Zero-cycles on geometrically rational surfaces (Variations upon a theme by Daniel Coray)

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References :

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Algebraic points on cubic hypersurfaces
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Zéro-cycles sur les surfaces de del Pezzo
(Variations sur un thème de Daniel Coray)
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Let X be an algebraic variety over a field k.
One lets $X(k)$ be the set of rational points of X.
If P is a closed point of X, we denote by $k(P)$ the residue field at P. This is a finite extension of k. Its degree $d_P := [k(P) : k]$ is called the degree of P.
A closed point P of degree 1 is a rational point.
The *index* $I(X) = I(X/k)$ is the g.c.d. of the degrees d_P for all closed points P of X.
It is also the g.c.d. of the degrees of the finite field extension K/k such that $X(K) \neq \emptyset$.
If $X(k) \neq \emptyset$, then clearly $I(X) = 1$.
Naive question : what about the converse ? One classical case : yes for quadrics (Artin 37, Springer 1952). One less classical case : yes for intersections of two quadrics (Amer 76, Coray 77, Brumer 78).

Zero-cycle on a k-variety X : finite linear combination with integral coefficients of closed points $\sum_P n_P P$, $n \in \mathbb{Z}$
Effective zero-cycle : all $n_P \geq 0$

Degree of the zero-cycle (over k) : $\sum_P n_P [k(P) : k] \in \mathbb{Z}$

Rational equivalence on the group $Z_0(X)$ of zero-cycles :
for any proper morphism $p : C \to X$ from a normal integral k-curve and any rational function $f \in k(C)^\times$, mod out by $p_*(div_C(f))$.
If X/k is proper, then get induced degree map

$$CH_0(X) = Z_0(X)/\mathrm{rat} \to \mathbb{Z}$$

from the Chow group of degree zero-cycles to $\mathbb{Z}$.
The image is $\mathbb{Z}.I(X) \subset \mathbb{Z}$, where $I(X)$ is the index.
The kernel $A_0(X)$ is the reduced Chow group of zero-cycles.

Smooth projective curves (zero-cycle = divisor)

Let z be a zero-cycle, $\ell(z) = h^0(C, O_C(z))$.
Riemann's inequality $\ell(z) \geq deg_k(z) + 1 - g$ for a divisor z on a smooth, projective, geometrically connected curve C/k of genus g implies :

• If $deg(z) \geq g$ then z is rationally equivalent to an effective cycle.

• For $g > 1$, if $deg(z) = 1$, then there exist effective zero-cycles of degree g and of degree $g + 1$, hence closed points of coprime degrees $\leq g + 1$.

• If $g \geq 1$ and $C(k) \neq \emptyset$, then $CH_0(C)$ is generated by closed points of degree at most g.

• For $g = 0$ and $g = 1$, if $I(C) = 1$, then $C(k) \neq \emptyset$.

Naive question. For higher dimensional varieties, do we have similar results for zero-cycles, with a suitable integer in place of g ?

Over $k = \mathbb{C}$ the complex field, only the first property is relevant. The answer is well known to be NO in general. This is the famous result of Mumford 1969 on a problem of Severi, expanded by Roitman 1971, with the proof by Spencer Bloch 1979 by means of algebraic correspondances, expanded by Bloch and Srinivas and then by many other authors.

Theorem. Let $X/\mathbb{C}$ be smooth, projective. If there exists an integer $N \geq 1$ such that any zero-cycle of degree at least N on X is rationally equivalent to an effective zero-cycle, then $H^0(X, \Omega^i_X) = 0$ for $i \geq 2$.

One is then led to restrict our naive questions and only consider smooth, projective geometrically connected varieties X/k such that, over (arbitrary) algebraically closed extensions Ω/k, the geometry of $Y := X \times_k \Omega$ is "reasonable". In decreasing order of generality :

(1) CH_0-representable : there exists a proper curve C and a proper morphism $p : C \to Y$ such that $p_* : CH_0(C) \to CH_0(Y)$ is onto
(2) CH_0-trivial : the degree map $CH_0(Y) \to \mathbb{Z}$ is an isomorphism.
(3) Rationally connected varieties (Kollár-Miyaoka-Mori), e.g. Fano varieties.
(4) Unirational varieties

For surfaces, classes (3) and (4) coincide with the class of rational surfaces.

Smooth compactifications X of a homogeneous space E of a connected linear algebraic group G over a field k are geometrically unirational.

