Ann. Henri Poincaré 7 (2006) 161 – 198

1424-0637/06/010161-38, *Published online* 2006-01-16
DOI 10.1007/s00023-005-0246-4

Annales Henri Poincaré

Relaxation Time of Quantized Toral Maps

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Abstract. We introduce the notion of *relaxation time* for noisy quantum maps on the $2d$-dimensional torus – generalization of previously studied *dissipation time*. We show that the relaxation time is sensitive to the chaotic behavior of the corresponding classical system if one simultaneously considers the semiclassical limit ($\hbar \to 0$) together with the limit of small noise strength ($\epsilon \to 0$).

Focusing on quantized smooth Anosov maps, we exhibit a semiclassical régime $\hbar < \epsilon^E \ll 1$ (where $E > 1$) in which classical and quantum relaxation times share the same asymptotics: in this régime, a quantized Anosov map relaxes to equilibrium fast, as the classical map does. As an intermediate result, we obtain rigorous estimates of the quantum-classical correspondence for noisy maps on the torus, up to times logarithmic in $\hbar^{-1}$. On the other hand, we show that in the "quantum régime" $\epsilon \ll \hbar \ll 1$, quantum and classical relaxation times behave very differently. In the special case of ergodic toral symplectomorphisms (generalized "Arnold's cat" maps), we obtain the exact asymptotics of the quantum relaxation time and precise the régime of correspondence between quantum and classical relaxations.

1 Introduction

The notion of dissipation time for classical systems has been introduced in various contexts in [21, 22, 23, 24] to study the speed at which a conservative dynamical system converges to some equilibrium, when subjected to noise (e.g., due to interactions with the 'environment').

In those references, the state of the system was represented by a probability density function, and the distance of the system from equilibrium was measured by the mean-square fluctuations of the density w.r.to the equilibrium density. The term dissipation referred in those works to the process of the decay of density fluctuations during the noisy evolution.

In the present work we generalize our results to the quantum-mechanical setting and introduce the *relaxation time*, which in the context of the above mentioned papers exactly coincides with the *dissipation time*, and generalizes it to the setting where relaxation of the system towards its equilibrium need not involve an energy exchange. To uniformize the terminology, we will only use the term relaxation time in the sequel.

The relaxation time τ_c will now refer both to the time scale after which the density fluctuations are reduced by a fixed factor, and in general to the time

*The research of AF is partially supported by the grant from U.S. National Science Foundation, DMS-9971322 and UC Davis Chancellor's Fellowship

scale on which the system finds itself in an intermediate state, roughly speaking, 'half-way' between the initial state and the final equilibrium.

The results obtained in [23, 24] yielded the information about the asymptotic behavior of the relaxation time (in the limit when the noise strength ϵ tends to zero) for a particular type of dynamics, namely volume-preserving maps on a d-dimensional torus phase space, for which the "natural" equilibrium density is the constant function. Such torus maps constitute simple examples of dynamical systems with proven chaotic behavior. Our main conclusion was that the asymptotic behavior of $\tau_c(\epsilon)$ strongly depends on the ergodic properties of the underlying noiseless map. We found that the relaxation toward the equilibrium occurs much faster in the case of a chaotic dynamics, than for a "regular" one. More precisely, the relaxation time displays two main behaviors in the small-ϵ limit:

Logarithmic-law $\tau_c \sim \ln(\epsilon^{-1})$. In this case one speaks of *fast relaxation* (short relaxation time). This behavior is characteristic of strongly chaotic systems, e.g., maps with exponential mixing, including uniformly expanding or hyperbolic systems [23]. When the map is an (irreducible) linear hyperbolic automorphism of the torus, the constant in front of the logarithm (the "relaxation rate constant") can be computed explicitly, and is related with the Kolmogorov-Sinai (KS) entropy of the map [24].

Power-law $\tau_c \sim \epsilon^{-\beta}$. One then speaks of *slow relaxation* (long relaxation time). This behavior virtually concerns all non-weakly-mixing systems (non-ergodic maps, Kronecker maps on the torus); it may also apply to systems with sufficiently slow (power-law) decay of correlations, like intermittent maps [4].

One can intuitively understand these opposite asymptotics through the way the noiseless dynamics connects different spatial scales (or "wavelengths"). A chaotic map typically transforms modes of wavelength $\approx \ell$ into modes of wavelength $\approx e^{\pm\lambda}\ell$, where λ is the (largest) Lyapounov exponent. By iteration, it will transfer density fluctuations at scale ℓ into fluctuations at scale ℓ' in a time $\sim |\log(\ell/\ell')|$. On the other hand, a noise of "strength" ϵ strongly reduces fluctuations at wavelengths $\leq \epsilon$, acting effectively as a ultraviolet cutoff. Therefore, $|\log \epsilon|$ is the minimal time needed for the system to bring fluctuations from all scales $1 \geq \ell \geq \epsilon$ down to the scale ϵ, where they get damped. On longer time scales the system can be thought of as in equilibrium. On the opposite, a non-weakly-mixing system will mix different scales at a much smaller speed, so fluctuations at wavelengths $\ell \gg \epsilon$ will take a longer time to get damped. We believe that these various behaviors of the relaxation time hold as well in the case of flows on compact phase spaces (the noise then acts continuously in time, instead of "stroboscopically" for the case of maps [37]).

In the present paper, we apply the notion of relaxation time to *quantum* dynamical systems. To be able to use our "classical" results of [23], we will focus on the quantum systems corresponding to volume-preserving maps on the torus, namely quantized maps on the torus. Besides being volume-preserving, the maps need to be invertible and preserve the symplectic structure on the (necessarily even-dimensional) torus, that is, be *canonical*. Quantum maps have been much studied in the last 25 years as convenient toy models of "quantum chaos" [32, 31]. According to the "standard" quantization schemes, compactness of the torus phase space leads to finite-dimensional quantum Hilbert spaces, where the quantum maps takes the form of a unitary propagator. Such finite-dimensional operators are obviously much easier to study numerically than Schrödinger operators on $L^2(\mathbb{R}^d)$. The semiclassical limit is recovered when the dimension $N = (2\pi\hbar)^{-1}$ of the Hilbert space diverges.

The influence of "noise" on an otherwise unitary quantum evolution has already attracted much attention, both in the mathematical [38] and physics literature [13, 29, 44]. Noise can be due to interactions of the quantum system under study with uncontrolled degrees of freedom, like those of the "environment" of the system, or on the contrary internal degrees of freedom not accounted for. The form of quantum noise we will consider is not the most general one, it is obtained by quantizing the noise affecting the corresponding classical system (Section 2.3): the quantum equilibrium state is then the fully mixed state with maximal Von Neumann entropy. Several works have studied the problem of relaxation in the framework of quantized maps, especially when the classical dynamics is chaotic [12, 26, 40, 6]. The effect of noise can be measured through various ways (growth of the Von Neumann entropy, decay of purity, decay of "fidelity" etc.). One can also observe how the *spectrum* of the quantum noisy propagator departs from unitarity [12, 40, 43, 27]; since the noisy propagator is a non-normal operator, its spectral radius only influences the *long-time* evolution of the system. On the opposite, the behavior for shorter times could possibly be analyzed through the *pseudospectrum* of the propagator [17]. Our present study bypasses this spectral approach, by directly estimating the "quantum relaxation time" τ_q: this quantity indicates at which time the system has significantly relaxed to the equilibrium state, uniformly over all possible initial conditions.

The problematic of quantum chaos ("where does a quantum system encode the information that its classical limit is chaotic?") yields another (more formal) reason to study the quantum relaxation time. Indeed, the above-described dichotomy between the two possible small-noise behaviors of τ_c shows that the logarithmic-law is a decent indicator of chaotic dynamics. Therefore, it seems reasonable to try using the small-ϵ behavior of the quantum relaxation time τ_q to characterize a quantum chaotic system. Yet, we are now dealing with *two limits*: on the one hand, one expects the quantum system to mimic the classical one only in the *semiclassical limit* $\hbar \to 0$; on the other hand, to characterize the classical dynamics we also want to consider the *small-noise limit* $\epsilon \to 0$. The major part of this article will study the interplay between these two limits, which do not commute with each other.

In order to carry out this program rigorously, we will focus our attention on a small subclass of the maps studied in [23], namely the smooth Anosov maps, which include the hyperbolic linear symplectomorphisms (or generalized "Arnold's cat" maps). As mentioned above, for such systems one can understand the logarithmic behavior of the classical relaxation time through the "mixing of scales" performed by the dynamics. Quantum mechanics contains an intrinsic scale, namely Planck's constant $\hbar$: it gives the size of the "quantum mesh" on the torus which supports the Hilbert space (see Section 2). This irreducible scale allows one to estimate the *breaking time* for the quantum-classical correspondence, namely the time when the evolution (through the noiseless dynamics) of quantum observable starts to strongly deviate from the evolution of the corresponding classical observable (this time is often called *Ehrenfest time*, and we will denote it by τ_E) [50, 14]. For a hyperbolic system, this time also satisfies a logarithmic law $\tau_E \approx \frac{\ln(\hbar^{-1})}{\lambda}$, which can be understood similarly as for $\tau_c(\epsilon)$: τ_E is the shortest time needed for the system to transfer all scales $1 \geq \ell \geq \hbar$ down to the "quantum scale" $\hbar$, where classical and quantum dynamics depart from each other.

When switching on the noise, quantum and classical dynamics will also correspond to each other at least until the Ehrenfest time τ_E, whatever the noise strength ϵ. Therefore, if the classical system decays before the Ehrenfest time ($\tau_c < \tau_E$), then the quantum system will decay around the same time: $\tau_q \approx \tau_c$. This situation is described in Proposition 5 and Corollary 1. This régime was already studied in various semiclassical approaches to study convergence to equilibrium in a quantum system subject to some type of noise (see, e.g., results regarding the spectrum of noisy quantum propagators [12, 40, 43, 27], the rate of decoherence [44, 6, 28] and its relation with quantum dynamical entropy [1, 2, 5]).

When one allows the noise strength to decrease together with Planck's constant, the correspondence $\tau_q \approx \tau_c \sim \ln(\epsilon^{-1})$ remains valid as long as those times are smaller than the breaking time τ_E. Such a "semiclassical régime" is partially analyzed in Section 4.1 for the case of smooth Anosov maps: Theorem 2 identifies a condition of the form $\epsilon > \hbar^{1/E}$, which ensures that $\tau_q \approx \tau_c$ (the exponent $1/E < 1$ depends on the expanding rates of the classical map). More precise estimates are obtained in Section 4.2 for the case of Anosov linear automorphisms of the torus. Theorem 3 and Corollary 3 state that the correspondence $\tau_q \approx \tau_c$ holds under the milder condition $\epsilon \geq C\hbar$. One can check in this linear case that this condition ensures $\tau_c \leq \tau_E$, which justifies the correspondence. The correspondence between quantum and classical relaxation times includes the prefactor in front of $\log(\epsilon^{-1})$. As mentioned above, this constant is related to the KS entropy of the classical map, which also coincides with various types of *quantum* dynamical entropies introduced in the algebraic quantization schemes [2, 5].

In Section 3 we investigate the opposite situation (dubbed as the "quantum limit") where the classical relaxation time is longer than the Ehrenfest time. Beyond that time the quantum system will approach equilibrium much more slowly than its classical counterpart, and rather independently of the noiseless dynamics. Precisely, we show in Proposition 4 that under the condition $\epsilon/\hbar \ll 1$ (meaning that the noise scale is smaller than the quantum scale), the quantum relaxation time is bounded from below as $\tau_q \geq f(\hbar/\epsilon)$, where the function f grows at a rate only depending on the "shape" of the noise. In Remark 3, we notice that a slightly stronger condition on the decay of $\epsilon/\hbar$ ensures that $\tau_q \gg \tau_c$ independently of the unitary quantum dynamics. In such a régime, the noise scale is much smaller than the quantum mesh size, so the quantum evolution is insensitive to the noise, and propagates almost unitarily. It is indeed irrelevant to cutoff fluctuations at a scale ϵ when the smallest possible scale of the system is $\hbar \gg \epsilon$.

As in the classical case, we believe that our results should extend to quantized Anosov flows (for which exponential decay of correlations has been recently proven in [39]), like for instance the Laplace operator on a compact manifold of negative curvature.

To finish this section, we will compare our results on the relaxation time with the related decay of *fidelity*, which has recently received much attention in the physics literature. Fidelity measures the discrepancy between, on the one hand, the "unperturbed" evolution of an initial state $|\psi_o\rangle$ under some quantum dynamics (say, a quantum map U_N, see Section 2.2), on the other hand, the evolution of the same initial state, but under a "perturbed dynamics" (say, the map $U_N\, e^{-i2\pi N\epsilon Op_N(H)}$). The perturbing Hamiltonian H is chosen randomly, but is *independent of time*: this constitutes the major difference from our "noise", which is equivalent with a random perturbation changing at each time step. The fidelity is then defined as

$$F(n) = \left|\langle\psi_o|(U_N\, e^{-i2\pi N\epsilon Op_N(H)})^{-n}\, U_N^n|\psi_o\rangle\right|^2 .$$

This quantity was first introduced in [45], and several regimes of its decay have been identified [34, 46, 49, 15], depending of the type of classical dynamics (chaotic vs. regular), and of the relative values of the perturbation strength ϵ and Planck's constant $\hbar = (2\pi N)^{-1}$. In general, the fidelity starts to decay around a certain "fidelity time" $n \approx \tau_F$, down to a saturation where it oscillates around values $\mathcal{O}(\hbar)$. We will recall below how τ_F depends on ϵ and $\hbar$ (when both are small), in the case where the classical dynamics is an Anosov map on the 2-dimensional torus, and the initial state $|\psi_o\rangle$ is a Gaussian wavepacket (coherent state) of width $\sqrt{\hbar}$. We were able to identify at least four régimes from the physics literature:

- for large enough perturbations, namely $\epsilon \gg \sqrt{\hbar}$, the fidelity decays instantaneously, $\tau_F = 1$.
- in the range $\hbar \ll \epsilon \ll \sqrt{\hbar}$, the fidelity starts to decay at the time $\tau_F \approx \frac{2\log(\epsilon^{-1})-(\log\hbar^{-1})}{2\lambda}$, which is comparable with our "log-time decay".
- for $\hbar^{3/2} \ll \epsilon \ll \hbar$, we are in the "Fermi golden rule régime", and $\tau_F \sim \left(\frac{\hbar}{\epsilon}\right)^2$.
- $\epsilon \ll \hbar^{3/2}$ corresponds to the "perturbative régime", where $\tau_F \sim \frac{\sqrt{\hbar}}{\epsilon}$.

Subsequent régimes are connected through crossovers, some of which have been analyzed [15]. The two last régimes of weak perturbations are analog with our "quantum limit" for the relaxation time. In these régimes, the fidelity time is much longer than the Ehrenfest time τ_E. Around τ_F, the initial wavepacket is then spread across the full torus, looking like a "random state"; the same decay occurs if we take for $|\psi_o\rangle$ an arbitrary state.

In the first two régimes of strong perturbation, the fidelity time satisfies $\tau_F \lesssim \frac{\tau_E}{2}$; therefore, an evolved coherent state is still localized in phase space around τ_F. This shows that in these régimes, the decay of fidelity crucially depends on the choice for $|\psi_o\rangle$ of an $\sqrt{\hbar}$-localized wavepacket. The inequality $\tau_F \leq \frac{\tau_E}{2}$ implies that the quantum-classical correspondence still holds at the time τ_F: this time is thus asymptotically equal to the "classical fidelity time", which is the time when an initial classical density of width $\sqrt{\hbar}$, evolved by the perturbed classical dynamics, departs from the same density evolved by the unperturbed dynamics. Because the classical fidelity instantaneously decays for strong perturbations (as opposed to the logarithmic law for the classical relaxation time), the quantum fidelity time τ_F does so too, thus behaving differently from the quantum relaxation time τ_q.

