

The Color–Flavor Transformation of induced QCD

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Abstract

The color–flavor transformation is applied to the $U(N_c)$ lattice gauge model, in which the gauge theory is induced by a heavy chiral scalar field sitting on lattice sites. The flavor degrees of freedom can encompass several ‘generations’ of the auxiliary field, and for each generation, remaining indices are associated with the elementary plaquettes touching the lattice site. The effective, color-flavor transformed theory is expressed in terms of gauge singlet matrix fields carried by lattice links. The effective action is analyzed for a hypercubic lattice in arbitrary dimension. The saddle points equations of the model in the large- N_c limit are discussed.

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1 Introduction

The complexity of quantum chromodynamics (QCD) originates from the random character of the gauge field in the low-energy regime, while at high energy (small scales) the theory is asymptotically free. One of the most successful approaches to analyse QCD in the non-perturbative domain is the lattice formulation due to Wilson [?], where the strong coupling regime becomes natural. On the other hand, the continuum limit of lattice QCD corresponds to the weakly-coupled regime. It was shown by Gross and Witten [?] that two-dimensional lattice QCD can be solved exactly in the large- N_c limit and there is a third-order weak- to strong-coupling phase transition. The problem, which persists over the years is, if such a phase transition occurs in the realistic $SU(3)$ four-dimensional gauge theory.

One of the ways to attack this problem is to use some form of *duality* [?] which appears in the lattice theory. The general idea of duality is that a given theory can have two, or more equivalent formulations, with different sets of fundamental variables. Usually these dual formulations are related by interchanging a parameter, e.g. the electromagnetic coupling constant e^2 , with its inverse $1/e^2$. For example, in the weak-coupling limit, the action of the Abelian lattice gauge model can be approximated by the Villain form (see e.g. [?, ?]) which allows to define the dual variables. Similarly, it was shown that there is a duality transformation from the compact $U(1)$ gauge theory into a non-compact Abelian Higgs model [?].

The dual variables are in general not defined on the original lattice but on a *dual lattice*. For a hypercubic lattice, it is obtained by shifting the original lattice by half of the lattice spacing in all dimensions. Thus, the lattice duality not only transforms the variables of the functional integral, but also incorporates a transfer to the dual lattice.

In the case of a nonabelian lattice gauge theory, one can reformulate the model in terms of plaquette variables [?, ?]. Recently, following the idea of [?], Diakonov and Petrov [?] applied a Fourier transformation to write down the Jacobian from the link variables to the plaquette variables in $d = 3$ gluodynamics for the gauge group $SU(2)$; *in fine* the dual lattice is composed of tetrahedra representing $6j$ -symbols, with links of arbitrary lengths. In the continuum limit this effective theory is equivalent with quantum gravity with the Einstein-Hilbert action. Unfortunately, this scheme seems difficult to apply to higher-rank gauge groups and in $d > 3$ dimensions.

Another approach to analyze non-perturbative QCD is to start from a “simpler” theory which contains auxiliary fields coupled to the gauge fields, and “induce” the nonabelian lattice gauge action by integrating over the auxiliary fields. Several QCD-inducing models were proposed, with an auxiliary field living either in the fundamental representation of the gauge group [?, ?], or in the adjoint representation [?, ?]. In all cases, one recovers Wilson’s action when the mass of the auxiliary field goes to infinity. In the latter case, one can solve the large- N_c limit, but the adjoint representation implies an extra local Z_{N_c} symmetry which leads to an infinite string tension [?, ?].

In the present note we discuss another approach to treat a similar type of inducing model. Our construction starts from an inducing theory similar with [?, ?], already introduced in [?], and applies a certain duality transformation, namely the “color-flavor transformation” [?]. We have recently applied this transformation to the lattice $SU(N_c)$ model in the strong-coupling limit, which describes quarks coupled with a background gauge field [?].

After this work was completed, we learnt that Schlittgen and Wettig independently applied the $SU(N)$ color-flavor transformation to a similar, yet different QCD-inducing model [?].

2 A model of induced lattice gauge theory

2.1 Wilson's lattice action

We consider a Euclidean $U(N_c)$ pure gauge action (no quarks) in d dimensions, placed on a hypercubic lattice with lattice constant a . The choice of $U(N_c)$ instead of the realistic $SU(N_c)$ highly simplifies the subsequent color-flavor transformation.

The lattice sites are labeled by integer vectors $n = (1, \dots, n_d)$, the gauge matrix variables

$$U_\mu(n) \equiv U_{n, n+\hat{\mu}} \equiv U(n+\frac{\hat{\mu}}{2}) = \exp\left(ia g A_\mu(na + \frac{a\hat{\mu}}{2})\right) \in U(N_c) \quad (1)$$

are placed on the lattice links $n + \hat{\mu}/2$ (we label links by their *middle points*), leaving the site n in any of the “positive” directions $\mu = 1, \dots, d$. The plaquettes are either labeled by an independent index p , or by triplets of the form $(n, \pm\mu, \pm\nu)$. For instance, the plaquette (n, μ, ν) contains the links $n + \hat{\mu}/2$ and $n + \hat{\nu}/2$. To fix an ordering between the directions, we will in general assume that $1 \leq \mu < \nu \leq d$. Notice that the same plaquette corresponds to the triplets (n, μ, ν) and $(n + \hat{\mu}, -\mu, \nu)$ (as well as two other triplets).