For such X over nonalgebraically closed k, the question whether $I(X) = 1$ implies $X(k) \neq \emptyset$ has been much investigated.
For E a principal homogeneous space of G, the question is open in general. There are positive results (Serre, Sansuc, Bayer–Lenstra).
For projective homogeneous spaces, we have Springer's theorem on quadrics.
But in the general case, one may have $I(X) = 1$ and $X(k) = \emptyset$ (Florence, Parimala).

In this talk, we shall concentrate on the class of geometrically rational surfaces over a field k of characteristic zero. We shall see that analogues of the properties for curves hold for such surfaces.

The first theorem is a substitute for the would-be statement : $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$. It generalizes the result for cubic surfaces obtained by Coray (1974) in his thesis (to be discussed further below).

Theorem A

Let X/k be a smooth, projective, geometrically rational surface. There exists an integer $N(X) \geq 1$, which depends only on the geometry of X, such that, if $I(X) = 1$, then there exist closed points of coprime degrees all less than $N(X)$.

$N(X) =$ explicit formula in terms of $(K_X.K_X)$

The second theorem generalizes a result of Coray and mine (1979) on conic bundles over the projective line.

Theorem B

Let X/k be a smooth, projective, geometrically rational surface with a k-rational point. There exists an integer $M(X) \geq 1$, which depends only on the geometry of X, such that any zero-cycle on X of degree at least $M(X)$ is rationally equivalent to an effective zero-cycle. In particular, the Chow group of zero-cycles is generated by closed points of degree at most $M(X)$.

$M(X) =$ explicit formula in terms of $(K_X.K_X)$

Index 1 for del Pezzo surfaces of degree 3 : Coray's thesis

In his Ph.D. thesis (Cambridge, UK 1974), Daniel Coray (1947-2015) studied a question raised by Cassels and Swinnerton-Dyer :

Let k be a field. If a cubic hypersurface X in $\mathbb{P}^n_k$ has a rational point in a finite extension K/k of degree prime to 3, does it have a rational point in k ?
That is, if $I(X) = 1$, do we have $X(k) \neq \emptyset$?
This question is still open.

There are two main theorems in this thesis.

Theorem (Coray). *If k is the field of fractions of a complete DVR with residue field κ, if $I(Y) = 1 \Longrightarrow Y(\kappa) \neq \emptyset$ holds for cubic hypersurfaces Y/κ in any dimension, then $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ for cubic hypersurfaces X/k in any dimension.*

This is proved by a delicate study of possible bad reduction of cubic hypersurfaces, extending earlier work of Demjanov, Lewis, Springer.

Corollary (Coray). *Let k be a p-adic field. Then $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ holds for arbitrary cubic hypersurfaces of arbitrary dimension over a p-adic field.*
Indeed, it is easy to prove $I(Y) = 1 \Longrightarrow Y(\kappa) \neq \emptyset$ for (arbitrary) cubic hypersurfaces Y of arbitrary dimension over a finite field κ.

This talk is concerned with the second main theorem in Coray's thesis, his method, and some improvements which lead to Theorems A and B. Here k is an arbitrary field.

Theorem (Coray 1974)
Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface. If it has a rational point in a fiinite extension of k of degree prime to 3, then it has a rational point in an extension of degree 1, or of degree 4, or of degree 10.
("or "not exclusive)

I shall describe the main points of Coray's proof. It uses curves of low genus lying on the surface. The basic tools are classical : Riemann-Roch for line bundles on a surface and the formula for the arithmetic genus of a curve on a surface. One does not know whether the curves one produces are smooth or even irreducible.

One must then envision possible degeneracy cases. I shall then explain a general method to dodge this part of the argument, and, with the added flexibility, produce new results without too much pain.

From now on, to be on the safe side, I assume $\mathrm{char}(k) = 0$.

We assume that the smooth cubic surface $X \subset \mathbb{P}^3_k$ has a closed point of degree d prime to 3. Let d be the least such integer.
If $d = 1$, there is nothing to do. If $d = 2$, then taking the line through a quadratic point and its conjugate, and intersecting with X, we get a rational point, thus in fact $d = 1$.
Let us thus assume d prime to 3 and $d \geq 4$. Let $P \in X$ be a closed point of degree d.
On the surface X we find a closed point Q of degree 3 by intersecting with a line $\mathbb{P}^1_k \subset \mathbb{P}^3_k$.

