$. In light of the formula $v_n = -u_n(G_n^{-1}) + \sqrt{n}\,(G_n \circ G_n^{-1} - I)$, it is easily seen that the uniform quantile process converges in law towards a Brownian bridge, when n tends to infinity. Furthermore, the present work also relies on some more advanced properties on the ordered statistics related to the modulus of continuity of the uniform quantile process, cf. section 14-7 in [8].

3 Last Collision

A consequence of the confining of the system is that every particles have merged into a single cluster after a finite time. We have already recall that when particles are initially located on the lattice sites i/n, they all collide simultaneously at time 1. We study in this section the effect on the last collision of a randomization of the initial location. We show that in this case the last collision occurs between two macroscopic clusters at time of order $1 + O(1/\sqrt{n})$.

Theorem 1 *In the n-particles system, the last collision occurs a.s. between two macroscopic clusters at time $T_n^{l.c.}$ which follows the convergence in law when $n \to \infty$*

$$\sqrt{n}\left(T_n^{l.c.} - 1\right) \overset{law}{\to} \sup_{x\in[0,1]} \left(\frac{1}{1-x}\int_x^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt \right),$$

where b denotes a Brownian bridge.

It should be noticed that the law of

$$\sup_{x\in[0,1]} \left(\frac{1}{1-x}\int_x^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt \right)$$

is not degenerated since we have the inequalities

$$|\int_0^1 b(t)\,dt| \leq \sup_{x\in[0,1]} \left(\frac{1}{1-x}\int_x^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt \right) \leq 2 \sup_{x\in[0,1]} |b(t)|\,.$$

Proof of Theorem 1
We first focus on the time $T_n^{l.c.}$ of last collision. According to condition (32), the last collision occurs before time t if and only if

$$\text{for } r = 1,\dots,n-1\,, \qquad \frac{1}{n-r}\sum_{i=r+1}^{n} X_{n:i} - \frac{1}{r}\sum_{i=1}^{r} X_{n:i} \leq \frac{t^2}{2}\,,$$

which implies

$$\left(T_n^{l.c.}\right)^2 = 2 \sup_{r=1,\dots,n-1} \left(\frac{1}{n-r}\sum_{i=r+1}^{n} X_{n:i} - \frac{1}{r}\sum_{i=1}^{r} X_{n:i} \right).$$

We want to express the time of last collision in terms of the uniform quantile process that has been introduced in section 2-2. It follows from the equalities

$$\int_0^{r/n} v_n(t)\,dt = \sqrt{n}\sum_{i=1}^{r} \frac{1}{n} X_{n:i} - \sqrt{n}\,\frac{r^2}{2n^2}$$

and

$$\int_{r/n}^{1} v_n(t)\,dt = \sqrt{n}\sum_{i=r+1}^{n}\frac{1}{n}X_{n:i} - \sqrt{n}\left(\frac{1}{2}-\frac{r^2}{2n^2}\right)$$

that

$$\left(T_n^{l.c.}\right)^2 = 1 + \frac{2}{\sqrt{n}}\sup_{r=1,\dots,n-1}\left(\frac{1}{1-r/n}\int_{r/n}^{1} v_n(t)\,dt - \frac{n}{r}\int_0^{r/n} v_n(t)\,dt\right).$$

Recall that v_n converges in law to a Brownian bridge b when $n \to \infty$ (see e.g. [8] Chap. 3). We shall prove now the convergence

$$\sqrt{n}\left(\left(T_n^{l.c.}\right)^2 - 1\right) \overset{\text{law}}{\to} 2\sup_{x\in[0,1]}\left(\frac{1}{1-x}\int_x^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt\right) \qquad \text{when } n\to\infty\,, \quad (39)$$

from which follows the convergence given in Theorem 1, since $\left(T_n^{l.c.}\right)^2 - 1 = \left(T_n^{l.c.}+1\right)\left(T_n^{l.c.}-1\right)$ and $T_n^{l.c.} \to 1$ in probability.