The Wilson pure gauge action is given by a sum over all elementary plaquettes:

$$-S_{\text{gluons}} = \beta_W \sum_p \text{Tr} \left(U_P(p) + U_P^\dagger(p) \right). \quad (2)$$

Here β_W is the lattice coupling, which is related to the bare continuum coupling constant g through

$$\beta_W = \frac{a^{d-4}}{2g^2}. \quad (3)$$

The plaquette field U_P is defined as an ordered product of the link variables along the boundary of the given plaquette:

$$U_P(n, \mu, \nu) = U(n+\frac{\hat{\mu}}{2})U(n+\hat{\mu}+\frac{\hat{\nu}}{2})U^{-1}(n+\hat{\nu}+\frac{\hat{\mu}}{2})U^{-1}(n+\frac{\hat{\nu}}{2}) \quad (4)$$

The partition function is defined as

$$Z = \int \mathcal{D}U e^{-S_{\text{gluons}}}, \quad (5)$$

the invariant measure of integration is defined as a product over all links $\mathcal{D}U = \prod_{n, \mu} dU(n+\frac{\hat{\mu}}{2})$ and $dU(n+\frac{\hat{\mu}}{2})$ is the Haar measure on the group $U(N_c)$.

Using a generalized Baker-Campbell-Hausdorff formula, one can relate the continuum field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

with the plaquette matrices as follows:

$$U_P(n, \mu, \nu) = e^{iga^2 F_{\mu\nu}(n) + O(a^3)}. \quad (6)$$

Expansion in the lattice spacing up to the second order then yields

$$\text{Tr} U_P(n, \mu, \nu) \approx N_c + iga^2 \text{Tr} F_{\mu\nu} - \frac{g^2 a^4}{2} \text{Tr} F_{\mu\nu}^2$$

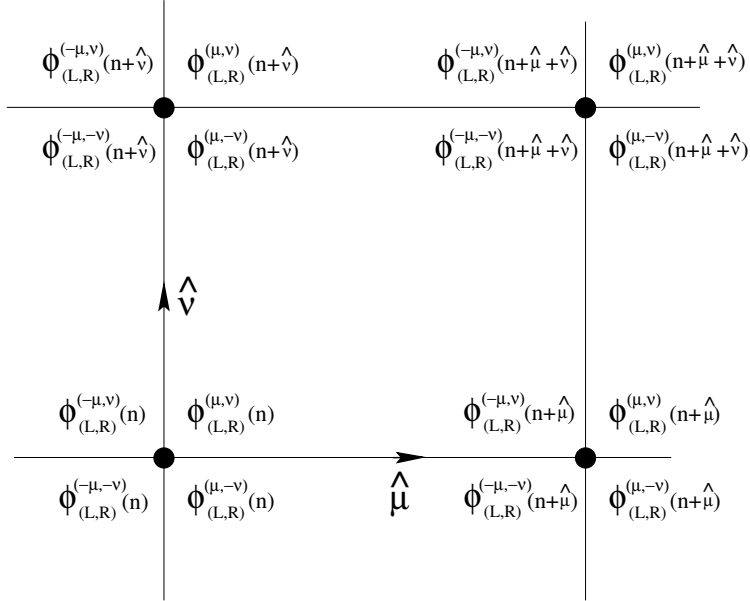


Figure 1: Auxiliary chiral fields in the plane $(\mu\nu)$. Each field component is written inside the plaquette it is associated with.

so that the standard Yang-Mills action is recovered in the continuum limit:

$$S_{cont} = \frac{1}{2} \int d^d x \text{Tr} F_{\mu\nu}^2. \quad (7)$$

Wilson's lattice gauge action (??) is written via the plaquette matrices $U_P(p)$ while the variables of the integration measure are the link matrices $U(n+\frac{\hat{\mu}}{2})$. It is possible to explicitly transfer the integration from link to plaquette matrices [?, ?, ?], but this procedure is technically involved and can be applied only in a few simple cases. We would like to investigate a possibility to apply another approach, which starts from a modification of the QCD inducing model of Bander and Hamber [?, ?].

2.2 Our model and its induced gauge action

Let us consider a massive complex bosonic field $\phi(n)$ placed on the lattice sites n . This field has “flavor” components $\phi^{(\pm\mu, \pm\nu)}(n)$ associated to each of the $2d(d-1)$ plaquette $\{(n, \pm\mu, \pm\nu); 1 \leq \mu < \nu \leq d\}$ adjacent to the site n (see Figures ??,??). The field furthermore decomposes into two ‘chiral components’ $\phi_R^{(\mu, \nu)}(n)$ and $\phi_L^{(\mu, \nu)}(n)$, which are hopping in opposite directions. These fields will be referred as ‘left’ and ‘right’ respectively.