Let $n \geq 1$ be an integer. On the surface X we easily compute

$$h^0(X, O_X(n)) \geq 3n(n+1)/2 + 1.$$

If $3n(n+1)/2 - 3 \geq d$ then

$$3n(n+1)/2 + 1 \geq d + 3 + 1$$

and one may find a surface of degree n cutting out on X a curve Γ passing through the closed points P (of degree d, prime to 3) and Q (of degree 3). Assume Γ is smooth and geometrically irreducible. Its genus is $g = p_a(\Gamma) = 3n(n-1)/2 + 1$. On Γ, there is a zero-cycle of degree 1.

If $d \geq 3n(n-1)/2+4$, then

$$d-3 \geq 3n(n-1)/2+1 = g(\Gamma),$$

thus on the smooth curve Γ, the zero-cycle $P-Q$ is rationally equivalent to an effective zero-cycle of degree $d-3<d$. Thus there exists a closed point of degree prime to 3 and smaller than d, contradiction.

This argument works for any integer d prime to 3 which lies in an interval
$$3n(n+1)/2 - 3 \geq d \geq 3n(n-1)/2 + 4.$$
For other values of d, a complementary argument is needed.
In particular, for integers of the shape $d = 3n(n-1)/2 + 1$, one uses a curve Γ which is the normalisation of a curve $\Gamma_0 \subset X$ cut out by a surface of degree n passing through P and having a double point at the point Q of degree 3. The genus of the curve drops down by 3, and the dimension of the linear system of interest drops down by 9.

For $d = 3n(n-1)/2 + 1$ with $n \geq 4$, there is enough room. But there is not enough room in the case $n = 2, d = 4$ and in the case $n = 3, d = 10$.

CONCLUSION (up to good position argument)

On a smooth cubic surface X/k with a closed point of degree d prime to 3, the least such d lies in $\{1, 4, 10\}$.

48 years old question : Can one eliminate 10, 4, both ?

The above argument for cubic surfaces assumes that the curves Γ found in the linear system are geometrically irreducible and smooth. In his paper, Coray then discusses the possible singular and even reducible curves which may turn up, and manages to go down to 1, 4 or 10 also in these cases.

It is clear that such cases may occur : consider the simpler question of finding a smooth plane conic through a closed point of degree 3 in $\mathbb{P}^2_k$. If the closed point happens to lie on a $\mathbb{P}^1_k \subset \mathbb{P}^2_k$, this is not possible.

Making the method flexible

I now explain how to avoid such a discussion of degenerate cases.

Ideas :

• When available, use results of the type : if there is a k-rational point on a k-variety X of the type under study, then the k-rational points are Zariski dense.

• use the Bertini theorems (not very original !)

• replace k by the "large" field $F = k((t))$, so that there are many F-points on whichever smooth variety appears in the process (the original variety, or some parameter space) as soon as there is at least one F-point.

• For each of the problems under consideration here, to prove a result for a k-variety X, it is enough to solve it positively for the $k((t))$-variety $X \times_k k((t))$.

Theorem (a variation on the Bertini theorems, as found in Jouanolou's book)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$.
Let $r \leq n$ be an integer. There exists a nonempty open set $U \subset X^r$ such that, for any field L containing k and any L-point $(P_1, \dots, P_r) \in U(L)$, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points $\{P_1, \dots, P_r\}$.

Here we just say : " If there is a point in $U(L)$, then ...". But for a given L, $U(L)$ could be empty.

Let X be a smooth k-variety and $m > 0$ be an integer. Consider the open set W of X^m consisting of m-tuples $(x_1, \dots, x_m)$ with $x_i \neq x_j$ for $i \neq j$.

The symmetric group $\mathfrak{S}_m$ acts on W, the quotient is a smooth k-variety $Sym^m_{sep}X$. It parametrizes effective zero-cycles of degre m which are "separable".

Theorem (zero-cycles version of previous theorem)

Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $s_1, \dots, s_t$ be natural integers such that $\sum_i s_i \leq n$. There exists a nonempty open set U of the product $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$ such that, for any field L containing k and any L-point of U, corresponding to a family $\{z_i\}$ of separable effective zero-cycles of respective degrees s_i, there exists a hyperplane $h \subset \mathbb{P}^n_L$ whose inverse image $X_h = f^{-1}(h) \subset X_L$ is a smooth, geometrically integral L-variety which contains the points of the supports of the cycle $\sum_i z_i$.