In order to prove (39) we define for any left continuous with right limits functions v

$$f_n(v) :\,= \sup_{r=1,\dots,n-1}\frac{1}{1-r/n}\int_{r/n}^{1} v(t)\,dt - \frac{n}{r}\int_0^{r/n} v(t)\,dt$$

$$\text{and} \quad f_\infty(v) :\,= \sup_{x\in[0,1]}\frac{1}{1-x}\int_x^1 v(t)\,dt - \frac{1}{x}\int_0^x v(t)\,dt\,.$$

We shall prove that for any Lipschitz bounded function g,

$$\mathbb{E}(g(f_n(v_n))) \overset{n\to\infty}{\to} \mathbb{E}(g(f_\infty(b)))\,.$$

By the triangle inequality

$$\begin{aligned}&|\mathbb{E}(g(f_n(v_n))) - \mathbb{E}(g(f_\infty(b)))|\\ &\le |\mathbb{E}(g(f_\infty(v_n)) - g(f_n(v_n)))| + |\mathbb{E}(g(f_\infty(v_n)) - g(f_\infty(b)))|\,.\end{aligned} \quad (40)$$

The second term tends to 0 when $n\to\infty$, since $g\circ f_\infty$ is a continuous bounded functional and $v_n \overset{\text{law}}{\to} b$. We focus now on the first term. For $\frac{r-1}{n} < x \le \frac{r}{n}$ we have the inequalities

$$\begin{aligned}|\frac{1}{x}\int_0^x v_n(t)\,dt - \frac{n}{r}\int_0^{r/n} v_n(t)\,dt| &\le |\frac{1}{x}-\frac{n}{r}|\int_0^x |v_n(t)|\,dt + \frac{n}{r}\int_x^{r/n}|v_n(t)|\,dt\\
&\le \left(\frac{n}{r-1}-\frac{n}{r}\right)\int_0^{r/n}|v_n(t)|\,dt + \frac{n}{r}\int_{(r-1)/n}^{r/n}|v_n(t)|\,dt\\
&\le \frac{1}{r-1}\sup_{t\in[0,r/n]}|v_n(t)| + \frac{1}{r}\sup_{t\in[(r-1)/n,r/n]}|v_n(t)|\\
&\le \begin{cases} 2\sup_{t\in[0,\sqrt{n}/n]}|v_n(t)| & \text{for } 2\le r\le\sqrt{n}\\ 2\sup_{t\in[0,1]}|v_n(t)|/\sqrt{n} & \text{for } r\ge\sqrt{n}+1\end{cases}\\
&\le \frac{2}{\sqrt{n}}\sup_{t\in[0,1]}|v_n(t)| + 2\sup_{t\in[0,\sqrt{n}/n]}|v_n(t)| =: K_n\,,\end{aligned}$$

and this upper bound remains true for $r = 1$. For any $\varepsilon > 0$, we can write

$$\begin{aligned}|\mathbb{E}(g(f_n(v_n)) - g(f_\infty(v_n)))| &\le |\mathbb{E}(g(f_n(v_n)) - g(f_\infty(v_n)); K_n \le \varepsilon)| + 2\,||g||_\infty \mathbb{P}(K_n > \varepsilon) \\ &\le M\varepsilon + 2\,||g||_\infty \mathbb{P}(K_n > \varepsilon)\,,\end{aligned}$$

where $||g||_\infty = \sup_{t\in\mathbb{R}} |g(t)|$ and M is the Lipschitz modulus of g. We take first the limit $n \to \infty$, and then $\varepsilon \to 0$ to obtain

$$\lim_{n\to\infty} \mathbb{E}(g(f_n(v_n)) - g(f_\infty(v_n))) = 0.$$

Substituting this result in the inequality (40) gives formula (39). The proof of the convergence in law of $T_n^{l.c.}$ is complete.

Let us check now the first assertion of Theorem 1. The asymptotic masses of the two last clusters are given by the abscissa x_0 for which

$$\alpha(x) = \frac{1}{1-x}\int_x^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt$$

reaches its maximum. Indeed, the asymptotic masses of the two last clusters are given by x_0 and $1 - x_0$. All that we need is to show that x_0 is different from 0 and 1 with probability 1. We can write

$$\alpha(x) = \frac{1}{1-x}\left(\int_0^1 b(t)\,dt - \frac{1}{x}\int_0^x b(t)\,dt\right).$$

Suppose for example that $\int_0^1 b(t)\,dt$ is positive. Then $\alpha(1) \le 0 \le \alpha(0)$. We shall prove that with probability 1 there exists $X \in]0,1[$ such that $\int_0^X b(t)\,dt < 0$, which implies in particular that $\alpha(X) > \alpha(0)$ and finally that x_0 is different from 0 and 1.