2 Setup and notation

In all that follows, we use the following conventions to compare asymptotic behaviors of two quantities, for instance $a(\epsilon)$ and $b(\epsilon)$ in the limit $\epsilon \to 0$:

- $a(\epsilon) \ll b(\epsilon)$ iff $\frac{a(\epsilon)}{b(\epsilon)} \to 0$.
- $a(\epsilon) \lesssim b(\epsilon)$ iff there is a constant $C > 0$ such that $\frac{a(\epsilon)}{b(\epsilon)} \leq C$.
- $a(\epsilon) \sim b(\epsilon)$ iff there are constants $C \geq c > 0$ such that $c \leq \frac{a(\epsilon)}{b(\epsilon)} \leq C$.
- $a(\epsilon) \approx b(\epsilon)$ iff $\frac{a(\epsilon)}{b(\epsilon)} \to 1$.

2.1 Quantization on the torus

The quantization on $\mathbb{T}^{2d}$ presented below strictly follows that considered in [33] and [19] in the $d = 1$ case. The generalization to arbitrary d is in most aspects straightforward, and has been presented, in a slightly different notational setting, in [51, 47, 9].

2.1.1 State space and observables

Let $T_{\boldsymbol{v}} = e^{\frac{i}{\hbar}\boldsymbol{v}\wedge\boldsymbol{Z}}$ denote the standard Weyl translation operators on $L^2(\mathbb{R}^d)$, with $\boldsymbol{v} = (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2d}$, $\boldsymbol{Z} = (\boldsymbol{Q}, \boldsymbol{P})$ and $\boldsymbol{v} \wedge \boldsymbol{Z} = \boldsymbol{p} \cdot \boldsymbol{Q} - \boldsymbol{q} \cdot \boldsymbol{P}$. Here $\boldsymbol{Q} = (Q_1, \ldots, Q_d)$ and $\boldsymbol{P} = (P_1, \ldots, P_d)$ denote the quantum position and momentum operators, i.e., $Q_j\psi(\boldsymbol{x}) = x_j\psi(\boldsymbol{x})$, $P_j\psi(\boldsymbol{x}) = -i\hbar\partial_{x_j}\psi(\boldsymbol{x})$.

To quantize the torus, one extends the domain of $T_{\boldsymbol{v}}$ to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, and considers its action on the $\boldsymbol{\theta}$-quasiperiodic elements (wavefunctions) of $\mathcal{S}'(\mathbb{R}^d)$, that is distributions $\psi(\boldsymbol{q})$ satisfying:

$$\psi(\boldsymbol{q} + \boldsymbol{m}_1) = e^{2\pi i\boldsymbol{\theta}_p\cdot\boldsymbol{m}_1}\psi(\boldsymbol{q}), \qquad (\mathcal{F}_h\psi)(\boldsymbol{p} + \boldsymbol{m}_2) = e^{-2\pi i\boldsymbol{\theta}_q\cdot\boldsymbol{m}_2}(\mathcal{F}_h\psi)(\boldsymbol{p}). \tag{1}$$

Here, the "Bloch angle" $\boldsymbol{\theta} = (\boldsymbol{\theta}_q, \boldsymbol{\theta}_p) \in \mathbb{T}^{2d}$ is fixed, while $\boldsymbol{m} = (\boldsymbol{m}_1, \boldsymbol{m}_2)$ takes any value in $\mathbb{Z}^{2d}$. $\mathcal{F}_h$ denotes the usual quantum Fourier transform

$$(\mathcal{F}_h\psi)(\boldsymbol{p}) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \psi(\boldsymbol{q}) e^{-i\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{\hbar}} d\boldsymbol{q}.$$

For any angle $\boldsymbol{\theta}$, the space of such quasiperiodic distributions is nontrivial iff $2\pi\hbar = h = 1/N$ for a certain $N \in \mathbb{Z}_+$. From now on we only consider such values of Planck's constant. The corresponding space of wavefunctions will be denoted by $\mathcal{H}_N(\boldsymbol{\theta})$. It forms a finite dimensional subspace of $\mathcal{S}'(\mathbb{R}^d)$ and can be identified with $\mathbb{C}^{N^d}$. The quasiperiodicity conditions (1) can be restated in terms of the action of translation operators:

$$\psi \in \mathcal{H}_N(\boldsymbol{\theta}) \quad \Longleftrightarrow \quad \forall \boldsymbol{m} \in \mathbb{Z}^{2d}, \qquad T_{\boldsymbol{m}}\psi = e^{2\pi i\left(\frac{N}{2}\boldsymbol{m}_1\cdot\boldsymbol{m}_2 + \boldsymbol{m}\wedge\boldsymbol{\theta}\right)}\psi. \tag{2}$$

That is, $\mathcal{H}_N(\boldsymbol{\theta})$ consists of simultaneous eigenstates of all translations on the $\mathbb{Z}^{2d}$ lattice.

A translation $T_{\boldsymbol{v}}$ acts inside $\mathcal{H}_N(\boldsymbol{\theta})$ iff $\boldsymbol{v} \in N^{-1}\mathbb{Z}^{2d}$, and a natural Hermitian structure can be set on $\mathcal{H}_N(\boldsymbol{\theta})$ such that all these operators act unitarily. This observation motivates the introduction of microscopic quantum translations on $\mathcal{H}_N(\boldsymbol{\theta})$:

$$W_{\boldsymbol{k}} = W_{\boldsymbol{k}}(N, \boldsymbol{\theta}) := T_{\boldsymbol{k}/N|\mathcal{H}_N(\boldsymbol{\theta})} = \left(e^{2\pi i \boldsymbol{k}\wedge\boldsymbol{Z}}\right)_{|\mathcal{H}_N(\boldsymbol{\theta})}.$$

The operators $W_{\boldsymbol{k}}$ are indexed by points $\boldsymbol{k}$ on the "Fourier" or "reciprocal" lattice $\mathbb{Z}^{2d}$. Since they quantize the classical Fourier modes $w_{\boldsymbol{k}}(\boldsymbol{x}) = e^{2\pi i\boldsymbol{k}\wedge\boldsymbol{x}}$, they can be thought of as *Quantum Fourier Modes*. The canonical commutation relations (CCR) take the form

$$W_{\boldsymbol{k}}W_{\boldsymbol{m}} = e^{\frac{\pi i}{N}\boldsymbol{k}\wedge\boldsymbol{m}}W_{\boldsymbol{k}+\boldsymbol{m}}, \qquad W_{\boldsymbol{k}}W_{\boldsymbol{m}} = e^{\frac{2\pi i}{N}\boldsymbol{k}\wedge\boldsymbol{m}}W_{\boldsymbol{m}}W_{\boldsymbol{k}}. \tag{3}$$

Furthermore, the quasiperiodicity of the elements of $\mathcal{H}_N(\boldsymbol{\theta})$ induces a quasiperiodicity of the Quantum Fourier Modes acting on that space. Namely, for any $\boldsymbol{m} \in \mathbb{Z}^{2d}$ we have

$$W_{\boldsymbol{k}+N\boldsymbol{m}}(N, \boldsymbol{\theta}) = e^{2\pi i\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta})}\, W_{\boldsymbol{k}}(N, \boldsymbol{\theta}), \tag{4}$$

with the phase

$$\alpha(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{k}\wedge\boldsymbol{m} + \frac{N}{2}\boldsymbol{m}_1\cdot\boldsymbol{m}_2 + \boldsymbol{m}\wedge\boldsymbol{\theta}.$$

The algebra of observables on the quantum space $\mathcal{H}_N(\boldsymbol{\theta})$ is generated by the set of operators $\{W_{\boldsymbol{k}}(N, \boldsymbol{\theta})\}_{\boldsymbol{k}\in\mathbb{Z}^{2d}}$ and will be denoted by $\mathcal{A}_N(\boldsymbol{\theta})$. Due to quasiperiodicity, $\mathcal{A}_N(\boldsymbol{\theta})$ is finite dimensional and can be identified (as a linear space) with the set of matrices $\mathcal{L}(\mathcal{H}_N(\boldsymbol{\theta})) \cong \mathcal{M}_{N^d\times N^d} \cong \mathbb{C}^{N^{2d}}$.

We select a fundamental domain $\mathbb{Z}_N^{2d}$ of the quantum Fourier lattice. The choice centered around the origin seems to be the most natural one for our purposes (cf. [43]). Namely, we take for fundamental domain the set of lattice points $\boldsymbol{k} = (k_1, \ldots, k_{2d}) \in \mathbb{Z}^{2d}$ such that

$$\forall j \in \{1, \ldots, 2d\}, \quad k_j \in \begin{cases} \{-N/2+1, \ldots, N/2\}, & \text{for } N \text{ even} \\ \{-(N-1)/2+1, \ldots, (N-1)/2\}, & \text{for } N \text{ odd.} \end{cases}$$

The set $\{W_{\boldsymbol{k}}(N, \boldsymbol{\theta}),\ \boldsymbol{k} \in \mathbb{Z}_N^{2d}\}$ forms a basis for $\mathcal{A}_N(\boldsymbol{\theta})$. Using the tracial state $\tau(A) := N^{-d}\mathrm{Tr}(A)$ on this algebra of matrices, we induce the Hilbert-Schmidt scalar product

$$\langle A, B \rangle = \tau(A^* B), \qquad A, B \in \mathcal{A}_N(\boldsymbol{\theta}).$$

The corresponding norm will be denoted by $\|\cdot\|_{HS}$. Equipped with this norm, the above basis is orthonormal. One needs to keep in mind that $\|\cdot\|_{HS}$ does not coincide with the standard operator norm, hence $\mathcal{A}_N(\boldsymbol{\theta})$ is not considered here as a C^*-algebra.

We can now easily quantize classical observables on $\mathbb{T}^{2d}$. To any smooth observable $f \in C^\infty(\mathbb{T}^{2d})$ with Fourier expansion $f = \sum_{\boldsymbol{k} \in \mathbb{Z}^{2d}} \hat{f}(\boldsymbol{k})\, w_{\boldsymbol{k}}$, corresponds an element of $\mathcal{A}_N(\boldsymbol{\theta})$, called its *Weyl quantization*, denoted by $Op_{N,\boldsymbol{\theta}}(f)$, and defined as:

$$Op_{N,\boldsymbol{\theta}}(f) = \sum_{\boldsymbol{k} \in \mathbb{Z}^{2d}} \hat{f}(\boldsymbol{k})\, W_{\boldsymbol{k}}(N, \boldsymbol{\theta}) = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^{2d}} \left(\sum_{\boldsymbol{m} \in \mathbb{Z}^{2d}} e^{2\pi i \alpha(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{\theta})} \hat{f}(\boldsymbol{k} + N\boldsymbol{m}) \right) W_{\boldsymbol{k}}(N, \boldsymbol{\theta}). \quad (5)$$

This quantization can be extended to observables $f \in L^2(\mathbb{T}^{2d})$ satisfying $\sum_{\boldsymbol{k}} |\hat{f}(\boldsymbol{k})| < \infty$.

The map $Op_{N,\boldsymbol{\theta}} \colon C^\infty(\mathbb{T}^{2d}) \to \mathcal{A}_N(\boldsymbol{\theta})$ is not injective. One can nevertheless define an *isometric* embedding $W^P \colon \mathcal{A}_N(\boldsymbol{\theta}) \mapsto L^2(\mathbb{T}^{2d})$, which associates with each quantum observable $A \in \mathcal{A}_N(\boldsymbol{\theta})$ its polynomial Weyl symbol [20]

$$A = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^{2d}} a_{\boldsymbol{k}}\, W_{\boldsymbol{k}}(N, \boldsymbol{\theta}) \mapsto W^P(A) = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^{2d}} a_{\boldsymbol{k}}\, w_{\boldsymbol{k}}. \quad (6)$$

The range of W^P is the subspace $\mathcal{I}_N = \mathrm{Span}_{\mathbb{C}}\{w_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}_N^{2d}\}$. The quantization map $Op_{N,\boldsymbol{\theta}}$ restricted to $\mathcal{I}_N$ is the inverse of W^P.

The choice to work with the Hilbert structure on $\mathcal{A}_N(\boldsymbol{\theta})$ corresponds to the choice made in the classical setting to measure classical observables through their L^2 norm, rather than their L^∞ norm. With this choice, the notion of classical relaxation (dissipation) time [23, 24] can be straightforwardly extended to the quantum dynamics, and is suitable for semiclassical analysis.

2.2 Quantization of toral maps

Let Φ denote a canonical map on $\mathbb{T}^{2d}$, more precisely a C^∞ diffeomorphism preserving the symplectic form $\sum_j dp_j \wedge dq_j$. Any such map can be decomposed into the product of three maps:

$$\Phi = F \circ t_{\boldsymbol{v}} \circ \Phi_1,$$

where $F \in SL(2d, \mathbb{Z})$ is a linear automorphism of the torus, $t_{\boldsymbol{v}}$ denotes the translation $t_{\boldsymbol{v}}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{v}$, and the function $\Phi_1(\boldsymbol{x}) - \boldsymbol{x}$ is periodic and has zero mean on the torus.

We will assume that the canonical map Φ_1 is the time-1 flow map associated with a Hamiltonian function on $\mathbb{T}^{2d}$ (this Hamiltonian may depend on time). In the case $d = 1$, this assumption is automatically satisfied [16].

To quantize Φ, one first quantizes F, $t_{\boldsymbol{v}}$ and Φ_1 separately on $\mathcal{H}_N(\boldsymbol{\theta})$. The quantization of Φ is then defined as a composition of corresponding quantum maps $U(\Phi) = U(F)\, U(t_{\boldsymbol{v}})\, U(\Phi_1)$ [36]. To each quantum map $U(\Phi)$ on $\mathcal{H}_N(\boldsymbol{\theta})$ there corresponds a quantum Koopman operator $\mathcal{U}(\Phi) = \mathcal{U}_{N,\boldsymbol{\theta}}(\Phi)$ acting on $\mathcal{A}_N(\boldsymbol{\theta})$ through the adjoint map

$$\mathcal{A}_N(\boldsymbol{\theta}) \ni A \mapsto \mathcal{U}(\Phi)\, A = ad(U(\Phi))\, A = U(\Phi)^* A\, U(\Phi).$$

In the next subsections we describe the quantizations of F, $t_{\boldsymbol{v}}$ and Φ_1 in some detail. The quantization procedure will ensure that the correspondence principle holds. In our case this is expressed by the Egorov property, which states that for every $f \in C^\infty(\mathbb{T}^{2d})$ there exists $C_f > 0$ such that for any angle $\boldsymbol{\theta}$ and large enough N,

$$\|\mathcal{U}_{N,\boldsymbol{\theta}}(\Phi)\, Op_{N,\boldsymbol{\theta}}(f) - Op_{N,\boldsymbol{\theta}}(f \circ \Phi)\|_{HS} \leq \frac{C_f}{N}. \tag{7}$$

A more explicit estimate of the remainder is given in Proposition 6.