The chiral bosonic field can be thought of as an N_b -component vector in an auxiliary ‘flavor’ space (here the index b stands for ‘bosonic’). The number of ‘flavors’ has to be a multiple of $2d(d-1)$, that is the dimension of the ‘flavor’ space is $N_b = n_b \times 2d(d-1)$, where $n_b \in \mathbb{N}^*$ is the number of ‘generations’ of the bosonic field. The bosonic field ϕ also transforms as a vector through the gauge group $U(N_c)$, so it contains ‘color indices’ $i = 1, \dots, N_c$ besides the ‘flavor’ indices $a = 1, \dots, N_b$. All flavor components have the same mass m_b .

This model (already considered in [?, Chap. 5]) is different from the model usually considered in induced QCD [?, ?, ?], where the flavor degrees of freedom of the auxiliary

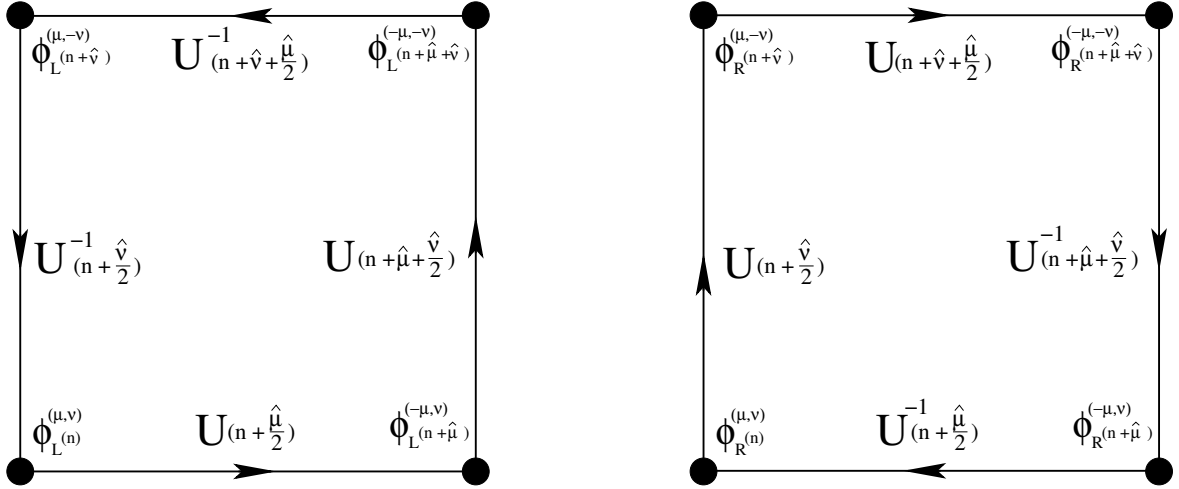


Figure 2: Coupling of the ‘left’ and ‘right’ chiral components of the bosonic field with the gauge field along the elementary plaquette (n, μ, ν) .

fields are not associated with plaquettes. As we will show below, this structure will induce a Wilson-type action in a cleaner way than in the previous models.

To complete our notations, we shall fix the orientation of the plaquettes, in order to define the hopping and ‘left’ and the ‘right’ components in all 2-dimensional planes on the lattice. On any plane (μ, ν) with $1 \leq \mu < \nu \leq d$ the ‘left’ chiral component hops on the plaquette (n, μ, ν) as

$$\phi_L(n) \rightarrow \phi_L(n + \hat{\mu}) \rightarrow \phi_L(n + \hat{\mu} + \hat{\nu}) \rightarrow \phi_L(n + \hat{\nu}) \rightarrow \phi_L(n), \quad (8)$$

and the ‘right’ chiral component hops in the opposite direction (see Fig. ??). This practically means that the ‘kinetic’ part of the action contains the term $-\phi_L^{(-\mu,\nu)\dagger}(n + \hat{\mu})U(n + \frac{\hat{\mu}}{2})\phi_L^{(\mu,\nu)}(n)$ (cf. Eq. (??)).