The process $I(x) = \int_0^x b(t)\,dt$ is symmetric and adapted to the filtration $\mathcal{F}_x = \sigma(b(t)\,;\ t \le x)$. Let A denotes the event

$$A :\, = \bigcap_{p\ge 1} \bigcup_{n\ge p} \{I(1/n) < 0\}\,,$$

that belongs to the σ-field

$$\bigcap_{x>0} \mathcal{F}_x$$

which is trivial. The probability of the event A is therefore 0 or 1. Since A is the decreasing limit when $p \to \infty$ of the events

$$\bigcup_{n\ge p} \{I(1/n) < 0\}$$

and since

$$\mathbb{P}\left(\bigcup_{n\ge p} \{I(1/n) < 0\}\right) \ge \mathbb{P}(I(1/p) < 0) = \frac{1}{2}$$

the probability of the event A is larger than 1/2 and hence is 1. As a consequence I takes negative value after 0 with probability 1.

4 First aggregations

We now turn our attention on the first aggregations. We start by determining the scale of size of the time of appearance of a cluster of size k in a n-particles system.

Theorem 2 *Let k be an integer, and t_n a sequence of positive time. When $n \to \infty$, the probability that there exists a cluster of size at least k at time t_n among a n-particles system tends to 0 if $n^{1/2(k-1)} t_n \to 0$ and tends to 1 if $n^{1/2(k-1)} t_n \to \infty$.*

A notable consequence of Theorem 2 is the asymptotic laws of the times $T_{n:j}$ of j-th collision.

Corollary 2.1 *For any integer k, the k-uplet $(\sqrt{n}T_{n:1}, \ldots, \sqrt{n}T_{n:k})$ converges in law to $(\sqrt{\mathsf{e}_1}, \ldots, \sqrt{\mathsf{e}_1 + \cdots + \mathsf{e}_k})$, where $\mathsf{e}_1, \ldots, \mathsf{e}_k$ are independent random variables with exponential law of parameter 1.*

The evolution of the system at small times may be thus describe for large n as follows. Particles start to aggregate pairwise at time $\asymp n^{-1/2}$. At time $\asymp n^{-1/4}$ clusters of size 3 appear, whereas we shall wait time of order $n^{-1/6}$ to see clusters of size 4, and so on. The fact that clusters of size 3 appear before clusters of size 4 may be physically explained by the few number ($\asymp n^{1/4}$) of clusters at time $\asymp n^{-1/4}$, so that the probability they meet together is infinitesimal.

Proof of Theorem 2
We first give an upper bound to the probability of existence of a cluster of size at least k at time t_n. Recall by condition (38), that this probability is less than the probability of the event

$$\exists\, k \le p \le 2k \ \text{ and } i \in \{0, \ldots, n-p\} \ \text{ such that } \ X_{n:i+1} - X_{n:i+p} + \frac{pt_n^2}{n} \ge 0\,.$$

It is known (see e.g. prop. 8-2-1 in [8]) that

$$(X_{n:i}\,; i = 1, \ldots, n) \overset{\text{law}}{\sim} \left(\frac{\mathsf{e}_1 + \cdots + \mathsf{e}_i}{\Gamma_{n+1}}\,; i = 1, \ldots, n \right),$$

where $(\mathsf{e}_i\,; i = 1, \ldots, n+1)$ are independent exponential variables of parameter 1 and $\Gamma_{n+1} = \mathsf{e}_1 + \cdots + \mathsf{e}_{n+1}$. We rexpress the previous condition in terms of e_i :

$$\exists\, k \le p \le 2k \ \text{ and } i \in \{0, \ldots, n-p\} \ \text{ such that } \mathsf{e}_{i+2} + \cdots + \mathsf{e}_{i+p} \le \frac{\Gamma_{n+1}}{n} pt_n^2.$$

We thus have for upper bound to the probability of existence of a cluster of size at least k at time t_n

$$\mathbb{P}\left(\frac{\Gamma_{n+1}}{n} > 2\right) + \sum_{p=k}^{2k} \sum_{i=0}^{n-p} \mathbb{P}\left(\mathsf{e}_{i+2} + \cdots + \mathsf{e}_{i+p} \le 2pt_n^2\right) \le$$

$$n \sum_{p=k}^{2k} \mathbb{P}\left(\mathsf{e}_1 + \cdots + \mathsf{e}_{p-1} \le 2pt_n^2\right) + \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} > 2\right).$$

Since $\Gamma_{n+1}/n \overset{\text{a.s.}}{\to} 1$ according to the law of large number and

$$\mathbb{P}\left(\mathsf{e}_1 + \cdots + \mathsf{e}_p \le \lambda\right) = e^{-\lambda} \sum_{j=p}^{\infty} \frac{\lambda^j}{j!} \overset{\lambda \to 0}{\sim} \frac{\lambda^p}{p!}\,, \tag{41}$$

the probability of existence of a cluster of size at least k at time t_n tends to 0 when $n\, t_n^{2(k-1)} \to 0$ (or in other words $n^{1/2(k-1)}\, t_n \to 0$). The first part of Theorem 2 follows.