2.2.1 Quantization of toral automorphisms

The symplectic map $F \in SL(2d, \mathbb{Z})$ acts on the algebra of observables by means of its Koopman operator $K_F f = f \circ F$. In the basis $\{w_{\boldsymbol{k}}\}$ of classical Fourier modes, this operator acts as a permutation: $K_F w_{\boldsymbol{k}} = w_{F^{-1}\boldsymbol{k}}$. To define the quantum counterpart of this dynamics, we will bypass the description of the quantum map $U(F)$ on $\mathcal{H}_N(\boldsymbol{\theta})$, and directly construct the quantum Koopman operator $\mathcal{U}_{N,\boldsymbol{\theta}}(F)$ acting on $\mathcal{A}_N(\boldsymbol{\theta})$:

$$\mathcal{U}_{N,\boldsymbol{\theta}}(F)\, W_{\boldsymbol{k}} = W_{F^{-1}\boldsymbol{k}}. \tag{8}$$

For the dynamics to be well defined, $\mathcal{U}_{N,\boldsymbol{\theta}}(F)$ has to be a *-automorphism of $\mathcal{A}_N(\boldsymbol{\theta})$, i.e., its action must be consistent with the algebraic (CCR) and quasiperiodic structures. The map F is called quantizable, if for every N there exist $\boldsymbol{\theta}$ such that these consistency conditions are satisfied. The appropriate condition can be formulated as follows (see [33, 19, 47, 9]):

Proposition 1 *A toral automorphism $F \in SL(2d, \mathbb{Z})$ is quantizable iff it is symplectic, that is, $F \in Sp(2d, \mathbb{Z})$. For any given N, an angle $\boldsymbol{\theta}$ is admissible iff it satisfies the following condition:*

$$\frac{N}{2}\begin{pmatrix} A \cdot B \\ C \cdot D \end{pmatrix} + F\boldsymbol{\theta} = \boldsymbol{\theta} \mod 1, \tag{9}$$

where A, B, C, D denote block-matrix elements of F:

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

and $A \cdot B$ denotes the contraction of the two matrices into a (column) vector:

$$(A \cdot B)_i = \sum_j A_{ij} B_{ij}.$$

The existence of admissible angles is easy to establish. If N is even, one can simply choose $\boldsymbol{\theta} = 0$. This solution can be chosen whenever all components of the vector $\left(\begin{smallmatrix} A \cdot B \\ C \cdot D \end{smallmatrix}\right)$ are even ('checkerboard' condition [33]). Otherwise one considers two cases. If $F - I$ is invertible, then for any $\boldsymbol{k} \in \mathbb{Z}^{2d}$ the following angle is admissible:

$$\boldsymbol{\theta} = (F - I)^{-1}\left(\frac{N}{2}\begin{pmatrix} A \cdot B \\ C \cdot D \end{pmatrix} + \boldsymbol{k}\right).$$

This leads to $|\det(F - I)|$ distinct admissible angles. If $F - I$ is singular, one can construct an appropriate $\boldsymbol{\theta}$ by applying the above considerations to the non-singular block. We finally remark that in view of the defining condition (8), the Egorov property (7) is automatically satisfied (with no error term).

2.2.2 Quantization of a translation $t_{\boldsymbol{v}}$

As explained in Section 2.1.1, a translation $t_{\boldsymbol{v}}$ is quantized on $L^2(\mathbb{R}^d)$ through a Weyl operator $T_{\boldsymbol{v}}$. It was noticed that such a quantum translation acts inside the algebra $\mathcal{A}_N(\boldsymbol{\theta})$ only if $\boldsymbol{v} \in N^{-1}\mathbb{Z}^{2d}$. In the opposite case, there are several possibilities to quantize the translation [10]. We will choose the prescription given in [41]: we take the vector $\boldsymbol{v}^{(N)} \in N^{-1}\mathbb{Z}^{2d}$ closest to $\boldsymbol{v}$ (in Euclidean distance), which can be obtained by taking, for each $j = 1, \ldots, 2d$, the component $v_j^{(N)} = \frac{[N v_j]}{N}$, where $[x]$ denotes the integer closest to x. One then quantizes $t_{\boldsymbol{v}}$ on $\mathcal{H}_N(\boldsymbol{\theta})$ through the restriction of $T_{\boldsymbol{v}^{(N)}}$ on that space (this is the same operator as $W_{[N\boldsymbol{v}]}(N, \boldsymbol{\theta})$). The corresponding *-automorphism on $\mathcal{A}_N(\boldsymbol{\theta})$ is provided by $\mathcal{U}_{N,\boldsymbol{\theta}}(t_{\boldsymbol{v}}) = ad(T_{\boldsymbol{v}^{(N)}})$. The Egorov property (7) holds for this quantization [41] (see also Appendix B.1).

2.2.3 Quantization of time-1 flow maps of periodic Hamiltonians

Let Φ_1 denote the time-1 flow map associated with the periodic Hamiltonian $H(\boldsymbol{z},t)$, meaning that $\Phi_t\colon \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ satisfies the Hamilton equations:

$$\frac{\partial \Phi_t(\boldsymbol{z})}{\partial t} = \nabla^{\perp} H(\Phi_t(\boldsymbol{z}),t), \qquad \Phi_0 = I.$$

To quantize Φ_1, one applies the Weyl quantization to the Hamiltonian $H(t)$, obtaining a time-dependent Hermitian operator $Op_{N,\boldsymbol{\theta}}(H(t))$. From there, one constructs the time-1 quantum propagator on $\mathcal{H}_N(\boldsymbol{\theta})$ associated with the Schrödinger equation of Hamiltonian $Op_{N,\boldsymbol{\theta}}(H(t))$:

$$U_{N,\boldsymbol{\theta}}(\Phi_1) := \mathcal{T}\, e^{-2\pi i N \int_0^1 Op_{N,\boldsymbol{\theta}}(H(t))\,dt}$$

($\mathcal{T}$ represents the time ordering). As above, the corresponding *-automorphism on $\mathcal{A}_N(\boldsymbol{\theta})$ is defined as $\mathcal{U}(\Phi_1)A = ad(U(\Phi_1))\,A$. The Egorov property for such a propagator is proven in Appendix B.1.

2.3 Quantum noise

We briefly review the construction and properties of convolution-type noise operators in the classical setting. For more detailed description we refer to [37, 4, 23]. The construction starts with a continuous, even-parity probability density $g(\boldsymbol{x}) \in L^1(\mathbb{R}^{2d})$ representing the "shape" of the noise. This function is sometimes assumed to be of higher regularity, and/or localized in a compact neighbourhood of the origin, and we will also require that $g(0) > 0$. The noise strength (or magnitude) is then adjusted through a single parameter $\epsilon > 0$, namely by defining the noise kernel using the rescaled density:

$$g_\epsilon(\boldsymbol{x}) := \frac{1}{\epsilon^{2d}}\, g\left(\frac{\boldsymbol{x}}{\epsilon}\right) \text{ on } \mathbb{R}^{2d}, \qquad \tilde{g}_\epsilon(\boldsymbol{x}) := \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} g_\epsilon(\boldsymbol{x}+\boldsymbol{n}) \text{ on } \mathbb{T}^{2d}.$$

In the sequel we use the following notation for the Fourier transform on $\mathbb{R}^{2d}$ and $\mathbb{T}^{2d}$:

$$\forall \boldsymbol{\xi} \in \mathbb{R}^{2d}, \quad \hat{g}(\boldsymbol{\xi}) := \int_{\mathbb{R}^{2d}} g(\boldsymbol{x})\, e^{-2\pi i \boldsymbol{\xi}\wedge\boldsymbol{x}} d\boldsymbol{x} \tag{10}$$

$$\forall \boldsymbol{k} \in \mathbb{Z}^{2d}, \quad \hat{\tilde{g}}(\boldsymbol{k}) := \int_{\mathbb{T}^{2d}} \tilde{g}(\boldsymbol{x})\, e^{-2\pi i \boldsymbol{k}\wedge\boldsymbol{x}} d\boldsymbol{x} = \langle w_{\boldsymbol{k}}, \tilde{g}\rangle. \tag{11}$$

One obviously has $\hat{\tilde{g}}_\epsilon(\boldsymbol{k}) = \hat{g}_\epsilon(\boldsymbol{k}) = \hat{g}(\epsilon\boldsymbol{k})$. Therefore, the Fourier expansion of $\tilde{g}$ reads

$$\tilde{g}_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} \hat{g}(\epsilon\boldsymbol{k})\, w_{\boldsymbol{k}}(\boldsymbol{x}). \tag{12}$$

The classical noise operator is defined on $L^2(\mathbb{R}^{2d}) \ni f$ as the convolution $G_\epsilon f := \tilde{g}_\epsilon * f$. The Fourier modes $\{w_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}^{2d}\}$ form a basis of eigenvectors of G_ϵ. The operator is compact, self-adjoint and admits the following spectral decomposition

$$G_\epsilon f = \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} \hat{g}(\epsilon \boldsymbol{k})\, \hat{f}(\boldsymbol{k})\, w_{\boldsymbol{k}}. \tag{13}$$

For any noise strength $\epsilon > 0$, the operator G_ϵ leaves invariant the constant density (conservation of the total probability), but is strictly contracting on $L^2_0(\mathbb{T}^{2d})$, the subspace of $L^2(\mathbb{T}^{2d})$ orthogonal to the constant functions. Using the parity of g, we notice that G_ϵ can be represented as:

$$G_\epsilon f = \int_{\mathbb{T}^{2d}} \tilde{g}_\epsilon(\boldsymbol{v})\, K_{\boldsymbol{v}} f\, d\boldsymbol{v},$$

where, $K_{\boldsymbol{v}}$ is the Koopman operator associated with the translation $t_{\boldsymbol{v}}$.

Using this formula, we can easily quantize the noise operator on $\mathcal{A}_N(\boldsymbol{\theta})$ [43]. For this, we formally replace in the above integral the Koopman operator $K_{\boldsymbol{v}}$ by its quantization $\mathcal{U}_{N,\boldsymbol{\theta}}(t_{\boldsymbol{v}})$ described in Subsection 2.2.2. Since $\mathcal{U}_{N,\boldsymbol{\theta}}(t_{\boldsymbol{v}})$ is constant when $\boldsymbol{v}$ varies on a "cube" of edges of length $\frac{1}{N}$, it is more convenient to adopt a different definition, and replace the above integral by a discrete sum, therefore defining the quantum noise operator as:

$$\begin{aligned}\mathcal{G}_{\epsilon,N,\boldsymbol{\theta}} &:= \frac{1}{N^{2d}\,Z} \sum_{\boldsymbol{n}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{n}}{N}\right) \mathcal{U}_{N,\boldsymbol{\theta}}(t_{\boldsymbol{n}/N}) \\ &= \frac{1}{N^{2d}\,Z} \sum_{\boldsymbol{n}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{n}}{N}\right) ad(W_{\boldsymbol{n}}(N,\boldsymbol{\theta})).\end{aligned}$$

We note that the assumption of continuity of g is used in the above formula in an essential way. Indeed, the quantum noise operator depends only on a discrete set of values of g (evaluated on the quantum lattice $\mathbb{Z}^{2d}/N$) and cannot be unambiguously defined for a general L^1 density.

The role of the prefactor $\frac{1}{Z}$ is to ensure that $\mathcal{G}_{\epsilon,N,\boldsymbol{\theta}}$ preserves the trace (the quantum version of the classical conservation of probability). One can easily check (see Appendix A.1) that $Z = \tilde{g}_{\epsilon N}(0)$, which cannot vanish from our assumption $g(0) > 0$. The spectrum of $\mathcal{G}_{\epsilon,N,\boldsymbol{\theta}}$ is similar to that of its classical counterpart:

Proposition 2 *$\mathcal{G}_{\epsilon,N,\boldsymbol{\theta}}$ admits as eigenstates the Quantum Fourier modes $\{W_{\boldsymbol{k}}(N,\boldsymbol{\theta}),$ $\boldsymbol{k} \in \mathbb{Z}_N^{2d}\}$, associated with the eigenvalues*

$$\gamma_{\epsilon,N}(\boldsymbol{k}) := \frac{\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{n})\, e^{-2\pi i \boldsymbol{k}\wedge\boldsymbol{n}/N}}{\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{n})}. \tag{14}$$

In the sequel we will often require higher regularity ($g \in C^M$ with $M \geq 1$) and fast decay properties of the noise generating density g. In such cases we will often use the representation of $\gamma_{\epsilon,N}$ obtained by applying the Poisson summation formula:

$$\gamma_{\epsilon,N}(\boldsymbol{k}) = \frac{\sum_{\boldsymbol{m}\in\mathbb{Z}^{2d}} \hat{g}_{\epsilon N}\left(\frac{\boldsymbol{k}}{N} + \boldsymbol{m}\right)}{\sum_{\boldsymbol{m}\in\mathbb{Z}^{2d}} \hat{g}_{\epsilon N}(\boldsymbol{m})}. \tag{15}$$

The conservation of the trace is embodied in the fact that $\gamma_{\epsilon,N}(\boldsymbol{0}) = 1$. Since the eigenvalues do not depend on the angle $\boldsymbol{\theta}$, we will call the noise operator $\mathcal{G}_{\epsilon,N}$ from now on. Let $\mathcal{A}_N^0(\boldsymbol{\theta})$ be the space of observables of vanishing trace, that is the quantum version of $L_0^2(\mathbb{T}^{2d})$. We then introduce the following norm for operators acting on $\mathcal{A}_N^0(\boldsymbol{\theta})$ (these are sometimes called *superoperators* in the physics literature):

$$\|\mathcal{G}_{\epsilon,N}\| := \sup_{A\in\mathcal{A}_N^0(\boldsymbol{\theta}),\ \|A\|_{HS}=1} \|\mathcal{G}_{\epsilon,N}A\|_{HS}. \tag{16}$$

Since $\mathcal{G}_{\epsilon,N}$ is Hermitian, we get from its spectral decomposition

$$\|\mathcal{G}_{\epsilon,N}\| = \max_{0\neq\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \gamma_{\epsilon,N}(\boldsymbol{k}).$$

The explicit formula for $\gamma_{\epsilon,N}(\boldsymbol{k})$, together with the fact that $g(\boldsymbol{x}) \geq 0$, show that the quantum noise operator acts as a strict contraction on $\mathcal{A}_N^0(\boldsymbol{\theta})$ (if g is compactly supported, strict contractivity is guaranteed only for large enough ϵN).

2.4 Noisy quantum evolution operator and its relaxation time

For a given quantizable map Φ of the torus, we define the noisy quantum propagator by the composition [6, 28, 43]

$$\mathcal{T}_{\epsilon,N} := \mathcal{G}_{\epsilon,N} \circ \mathcal{U}_{N,\boldsymbol{\theta}}(\Phi).$$

This model assumes that noise is present at each step of the evolution, and acts as a memoryless Markov process.

We will also consider the family of coarse-grained quantum propagators:

$$\tilde{\mathcal{T}}_{\epsilon,N}^{(n)} \quad := \quad \mathcal{G}_{\epsilon,N} \circ \mathcal{U}_{N,\boldsymbol{\theta}}(\Phi)^n \circ \mathcal{G}_{\epsilon,N}. \tag{17}$$

The latter type of dynamics assumes that some uncertainty is present at the initial and final steps (preparation and measurement of the system), but not during the evolution. All these operators are trace-preserving, and are strictly contracting on $\mathcal{A}_N^0(\boldsymbol{\theta})$ (except for the case mentioned at the end of Section 2.3), but in general they are not normal (their eigenstates are not orthogonal to each other).

We will study the action of these operators on the space $\mathcal{A}_N^0(\boldsymbol{\theta})$, using the norm (16). Mimicking the classical setting, we introduce the notion of quantum relaxation time associated with these two types of noisy dynamics:

$$\begin{aligned} \tau_q(\epsilon, N) &:= \min\{n \in \mathbb{Z}_+ : \|\mathcal{T}_{\epsilon,N}^n\| < e^{-1}\}, \\ \tilde{\tau}_q(\epsilon, N) &:= \min\{n \in \mathbb{Z}_+ : \|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| < e^{-1}\}. \end{aligned} \tag{18}$$

As in the classical case, the relaxation time provides an intermediate scale between the initial stage of the evolution (where the conservative dynamics is little affected by the noise) and the "final" stage when the noise has driven the system to its equilibrium (an initial observable A evolves towards $\tau(A)I$, which corresponds to a totally mixed state in the Schrödinger picture).

In the remaining part of the paper we will analyze the behavior of the quantum relaxation time in various régimes. To avoid any confusion we will reserve the symbols T_ϵ, $\tilde{T}_\epsilon^{(n)}$, $\tau_c(\epsilon)$, $\tilde{\tau}_c(\epsilon)$ for the corresponding propagators and times studied in [24, 23].