Let us now group the fields surrounding a given plaquette $p = (n, \mu, \nu)$ into the following *plaquette quadruplets*:

$$\phi_L(p) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_L^{(\mu,\nu)}(n) \\ \phi_L^{(-\mu,\nu)}(n + \hat{\mu}) \\ \phi_L^{(-\mu,-\nu)}(n + \hat{\mu} + \hat{\nu}) \\ \phi_L^{(\mu,-\nu)}(n + \hat{\nu}) \end{pmatrix}; \quad \phi_R(p) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_R^{(\mu,\nu)}(n) \\ \phi_R^{(-\mu,\nu)}(n + \hat{\mu}) \\ \phi_R^{(-\mu,-\nu)}(n + \hat{\mu} + \hat{\nu}) \\ \phi_R^{(\mu,-\nu)}(n + \hat{\nu}) \end{pmatrix} \quad (9)$$

Then the plaquette action of the ‘left’ bosonic massive field may be written in the concise form

$$S_L(p) = \phi_L^\dagger(p)M_L(p)\phi_L(p) \quad (10)$$

with the $4N_c \times 4N_c$ matrix

$$M_L(n, \mu, \nu) \stackrel{\text{def}}{=} \begin{pmatrix} m_b & 0 & 0 & -U^{-1}(n+\frac{\nu}{2}) \\ -U(n+\frac{\mu}{2}) & m_b & 0 & 0 \\ 0 & -U(n+\hat{\mu}+\frac{\nu}{2}) & m_b & 0 \\ 0 & 0 & -U^{-1}(n+\hat{\nu}+\frac{\mu}{2}) & m_b \end{pmatrix} \quad (11)$$

Similarly, ‘right’ bosonic action associated with the plaquette p reads:

$$S_R(p) = \phi_R^\dagger(p) M_L^\dagger(p) \phi_R(p). \quad (12)$$

We define the full partition function as:

$$\mathcal{Z} = \int \mathcal{D}\bar{\phi}_{L,R} \mathcal{D}\phi_{L,R} \mathcal{D}U \exp \left\{ - \sum_p S_L(p) + S_R(p) \right\}. \quad (13)$$

2.3 Integration over auxiliary fields

We now show that the model (??) induces a lattice gluodynamics which reduces to Wilson’s pure gauge action for suitably chosen parameters. We can treat plaquettes separately, since each field component is associated with one and only plaquette. The integration over the ‘left’ auxiliary fields in (??) yields

$$\int d\bar{\phi}_L(p) d\phi_L(p) \exp\{-S_L(p)\} = \text{Det}(M_L(p))^{-n_b} = \text{Det}(m_b^4 - U_P(p))^{-n_b}, \quad (14)$$

where n_b is the number of ‘generations’ of the auxiliary fields. The integration over the ‘right’ components is similar, with U_P replaced by U_P^\dagger . Thus, the integration over the auxiliary bosonic fields exactly yields the pure gauge effective action

$$\begin{aligned} S_{\text{plaq}} &= n_b \sum_p [\ln \text{Det}(1 - \beta_b U_P(p)) + \ln \text{Det}(1 - \beta_b U_P^\dagger(p))] \\ &= n_b \sum_p \text{Tr}[\ln(1 - \beta_b U_P(p)) + \ln(1 - \beta_b U_P^\dagger(p))], \end{aligned}$$

where we skipped a mass-dependent prefactor, and set $\beta_b = m_b^{-4}$. As opposed to the former inducing models [?, ?], this action does not contain any term related with larger loops. Expanding this action for small parameter β_b (that is, large m_b), we get

$$S_{\text{plaq}} = -n_b \beta_b \text{Tr} \sum_p \left(U_P(p) + U_P^\dagger(p) \right) + \mathcal{O}(n_b \beta_b^2). \quad (15)$$

This coincides with Wilson’s action (??) if we identify

$$\beta_W = n_b \beta_b \iff g^2 = \frac{a^{d-4}}{2n_b \beta_b} = \frac{a^{d-4} m_b^4}{2n_b}. \quad (16)$$

Remarks on the continuum limit Let us say a few words about the continuum limit of the model. In $d = 2$ and $d = 3$, the physical coupling constant g can remain fixed as one lets the lattice spacing a go to 0 and simultaneously $\beta_W = n_b \beta_b \rightarrow \infty$. In $d = 4$, this limit corresponds to the asymptotically free continuum theory. Recall that the mass of the auxiliary fields is measured in units of the lattice spacing: $m_b = m \times a$. Therefore the auxiliary field becomes non-observable in the continuum limit if the corresponding correlation length $\xi = (am)^{-1} = \beta_b^{1/4}$ stays finite. We assume it remains small enough to justify the expansion (??).

We end this section with one more comment. It is possible to consider a fermionic counterpart of the bosonic action (??), which induces a similar pure gauge effective action. One has to replace the bosonic auxiliary fields by fermionic (anticommuting) fields $\psi_{L,R}, \bar{\psi}_{L,R}$ which carry color indices and flavor indices related to plaquettes, exactly as for the multiplets (??); one can consider n_f ‘generations’ of these fermions. After integrating over them, one obtains the effective pure gauge action

$$-S = n_f \sum_p \text{Tr}[\ln(1 + \beta_f U_P(p)) + \ln(1 + \beta_f U_P^\dagger(p))]. \quad (17)$$

Here $\beta_f = m_f^{-4}$, where m_f is the fermion mass. Clearly, we once more recover the conventional Wilson action (??) for small values of β_f .