Same comment as before on $U(L)$ being possibly empty.
Note : Let $s = s_1 + \cdots + s_t$. For the proofs of Theorems A,B,C, we use $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$ and not only $\mathrm{Sym}^{s}_{sep} X$.

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety.

In this talk, we say that X has the *density property* if it satisfies : for any finite field extension L/k with $X(L) \neq \emptyset$, the set $X(L)$ is Zariski dense in X_L.

R-equivalence on $X(k)$ is the equivalence relation generated by the elementary relation : $A, B \in X(k)$ both lie in the image of $\mathbb{P}^1(k)$ under a k-morphism $\mathbb{P}^1_k \to X$.

In this talk, we say that X has the *R-density property* if it satisfies : for any finite field extension L/k and $P \in X(L)$, the set of points of $X(L)$ which are R-equivalent to P on X_L is Zariski dense on X_L.

Smooth cubic hypersurfaces in $\mathbb{P}^n_k$, $n \geq 3$, satisfy both properties.

Theorem (Bertini for varieties with density properties)

Let k be a field, $\mathrm{char}(k) = 0$. Let X be a smooth, projective, geom. connected k-variety. Let $f : X \to \mathbb{P}^n_k$ be a k-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_k$. Let $P_1, \ldots, P_t$ be closed points of X of respective degrees $s_1, \ldots, s_t$ such that $\sum_i s_i \leq n$.

(a) If X satisfies the density property, then there exists a hyperplane $h \subset \mathbb{P}^n_k$ defined over k such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \ldots, z_t$ of respective degrees $s_1, \ldots, s_t$.

(b) If X is satisfies the R-density property, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

Definition (F. Pop)
A field F is said to be a *large field* (in French, *corps fertile*) if, for any smooth geometrically connected variety X over F, if $X(F) \neq \emptyset$ then the set $X(F)$ of F-rational points is Zariski dense in X.

If a field F is large, then any finite extension of F is large.

Thus any smooth geom. connected variety over a large field satisfies the density property.

The formal power series field $F = k((t))$ over any field k is a large field.

Theorem (Bertini over a large field)

Let F be a large field, $\mathrm{char}(F) = 0$. Let X be a smooth, projective, geom. connected F-variety. Let $f : X \to \mathbb{P}^n_F$ be an F-morphism. Assume its image has dimension at least 2 and generates $\mathbb{P}^n_F$. Let $P_1, \dots, P_t$ be closed points of X of respective degrees $s_1, \dots, s_t$ such that $\sum_i s_i \leq n$.

(a) There exists a hyperplane $h \subset \mathbb{P}^n_F$ defined over F such that $X_h = f^{-1}(h) \subset X$ is smooth, geom. integral and contains effective zero-cycles $z_1, \dots, z_t$ of respective degrees $s_1, \dots, s_t$.

(b) If X is geometrically rationally connected, then one may moreover ensure that, for each i, the zero-cycle z_i is rationally equivalent to the zero-cycle P_i.

For the proof of (a) :
The family $P_1, \ldots, P_t$ defines an F-point of the smooth, connected k-variety $\mathrm{Sym}^{s_1}_{sep} X \times \cdots \times \mathrm{Sym}^{s_t}_{sep} X$. Since F is large, any nonempty Zariski open set of that k-variety contains an F-point.

For the proof of (b), one moreover uses a result due to Kollár (1999) (deformation method) : for any F-point P on a smooth, projective geometrically (separably) rationally connected variety X over a large field F, the set of F-points which are R-equivalent to P, hence in particular are rationally equivalent to P, is Zariski dense in X.

(Easy) Proposition

Let k be a field and $F = k((t))$. Let X be a proper k-variety.

(a) The gcd of degrees of closed points coincides for X/k and X_F/F.

(b) For any integer $r \geq 1$, the smallest degree of a closed point of degree prime to r, which is also the smallest degree of an effective zero-cycle of degree prime to r, coincides for X/k and X_F/F.

(c) Let S be a set of natural integers. If the Chow group of zero-cycles on X_F may be generated by the classes of effective cycles of degree $d \in S$, then the same holds for X.

(d) Let $d \geq 0$ be an integer. If every zero-cycle on X_F of degree at least d is rationally equivalent to an effective cycle, then the same holds for X.

One may then run Coray's proof using only smooth projective curves in the linear systems of interest.There are two ways to do this.

One may use the density property of smooth cubic surfaces and apply Bertini's theorem (a) for varieties with this property.