3 Relaxation times in the "quantum limit"

The main goal of this section is the analysis of the relaxation time of noisy quantum maps on the torus, for fixed Planck's constant $h = N^{-1}$ and small noise strength ϵ. As we explained in Section 2.2, the quantum Koopman operator $\mathcal{U}_N(\Phi)$ on $\mathcal{A}_N(\boldsymbol{\theta})$ associated with a canonical map Φ on the torus was constructed as the adjoint action of a unitary map $U_N(\Phi)$ on $\mathcal{H}_{N,\boldsymbol{\theta}}$:

$$\mathcal{U}_N(\Phi)A = ad\big(U_N(\Phi)\big) = U_N(\Phi)^*\, A\, U_N(\Phi), \qquad A \in \mathcal{A}_N(\boldsymbol{\theta}).$$

The unitary matrix $U_N(\Phi)$ admits an orthonormal basis of eigenfunctions $\psi_k^{(N)} \in \mathcal{H}_N(\boldsymbol{\theta})$. Each projector $|\psi_k^{(N)}\rangle\langle\psi_k^{(N)}|$ is invariant through $\mathcal{U}_N(\Psi)$. Therefore:

Proposition 3 *Any quantum Koopman operator $\mathcal{U}_N$ on $\mathcal{A}_N(\boldsymbol{\theta})$ admits unity in its spectrum, with a degeneracy at least N^d. As a consequence, for fixed N, the dynamics generated by $\mathcal{U}_N$ on $\mathcal{A}_N(\boldsymbol{\theta})$ is non-ergodic.*

In [23, Corollary 3], we showed that the classical relaxation time behaves as a power-law in ϵ if the Koopman operator K_Φ has a nontrivial eigenfunction with a modicum of Hölder regularity. Although in the quantum setting the corresponding regularity assumption on eigenstates of $\mathcal{U}_N(\Phi)$ would be satisfied automatically (every observable is expressible as a finite combination of Fourier modes), one cannot apply this corollary directly here due to the different (discrete) nature of the noise operator (cf. the remark ending this section). Nevertheless the main argument leading to the slow relaxation result is still valid.

Proposition 4 *Assume that the noise generating density g decays sufficiently fast at infinity: $\exists\gamma > 2d$ s.t. $g(\boldsymbol{x}) = \mathcal{O}(|\boldsymbol{x}|^{-\gamma})$ as $|\boldsymbol{x}| \to \infty$ (resp. $g(\boldsymbol{x}) = \pi^{-d}\exp(-\boldsymbol{x}^2)$, resp. g has compact support).*

Then, for any angle $\boldsymbol{\theta}$, and for any ϵ, N, the quantum noise operator on $\mathcal{A}_N(\boldsymbol{\theta})$ satisfies

$$\begin{aligned} \|1-\mathcal{G}_{\epsilon,N}\| \leq C\,(\epsilon N)^{\gamma}, \qquad \textit{resp.} \qquad \|1-\mathcal{G}_{\epsilon,N}\| \leq C\,e^{-\frac{1}{(\epsilon N)^2}}, \\ \textit{resp.} \qquad \|1-\mathcal{G}_{\epsilon,N}\| = 0 \qquad \textit{if} \quad \epsilon N < 1/C. \end{aligned} \tag{19}$$

All these bounds are meaningful in the limit $\epsilon N \ll 1$. As a result, the quantum relaxation time associated with any quantized map $\mathcal{U}_N(\Phi)$ is bounded as

$$\begin{aligned} \tau_q(\epsilon,N) \geq C(\epsilon N)^{-\gamma}, \qquad \textit{resp.} \qquad C\,N^2\,e^{\frac{1}{(\epsilon N)^2}} \geq \tau_q(\epsilon,N) \geq c\,e^{\frac{1}{(\epsilon N)^2}}, \\ \textit{resp.} \qquad \tau_q(\epsilon,N) = \infty \qquad \textit{if} \quad \epsilon N < 1/C. \end{aligned} \tag{20}$$

The constants only depend on g, and are independent of the map Φ.

Furthermore, for all these types of noise, there is a constant $\tilde{c} > 0$ such that if $\epsilon N < \tilde{c}$, the coarse-grained quantum dynamics does not undergo relaxation: $\tilde{\tau}_q(\epsilon,N) = \infty$.

Proof. We use the RHS of the explicit expression (14) for the eigenvalues $\gamma_{\epsilon,N}(\boldsymbol{k})$ of $\mathcal{G}_{\epsilon,N}$. From the decay assumption on g, we see that in the limit $\epsilon N \to 0$,

$$\sum_{0\neq \boldsymbol{n}\in\mathbb{Z}^{2d}} g\left(\frac{\boldsymbol{n}}{\epsilon N}\right) \leq C(\epsilon N)^{\gamma} \sum_{0\neq \boldsymbol{n}\in\mathbb{Z}^{2d}} \frac{1}{|\boldsymbol{n}|^{\gamma}}. \tag{21}$$

The sum on the RHS converges because $\gamma > 2d$. Therefore, we get $0 \leq 1-\gamma_{\epsilon,N}(\boldsymbol{k}) \leq C(\epsilon N)^{\gamma}$ uniformly w.r.to $\boldsymbol{k} \in \mathbb{Z}_N^{2d}$. Since $\mathcal{G}_{\epsilon,N}$ is Hermitian, this yields the estimate (19).

This implies that the noisy propagators contract very slowly, independently of the map Φ:

$$\begin{aligned} \forall n \geq 0, \quad \|\mathcal{T}_{\epsilon,N}^{n}\| \geq \big(\min_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \gamma_{\epsilon,N}(\boldsymbol{k})\big)^n \geq \big(1-C(\epsilon N)^{\gamma}\big)^n, \\ \|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| \geq \big(1-C(\epsilon N)^{\gamma}\big)^2. \end{aligned}$$

These inequalities prove the lower bound on τ_q in the case of a power-law decay of g. If g has compact support, the sum on the LHS of (21) clearly vanishes if ϵN is small enough, so that $\mathcal{G}_{\epsilon,N} = 1$ in this case.

The case of Gaussian noise is treated similarly, the LHS of Eq. 21 being clearly bounded above by $C\,e^{-1/(\epsilon N)^2}$. Besides, in that case the largest $\gamma_{\epsilon,N}(\boldsymbol{k})$ (e.g., for $\boldsymbol{k} = (1,0,\ldots,0)$) can be precisely estimated as $1 - C\,N^{-2}\,e^{-1/(\epsilon N)^2}$, yielding the upper bound for $\tau_q(\epsilon,N)$. □

Remark. In the case of Gaussian noise, we proved in [23, Corollary 1] that the classical relaxation time always satisfies the upper bound $\tau_c \lesssim \epsilon^{-2}$, independently of the map. Therefore, for this Gaussian noise, the bounds for τ_q obtained in the above proposition show that the quantum relaxation time is much larger than the classical one, regardless of the dynamics, as long as $\epsilon N \leq \frac{c}{\sqrt{\ln(\epsilon^{-1})}}$ for $c < 1/\sqrt{2}$. In this régime, the noise width ϵ is smaller than the quantum mesh size $\sim \hbar$, therefore the quantum dynamics does not feel the noise, and propagates (almost) unitarily.

4 Semiclassical analysis of the relaxation time

To extract information about the classical dynamics from the quantum relaxation time, one needs to consider a different régime from the one described in last section: what we need is a semiclassical régime where Planck's constant goes to zero together with the noise strength (cf. a similar discussion on the spectrum of $\mathcal{T}_{\epsilon,N}$ in [43, Section 5]).

The semiclassical analysis relates the quantum and classical propagators to one another. Following the notation introduced in Section 2.1.1, for any $N \in \mathbb{Z}_+$ we denote by $\Pi_{\mathcal{I}_N^0}$ the orthogonal (Galerkin-type) projector of $L^2_0(\mathbb{T}^{2d})$ onto its subspace $\mathcal{I}_N^0 = \mathrm{Span}\{w_{\boldsymbol{k}},\ \boldsymbol{k} \in \mathbb{Z}_N^{2d} - 0\}$. Using the fact that Op_N and its inverse W^P realize isometric bijections between $\mathcal{I}_N^0$ and $\mathcal{A}_N^0$, to any operator $\mathcal{T}_N \in \mathcal{B}(\mathcal{A}_N^0(\boldsymbol{\theta}))$ we associate the operator

$$\sigma_N(\mathcal{T}_N) := W^P \mathcal{T}_N Op_N \Pi_{\mathcal{I}_N^0}$$

acting on $L^2_0(\mathbb{T}^{2d})$. This operator is trivial on $(\mathcal{I}_N^0)^\perp$, and its restriction on $\mathcal{I}_N^0$ is isometric to $\mathcal{T}_N$. σ_N therefore defines an isometric embedding of the finite dimensional algebra $\mathcal{B}(\mathcal{A}_N^0(\boldsymbol{\theta}))$ into the infinite dimensional one $\mathcal{B}(L^2_0(\mathbb{T}^{2d}))$.

It has been shown in [43] (see Lemma 1 and its proof there) that for any quantizable smooth map Φ and any fixed $\epsilon > 0$, the operator $\sigma_N(\mathcal{T}_{\epsilon,N})$ (isometric to $\mathcal{T}_{\epsilon,N} = \mathcal{G}_{\epsilon,N}\mathcal{U}_N(\Phi)$) converges in the limit $N \to \infty$ to the classical noisy propagator $T_\epsilon = G_\epsilon\, K_\Phi$. This convergence holds in the norm of bounded operators on $L^2_0(\mathbb{T}^{2d})$. This implies in particular that for any fixed $\epsilon > 0$ and $n \in \mathbb{N}$ the sequence $\sigma_N(\mathcal{T}^n_{\epsilon,N})$ converges to T^n_ϵ in the semiclassical limit. The semiclassical convergence also holds for the coarse-grained propagators $\sigma_N(\tilde{\mathcal{T}}^{(n)}_{\epsilon,N})$. This convergence obviously implies the following behavior of the quantum relaxation time:

Proposition 5 *Let Φ be a smooth quantizable diffeomorphism on $\mathbb{T}^{2d}$, and g any noise generating density. Then for any fixed noise strength $\epsilon > 0$, the quantum relaxation time $\tau_q(\epsilon, N)$ (resp. $\tilde{\tau}_q(\epsilon, N)$) converges to the classical one $\tau_c(\epsilon)$ (resp. $\tilde{\tau}_c(\epsilon)$) in the semiclassical limit.*

Using a standard diagonal argument, one obtains:

Corollary 1 *Under the conditions of the proposition, there exists a régime $\epsilon \to 0$, $N(\epsilon) \to \infty$ such that $\tau_q(\epsilon, N(\epsilon)) \approx \tau_c(\epsilon)$ (resp. $\tilde{\tau}_q(\epsilon, N(\epsilon)) \approx \tilde{\tau}_c(\epsilon)$). Notice that these times necessarily diverge in this limit (cf. Propositions 2 and 3 in [23]).*

Proof of the Proposition. We treat the case of the noisy relaxation times τ_c and τ_q. For given $\epsilon > 0$, one has by definition $\|T_\epsilon^{\tau_c}\| < e^{-1}$, $\|T_\epsilon^{\tau_c - 2}\| > e^{-1}$ (the second inequality is strict because T_ϵ is strictly contracting on $L_0^2(\mathbb{T}^{2d})$). Therefore, the semiclassical convergence of $\sigma_N(\mathcal{T}_{\epsilon,N}^n)$ towards T_ϵ implies the existence of an integer $N(\epsilon)$ such that for any $N \geq N(\epsilon)$, one has simultaneously $\|\sigma_N(\mathcal{T}_{\epsilon,N}^n)^{\tau_c}\| < e^{-1}$ and $\|\sigma_N(\mathcal{T}_{\epsilon,N}^n)^{\tau_c-2}\| > e^{-1}$. This means that for $N \geq N(\epsilon)$, $\tau_q(\epsilon, N) = \tau_c(\epsilon)$ or $\tau_q(\epsilon, N) = \tau_c(\epsilon) - 1$.

The proof concerning the coarse-graining relaxation time is identical. □

Despite its generality, the above statement gives no information about the behavior of the quantum relaxation time unless the behavior of the classical one is known. The latter has been investigated in [23] for area-preserving maps on $\mathbb{T}^{2d}$. In particular, we have established logarithmic small-noise asymptotics $\tau_c(\epsilon) \sim \ln(\epsilon^{-1})$ (resp. $\tilde{\tau}_c(\epsilon) \sim \ln(\epsilon^{-1})$) for a class of Anosov diffeomorphisms [23, Theorem 4].

Our aim in the next subsection is to apply these results and some of their refinements to obtain quantitative estimates on the semiclassical régime for which quantum and classical relaxation times are of the same order.

4.1 Uniform semiclassical régimes

In this section we derive an estimate on the growth of the function $N(\epsilon)$ for which the classical-quantum correspondence of the relaxation times can be rigorously established. To this end we derive and apply more precise Egorov estimates than the one expressed in Eq. (7). The main idea was already outlined in the Introduction: for a generic map Φ, the correspondence between classical and quantum (noiseless) evolutions holds at least until the Ehrenfest time, the latter being of order $|\log \hbar|$ if the map Φ is chaotic. Therefore, if the classical relaxation takes place *before* this Ehrenfest time, then the quantum relaxation should occur simultaneously with the classical one.

We will restrict ourselves to the case of Anosov maps on $\mathbb{T}^{2d}$, which enjoy strong mixing properties:

Theorem 1 (Gouëzel-Liverani, [30]) *Let Φ be an Anosov C^∞ diffeomorphism on $\mathbb{T}^{2d}$, and let the noise generating function g be C^∞ and compactly supported. Then, for any pair of indices $s, s^* \in \mathbb{Z}_+$ there exists $0 < \sigma_{s,s^*} < 1$ and $C > 0$, defining a function $\Gamma(n) = C\,\sigma_{s,s^*}^n$, such that for small enough $\epsilon > 0$, the correlations between any pair of smooth observables f, h with $\int f = 0$ decay as follows:*

$$\begin{aligned} \forall n > 0, \quad & \left| \int_{\mathbb{T}^{2d}} f(\boldsymbol{x})\, h \circ \Phi^n(\boldsymbol{x})\, d\boldsymbol{x} \right| \leq \Gamma(n)\, \|f\|_{C^{s_*}}\, \|h\|_{C^s}, \\ \forall n > 0, \quad & \left| \int_{\mathbb{T}^{2d}} f(\boldsymbol{x})\, T_\epsilon^n h(\boldsymbol{x})\, d\boldsymbol{x} \right| \leq \Gamma(n)\, \|f\|_{C^{s_*}}\, \|h\|_{C^s}. \end{aligned} \tag{22}$$

This classical mixing allows us to slightly generalize our results of [23]. In particular, one does not need to assume any regularity condition on the invariant foliation of the map Φ. The condition of compact support for the noise generating kernel can probably be relaxed to functions g in the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$ (C. Liverani, private communication).

For such Anosov maps, we will exhibit a joint semiclassical régime and small-noise régime, for which quantum and classical relaxation rates are similar.