Having considered both bosonic and fermionic induced lattice gluodynamics, we can also represent the effective action as the ratio of ‘fermionic’ and ‘bosonic determinants’:

$$\exp\left[\frac{a^{d-4}}{2g^2} \text{Tr}(U_P + U_P^\dagger)\right] \approx \frac{[\text{Det}(1 + \beta_f U_P) \text{Det}(1 + \beta_f U_P^\dagger)]^{n_f}}{[\text{Det}(1 - \beta_b U_P) \text{Det}(1 - \beta_b U_P^\dagger)]^{n_b}}. \quad (18)$$

Thus, for small couplings β_b, β_f , the partition function can be represented by the following superintegral

$$\mathcal{Z} = \int \mathcal{D}U \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[-\bar{\psi}_{L,a}^i (\delta^{ij} + \beta_L (U_P)^{ij}) \psi_{L,a}^j - \bar{\psi}_{R,a}^i (\delta^{ij} + \beta_R (U_P^\dagger)^{ij}) \psi_{R,a}^j\right]. \quad (19)$$

The composite field $\psi, \bar{\psi}$ includes both bosonic and fermionic variables, which are distinguished by the ‘flavor’ index a .

There are therefore several ways to induce Wilson’s lattice gauge action. In all cases, the action (??) with fixed β_W can be recovered in the limit of large mass and large number of generations of the auxiliary fields (cf. Eq. (??)). Below we will restrict our considerations to the bosonic model (??).

3 Color-flavor transformation of the inducing theory

Though the equivalence between the model (??) and Wilson’s gluodynamics can be established only in the limit of large number of generations of the auxiliary field, we would like to study some properties of the underlying theory with a single bosonic ‘generation’ ($n_b = 1$), that is with a flavor space of dimension $N_b = 2d(d-1)$. This assumption simplifies the application of the color-flavor transformation [?].

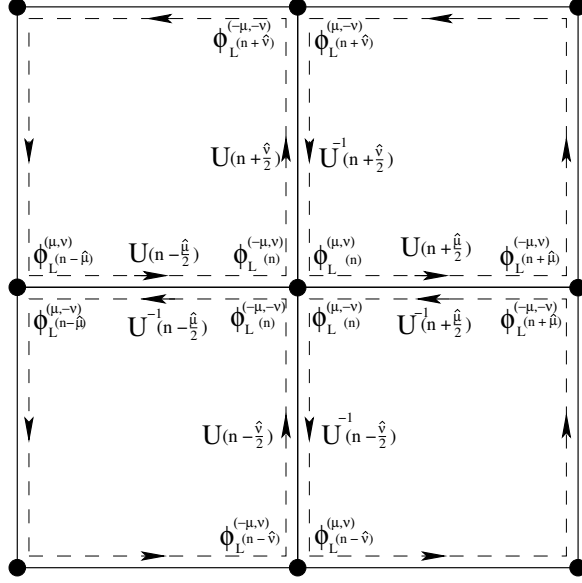


Figure 3: Gauge couplings of the ‘left’ bosonic fields carried by the 4 plaquettes in the plane $(\mu\nu)$ around the lattice site n .

Let us consider the interaction term of the bosonic action (??) on a given link $(n + \hat{\mu}/2)$ of the d -dimensional hypercubic lattice. The path ordered product of the link matrices defined by the ‘left’ action (??) is depicted in Fig. ???. There are $2d - 2$ plaquettes which share this common link. Above we have used the plaquette quadruplets (??) in order to write the plaquette action concisely. Now we will rather decompose the full action into a sum over *links*, which forces us to gather the auxiliary bosonic fields into two series of *site-link multiplets*, in order to include all fields coupled by $U(n + \hat{\mu}/2)$ or $U^\dagger(n + \hat{\mu}/2)$. We thus define two chirally-conjugated multiplets associated with the site n and the link $(n + \hat{\mu}/2)$:

$$\Psi(n; \mu) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_R^{(\mu, \mu+1)}(n) \\ \phi_L^{(\mu, -\mu-1)}(n) \\ \dots \\ \phi_R^{(\mu, d)}(n) \\ \phi_L^{(\mu, -d)}(n) \\ \phi_R^{(\mu, -1)}(n) \\ \phi_L^{(\mu, 1)}(n) \\ \dots \\ \phi_R^{(\mu, -\mu+1)}(n) \\ \phi_L^{(\mu, \mu-1)}(n) \end{pmatrix}; \quad \Phi(n; \mu) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_L^{(\mu, \mu+1)}(n) \\ \phi_R^{(\mu, -\mu-1)}(n) \\ \dots \\ \phi_L^{(\mu, d)}(n) \\ \phi_R^{(\mu, -d)}(n) \\ \phi_L^{(\mu, -1)}(n) \\ \phi_R^{(\mu, 1)}(n) \\ \dots \\ \phi_L^{(\mu, -\mu+1)}(n) \\ \phi_R^{(\mu, \mu-1)}(n) \end{pmatrix} \quad (20)$$

The multiplets $\Psi(n; -\mu)$ and $\Phi(n; -\mu)$ associated with the site n and link $n - \hat{\mu}/2$ are obtained from the ones above by flipping the *first* superscript μ into $-\mu$ in all components, while changing neither the second superscript nor the ordering of the fields.