We give now a lower bound to the probability of existence of a cluster of size at least k at time t_n. Using the condition (36), one notices that the latter is larger than the probability of the event

$$\exists\, i \le \frac{n-k}{k} \;\text{ such that }\; X_{n:ki+1} - X_{n:ki+k} + \frac{kt_n^2}{2n} \ge 0\,,$$

which itself is at least

$$\mathbb{P}\left(\exists\, i \le \frac{n-k}{k} / \sum_{j=2}^{k} \mathsf{e}_{ki+j} \le \frac{kt_n^2}{4}\right) - \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \le \frac{1}{2}\right).$$

Since for $i \le (n-k)/k$ the events

$$\left\{\sum_{j=2}^{k} \mathsf{e}_{ki+j} \le \frac{kt_n^2}{4}\right\}$$

are independent and identically distributed, we have

$$\mathbb{P}\left(\exists\, i \le \frac{n-k}{k} / \sum_{j=2}^{k} \mathsf{e}_{ki+j} \le \frac{kt_n^2}{4}\right) = 1 - \left(1 - \mathbb{P}\left(\mathsf{e}_1 + \cdots + \mathsf{e}_{k-1} \le kt_n^2/4\right)\right)^{[(n-k)/k]},$$

where $[x]$ denotes the integer part of x. If t_n tends to 0 and $n\, t_n^{2(k-1)} \to \infty$ we obtain with formula (41)

$$\mathbb{P}\left(\exists\, i \le \frac{n-k}{k}, \quad \sum_{j=2}^{k} \mathsf{e}_{ki+j} \le \frac{kt_n^2}{4}\right) \overset{n\to\infty}{=} 1 - \exp\left(-\left[\frac{n-k}{k}\right] \frac{(kt_n^2/4)^{k-1}}{(k-1)!}(1+o(1))\right)$$
$$\overset{n\to\infty}{\to} 1\,.$$

Now the law of large numbers ensures that

$$\mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \le \frac{1}{2}\right) \to 0, \qquad \text{when } n \to \infty\,,$$

so putting pieces together we obtain the second part of Theorem 2.

Proof of Corollary 2.1

Let us call $T_n^{(3)}$ the time of appearance of the first cluster of size at least 3. The particles j and $j+1$ merge into a single cluster at time $t < T_n^{(3)}$ if and only if

$$X_{n:j+1} - X_{n:j} \leq \frac{t^2}{n}\,.$$

If we write $\delta_{n:i}$ for the i-th smallest spacing between the $X_{n:j+1}$ and $X_{n:j}$, we have the equalities $T_{n:1}^2 = n\delta_{n:1}, \dots, T_{n:k}^2 = n\delta_{n:k}$ on the event $\left\{T_{n:k} < T_n^{(3)}\right\}$ or equivalently on the event $\left\{n\delta_{n:k} < \left(T_n^{(3)}\right)^2\right\}$. Recall the identity in law

$$(X_{n:i}\,;i=1,\dots,n) \overset{\text{law}}{\sim} \left(\frac{\mathsf{e}_1+\cdots+\mathsf{e}_i}{\Gamma_{n+1}}\,;i=1,\dots,n\right)\,,$$

where $(\mathsf{e}_i\,;i=1,\dots,n+1)$ are independent exponential variables of parameter 1 and $\Gamma_{n+1} = \mathsf{e}_1+\cdots+\mathsf{e}_{n+1}$. The spacing $\delta_{n:i}$ has the same law as $m_n^{[i]}/\Gamma_{n+1}$, where $m_n^{[i]}$ denotes the i-th smallest variable e_j. In order to evaluate the asymptotic law of $(n^2\delta_{n:1},\dots,n^2\delta_{n:k})$, we shall study the asymptotics of $(nm_n^{[1]},\dots,nm_n^{[k]})$ since $\Gamma_{n+1}/n \to 1$ a.s. The law of $(m_n^{[1]},\dots,m_n^{[k]})$ is given for $0<s_1<\cdots<s_k$ by