Theorem 2 *Let Φ be a quantizable Anosov C^∞ diffeomorphism on $\mathbb{T}^{2d}$, and let the noise generating function g be in the Schwartz space $\mathcal{S}(\mathbb{R}^{2d})$, so that the classical correlations decay as in the previous theorem.*

Then there exists an exponent $E = E(\Phi)$ such that in the régime $\epsilon \to 0$, $N = N(\epsilon) > \epsilon^{-E}$, the quantum relaxation times satisfy the same bounds as their classical counterparts:

I) *There exist $\tilde{\Gamma} > 0$, $\tilde{C} > 0$ such that the quantum coarse-grained relaxation time is bounded as:*

$$\frac{1}{\tilde{\Gamma}} \ln(\epsilon^{-1}) - \tilde{C} \le \tilde{\tau}_q(\epsilon, N) \le \frac{2d + s + s^*}{|\ln \sigma_{s,s^*}|} \ln(\epsilon^{-1}) + \tilde{C},$$

II) (*Assume furthermore that the noise kernel g is compactly supported.*) *There exists $\Gamma > 0$, $C > 0$ such that the quantum noisy relaxation time satisfies:*

$$\frac{1}{\Gamma} \ln(\epsilon^{-1}) - C \le \tau_q(\epsilon, N) \le \frac{2d + s + s^*}{|\ln \sigma_{s,s^*}|} \ln(\epsilon^{-1}) + C$$

As mentioned above, the restriction to compactly-supported noise kernel in statement (II) is probably unnecessary, so we put it between parentheses.

The semiclassical régime $N\epsilon^E > 1$ of this theorem is quite distant from the "quantum régime" ($N\epsilon \ll 1$) described in Proposition 4. Inbetween we find a "crossover range"

$$\epsilon^{-1} \ll N \le \epsilon^{-E} \tag{23}$$

for which we do not control the quantum relaxation rates. However, at the level of characteristic times, this range corresponds to differences between prefactors, as we summarize in the following corollary. There we define the "Ehrenfest time" precisely as $\tau_E = \frac{\ln(N)}{\Gamma}$, where Γ is the largest expansion rate of the Anosov map (see Lemma 1) instead of using the Lyapounov exponent λ (in general, λ and Γ do not differ too much).

Corollary 2 *Assume the conditions of Theorem 2.*

i) *In the semiclassical régime $N \ge \epsilon^{-E}$, the Ehrenfest time is strictly larger than the classical and quantum relaxation times:*

$$\tau_E = \frac{\ln(N)}{\Gamma} \ge \frac{E \ln(\epsilon^{-1})}{\Gamma} \ge \begin{cases} K\, \tau_c(\epsilon) \\ K\, \tau_q(\epsilon, N) \end{cases},$$

with a constant $K > 1$.

ii) *In the quantum régime $N\epsilon \ll 1$, we have on the contrary*

$$\tau_E \leq \frac{\ln(\epsilon^{-1})}{\Gamma} \leq \tau_c(\epsilon).$$

iii) *For any $\gamma > 0$, the noise kernel $g \in \mathcal{S}(\mathbb{R}^{2d})$ decays as $|\boldsymbol{x}|^{-\gamma}$. Then, in the "deeply quantum" régime $N\epsilon \ll |\ln \epsilon|^{-1/\gamma}$ one has*

$$\tau_E \leq \frac{\ln(\epsilon^{-1})}{\Gamma} \leq \tau_c(\epsilon) \ll \tau_q(\epsilon, N).$$

This corollary is easily proven by using the bounds in the above theorem as well as in its classical counterpart [23, Th. 4 (II)], the explicit formulas (40, 42) for the exponent E and Proposition 4. It confirms the argument presented in the Introduction: the quantum relaxation behaves like the classical one if both are shorter than the Ehrenfest time; on the opposite, quantum relaxation becomes much slower than the classical one if the classical relaxation time is larger than τ_E. Inbetween, the "crossover range" (23) corresponds to a situation where the classical relaxation time is of the same order as the Ehrenfest time, but where we do not precisely control the quantum relaxation time.

Remark. The above theorem does only specify a régime for which the quantum and classical relaxation times are *of the same order*, $\tau_q(\epsilon, N(\epsilon)) \sim \tau_c(\epsilon) \sim \ln(\epsilon^{-1})$. For a general Anosov map Φ we are unable to exhibit a régime for which $\tau_q(\epsilon, N(\epsilon)) \approx \tau_c(\epsilon)$, that is for which the relaxation times are *asymptotic to each other* (cf. Corollary 1). The reason for this failure resides in our insufficient knowledge of the observables which maximize the norms $\frac{\|T_\epsilon^n f\|}{\|f\|}$ (or $\frac{\|\tilde{T}_\epsilon^{(n)} f\|}{\|f\|}$). These observables become quite singular when n becomes large, so we do not know whether the quantum-classical correspondences stated in Propositions 6–7 are helpful when applied to these "maximizing" observables, if n is close to the classical relaxation time.

More precise estimates will be obtained in Section 4.2 in the special case of *linear* Anosov diffeomorphisms of the torus.

Proof of Theorem 2. The proof will proceed in several steps. We start with refinements of the Egorov property (7) for general maps Φ. Then, we prove lower bounds for the quantum relaxation times in the case of an expansive map, and upper bounds if the map is mixing, so that both bounds can be applied if Φ is Anosov.

4.1.1 Egorov estimates

The two following estimates (proven in Appendix B.1) are obtained by adapting the methods of [11] to quantum mechanics on $\mathbb{T}^{2d}$. To alleviate the notations we omit to indicate the dependence on the angle $\boldsymbol{\theta}$.

Proposition 6 *Let Φ be a smooth quantizable map on $\mathbb{T}^{2d}$, and $\mathcal{U}_N(\Phi)$ its quantization on $\mathcal{A}_N$. Then there exists a constant $C > 0$ such that for any $N > 0$, any classical observable $f \in C^\infty(\mathbb{T}^{2d})$ and any $n \in \mathbb{N}$, one has*

$$\|\mathcal{U}_N(\Phi)^n \, Op_N(f) - Op_N(f \circ \Phi^n)\|_{HS} \le \frac{C}{N} \sum_{m=0}^{n-1} \|f \circ \Phi^m\|_{C^{2d+3}}. \tag{24}$$

For a generic map Φ, the norm on the RHS will grow exponentially, with a rate $e^{\Gamma n}$ where Γ depends on the local hyperbolicity of the map. For more "regular" maps, the derivatives may grow as a power law (cf. the discussion on the differential $D\Phi^n$ in [23, Section 4]).

We will also need the following noisy version of the classical-quantum correspondence (proven in Appendix B.2):

Proposition 7 *Assume that for some power $M \ge 2d + 1$, the noise generating function $g \in C^M(\mathbb{R}^{2d})$ and all its derivatives up to order M decay fast at infinity.*

Let Φ be a quantizable map and T_ϵ, $\mathcal{T}_{\epsilon,N}$ the associated classical and quantum noisy propagators. Then there exists $\tilde{C} > 0$ such that, for any $f \in C^\infty(\mathbb{T}^{2d})$ and any $n \ge 0$,

$$\|\mathcal{T}^n_{\epsilon,N} \, Op_N(f) - Op_N(T^n_\epsilon \, f)\|_{HS} \le \tilde{C}\Big(\sum_{m=0}^{n-1} \frac{\|T^m_\epsilon \, f\|_{C^{2d+3}}}{N}\Big) + \tilde{C}\, \frac{\|T^n_\epsilon \, f\|_{C^M}}{(\epsilon N)^M}. \tag{25}$$

Using these two propositions, we will now to adapt the proofs given in [23] for lower and upper bounds of the classical relaxation times, to the quantum framework.

4.1.2 Lower bounds for expansive maps

The lower bounds for the noisy relaxation time $\tau_c(\epsilon)$ rely on the following identity [23, Section 4]. Let f be an arbitrary function in $C^1_0(\mathbb{T}^{2d})$, e.g., the Fourier mode $f = w_{\boldsymbol{k}}$ for $\boldsymbol{k} = (1, 0, \dots, 0)$. For g decaying fast at infinity, we showed that for a certain $C > 0$,

$$\|T^n_\epsilon w_{\boldsymbol{k}}\|_{L^2_0} \ge 1 - C\epsilon \sum_{m=1}^{n} \|\nabla(T^m_\epsilon w_{\boldsymbol{k}})\|_{L^2_0} \ge 1 - C\epsilon \|\nabla w_{\boldsymbol{k}}\|_{C^0} \sum_{m=1}^{n} \|D\Phi\|^m_{C^0}.$$

We will now use this formula to get a lower bound on the corresponding quantum quantity, $\|\mathcal{T}^n_{\epsilon,N} W_{\boldsymbol{k}}\|_{HS}$. Indeed, from Eq. (25), we have for $M \ge 2d+3$:

$$\|\mathcal{T}^n_{\epsilon,N} W_{\boldsymbol{k}}\|_{HS} \ge 1 - C\epsilon \|\nabla w_{\boldsymbol{k}}\|_{C^0} \sum_{m=1}^{n} \|D\Phi\|^m_{C^0} - \frac{C}{\min\big(N, (\epsilon N)^M\big)} \sum_{m=0}^{n} \|T^n_\epsilon \, w_{\boldsymbol{k}}\|_{C^M}. \tag{26}$$

We need to control the higher derivatives of $T^m_\epsilon w_{\boldsymbol{k}}$. This can be done quite easily applying the chain rule (see [11, Lemma 2.2] and Appendix A.2):

Lemma 1 *For any C^∞ diffeomorphism Φ, denote by $\Gamma = \ln\big(\sup_x \|D\Phi_{|x}\|\big)$ the local expansion parameter of Φ. Then for any index $M \in \mathbb{N}_0$, there exists a constant $C_M > 0$ such that*

$$\forall f \in C^\infty(\mathbb{T}^{2d}), \quad \forall n \geq 1, \qquad \|f \circ \Phi^n\|_{C^M} \leq C_M\, e^{nM\Gamma}\, \|f\|_{C^M}.$$

Furthermore, for any $\epsilon > 0$, the noisy evolution is also under control:

$$\forall n \geq 1, \qquad \|T_\epsilon^n f\|_{C^M} \leq C_M\, e^{nM\Gamma}\, \|f\|_{C^M}.$$

We will only consider the generic case of an expansive map, for which $\Gamma > 0$. The inequality (26) yields, for $M \geq 2d+3$, the lower bound

$$\|\mathcal{T}_{\epsilon,N}^n\| \geq 1 - C_M\big(\epsilon e^{n\Gamma} + \big(N^{-1} + (\epsilon N)^{-M}\big)e^{nM\Gamma}\big). \tag{27}$$

The same lower bound can be obtained for the coarse-grained evolution. Indeed,

$$\tilde{\mathcal{T}}_{\epsilon,N}^{(n)} W_{\boldsymbol{k}} = \gamma_{\epsilon,N}(\boldsymbol{k})\, \tilde{\mathcal{G}}_{\epsilon,N}\mathcal{U}_N^n W_{\boldsymbol{k}}.$$

Using the Egorov estimate in Proposition 6 and the bound (66), the norm of the RHS is bounded from below by

$$|\gamma_{\epsilon,N}(\boldsymbol{k})| \Big(\|G_\epsilon w_{\boldsymbol{k}} \circ \Phi^n\|_{L_0^2} - Ce^{nM\Gamma}(N^{-1} + (\epsilon N)^{-M})\Big).$$

Since g decays fast, the classical lower bound [23, Eq. (36)] yields:

$$\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| \geq 1 - C\epsilon\|D\Phi^n\|_{C^0} - Ce^{nM\Gamma}(N^{-1} + (\epsilon N)^{-M}), \tag{28}$$

which is of the same type as the lower bound (27). We assume that the derivative of Φ^n grows with a rate $\tilde{\Gamma} > 0$ (with $\tilde{\Gamma} \leq \Gamma$): there is a constant $A > 0$ such that for all $n > 0$, $\|D\Phi^n\|_{C^0} \leq A\, e^{n\tilde{\Gamma}}$.

Proposition 8 *Assume that the noise generating function $g \in C^M$ with $M \geq 2d+3$, and all its derivatives decay fast at infinity. For any smooth expansive diffeomorphism Φ, we have in the joint limit $\epsilon \to 0$, $\epsilon N \to \infty$, the following lower bounds for the quantum relaxation times:*

$$\tau_q(\epsilon, N) \geq \min\Big(\frac{\ln(\epsilon^{-1})}{\Gamma}, \frac{\ln N}{M\Gamma}, \frac{\ln(\epsilon N)}{\Gamma}\Big) + C \tag{29}$$

$$\tilde{\tau}_q(\epsilon, N) \geq \min\Big(\frac{\ln(\epsilon^{-1})}{\tilde{\Gamma}}, \frac{\ln N}{M\Gamma}, \frac{\ln(\epsilon N)}{\Gamma}\Big) + C \tag{30}$$

Since $M > 2$, we conclude that in a régime satisfying $N > \epsilon^{-M}$ (and respectively $N > \epsilon^{-\frac{\Gamma}{\tilde{\Gamma}}M}$), the above lower bounds for the quantum relaxation times are identical with the ones obtained for the classical relaxation times.

4.1.3 Upper bounds for mixing maps

In the classical framework [23, Section 5], we used the Fourier decomposition to get an upper bound on $\|T_\epsilon^n f\|$ for all possible $f \in L_0^2$, and then applied the classical mixing (which holds for *differentiable* observables) to the individual Fourier modes. Since our estimates of the quantum-classical correspondence (Props 6, 7) apply to observables with some degree of differentiability, this Fourier decomposition is well adapted to the generalization to the quantum framework.

Consider an arbitrary quantum observable $A \in \mathcal{A}_N^0$, $\|A\| = 1$ with Fourier coefficients $\{a_{\boldsymbol{k}}\}$. Using Fourier decomposition, we easily get for the coarse-grained evolution:

$$\tilde{\mathcal{T}}_{\epsilon,N}^{(n)} A = \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}_N^{2d}} \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}_N^{2d}} a_{\boldsymbol{k}}\, \gamma_{\epsilon,N}(\boldsymbol{j})\gamma_{\epsilon,N}(\boldsymbol{k})\langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}}\rangle\, W_{\boldsymbol{j}} \tag{31}$$

$$\Longrightarrow \|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)} A\|_{HS}^2 \leq \|A\|_{HS}^2 \sum_{0\neq \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}_N^{2d}} |\gamma_{\epsilon,N}(\boldsymbol{j})\gamma_{\epsilon,N}(\boldsymbol{k})|^2\, |\langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}}\rangle|^2 \tag{32}$$

The overlaps $\langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}}\rangle$ can be seen as quantum correlation functions. From the Egorov estimate of Proposition 6, this correlation can be related to the classical correlation function $\langle w_{\boldsymbol{j}}, w_{\boldsymbol{k}} \circ \Phi^n\rangle$:

$$\begin{aligned} \langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}}\rangle &= \langle W_{\boldsymbol{j}}, Op_N(w_{\boldsymbol{k}} \circ \Phi^n)\rangle + \mathcal{O}\Big(\frac{1}{N}\sum_{m=0}^{n-1} \|w_{\boldsymbol{k}}\circ\Phi^m\|_{C^{2d+3}}\Big) \\ &= \langle w_{\boldsymbol{j}}, w_{\boldsymbol{k}}\circ\Phi^n\rangle + \sum_{0\neq \boldsymbol{m}\in\mathbb{Z}^{2d}} (\pm)\langle w_{\boldsymbol{j}+N\boldsymbol{m}}, w_{\boldsymbol{k}}\circ\Phi^n\rangle + \mathcal{O}\Big(\frac{1}{N}\sum_{m=0}^{n-1} \|w_{\boldsymbol{k}}\circ\Phi^m\|_{C^{2d+3}}\Big). \end{aligned} \tag{33}$$

To write the second line, we used the explicit expression (5) for $Op_N(f)$. From the smoothness of $w_{\boldsymbol{k}}\circ\Phi^n$, the sum over $\boldsymbol{m} \neq 0$ on the RHS is an $\mathcal{O}(N^{-M}\, \|w_{\boldsymbol{k}}\circ\Phi^n\|_{C^M})$ for any $M > 2d$. Therefore,

$$\langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}}\rangle = \langle w_{\boldsymbol{j}}, w_{\boldsymbol{k}}\circ\Phi^n\rangle + \mathcal{O}\Big(\frac{1}{N}\sum_{m=0}^{n} \|w_{\boldsymbol{k}}\circ\Phi^m\|_{C^{2d+3}}\Big).$$

We can then use classical information on the derivatives of $w_{\boldsymbol{k}} \circ \Phi^m$ and the correlation functions $\langle w_{\boldsymbol{j}}, w_{\boldsymbol{k}} \circ \Phi^n\rangle$. The former are estimated in Lemma 1, while the latter depend on the dynamics generated by Φ.