Recall that the fields $\phi^{(\mu, \nu)}$ are vectors with respect to the colour group. Thus, the link multiplets $\Phi_a^i(n; \mu)$, $\Psi_a^i(n; \mu)$ have to be labeled by color (i) and flavor (a) indices. The latter

are given by the second superscript in the definition of the multiplets: for instance, the flavor indices of the multiplets (??) take the successive values $a = \mu + 1, -\mu - 1, \mu + 2, \dots$ etc. Had we included multiple generations, the dimension of the coupling matrices would have been $d' = n_b(2d - 2)$.

Since there are $2d$ links around the site n and for each link, two multiplets containing $2d - 2$ components, the full set of multiplets is of dimension $8d(d - 1)$. On the other hand, the number of independent flavor components at each site is $2 \times N_b = 4d(d - 1)$ (the factor 2 corresponds to the chirality). Therefore, the site-link multiplets are not linearly independent; indeed, each field component $\phi_{L/R}^{(\pm\mu, \pm\nu)}(n)$ is contained in exactly two multiplets, one associated with the link $(n + \hat{\mu}/2)$, the other with the link $(n + \hat{\nu}/2)$.

In terms of these multiplets, the interacting part of the action (??) associated with the link $(n + \hat{\mu}/2)$ can be written in a compact form as follows (repeated indices are summed over):

$$-S_U(n + \frac{\hat{\mu}}{2}) = \bar{\Phi}_a^i(n + \hat{\mu}; -\mu) U^{ij}(n + \frac{\hat{\mu}}{2}) \Phi_a^j(n; \mu) + \bar{\Psi}_a^i(n; \mu) U^{\dagger ij}(n + \frac{\hat{\mu}}{2}) \Psi_a^j(n + \hat{\mu}; -\mu). \quad (21)$$

Now that we isolated the part of the action associated with the matrix $U(n + \frac{\hat{\mu}}{2})$, we can apply the bosonic $U(N_c)$ color-flavor transformation on this action, that is replace the integration over $U(n + \frac{\hat{\mu}}{2})$ by an integral over a complex matrix $Z(n + \frac{\hat{\mu}}{2})$ of dimension d' [?]:

$$\int_{U(N_c)} dU(n + \frac{\hat{\mu}}{2}) \exp[-S_U(n + \frac{\hat{\mu}}{2})] = \int_{D_{d'}} d\mu(Z, Z^\dagger) \text{Det}(1 - ZZ^\dagger)^{N_c} \times \exp \left[\bar{\Phi}_a^i(n + \hat{\mu}; -\mu) Z_{ab}(n + \frac{\hat{\mu}}{2}) \Psi_b^i(n + \hat{\mu}; -\mu) + \bar{\Psi}_b^i(n; \mu) Z^\dagger_{ba}(n + \frac{\hat{\mu}}{2}) \Phi_a^i(n; \mu) \right]. \quad (22)$$

$D_{d'}$ denotes the set of complex matrices Z of dimension d' such that the Hermitian matrix $1 - ZZ^\dagger$ is positive definite. This set is in one-to-one correspondence with the non-compact symmetric space $U(d', d')/U(d') \times U(d')$ [?], and the measure $d\mu(Z, Z^\dagger)$ is the (suitably normalized) invariant measure on this symmetric space:

$$d\mu(Z, Z^\dagger) = \text{const} \times \text{Det}(1 - ZZ^\dagger)^{-2d'} \prod_{a,b=1}^{d'} dZ_{ab} d\bar{Z}_{ab}.$$

The identity (??) makes sense iff

$$N_c \geq 2d' = 4(d - 1), \quad (23)$$

otherwise the integral over Z does not converge.

In the color-flavor transformed action, the auxiliary fields are coupled *ultralocally* via the Z matrices through their flavor indices. The indices of the matrix $Z(n + \frac{\hat{\mu}}{2})$ are associated with the plaquettes adjacent to the link $(n + \hat{\mu}/2)$, so that each entry of that matrix describes a correlation between these plaquettes. This is to be put in contrast with the original action (??), which described a parallel transport of the bosonic field along the links.

Grouping all auxiliary fields at the site n we get the interaction part of the local effective action

$$-S_Z[n] = \sum_{\mu=1}^d \left[\bar{\Psi}_b^i(n; \mu) \bar{Z}_{ab}(n + \frac{\hat{\mu}}{2}) \Phi_a^i(n; \mu) + \bar{\Phi}_a^i(n; -\mu) Z_{ab}(n - \frac{\hat{\mu}}{2}) \Psi_b^i(n; -\mu) \right], \quad (24)$$

which is diagonal in the color indices. Since the mass term is diagonal with respect to both the color and flavor indices, the color degrees of freedom are decoupled in the transformed action, so that the partition function can be factorized into N_c identical integrals, each corresponding to a given color. Still, the coupling between the $4d(d-1)$ flavor components at each site is not completely obvious, so we first analyze the simpler case of $d=2$ before turning to the general case.