$$\begin{aligned}
\mathbb{P}(m_n^{[1]}\in ds_1,\dots,m_n^{[k]}\in ds_k) &= \mathbb{P}\left(\bigcup_{i_1\neq\cdots\neq i_k}\{\mathsf{e}_{i_1}\in ds_1,\dots,\mathsf{e}_{i_k}\in ds_k\}\bigcap_{i\notin\{i_1,\dots,i_k\}}\{\mathsf{e}_i>s_k\}\right)\\
&= \sum_{i_1\neq\cdots\neq i_k}\mathbb{P}(\mathsf{e}_{i_1}\in ds_1)\cdots\mathbb{P}(\mathsf{e}_{i_k}\in ds_k)\prod_{i\notin\{i_1,\dots,i_k\}}\mathbb{P}(\mathsf{e}_i>s_k)\\
&= \frac{n!}{(n-k)!}e^{-s_1}\cdots e^{-s_k}e^{-(n-k)s_k}\,ds_1\dots ds_k\,,
\end{aligned}$$

where the second equality stems from the independence of the exponential variables. We obtain the convergence

$$\begin{aligned}
\mathbb{P}\left(n\,m_n^{[1]}\in ds_1,\dots,,n\,m_n^{[k]}\in ds_k\right) &= \frac{n!}{n^k(n-k)!}\,e^{-s_1/n}\cdots e^{-s_{k-1}/n}e^{-(n-k+1)s_k/n}\,ds_1\dots ds_k\\
&\overset{n\to\infty}{\to}\ e^{-s_k}\,ds_1\cdots ds_k\,,
\end{aligned}$$

which means that

$$(n\,m_n^{[1]},\dots,n\,m_n^{[k]}) \overset{\text{law}}{\to} (\mathsf{e}'_1,\dots,\mathsf{e}'_1+\cdots+\mathsf{e}'_k),$$

where $\mathsf{e}'_1,\dots,\mathsf{e}'_k$ are independent exponential variables of parameter 1. Since Theorem 2 ensures that $n\left(T_n^{(3)}\right)^2\to\infty$ a.s., and thus that $n\delta_{n:k}\ll\left(T_n^{(3)}\right)^2$ a.s., we finally obtain the convergence

$$(nT_{n:1}^2,\dots,nT_{n:k}^2) \overset{\text{law}}{\to} (\mathsf{e}'_1,\dots,\mathsf{e}'_1+\cdots+\mathsf{e}'_k).$$

Corollary 2.1 follows.

5 Size of the largest cluster at a fixed time t

We focus now on the system at a given time $t \in]0,1[$. It is of interest to estimate the size $L_n(t)$ of the largest cluster at time t. The following theorem ensures that $L_n(t) \asymp \log n$.

Theorem 3 *For any $t \in]0,1[$, there exists two constants $0 < C_t \leq C_t' < \infty$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(C_t \log n \leq L_n(t) \leq C_t' \log n\right) = 1\,.$$

This result implies in particular, that the first macroscopic cluster (i.e. cluster of size $\asymp n$) appears at a time which tends to the critical time 1, when n tends to infinity (see Theorem 1). A referee raised the question of describing the evolution of the gas at times closed to the critical time. This interesting problem remains open.

Proof of Theorem 3

We give in a first time an upper bound to the size $L_n(t)$ of the largest cluster. According to condition (37) a necessary condition for the existence of a cluster of size k at time t is

$$\exists\, k \leq p \leq 2k \ \text{ and } i \in \{0,\dots,n-p\} \ \text{ such that for } r = 1,\dots,p-1,$$
$$\frac{1}{r}\sum_{j=1}^{r} X_{n:i+j} - \frac{1}{p-r}\sum_{j=r+1}^{p} X_{n:i+j} + \frac{pt^2}{2n} \geq 0\,,$$

which may be written

$$\exists\, k \leq p \leq 2k \ \text{ and } i \in \{0,\dots,n-p\} \ \text{ such that for } r = 1,\dots,p-1,$$
$$\frac{1}{p-r}\sum_{j=r+1}^{p}\left(X_{n:i+j} - X_{n:i} - \frac{j}{n}\right) - \frac{1}{r}\sum_{j=1}^{r}\left(X_{n:i+j} - X_{n:i} - \frac{j}{n}\right) \leq p\,\frac{t^2-1}{2n}\,.$$