We now use the fact that the map Φ is mixing, both with and without noise, in a way stated in Eqs. (22) (for a moment we do not need to precise that $\Gamma(n)$ decays exponentially fast). Applied to the Fourier modes, Eqs. (22) read (with C depending only on the indices s, s^*):

$$\forall \boldsymbol{j}, \boldsymbol{k} \in \mathbb{Z}^{2d} - 0, \quad \forall n \in \mathbb{N}, \qquad |\langle w_{\boldsymbol{j}}, w_{\boldsymbol{k}} \circ \Phi^n \rangle| \leq C\, |\boldsymbol{j}|^s\, |\boldsymbol{k}|^{s_*} \Gamma(n), \tag{34}$$

$$\text{for any small enough } \epsilon > 0 \text{ and any } n \in \mathbb{N}, \qquad |\langle w_{\boldsymbol{j}}, T_\epsilon^n w_{\boldsymbol{k}} \rangle| \leq C\, |\boldsymbol{j}|^s\, |\boldsymbol{k}|^{s_*} \Gamma(n). \tag{35}$$

From this classical mixing, the quantum correlation functions are bounded from above as:

$$|\langle W_{\boldsymbol{j}}, \mathcal{U}_N^n(\Phi) W_{\boldsymbol{k}} \rangle| \leq C\, |\boldsymbol{j}|^s\, |\boldsymbol{k}|^{s_*} \Gamma(n) + C \frac{\left(e^{n\Gamma}\, |\boldsymbol{k}|\right)^{2d+3}}{N}. \tag{36}$$

We are now in a position to estimate the two sums in the RHS of Eq. (32). From the estimate (67) and the fast decay at infinity of g, we can approximate sums over the quantum noise eigenvalues by integrals [23, Lemma 4]:

$$\sum_{0 \neq \boldsymbol{j} \in \mathbb{Z}_N^{2d}} |\gamma_{\epsilon,N}(\boldsymbol{j})|^2\, |\boldsymbol{j}|^{2s} = \frac{1}{\epsilon^{2s+2d}} \left(\int |\hat{g}(\boldsymbol{\xi})|^2\, |\boldsymbol{\xi}|^{2s}\, d\boldsymbol{\xi} + \mathcal{O}(\epsilon) + \mathcal{O}\big((\epsilon N)^{2d+2s-2D}\big) \right). \tag{37}$$

The exponent D is related to the smoothness of g, and should satisfy $D \geq 2d+1$. We will also assume that $D > d+s$ to make the last remainder small. The estimate (37) can be used to control the other terms appearing when combining Eqs (32) and (36). The index s will be replaced by s_*, 0 and $2d+3$ respectively. In all cases, we will assume that $D > d + \mathit{index}$. The same methods can be applied to estimate the norm of the noisy evolution $\mathcal{T}_{\epsilon,N}$ (assuming a classical mixing of the type (35)).

Proposition 9 *Assume that the noiseless and noisy dynamics generated by the map Φ are mixing, as in Eqs. (22). Then the quantum coarse-grained and noisy propagators satisfy the following bounds, in the joint limits $\epsilon \to 0$, $\epsilon N \to \infty$:*

$$\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\|^2 \lesssim \frac{\Gamma(n)^2}{\epsilon^{2(2d+s+s_*)}} + \frac{e^{2n(2d+3)\Gamma}}{N^2\, \epsilon^{8d+6}}, \tag{38}$$

$$\|\mathcal{T}_{\epsilon,N}^n\|^2 \lesssim \frac{\Gamma(n)^2}{\epsilon^{2(2d+s+s_*)}} + \frac{e^{2n(2d+3)\Gamma}}{N^2\, \epsilon^{8d+6}} + \frac{e^{2nM\Gamma}}{N^{2M}\, \epsilon^{4d+4M}}. \tag{39}$$

In the second line, the upper bound holds for any exponent $M \geq 2d+1$.

The first term in the RHS of those two equations is of purely classical origin, it is identical to the classical upper bounds [23, Th. 3] (remember that the dimension of the phase space is now $2d$). This term decreases according to the function $\Gamma(n)$, that is, according to the speed of mixing. On the opposite, the remaining terms, due to quantum effects, grow exponentially in time.

End of the proof of the Theorem. We are now in position to combine our results for lower and upper bounds, in the case of a smooth Anosov diffeomorphism. Such a diffeomorphism is expansive, therefore it admits positive expansion parameters $\Gamma \geq \tilde{\Gamma} > 0$, as defined in Lemma 1 and before Proposition 8. From that proposition, the constant

$$E_1 := (2d+3)\frac{\Gamma}{\tilde{\Gamma}} \tag{40}$$

is such that in the régime $N > \epsilon^{-E_1}$, the lower bounds for the quantum and classical times are identical (in case of fully noisy dynamics it suffices to take $E_1 := 2d+3$).

On the other hand, from Theorem 1 the classical mixing is exponential, with a rate $\sigma_{s,s^*} < 1$. As a result, this theorem and the analysis of [23] imply that the *classical* relaxation times $\tilde{\tau}_c(\epsilon)$ and $\tau_c(\epsilon)$ are bounded from above by

$$\tau_c(\epsilon),\ \tilde{\tau}_c(\epsilon) \leq \frac{2d+s+s^*}{|\ln \sigma_{s,s^*}|} \ln(\epsilon^{-1}) + \text{const.} \tag{41}$$

We set $M = 2d+3$ in Proposition 9, and insert the upper bound (41) in the second and third terms in the RHS of Eqs. (38, 39): these terms are then of respective orders $\mathcal{O}\big((N\,\epsilon^{E_2})^{-2}\big)$ and $\mathcal{O}\big((N\,\epsilon^{E_3})^{-2(2d+3)}\big)$, where

$$E_2 = \Gamma\,\frac{(2d+3)(2d+s+s^*)}{|\ln \sigma_{s,s^*}|} + 3 + 4d, \quad E_3 = \Gamma\,\frac{2d+s+s^*}{|\ln \sigma_{s,s^*}|} + 2 + \frac{2d}{2d+3}. \tag{42}$$

The second exponent is clearly smaller than the first one. Therefore, in the régime $N \gg \epsilon^{-E_2} \gg \epsilon^{-E_3}$, these two terms are $\ll 1$ when n is smaller than the classical relaxation times. Therefore in this régime the *quantum* relaxation times $\tilde{\tau}_q(\epsilon, N)$, $\tau_q(\epsilon, N)$ are also bounded from above by the RHS of Eq. (41).

Finally, for any power $E > \max(E_1, E_3)$, the condition $N > \epsilon^{-E}$ provides the "semiclassical régime". Note that the exponent E is defined from purely classical quantities related to the map Φ. □

4.2 Relaxation time of quantum toral symplectomorphisms

In this section we analyze the quantum relaxation times when the map Φ is a quantizable symplectomorphism F of the torus $\mathbb{T}^{2d}$ (see Subsection 2.2.1). We will only restrict ourselves to the case where the matrix F is *ergodic* (none of its eigenvalues is a root of unity), and diagonalizable. Let us remind some notations we used in the classical setting [24]. Diagonalizability of F implies that there exists a rational basis of $\mathbb{R}^{2d}$ where F takes the form $diag(A_1, \ldots, A_r)$, where each block A_j is a $d_j \times d_j$ rational matrix, the characteristic polynomial of which is irreducible over $\mathbb{Q}$. The eigenvalues of A_j are denoted by $\{\lambda_{j,k},\ k = 1, \ldots, d_j\}$. We call $h_j = \sum_{|\lambda_{j,k}|>1} \log|\lambda_{j,k}|$ the Kolmogorov-Sinai (K-S) entropy of the block A_j, and $\hat{h}_j = \frac{h_j}{d_j}$ its "dimensionally-averaged K-S entropy". Finally, we associate to the full matrix F the "minimal dimensionally-averaged K-S entropy"

$$\hat{h} = \min_{j=1,\dots,r} \hat{h}_j. \tag{43}$$

Due to the simple action of the map $\mathcal{U}_N(F)$ on the quantum Fourier modes (Eq. (8)), many computations can be carried out explicitly, and yield precise asymptotics of the quantum relaxation times.

To focus attention and avoid unnecessary notational and computational complications, we restrict the considerations of this subsection to an isotropic Gaussian noise $\hat{g}(\boldsymbol{k}) = e^{-|\boldsymbol{k}|^2}$ (in [24] a slightly more general noise was considered, given by α-stable laws).

From the exact Egorov property (8) and the fact that the quantum Fourier modes $W_{\boldsymbol{k}}(N,\boldsymbol{\theta})$ are eigenstates of the quantum noise operator (cf. Proposition 2), one easily proves that any $A \in \mathcal{A}_N^0(\boldsymbol{\theta})$ with Fourier coefficients $\{a_{\boldsymbol{k}}\}$ (cf. Eq. (6)) evolves into

$$\mathcal{T}_{\epsilon,N}^n A = \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} a_{\boldsymbol{k}} \left(\prod_{l=1}^n \gamma_{\epsilon N}(F^{-l}\boldsymbol{k}) \right) W_{F^{-n}\boldsymbol{k}}.$$

Orthogonality of the $\{W_{\boldsymbol{k}}\}$ then induces the exact expression:

$$\|\mathcal{T}_{\epsilon,N}^n\| = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \left(\prod_{l=1}^n \gamma_{\epsilon N}(F^{-l}\boldsymbol{k}) \right) = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \left(\prod_{l=1}^n \gamma_{\epsilon N}(F^{l}\boldsymbol{k}) \right), \tag{44}$$

Similarly, in the coarse grained case we have

$$\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \big(\gamma_{\epsilon N}(\boldsymbol{k}) \gamma_{\epsilon N}(F^n \boldsymbol{k}) \big). \tag{45}$$

Using these exact formulas, we can precisely estimate the quantum relaxation times.

Theorem 3 *Let $F \in Sp(2d,\mathbb{Z})$ be ergodic and diagonalizable, and for all $N \in \mathbb{N}$ we select an admissible angle $\boldsymbol{\theta}$ for which F may be quantized on $\mathcal{H}_{N,\boldsymbol{\theta}}$. The noise is assumed to be Gaussian. Then the quantum relaxation times associated with the quantum dynamics satisfy the following estimates:*

I) *For any $\epsilon > 0$ and $N \in \mathbb{Z}_+$,*

$$\tau_q(\epsilon,N) \geq \tau_c(\epsilon), \qquad \tilde{\tau}_q(\epsilon,N) \geq \tilde{\tau}_c(\epsilon).$$

II) *There exists $M > 0$ (made explicit in Eq. (50)) such that in the joint limit $\epsilon \to 0$, $N > M\epsilon^{-1}$,*

$$\tau_q(\epsilon,N) \approx \tau_c(\epsilon) \approx \frac{1}{\hat{h}(F)} \ln(\epsilon^{-1}).$$

III) *Let* $\mu = \max(\|F\|, \|F^{-1}\|)$. *For any coefficient* $\beta > \frac{\ln\mu}{2\hat{h}(F)} + 1$, *one has in the joint limit* $\epsilon \to 0$, $N > \epsilon^{-\beta}$:

$$\tilde{\tau}_q(\epsilon, N) \approx \tilde{\tau}_c(\epsilon) \approx \frac{1}{\hat{h}(F)} \ln(\epsilon^{-1}).$$

Here $\hat{h}(F)$ *is the minimal dimensionally averaged K-S entropy of* F, *Eq.* (43).

As a direct corollary of the above theorem (and using Proposition 4), we obtain the following relations between, on one side, the "spatial" scales (namely ϵ for the noise, $\hbar$ for the scale of the "quantum mesh"), and on the other side the "time scales" (namely the relaxation and Ehrenfest times), for the case of linear ergodic (diagonalizable) symplectomorphisms. As in Corollary 2, we take for the Ehrenfest time $\tau_E = \frac{\ln N}{\Gamma}$, with now $\Gamma = \ln(\|F\|)$.

Corollary 3 *Under the assumptions of Theorem 3 the following relations hold between the noisy quantum relaxation time and the Ehrenfest time* τ_E, *in the joint limit* $\epsilon \to 0$, $N \to \infty$, *depending on the behavior of the product* ϵN:

i) *If* $N \gg \epsilon^{-1}$, *then*
$$\tau_E \gtrsim \tau_c(\epsilon) \approx \tau_q(\epsilon, N).$$
The first $\gtrsim$ *can be replaced by* $\geq$ *if* $N \gg \epsilon^{-\Gamma/\hat{h}(F)}$.

ii) *There exists* $M > 0$ (*see* (50)) *such that, for any finite* $M' > M$,
$$\text{if} \quad \epsilon N \to M' \quad \text{then} \quad \tau_c(\epsilon) \approx \tau_q(\epsilon, N) \sim \tau_E.$$

iii) *If* $\epsilon N \leq \frac{1-\delta}{\sqrt{\ln\ln(\epsilon^{-1})}}$ *for some* $\delta > 0$, *then*
$$\tau_E \leq \tau_c(\epsilon) \ll \tau_q(\epsilon, N).$$

The form of the "deeply quantum régime" iii) is due to the Gaussian noise (compare with Corollary 2 iii) for a more general noise). For linear automorphisms, the "crossover range" is much thinner than for a nonlinear Anosov map (see Corollary 2): here this crossover takes place when Planck's constant N crosses a window $[\frac{\epsilon^{-1}}{\sqrt{\ln\ln(\epsilon^{-1})}}, M\epsilon^{-1}]$, to be compared with a window $[\frac{\epsilon^{-1}}{\sqrt{\ln\ln(\epsilon^{-1})}}, \epsilon^{-E}]$ for a general Anosov map with Gaussian noise.

Proof of Theorem 3. To prove the theorem we will need the following estimates (proven in Appendix A.3), which relate the eigenvalues of the classical and quantum noise operators. We remind that here and below, $\hat{g}_\sigma(\boldsymbol{\xi}) = e^{-|\sigma\boldsymbol{\xi}|^2}$.

Lemma 2 *For any* $N \in \mathbb{N}_0$ *and* $\boldsymbol{\xi} \in \mathbb{R}^{2d}$, *we denote by* $\boldsymbol{\xi}^N$ *the unique vector in* $\mathbb{R}^{2d}$ *s.t. all its components satisfy* $\boldsymbol{\xi}_j^N \equiv \boldsymbol{\xi}_j \bmod N$ *and* $\boldsymbol{\xi}_j^N \in (-N/2, N/2]$.

Then for any $\epsilon > 0$, $N \in \mathbb{N}_0$ *and all* $\boldsymbol{\xi} \in \mathbb{R}^{2d}$,

$$\hat{g}_\epsilon(\boldsymbol{\xi}) \leq \hat{g}_\epsilon(\boldsymbol{\xi}^N) \leq \gamma_{\epsilon,N}(\boldsymbol{\xi}) \leq \frac{\hat{g}_\epsilon(\boldsymbol{\xi}^N)}{\tilde{g}_{\epsilon N}(0)} + 4d\, e^{-\frac{(\epsilon N)^2}{4}} \leq \hat{g}_\epsilon(\boldsymbol{\xi}^N) + 4d\, e^{-\frac{(\epsilon N)^2}{4}}. \tag{46}$$

Besides, we will need the following integer programming result [24], which measures the "minimal extension" of an F-trajectory on the Fourier lattice:

Proposition 10 *Let $F \in SL(2d,\mathbb{Z})$ be ergodic and diagonalizable. For any (small) $\delta > 0$, there exists $n(\delta) > 0$ s.t. for any $n \geq n(\delta)$, we have:*

$$e^{2(1-\delta)\hat{h}(F)n} < \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^{2d}} \left(|\boldsymbol{k}|^2 + |F^n\boldsymbol{k}|^2\right) < \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^{2d}} \sum_{l=0}^{n} |F^l\boldsymbol{k}|^2 < e^{2(1+\delta)\hat{h}(F)n} \quad (47)$$

As above, $\hat{h}(F)$ is the minimal dimensionally-averaged entropy (43).