Or one may reduce to the case of large fields F via replacing k by $k((t))$, use Bertini theorem (a) for large fields, and then use the fact that the statement of the theorem for $X_{k((t))}$ over $k((t))$ implies it for X over k.

In any case, an important point has been to be able to move the effective zero-cycles through which one wants curves of a given linear system to pass and simultaneously be smooth.

The gained flexibility enables us to prove the next theorems by Coray's method without too much effort.

Index 1 for del Pezzo surfaces of degree 2

“Bertini over a large field” (a) is enough to prove :

Theorem

Let X be a del Pezzo surface of degree 2, i.e. a double cover of $\mathbb{P}^2_k$ ramified along a smooth quartic. If there exists a closed point of odd degree on X, then there exists a closed point of degree 1, or 3, or 7.

In the proof, just as for cubic surfaces, in certain cases, one needs to blow up points on X. To apply the Bertini types of results, one needs to know that certain invertible sheaves are very ample. Here one may use Reider's criteria (1988).

For del Pezzo surfaces of degree 2 with a k-rational point not in a very special situation, k-unirationality is known. But the trick with large fields enables us to handle our problem without using k-unirationality.

Remark (Kollár-Mella 2017). *There exist examples of del Pezzo surfaces X of degree 2 with a closed point of degree 3, hence $I(X) = 1$, but with no rational point.*

Suppose F is a field with a quadratic field extension $F(\sqrt{a})/F$, a cubic field extension and a quintic field extension. Let $C \subset \mathbb{P}^2_F$ a conic with a smooth F-point. Let $Q \subset \mathbb{P}^2_F$ be a smooth quartic with $Q \cap C = \{A, B\}$, with A closed point of degree 3 over F and B closed point of degree 5 over F.
Let $k = F(t)$. Let X/k be the smooth del Pezzo surface of degree 2 defined by the equation

$$z^2 - aC(u, v, w)^2 + tQ(u, v, w) = 0.$$

It has obvious points of degree 3 and 5.
However congruences modulo powers of t show it has no k-point.

Effectivity results for zero-cycles on del Pezzo surfaces

Using "Bertini over a large field" (b) and "Bertini for varieties with the R-density property" (b), one moves effective zero-cycles while keeping their class in the Chow group. Together with the basic argument with curves of small degree, this leads to :

Theorem

Let $X \subset \mathbb{P}^3_k$ be a smooth cubic surface. Suppose $X(k) \neq \emptyset$ and $Q \in X(k)$.

(i) Every effective zero-cycle of degree at least 3 on X is rationally equivalent to an effective zero-cycle $z_1 + rQ$ with $r \geq 0$ and z_1 effective of degree ≤ 3.

(ii) Every zero-cycle of degree zero on X is rationally equivalent to the difference of two effective zero-cycles of degree 3.

(iii) Every zero-cycle on X of degree ≥ 10 is rationally equivalent to an effective zero-cycle.

Theorem

Let X/k be a del Pezzo surface of degree 2. Assume $X(k) \neq \emptyset$. Let $Q \in X(k)$.
(i) Every effective zero-cycle of degree at least 6 on X is rationally equivalent to an effective zero-cycle $z_1 + rQ$ with $r \geq 0$ and z_1 effective of degree ≤ 6.

(i)i Every zero-cycle of degree zero is rationally equivalent to the difference of two effective zero-cycles of degree 6.

(iii) Every zero-cycle of degree at least 43 is rationally equivalent to an effective cycle.

Since we do not know the R-density property for del Pezzo surfaces of degree 2, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties (proved using deformation theory).

Theorem

Let X/k be a del Pezzo surface of degree 1. It has a k-point, the fixed point of the anticanonical system.

(i) Every effective zero-cycle of degree at least 21 is rationally equivalent to an effective zero-cycle $z_1 + rQ$ with z_1 effective of degree at most 21.

(ii) Every zero-cycle of degree zero is rationally equivalent to the difference of effective zero-cycles of degree 21.

(iii) Every zero-cycle of degree at least 904 is rationally equivalent to an effective cycle.

Since we do not know the density property and even less the R-density property for del Pezzo surfaces of degree 1, the proof here relies on Bertini over a large field (b), the combination of the reduction trick from k to $k((t))$ and Kollár's result on R-density for geometrically rationally connected varieties over a large field.

With some effort, one should be able to give analogous theorems without assuming $X(k) \neq \emptyset$.
Here is one example (full details have not been written down).