We shall deal with the quantity

$$\delta_n(k) = \max_{1\leq j\leq k}\ \max_{0\leq i\leq n+1-j} |X_{n:i+j} - X_{n:i} - \frac{j}{n}|$$

which has been introduced by Mason [5] in order to describe the oscillation modulus of the uniform quantile process v_n. It follows from the inequality

$$\frac{1}{p-r}\sum_{j=r+1}^{p}\left(X_{n:i+j} - X_{n:i} - \frac{j}{n}\right) - \frac{1}{r}\sum_{j=1}^{r}\left(X_{n:i+j} - X_{n:i} - \frac{j}{n}\right) \leq 2\delta_n(p)\,,$$

that when the condition

$$2\delta_n(2k) < \frac{k}{n}\left(\frac{1-t^2}{2}\right)$$

holds, no cluster of size larger than k can exist. When $k_n \sim c\log n$ with $c > 0$, Mason has shown (see Theorem 2(II') in [5]) that

$$\frac{n\delta_n(k_n)}{k_n} \overset{n\to\infty}{\to} \alpha_c^+ - 1 \qquad \text{a.s.},$$

where α_c^+ is the unique solution larger than 1 of $\alpha_c^+ - \log\alpha_c^+ - 1 = 1/c$. Since $\alpha_c^+ \to 1$ when $c \to \infty$, there exists $C_t' < \infty$ such that

$$4\left(\alpha^+_{2C_t'} - 1\right) < \frac{1-t^2}{2},$$

so for n large enough

$$2\delta_n(2k_n) < \frac{k_n}{n}\left(\frac{1-t^2}{2}\right) \quad \text{a.s. },$$

and there exists a.s. no cluster of size larger than $C_t' \log n$.

We now give a lower bound to the size of the largest cluster at time t. Recall from (36) that a sufficient condition for the existence of a cluster of size larger than k at time t is

$$\exists\, i \in \{0,\dots,n-k\} \text{ such that } X_{n:i+1} - X_{n:i+k} + \frac{kt^2}{2n} \geq 0\,,$$

which leads us to consider

$$D_n^-(k) :\, = \min_{0\leq i\leq n-k} \left(X_{n:i+k} - X_{n:i+1}\right)\,.$$

A sufficient condition for the existence of a cluster of size at least k in terms of $D_n^-(k)$ is thus

$$D_n^-(k) \leq \frac{kt^2}{2n}\,. \tag{42}$$

The analog to formula (15) in [5] (see also Theorem 2(12) in [8]) for $D_n^-(k)$ is when $k_n \sim c\log n$

$$\frac{nD_n^-(k_n)}{k_n} \overset{n\to\infty}{\sim} \alpha_c^- \quad \text{a.s.},$$

where α_c^- is the unique solution less than 1 of $\alpha_c^- - \log\alpha_c^- - 1 = 1/c$. Since $\alpha_c^- \to 0$ when $c \to 0$, there exists a constant C_t such that $\alpha^-_{C_t} < t^2/2$, and for n large enough

$$D_n^-(k_n) \leq \frac{k_n t^2}{2n} \quad \text{a.s.}$$

Combining this with formula (42) one obtains for n large enough the a.s. existence of a cluster of size larger than $C_t \log n$, which completes the proof of Theorem 3.

6 Evolution of a marked particle

We should like to estimate the typical size of a cluster. In this direction, we study the size of the cluster which contains a marked particle, say for example the particle number $[n/2]$ ($[\cdot]$ denotes the integer part function). The reason we choose a particle "in the middle" is that we want to avoid the side effects. The following theorem claims that particle $i_n = [n/2]$

does not collide with the others at infinitesimal times and belongs to a finite cluster at any time $t \in]0,1[$.

Theorem 4 *Let $S_n(t)$ denotes the size of the cluster which contains particle $i_n = [n/2]$ at time t.*
(i) *If t_n is a sequence of times decreasing to 0, then*

$$\mathbb{P}(S_n(t_n) = 1) \overset{n\to\infty}{\to} 1\,.$$

(ii) *If k_n is sequence of integer increasing to ∞ and $t \in]0,1[$, then*

$$\mathbb{P}(S_n(t) \geq k_n) \overset{n\to\infty}{\to} 0\,.$$

Proof of Theorem 4

According to condition (34), a necessary condition for the merging of $(i+1,\dots,i+k)$ into a cluster of size k before time t is

$$X_{n:i+1} - X_{n:i+k} + \frac{kt^2}{n} \geq 0\,.$$

In particular, a necessary condition for $S_n(t)$ to be larger than k is

$$\exists\, k \leq p \leq n, \text{ and } 0 \leq j \leq p-1, \text{ such that } X_{n:i_n-j} - X_{n:i_n-j+p-1} + \frac{pt^2}{n} \geq 0\,.$$