We start to prove the statement I) of the Theorem. According to the explicit equations (44, 45) and their classical counterparts [24], the norms of the noisy and coarse-grained propagators are given in terms of products of coefficients $\gamma_{\epsilon,N}(\boldsymbol{k})$ (resp. coefficients $\hat{g}_\epsilon(\boldsymbol{k})$ for the classical propagators). Lemma 2 shows that for any $\boldsymbol{k}\in\mathbb{Z}^{2d}$, $\gamma_{\epsilon,N}(\boldsymbol{k}) \geq \hat{g}_\epsilon(\boldsymbol{k})$. Applying this inequality factor by factor in the explicit expressions for classical and quantum norms yields:

$$\forall n \geq 1, \quad \|\mathcal{T}^n_{\epsilon,N}\| \geq \|T^n_\epsilon\|, \qquad \|\tilde{\mathcal{T}}^{(n)}_{\epsilon,N}\| \geq \|\tilde{T}^{(n)}_\epsilon\|,$$

which yield the statement I).

The lower bounds of statements II) and III) follow from the general inequalities established in point I), together with small-noise results obtained in the classical setting [24].

To prove the upper bound of statement III), we bound from above the RHS of Eq. (45). Given a coefficient β as in the statement, we fix some (arbitrarily small) $\delta > 0$ satisfying $\beta - 1 > \frac{\ln\mu}{2(1-\delta)\hat{h}}$ (from here on, we abbreviate $\hat{h}(F)$ by $\hat{h}$). In the régime $\epsilon^\beta N > 1$, for sufficiently small $\epsilon > 0$ there exist integers n in the interval

$$\frac{1}{(1-\delta)\hat{h}}\ln(2\epsilon^{-1}) < n < n+1 < \frac{1}{(1-\delta)\hat{h} + \frac{1}{2}\ln\mu}\ln(N/2). \quad (48)$$

We take ϵ small enough such that the LHS of this equation is larger than the threshold $n(\delta)$ defined in Proposition 10. We want to control the product $\gamma_{\epsilon N}(\boldsymbol{k}_0)$ $\gamma_{\epsilon N}(F^n\boldsymbol{k}_0)$ for integers n in this interval, uniformly for all $0 \neq \boldsymbol{k}_0 \in \mathbb{Z}^{2d}_N$. We need to consider two cases.

- If both $\boldsymbol{k}_0$ and $F^n\boldsymbol{k}_0$ belong to the "fundamental cell" $\mathbb{Z}^{2d}_N$, then from Proposition 10, we have

$$|\boldsymbol{k}_0|^2 + |F^n\boldsymbol{k}_0|^2 \geq \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^{2d}}(|\boldsymbol{k}|^2 + |F^n\boldsymbol{k}|^2) > e^{2(1-\delta)\hat{h}n}. \quad (49)$$

Thus for any such $\boldsymbol{k}_0$, $\max\big(|\boldsymbol{k}_0|,\ |F^n\boldsymbol{k}_0|\big) > \frac{1}{\sqrt{2}}\, e^{(1-\delta)\hat{h}n}$. Using (46) and the fact that all $\gamma_{\epsilon,N}(\boldsymbol{k}) < 1$, we obtain:

$$\begin{aligned}\gamma_{\epsilon,N}(\boldsymbol{k}_0)\gamma_{\epsilon,N}(F^n\boldsymbol{k}_0) &< \min\big(\gamma_{\epsilon,N}(\boldsymbol{k}_0), \gamma_{\epsilon,N}(F^n\boldsymbol{k}_0)\big)\\ &\le \exp\Big\{-\frac{\epsilon^2}{2}\, e^{2(1-\delta)\hat{h}n}\Big\} + C\, e^{-\frac{(\epsilon N)^2}{4}}.\end{aligned}$$

From the left inequality in (48), the argument of the exponential in the above RHS is smaller than -2. Since $\epsilon N > \epsilon^{1-\beta} \gg 1$, the product on the LHS is $< e^{-1}$.

- assume the opposite situation: $\boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$ but its image $F^n\boldsymbol{k}_0 \notin \mathbb{Z}_N^{2d}$. In that case, we may assume that the set $S_0 = \{\boldsymbol{k}_0, F\boldsymbol{k}_0, \ldots, F^{l_0-1}\boldsymbol{k}_0\} \subset \mathbb{Z}_N^{2d}$, while $F^{l_0}\boldsymbol{k}_0 \notin \mathbb{Z}_N^{2d}$. Consider also $\boldsymbol{k}_n = (F^n\boldsymbol{k}_0)^N$ the representative of $F^n\boldsymbol{k}_0$ in the fundamental cell, and assume that $S_n = \{\boldsymbol{k}_n, F^{-1}\boldsymbol{k}_n, \ldots, F^{-l_n+1}\boldsymbol{k}_n\} \subset \mathbb{Z}_N^{2d}$, while $F^{-l_n}\boldsymbol{k}_n \notin \mathbb{Z}_N^{2d}$. Obviously, the sets S_0, S_n have no common point (this would let the full trajectory $\{F^j\boldsymbol{k}_0\}_{j=0}^n$ be contained in $\mathbb{Z}_N^{2d}$), so that $l_0 + l_n \le n+1$. The vectors k_0, k_n satisfy the obvious inequalities

$$\begin{aligned}\frac{N}{2} &\le |F^{l_0}\boldsymbol{k}_0| \le \|F\|^{l_0}\,|\boldsymbol{k}_0| \le \mu^{l_0}\,|\boldsymbol{k}_0|,\\ \frac{N}{2} &\le |F^{-l_n}\boldsymbol{k}_n| \le \|F^{-1}\|^{l_n}\,|\boldsymbol{k}_n| \le \mu^{l_n}\,|\boldsymbol{k}_n|.\end{aligned}$$

  Since $\min(l_0, l_n) \le \frac{n+1}{2}$, either $|k_0|$ or $|k_n|$ is bounded from below by $\frac{N}{2}\,\mu^{-\frac{n+1}{2}}$, and, from the right inequality in (48), also by $e^{(1-\delta)\hat{h}n}$. We are back to the lower bound of the previous case, leading to the same conclusion.

We have therefore proven that for sufficiently small $\epsilon > 0$ and $N > \epsilon^{-\beta}$, any integer n in the (nonempty) interval (48) satisfies $\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| < e^{-1}$, and is therefore $\ge \tilde{\tau}_q(\epsilon, N)$. As a result,

$$\frac{1}{(1-\delta)\hat{h}}\ln(2\epsilon^{-1}) + 1 \ge \tilde{\tau}_q(\epsilon, N).$$

Since δ can be taken arbitrarily small, we obtain the statement III) of the Theorem.

The upper bounds of statement II) is proven with similar methods. We want to bound from above the product (44). Let C denote the constant of the RHS of (46), and take $M = M(F)$ a constant such that both conditions below are satisfied:

$$Ce^{-\frac{M^2}{4}} < e^{-2}, \qquad \frac{1}{\hat{h}}\ln\left(\frac{M}{4\,\|F\|}\right) > 2. \tag{50}$$

Let us fix some $0 < \delta' < \delta < 1/2$. If $\epsilon N > M$, the second condition implies the existence of an integer n such that

$$\frac{1}{(1-\delta)\hat{h}}\ln(2\,\epsilon^{-1}) < n - 1 < \frac{1}{(1-\delta)\hat{h}}\ln\left(\frac{N}{2\|F\|}\right). \tag{51}$$

We take ϵ small enough so that any n in the above interval is larger than the threshold $n(\delta')$ of Proposition 10, and also satisfies $e^{2(\delta-\delta')\hat{h}n} > n$. For such an n, we can then estimate the products $\prod_{l=0}^{n-1} \gamma_{\epsilon N}(F^l \boldsymbol{k}_0)$, considering two cases for $0 \neq \boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$:

- Assume that $F^l \boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$ for all $l = 0, \dots, n-1$. From Proposition 10 and the assumptions on n, we have

$$\sum_{l=0}^{n-1} |F^l \boldsymbol{k}_0|^2 \geq \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^{2d}} \sum_{l=0}^{n-1} |F^l \boldsymbol{k}|^2 > e^{2(1-\delta')\hat{h}(n-1)} > n\, e^{2(1-\delta)\hat{h}(n-1)}. \quad (52)$$

  Thus for any such $\boldsymbol{k}_0$, there exists $l_0 \in \{0, \dots, n-1\}$ such that $|F^{l_0} \boldsymbol{k}_0| > e^{(1-\delta)\hat{h}(n-1)}$.

- Assume there exists $0 \leq l_0 \leq n-1$ such that $\{\boldsymbol{k}_0, \dots, F^{l_0} \boldsymbol{k}_0\} \in \mathbb{Z}_N^{2d}$, while $F^{l_0+1} \boldsymbol{k}_0 \notin \mathbb{Z}_N^{2d}$. Using the RHS of (51), we necessarily have $|F^{l_0} \boldsymbol{k}_0| \geq \frac{N}{2\|F\|} > e^{(1-\delta)\hat{h}(n-1)}$.

Gluing together both cases and using (46), we infer that for any $0 \neq \boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$, there is an index $0 \leq l_0 \leq n-1$ such that

$$\gamma_{\epsilon,N}(F^{l_0} \boldsymbol{k}) \leq \exp\big\{ -\epsilon^2\, e^{2(1-\delta)\hat{h}(n-1)} \big\} + C e^{-\frac{(\epsilon N)^2}{4}} < e^{-4} + C e^{-\frac{M^2}{4}}.$$

From the first condition in (50), the RHS is $< e^{-1}$, so that $n \geq \tau_q(\epsilon, N)$. This holds for any n satisfying (51). We have proven that in the régime $\epsilon N > M$, one has $\tau_q(\epsilon, N) \leq \frac{\ln(2\,\epsilon^{-1})}{(1-\delta)\hat{h}} + 2$. This is true for any $\delta > 0$ and sufficiently small ϵ, which ends the proof of II). ☐

A Proofs of some elementary facts

A.1 Proof of Proposition 2

The value of the normalization constant is computed as follows

$$Z = \sum_{\boldsymbol{n} \in \mathbb{Z}_N^{2d}} \tilde{g}_\epsilon(N^{-1}\boldsymbol{n}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^{2d}} g_\epsilon(N^{-1}\boldsymbol{n}) = N^{2d} \sum_{\boldsymbol{n} \in \mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{n}) = N^{2d} \tilde{g}_{\epsilon N}(0).$$

Using the periodicity $ad(W_{\boldsymbol{n}+N\boldsymbol{m}}) = ad(W_{\boldsymbol{n}})$, the quantum noise operator can be expressed as:

$$\begin{aligned}
\mathcal{G}_{\epsilon,N} &= \frac{1}{Z} \sum_{\boldsymbol{n} \in \mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{n}}{N}\right) ad(W_{\boldsymbol{n}}) = \frac{1}{N^{2d}\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{n} \in \mathbb{Z}^{2d}} g_\epsilon\left(\frac{\boldsymbol{n}}{N}\right) ad(W_{\boldsymbol{n}}) \\
&= \frac{1}{\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{n} \in \mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{n})\, ad(W_{\boldsymbol{n}}).
\end{aligned}$$

Applying the commutation relations (3), $\mathcal{G}_{\epsilon,N}$ acts on $W_{\boldsymbol{k}}$ as follows

$$\mathcal{G}_{\epsilon,N} W_{\boldsymbol{k}} = \frac{1}{\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{n})\, e^{\frac{2\pi i}{N}\boldsymbol{k}\wedge\boldsymbol{n}}\, W_{\boldsymbol{k}}. \qquad \square$$

A.2 Proof of Lemma 1

The first assertion can be proven along the lines of [11, Lemma 2.2], by an induction argument over the degree k of differentiation (the only difference is that our map is defined for discrete times).

Our induction hypothesis: for any $0 \le k' < k$ there exists $\tilde{C}_{k'}$ such that for any multiindex $|\gamma| = k'$, $|\partial^\gamma \Phi^t| \le \tilde{C}_{k'}\, e^{\Gamma k' t}$. The case $k = 1$ is obvious: $\|\Phi^t(\boldsymbol{x})\| \le C$ uniformly in time.

We now take a multiindex α, $|\alpha| = k$, and apply the chain rule:

$$\partial^\alpha(\Phi\circ\Phi^t) = \sum_{j=1}^{2d} (\partial_j\Phi)\circ\Phi^t \times \partial^\alpha(\Phi^t)_j + \sum_{\gamma\le\alpha, |\gamma|>1} (\partial^\gamma\Phi)\circ\Phi^t \times \mathcal{B}_{\alpha,\gamma}(\phi^t).$$

Here $\mathcal{B}_{\alpha,\gamma}(\phi^t)$ is a sum of products of derivatives of Φ^t of order $< k$; using the induction hypothesis, each product is $\le C\, e^{\Gamma k t}$. Now we use the discrete-time version of [11, Lemma 2.3]. Namely, for a given point $\boldsymbol{x}$, the above equation may be written

$$X(t+1) = M(t)X(t) + Y(t),$$

where $X(t) = \partial^\alpha(\Phi^t)(\boldsymbol{x})$ is "unknown", the matrix $M(t) = D\Phi(\Phi^t(\boldsymbol{x}))$ satisfies $\|M(t)\| \le e^\Gamma$ for all times, and we checked above that $\|Y(t)\| \le C\, e^{\Gamma k t}$. From the explicit expression

$$X(t+1) = \big(\prod_{s=1}^{t} M(s)\big)X(1) + \big(\prod_{s=2}^{t} M(s)\big)Y(1) + \big(\prod_{s=3}^{t} M(s)\big)Y(2) + \cdots + Y(t),$$

one easily checks that $\|X(t)\| \le \tilde{C}_k\, e^{\Gamma k t}$ for a certain constant $\tilde{C}_k$, which proves the induction at the order k. Composing Φ^t with an observable f, we easily get the first assertion of the lemma.