Let $X \subset \mathbb{P}^4_k$ be a del Pezzo surface of degree 4. Assume $I(X) = 2$.
Over number fields, Creutz and Viray have recently discussed whether this implies the existence of a closed point of degree 2.
Over an arbitary field of char. zero, using the technique described in this talk, one may prove (CT+Kollár) :
For X/k as above, if $I(X) = 2$, then there exists a closed point of degree $2d$ with d odd in $\{1, 3, 7\}$.

Putting everything together

Analogues of these theorems for conic bundles over the projective line, the other class of geometrically rational surfaces, analogues of the above theorems were proved by Coray and me in 1979 – the tedious way, discussing possible degenerations. I have not investigated whether one could simplify the argument by using the large field trick.
The case of del Pezzo surfaces of degree $5 \leq d \leq 9$, as usual, is easily handled (the case $d = 6$ being a little more difficult).
For del Pezzo surfaces of degree 4, one has $I(X) = 1 \Longrightarrow X(k) \neq \emptyset$ (Amer 1976, Coray 1977, Brumer 1978). If $X(k) \neq \emptyset$, then blowing up a suitable k-point leads to a conic bundle over $\mathbb{P}^1_k$.
Using the k-birational classification of geometrically rational surfaces (Enriques, Manin, Iskovskikh, Mori) one gets Theorems A and B – whose statements do not depend on the k-birational equivalence class.

Theorem A

Let X/k be a smooth, projective, geometrically rational surface. If $I(X) = 1$, then there exist closed points of coprime degrees less than

$$N(X) = \max(10, \lfloor 4 - (K_X.K_X)/2 \rfloor).$$

Theorem B

Let X/k be a smooth, projective, geometrically rational surface with a k-rational point. Any zero-cycle on X of degree at least

$$M(X) = \max(904, \lfloor 3 - (K_X.K_X)/2 \rfloor)$$

is rationally equivalent to an effective cycle. In particular, the Chow group of zero-cycles is generated by closed points of degree at most $M(X)$.

Problems

(i) Can one give a proof of Theorems A and B (even with worse bounds) avoiding the birational k-classification ?

(ii) Do these results extend to higher dimensional geometrically rationally connected varieties ?
Favourite 3-folds : smooth cubic hypersurfaces in $\mathbb{P}^4_k$, conic bundles over $\mathbb{P}^2_k$, quadric bundles over $\mathbb{P}^1_k$?

(iii) What about geometrically CH_0-trivial varieties ? Enriques surfaces ?

(iv) What about geometrically CH_0-representable varieties ?
Salberger (unpublished, 1985) proved an effectivity theorem (of type B) for conic bundles over curves of arbitrary genus.

THE END

The proof of the following result is independent of previous arguments.

Theorem *Let* $X \subset \mathbb{P}^3_k$ *be a smooth cubic surface with* $X(k) = \emptyset$*. If any closed point of degree 3 on* X *is cut out by a line* $\mathbb{P}^1_k \subset \mathbb{P}^3_k$*, then to a general line* $\mathbb{P}^1_k \subset \mathbb{P}^3_k$ *we may associate a del Pezzo surface* W *of degree 1 over* k *such that* $W(k)$ *is not Zariski dense in* W*.*

The question whether such cubic surfaces exist was recently raised by Qixiao Ma.

Whether rational points are always Zariski dense on a del Pezzo surface of degree 1 is a well known open question.

Idea of the proof

Take a line $L \subset \mathbb{P}^3_k$. By assumption it cuts out a closed point P of degree 3 on X. Blow up that point. This gives a fibration $Y \to \mathbb{P}^1_k$ whose fibres are the sections of X by planes containing L.
If any closed point of degree 3 on X is cut out by a line, then in particular for any $t \in \mathbb{P}^1(k)$ with smooth fibre Y_t, we have $\mathrm{Pic}(Y_t) = \mathbb{Z}P$ hence $\mathrm{Pic}^0(Y_t) = 0$
The regular, relatively minimal model $g : W \to \mathbb{P}^1_k$ associated to the Jacobian of the generic fibre Y_η of $Y \to \mathbb{P}^1_k$ is the announced del Pezzo surface of degree 1. For $t \in \mathbb{P}^1(k)$ with W_t smooth, any k-point on the elliptic curve W_t is a 3-torsion point. The k-points of W lie in the union of the singular fibres of g and the curve which is the closure of the 3-torsion subscheme of W_η.