Recall that

$$(X_{n:i}\,; i=1,\dots,n) \overset{\text{law}}{\sim} \left(\frac{\mathtt{e}_1 + \cdots + \mathtt{e}_i}{\Gamma_{n+1}}\,; i = 1,\dots,n\right)\,,$$

where $(\mathtt{e}_i\,; i = 1,\dots,n+1)$ are independent random variables with exponential law of parameter 1 and $\Gamma_{n+1} = \mathtt{e}_1 + \cdots + \mathtt{e}_{n+1}$. We can give an upper bound to $\mathbb{P}(S_n(t) \geq k)$ in terms of $\mathtt{e}_i$: for any $\mu > 1$

$$\begin{aligned}\mathbb{P}(S_n(t) \geq k) &\leq \mathbb{P}\left(\bigcup_{p=k}^{n}\bigcup_{j=0}^{p-1}\left\{X_{n:i_n-j} - X_{n:i_n-j+p-1} + \frac{pt^2}{n} \geq 0\right\}\right)\\ &\leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq \mu\right) + \sum_{p=k}^{n} p\,\mathbb{P}(\mathtt{e}_1 + \cdots + \mathtt{e}_{p-1} \leq \mu p t^2)\,.\end{aligned}$$

The Cramer's large deviation inequality (see [2]) yields

$$\mathbb{P}\left(\frac{\mathtt{e}_1 + \cdots + \mathtt{e}_p}{p} \leq \lambda\right) \leq \exp(-p\Lambda^*(\lambda))\,,$$

with

$$\Lambda^*(\lambda) = \sup_{s\leq 0}(\lambda s - \Lambda(s)) = \lambda - 1 - \log\lambda\,.$$

We thus obtain the upper bound for μ such that $\frac{k}{k-1}\mu t^2 < 1$:

$$\mathbb{P}(S_n(t) \geq k) \leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq \mu\right) + \sum_{p=k}^{n} p \, \exp\left(-(p-1)\Lambda^*\left(\frac{p}{p-1}\mu t^2\right)\right). \tag{43}$$

We first focus on the case $t_n \overset{n\to\infty}{\to} 0$. Under the assumption that $t_n^2 < 1/4$ formula (43) gives for $k = 2$ and $\mu = 2$

$$\begin{aligned}\mathbb{P}(S_n(t_n) \geq 2) &\leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq 2\right) + \sum_{p=2}^{n} p \, \exp\left(-(p-1)\Lambda^*\left(\frac{p}{p-1}2t_n^2\right)\right) \\ &\leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq 2\right) + \sum_{p=2}^{n} \exp\left(\log p - (p-1)\Lambda^*\left(4t_n^2\right)\right).\end{aligned}$$

Let us consider the exponential term. Expanding Λ^*, we obtain

$$\begin{aligned}\log p - (p-1)\Lambda^*(4t_n^2) &= \log p - (p-1)(4t_n^2 - 1 - 2\log(2t_n)) \\ &\leq \log p + (p-1)\log(2t_n) + (p-1)(1 + \log(2t_n)).\end{aligned}$$

As soon as $t_n \leq 1/4$, the term $\log p + (p-1)\log(2t_n)$ is negative for any $p \geq 2$, so we have the upper bound

$$\begin{aligned}\mathbb{P}(S_n(t_n) \geq 2) &\leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq 2\right) + \sum_{p=2}^{n} \exp((p-1)(1 + \log(2t_n)) \\ &\leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq 2\right) + \frac{\exp(1 + \log(2t_n))}{1 - \exp(1 + \log(2t_n))}.\end{aligned}$$

A consequence of the previous inequality is that $\mathbb{P}(S_n(t_n) \geq 2)$ tends to 0 when n tends to infinity. The first part of Theorem 4 is proved.

We now focus on the case $t \in]0, 1[$ and k_n is an increasing sequence of integer. Formula (43) may be written in this case as

$$\mathbb{P}(S_n(t) \geq k_n) \leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq \mu\right) + \sum_{p=k_n}^{n} p \, \exp\left(-(p-1)\Lambda^*\left(\frac{k_n}{k_n-1}\mu t^2\right)\right).$$

Since $t < 1$, we can choose $\mu > 1$ such that $\mu t^2 < 1$ and then for n large enough $\frac{k_n}{k_n-1}\mu t^2 \leq \delta < 1$. Under these assumptions we obtain

$$\mathbb{P}(S_n(t) \geq k_n) \leq \mathbb{P}\left(\frac{\Gamma_{n+1}}{n} \geq \mu\right) + \sum_{p=k_n}^{n} p \, \exp\left(-(p-1)\Lambda^*(\delta)\right),$$

with $\Lambda^*(\delta) > 0$. It follows that $\mathbb{P}(S_n(t) \geq k_n)$ tends to 0 when n tends to infinity. This conclude the proof of Theorem 4.