To get the second assertion, we notice that the noise operator consists in averaging over maps of the type $\Phi^t_{\{\boldsymbol{v}_j\}} = t_{\boldsymbol{v}_t}\Phi t_{\boldsymbol{v}_{t-1}}\Phi\cdots t_{\boldsymbol{v}_1}\Phi$. Now, one can easily adapt the above proof to show that each of those maps satisfies, for $|\alpha| = k$,

$$|\partial^\alpha(f\circ\Phi^t_{\{\boldsymbol{v}_j\}})| \le \tilde{C}_k\, \|f\|_{C^k}\, e^{\Gamma k t},$$

with $\tilde{C}_k$ independent of the realization $\{\boldsymbol{v}_j\}$. Averaging over the realizations does not harm the upper bound, yielding the second assertion. $\square$

A.3 Proof of Lemma 2

Since the $2d$-dimensional Gaussian $e^{-|\boldsymbol{\xi}|^2}$ factorizes into $\prod_i e^{-\xi_i^2}$, it is natural to first treat the one-dimensional case, that is consider the periodized Gaussian (a Jacobi theta function)

$$\theta_\sigma(\xi) = \sum_{\nu\in\mathbb{Z}} e^{-\sigma^2(\xi+\nu)^2}, \qquad \tilde{\theta}_\sigma(\xi) = \sum_{0\neq\nu\in\mathbb{Z}} e^{-\sigma^2(\xi+\nu)^2}.$$

If we assume that $\xi \in (-1/2, 1/2]$, one has $\nu + \xi > \nu - 1/2$ for $\nu > 0$ and $\nu + \xi < \nu + 1/2$ for $\nu < 0$. From the monotonicity of the Gaussian on $\mathbb{R}^{\pm}$, this implies

$$\tilde{\theta}_\sigma(\xi) \le \theta_\sigma(1/2) = 2\, e^{-\sigma^2/4} \sum_{\nu\ge 0} e^{-\sigma^2\nu(\nu+1)} \le 2\, e^{-\sigma^2/4}\, \theta_\sigma(0). \tag{53}$$

We will also use the lower bound:

$$\theta_\sigma(\xi) = e^{-\sigma^2\xi^2}\big(1 + \sum_{\nu>0} 2\cosh(2\sigma^2\nu\xi)\, e^{-\sigma^2\nu^2}\big) \ge e^{-\sigma^2\xi^2}\, \theta_\sigma(0). \tag{54}$$

We can now pass the the $2d$-dimensional case and consider $\boldsymbol{\xi}$, with all components in $(-1/2, 1/2]$. An easy bookkeeping shows that

$$\begin{aligned} \theta_\sigma(\boldsymbol{\xi}) := \prod_{i=1}^{2d} \theta_\sigma(\xi_i) &= e^{-\sigma^2|\boldsymbol{\xi}|^2} + \tilde{\theta}_\sigma(\xi_1) \prod_{i=2}^{2d} \theta_\sigma(\xi_i) + e^{-\sigma^2\boldsymbol{\xi}_1^2}\tilde{\theta}_\sigma(\xi_2) \prod_{i=3}^{2d} \theta_\sigma(\xi_i) \\ &\quad + e^{-\sigma^2(\boldsymbol{\xi}_1^2+\boldsymbol{\xi}_2^2)}\tilde{\theta}_\sigma(\xi_3) \prod_{i=4}^{2d} \theta_\sigma(\xi_i) + \cdots + e^{-\sigma^2(\boldsymbol{\xi}_1^2+\cdots+\boldsymbol{\xi}_{2d-1}^2)}\tilde{\theta}_\sigma(\xi_{2d}). \end{aligned} \tag{55}$$

Using the bound (53) and the fact that the maximum of θ_σ is $\theta_\sigma(0) > 1$, we obtain:

$$\theta_\sigma(\boldsymbol{\xi}) \le e^{-\sigma^2|\boldsymbol{\xi}|^2} + 4d\, e^{-\sigma^2/4}\, \theta_\sigma(0)^{2d} = e^{-\sigma^2|\boldsymbol{\xi}|^2} + 4d\, e^{-\sigma^2/4}\, \theta_\sigma(\mathbf{0}). \tag{56}$$

The quantum eigenvalues are expressed in terms of the function

$$\gamma_{\epsilon,N}(\boldsymbol{\xi}) = \gamma_{\epsilon,N}(\boldsymbol{\xi}^N) = \frac{\theta_{\epsilon N}(\boldsymbol{\xi}^N/N)}{\theta_{\epsilon N}(\mathbf{0})}.$$

From the estimates (54, 56), this function satisfies

$$e^{-\epsilon^2|\boldsymbol{\xi}^N|^2} \le \gamma_{\epsilon,N}(\boldsymbol{\xi}^N) \le \frac{e^{-\epsilon^2|\boldsymbol{\xi}^N|^2}}{\theta_{\epsilon N}(\mathbf{0})} + 4d\, e^{-(\epsilon N)^2/4} \le e^{-\epsilon^2|\boldsymbol{\xi}^N|^2} + 4d\, e^{-(\epsilon N)^2/4}.$$

□

B Egorov estimates

B.1 Proof of Proposition 6

We need to prove the statement for one iterate of the map ($n = 1$). As explained in Section 2.2, Φ is a combination of a linear automorphism F, a translation $t_{\boldsymbol{v}}$ and the time-1 flow map Φ_1: $\Phi = F \circ t_{\boldsymbol{v}} \circ \Phi_1$. The quantum propagator on $\mathcal{A}_N$ is given by the (contravariant) product:

$$\mathcal{U}(\Phi) = \mathcal{U}(\Phi_1)\mathcal{U}(t_{\boldsymbol{v}})\mathcal{U}(F). \tag{57}$$

We estimate the quantum-classical discrepancy of each component separately. The estimate will be valid for either the operator norm on $\mathcal{H}_{N,\boldsymbol{\theta}}$, or the Hilbert-Schmidt norm.

As explained in Section 2.2.1, the correspondence is exact for the linear automorphism:

$$\mathcal{U}(F)Op(f) = Op(f \circ F). \tag{58}$$

The translation $t_{\boldsymbol{v}}$ is quantized by a quantum translation of vector $\boldsymbol{v}^{(N)}$, which is at a distance $|\boldsymbol{v}-\boldsymbol{v}^{(N)}| \leq CN^{-1}$: $\mathcal{U}(\boldsymbol{v})Op(f) = Op(f \circ t_{\boldsymbol{v}^{(N)}})$. If we Fourier decompose $f = \sum_{\boldsymbol{k}} \hat{f}(\boldsymbol{k}) w_{\boldsymbol{k}}$, we have trivially $f \circ t_{\boldsymbol{v}} = \sum_{\boldsymbol{k}} e^{2i\pi \boldsymbol{k}\wedge\boldsymbol{v}} f(\boldsymbol{k})\, w_{\boldsymbol{k}}$. As a result, since for our norms $\|W_{\boldsymbol{k}}\| = 1$, we simply get

$$\|\mathcal{U}(\boldsymbol{v})Op(f) - Op(f \circ t_{\boldsymbol{v}})\| \leq \sum_{\boldsymbol{k}} |f(\boldsymbol{k})|\, |e^{2i\pi\boldsymbol{k}\wedge\boldsymbol{v}^{(N)}} - e^{2i\pi\boldsymbol{k}\wedge\boldsymbol{v}}|.$$

The last factor in the RHS is an $\mathcal{O}\big(\frac{|\boldsymbol{k}|}{N}\big)$. Since the Fourier coefficients decay as $|\hat{f}(\boldsymbol{k})| \leq C_M \frac{\|f\|_{C^M}}{(1+|\boldsymbol{k}|)^M}$ for any $M > 0$, we can take $M = 2d + 2$, which makes the sum over $\boldsymbol{k}$ finite, and we obtain

$$\|\mathcal{U}_N(\boldsymbol{v})Op_N(f) - Op_N(f \circ t_{\boldsymbol{v}})\| \leq C\frac{\|f\|_{C^{2d+2}}}{N}. \tag{59}$$

The quantum-classical discrepancy due to the nonlinear map Φ_1 is computed along the lines of [11]. Φ_1 is time-1 map generated by the flow of Hamiltonian $H(t)$. We want to compare $Op(f \circ \Phi_1)$ with the quantum-mechanically evolved observable $\mathcal{U}(\Phi_1)Op(f)$. To do so, we compare the infinitesimal evolutions. Let us call $\mathcal{U}(t,s) = ad\big(\mathcal{T}\, e^{-\frac{i}{\hbar}\int_s^t Op(H(r))dr}\big)$ the quantum propagator between times $s < t$, and $K(t,s)$ the corresponding classical propagator. Duhamel's principle lies in the following observation: from the identities

$$\frac{d}{dt}\mathcal{U}(t,s)A = i\hbar^{-1}\mathcal{U}(t,s)[Op(H(t)), A], \qquad \frac{d}{ds}K(t,s)f = -\{H(s), K(t,s)f\},$$

one constructs the following total derivative:

$$\begin{aligned}&\frac{d}{dt}\big(\mathcal{U}(t,0)Op(K(1,t)f)\big)\\ &\quad= \mathcal{U}(t,0)\left\{i\hbar^{-1}\big[Op(H(t)), Op(K(1,t)f)\big] - Op\big(\{H(t), K(1,t)f\}\big)\right\}.\end{aligned} \tag{60}$$

Integrating over $t \in [0,1]$ and taking the norm, using the unitarity of $\mathcal{U}(t,0)$, one gets:

$$\|\mathcal{U}(\Phi_1)Op(f) - Op(K(1,0)f)\| \leq \int_0^1 dt\, \|i\hbar^{-1}[Op(H(t)), Op(K(1,t)f)] - Op\big(\{H(t), K(1,t)f\}\big)\|. \quad (61)$$

We can easily estimate the norm of (60), using the Fourier decomposition of $H(t)$ and $K(1,t)f$: we write $H(t) = \sum_{\boldsymbol{k}} \hat{H}(\boldsymbol{k},t)\, w_{\boldsymbol{k}}$, $K(1,t)f = \sum_{\boldsymbol{m}} \hat{f}(\boldsymbol{m},t)\, w_{\boldsymbol{m}}$, and expand. The CCR (3) and their corresponding Poisson brackets read

$$[W_{\boldsymbol{k}}, W_{\boldsymbol{m}}] = 2i\sin(\pi \boldsymbol{k}\wedge\boldsymbol{m}/N)\, W_{\boldsymbol{k}+\boldsymbol{m}}, \quad \{w_{\boldsymbol{k}}, w_{\boldsymbol{m}}\} = -4\pi^2 \boldsymbol{k}\wedge\boldsymbol{m}\, w_{\boldsymbol{k}+\boldsymbol{m}}.$$

This gives us for the operator in the above integral:

$$\sum_{\boldsymbol{k},\boldsymbol{m}} \hat{H}(\boldsymbol{k},t)\, \hat{f}(\boldsymbol{m},t)\, 4\pi\big\{\pi\boldsymbol{k}\wedge\boldsymbol{m} - N\sin(\pi\boldsymbol{k}\wedge\boldsymbol{m}/N)\big\} W_{\boldsymbol{k}+\boldsymbol{m}}.$$

The term in the curly brackets is an $\mathcal{O}\big(\frac{(|\boldsymbol{k}||\boldsymbol{m}|)^2}{N}\big)$, while the product of Fourier coefficients decays like $(|\boldsymbol{k}|\,|\boldsymbol{m}|)^{-M}$ for any $M>0$, due to the smoothness of $H(t)$ and f. To be able to sum over $\boldsymbol{k}$, $\boldsymbol{m}$ we need to take $M \geq 2d+3$, and get for any $t\in[0,1]$:

$$\|i\hbar^{-1}[Op(H(t)), Op(K(1,t)f)] - Op\big(\{H(t), K(1,t)f\}\big)\| \leq C\frac{\|H(t)\|_{C^M}\|K(1,t)f\|_{C^M}}{N}. \quad (62)$$

Due to the smoothness of $H(t)$, the norm $\|K(1,t)f\|_{C^M}$ can only differ from $\|f\|_{C^M}$ by a finite factor independent of f [11]. We therefore get for any smooth f:

$$\|\mathcal{U}(\Phi_1)Op(f) - Op(f\circ\Phi_1)\| \leq C\frac{\|f\|_{C^{2d+3}}}{N}. \quad (63)$$

We now control the quantum-classical discrepancy stepwise. We use the discrete-time Duhamel principle to control the discrepancy for the full map (57):

$$\begin{aligned}\|\mathcal{U}(\Phi)Op(f) - Op(f\circ\Phi)\| &\leq \|\mathcal{U}(F)Op(f) - Op(f\circ F)\| + \\ &+ \|\mathcal{U}(t_{\boldsymbol{v}})Op(f\circ F) - Op(f\circ F\circ t_{\boldsymbol{v}})\| + \|\mathcal{U}(\Phi_1)Op(f\circ F\circ t_{\boldsymbol{v}}) - Op(f\circ F\circ t_{\boldsymbol{v}}\circ\Phi_1)\|,\end{aligned} \quad (64)$$

and for its iterates:

$$\|\mathcal{U}(\Phi)^n Op(f) - Op(f\circ\Phi^n)\| \leq \sum_{j=0}^{n-1} \|\mathcal{U}(\Phi)Op(f\circ\Phi^j) - Op(f\circ\Phi^{j+1})\|. \quad (65)$$

Putting together the estimates (58, 59, 63) we get the statement of the proposition, with either norm $\|\cdot\|_{\mathcal{B}(\mathcal{H}_N)}$ or $\|\cdot\|_{HS}$. □

B.2 Proof of Proposition 7

Compared with the previous appendix, we now also need to control the discrepancy between classical and quantum noise operators. This is quite easy to do in Fourier space: for any $f \in C^\infty(\mathbb{T}^{2d})$, we have:

$$\|\mathcal{G}_{\epsilon,N} Op(f) - Op(G_\epsilon f)\| \le \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} |\gamma_{\epsilon,N}(\boldsymbol{k}) - \hat{g}_\epsilon(\boldsymbol{k})|\,|\hat{f}(\boldsymbol{k})|. \tag{66}$$

Let us assume that the Fourier transform of g decays as $|\hat{g}(\boldsymbol{\xi})| \le \frac{C}{(1+|\boldsymbol{\xi}|)^D}$ as $\boldsymbol{\xi} \to \infty$, with $D \ge 2d+1$. From the explicit expression (15), we easily get the estimate (in the limit $\epsilon N \to \infty$):

$$\gamma_{\epsilon,N}(\boldsymbol{k}) = \frac{\hat{g}_\epsilon(\boldsymbol{k}) + \sum_{\boldsymbol{m}\neq 0} \mathcal{O}\big((\epsilon N|\boldsymbol{m}|)^{-D}\big)}{\hat{g}_\epsilon(0) + \sum_{\boldsymbol{m}\neq 0} \mathcal{O}\big((\epsilon N|\boldsymbol{m}|)^{-D}\big)} = \hat{g}_\epsilon(\boldsymbol{k}) + \mathcal{O}\big((\epsilon N)^{-D}\big), \tag{67}$$

and the estimate is uniform for $\boldsymbol{k} \in \mathbb{Z}_N^{2d}$. For $\boldsymbol{k}$ outside $\mathbb{Z}_N^{2d}$, we simply bound the difference by

$$|\gamma_{\epsilon,N}(\boldsymbol{k}) - \hat{g}_\epsilon(\boldsymbol{k})| \le 2.$$

Therefore, for any $f \in C_0^\infty(\mathbb{T}^{2d})$, one has:

$$\|\mathcal{G}_{\epsilon,N} Op(f) - Op(G_\epsilon f)\| \le \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}-0} \frac{C}{(\epsilon N)^D}\,|\hat{f}(\boldsymbol{k})| + 2\sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}\setminus\mathbb{Z}_N^{2d}} |\hat{f}(\boldsymbol{k})| \le C\frac{\|f\|_{C^D}}{(\epsilon N)^D}. \tag{68}$$

From the previous appendix we control the quantum-classical discrepancy of the unitary step $\mathcal{U}(\Phi)$. Both yield:

$$\begin{aligned}
&\|\mathcal{T}_{\epsilon,N} Op(f) - Op(T_\epsilon f)\| \\
&\qquad \le \|\mathcal{G}_{\epsilon,N}\big(\mathcal{U}(\Phi) Op(f) - Op(K_\Phi f)\big)\| + \|\mathcal{G}_{\epsilon,N} Op(K_\Phi f) - Op(G_\epsilon K_\Phi f)\| \\
&\qquad \le \|\mathcal{U}(\Phi) Op(f) - Op(K_\Phi f)\| + \|\mathcal{G}_{\epsilon,N} Op(K_\Phi f) - Op(G_\epsilon K_\Phi f)\| \\
&\qquad \le C\frac{\|f\|_{C^{2d+3}}}{N} + C\frac{\|f\|_{C^D}}{(\epsilon N)^D},
\end{aligned}$$

valid for any $D \ge 2d+1$. To obtain the proposition, we apply an obvious generalization of Duhamel's principle, using the fact that $\mathcal{T}_{\epsilon,N}$ is contracting on $\mathcal{A}_N^0$. □

Acknowledgments. The third author would like to thank Prof. B. Nachtergaele for helpful suggestions and discussions. We thank the referee for suggesting the comparison between the relaxation time scale and the decay of quantum fidelity.

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Communicated by Jens Marklof
submitted 4/01/05, accepted 2/02/05