7 Concluding remarks

We would like to emphasize the difference of behaviour between the system considered in the present paper, and the Gaussian one studied by Martin et al. in [1, 4]. The major difference is the scarcity of collisions in our case compared to the Gaussian case, due to the static initial condition of the gas. Computer numerical simulations lead Bonvin and al. to conjecture the existence in the Gaussian case of $\asymp \sqrt{n}$ aggregates of size $\asymp \sqrt{n}$ at fixed time $t < t^*$, whereas we have seen that in our case a typical cluster is of finite size at time $t < t^* = 1$, and has in any case a size bounded by $C'_t \log n$. The proliferation of collisions in the Gaussian case is a consequence of the existence of particles with high initial kinetic energy, which collect quickly many neighboring particles. We must underline at this point that in the case we consider, the scarcity of collisions is a characteristic phenomenon that explains as well the appearance of clusters of size k before those of size $k+1$, as the somewhat small size of the largest cluster at time $t < 1$. It is also to be noticed that the last collision occurs in different ways in the two cases. In the Gaussian one the time of last collision do not converge to the characteristic time t^* (see formula (37) in [4]) and the last collision involves a macroscopic together with a microscopic cluster (see [4] section 6 and also [1] section 3(ii)). This phenomenon results again from the existence of particles with high kinetic energy. Some of them are near the sides and they flee far away from the system at small times. We shall conclude with an interesting comment made by a referee. There is a qualitative gap between the evolution of the system starting at zero temperature[1] and the evolution starting at low temperature. Indeed, as soon as the temperature of the initial particles is not strictly zero, the gas follows the behaviour described in [1, 4].

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Références

[1] J.C. Bonvin, Ph.A. Martin, J. Piasecki, X. Zotos : *Statistics of Mass Aggregation in a Self-Gravitating One dimensional Gas.* J. Stat. Phys. **91** (1998) p. 177-197.

[2] J.D. Deuschel, D.W. Stroock, *Large Deviations.* Pure Appl. Math. **137**, Academic Press INC., 1989.

[3] W. E, Y.G. Rykov, Ya.G. Sinai : *Generalized Variational Principles, Global Weak Solutions and Behavior with Random Initial Data for Systems of Conservation Laws Arising in Adhesion Particle Dynamics.* Com. Math. Phys. **177** (1996) p. 349-380.

1. The zero temperature relies to the static initial state of the gas, which gets of course heated as time runs by the conversion of potential energy into kinetic energy. It will then cool down again by dissipative collisions.

[4] P.A. Martin, J. Piasecki: *Aggregation Dynamics in a Self-Gravitating One-Dimensional Gas.* J. Stat. Phys. **84** (1996) p. 837-857.

[5] D.M. Mason: *A Strong Limit Theorem for the Oscillation Modulus of the Empirical Quantile Process.* Stoch. Proc. and Appl. **17** (1984) p. 127-136.

[6] D.M. Mason, G.R. Shorack, J. A. Wellner: *Strong Limit Theorems for Oscillation Moduli of the Uniform Empirical Process.* Z. Wahrschein. verw. Gebiete **65** (1983) p. 127-136.

[7] J.S. Rao, M. Sobel: *Incomplete Dirichlet Integrals with Applications to Ordered Uniform Spacing.* J. Multivar. Anal **10** (1980) p. 603-610.

[8] G.R. Shorack, J.A. Wellner: *Empirical Processes with Applications to Statistics.* Wiley 1986.

[9] T. Suidan, *Adhesion Dynamics on the Line: the Mass Aggregation Process.* J. of Stat. Phys. **101** 3/4 (2000) p. 893-903.

[10] M. Vergassola, B. Dubrulle, U. Frisch, A. Noullez: *Burgers' equation, Devil's Staircases and the Mass Distribution Function for Large Scale Structures.* Astron. Astrophys. **289** (1994) p.325-356.

[11] Ya.B. Zeldovich: *Gravitational Instability: an Approximate Theory for Large Density Pertubations.* Astron. Astrophys. **5** (1970) p. 84-89.