3.1 $d=2$ effective action

The simplest possible situation corresponds to the model placed on the 2-dimensional square lattice spanned by two orthogonal unit vectors $\hat{1}$ and $\hat{2}$. Let us consider the four links having the lattice site n in common (see Fig ??). The space of auxiliary fields at n is of dimension 8 and, according to (??,??), the site-link multiplets (here, doublets) read

$$\begin{aligned}\Psi(n; 1) &= \begin{pmatrix} \phi_R^{(1,2)}(n) \\ \phi_L^{(1,-2)}(n) \end{pmatrix}; & \Psi(n; -1) &= \begin{pmatrix} \phi_R^{(-1,2)}(n) \\ \phi_L^{(-1,-2)}(n) \end{pmatrix} \\ \Psi(n; 2) &= \begin{pmatrix} \phi_R^{(2,-1)}(n) \\ \phi_L^{(2,1)}(n) \end{pmatrix}; & \Psi(n; -2) &= \begin{pmatrix} \phi_R^{(-2,-1)}(n) \\ \phi_L^{(-2,1)}(n) \end{pmatrix}.\end{aligned}\tag{25}$$

The 4 chirally conjugated doublets $\Phi(n; \pm\hat{\alpha})$ are obtained by exchanging $L \leftrightarrow R$. These doublets are coupled through the 2×2 matrices $Z^\dagger(n+\frac{\hat{1}}{2})$, $Z^\dagger(n+\frac{\hat{2}}{2})$, $Z(n-\frac{\hat{1}}{2})$ and $Z(n-\frac{\hat{2}}{2})$ carried by the four links adjacent to the site n . To give an example, the matrix $Z^\dagger(n+\frac{\hat{1}}{2})$ has the following index structure:

$$Z^\dagger(n+\frac{\hat{1}}{2}) = \begin{pmatrix} Z_{2,2}^\dagger(n+\frac{\hat{1}}{2}) & Z_{2,-2}^\dagger(n+\frac{\hat{1}}{2}) \\ Z_{-2,2}^\dagger(n+\frac{\hat{1}}{2}) & Z_{-2,-2}^\dagger(n+\frac{\hat{1}}{2}) \end{pmatrix}\tag{26}$$

Together with the link carrying the matrix, the lower pair of indices represent the plaquettes associated with the field components coupled by the matrix element: the diagonal element $Z_{2,2}^\dagger(n+\frac{\hat{1}}{2})$ couples different fields associated with the same plaquette $(n, 1, 2)$, while the non-diagonal element $Z_{-2,2}^\dagger(n+\frac{\hat{1}}{2})$ couples fields associated with the two plaquettes $(n, 1, 2)$ and $(n, 1, -2)$.

We want to write an effective action uniquely in terms of the Z fields, by integrating over the bosonic fields. For this aim, we need to describe the coupling between each pair of flavors in the action (??). As we already mentioned, the site-link multiplets (??) are not independent of each other, so we now group the auxiliary fields at the lattice site n into chirally conjugated *site quadruplets*:

$$\Phi(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_R^{(1,2)}(n) \\ \phi_L^{(1,-2)}(n) \\ \phi_L^{(-1,2)}(n) \\ \phi_R^{(-1,-2)}(n) \end{pmatrix}; \quad \Psi(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_L^{(1,2)}(n) \\ \phi_R^{(1,-2)}(n) \\ \phi_R^{(-1,2)}(n) \\ \phi_L^{(-1,-2)}(n) \end{pmatrix}.\tag{27}$$

The union of these two quadruplets contain each bosonic component once. The color-flavor transformed action (??) can be written in terms of these quadruplets via two complex 4×4 matrices in the flavor space, $V(n)$ and $W(n)$, which contain the components of the Z -fields:

$$-S_Z[n] = \Phi^\dagger(n)V(n)\Psi(n) + \Psi^\dagger(n)W(n)\Phi(n).\tag{28}$$

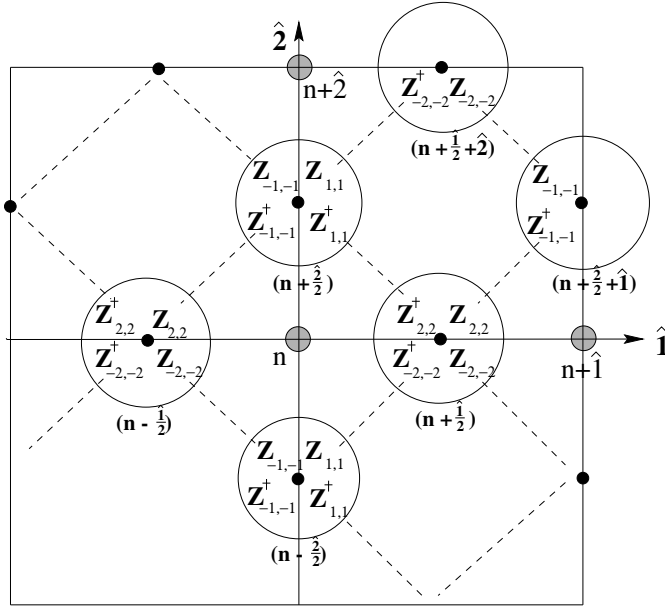


Figure 4: Schematic representation of the quadratic action (??). Each large circle contains diagonal matrix elements situated at some link. The correlations between matrix elements of different links are depicted by broken lines.

The matrices $V(n)$ and $W(n)$ can be compactly written

$$V(n) \stackrel{\text{def}}{=} \begin{pmatrix} Z^\dagger(n+\frac{\hat{1}}{2}) & 0 \\ 0 & Z(n-\frac{\hat{1}}{2}) \end{pmatrix}; \quad W(n) \stackrel{\text{def}}{=} \tau_{(1,4)} \begin{pmatrix} Z(n-\frac{\hat{2}}{2}) & 0 \\ 0 & Z^\dagger(n+\frac{\hat{2}}{2}) \end{pmatrix} \tau_{(1,4)} \quad (29)$$

where the permutation matrix $\tau_{(1,4)}$ interchanges the first and fourth indices. The integral over auxiliary fields at the site n (including one color component) reads

$$\begin{aligned} \mathcal{Z}[n] &= \int d\Psi^\dagger(n) d\Psi(n) d\Phi^\dagger(n) d\Phi(n) \exp \left[-m_b(\Psi^\dagger \Psi + \Phi^\dagger \Phi) + \Phi^\dagger V \Psi + \Psi^\dagger W \Phi \right] \\ &\propto \text{Det} \begin{pmatrix} m_b & -V \\ -W & m_b \end{pmatrix}^{-1} = \text{Det}(m_b^2 - VM)^{-1} \\ &= \exp \left[-\text{Tr} \ln(1 - m_b^{-2} VW) \right] \approx \exp \left[m_b^{-2} \text{Tr}(VW) \right]. \end{aligned} \quad (30)$$

In the last line we expanded the logarithm to first order in $1/m_b$. The trace of the product VW can be easily computed:

$$\begin{aligned} \text{Tr}(V(n)W(n)) &= Z_{2,2}^\dagger(n+\frac{\hat{1}}{2}) Z_{1,1}^\dagger(n+\frac{\hat{2}}{2}) + Z_{-2,-2}^\dagger(n+\frac{\hat{1}}{2}) Z_{1,1}(n-\frac{\hat{2}}{2}) \\ &\quad + Z_{2,2}(n-\frac{\hat{1}}{2}) Z_{-1,-1}^\dagger(n+\frac{\hat{2}}{2}) + Z_{-2,-2}(n-\frac{\hat{1}}{2}) Z_{-1,-1}(n-\frac{\hat{2}}{2}). \end{aligned} \quad (31)$$

Notice that only the diagonal elements of the Z matrices appear in this leading-order term, which represent couplings between auxiliary fields carried by the same plaquette. In each term of the sum (??), the two matrix elements are carried by different links, but they correspond to fields related to the same plaquette, precisely the plaquette which shares these two links. One can represent the correlations embodied in (??) by *dual links* joining the middles of the two coupled links (see Fig ??).

To summarize, to leading order in $1/m_b$ the full partition function is given by:

$$\mathcal{Z} = \int \left\{ \prod_n \prod_{\alpha=1,2} d\mu(Z, Z^\dagger_{(n+\frac{\alpha}{2})}) \right\} \exp(-N_c S[Z]), \quad (32)$$

with the effective action depending on the ‘flavor’ matrices Z :

$$-S[Z] = \sum_n \left[m_b^{-2} \text{Tr}(V(n)W(n)) + \sum_{\alpha=1,2} \text{Tr} \ln \left(1 - Z_{(n+\frac{\alpha}{2})} Z^\dagger_{(n+\frac{\alpha}{2})} \right) \right]. \quad (33)$$

3.2 Effective action in arbitrary dimension

Our aim in this section is the same as in the last one, that is integrate the action (??) over the auxiliary fields at the site n , and compute the resulting effective action in the matrices Z, Z^\dagger in arbitrary dimension¹. We will only consider the case of one generation of auxiliary fields, that is, $n_b = 1$. As we already pointed out, the difficulty comes from the fact that the same field $\phi_{L/R}^{(\pm\mu, \pm\nu)}$ is contained in two different site-links multiplets (??). In order to integrate over these fields, we first need to regroup them, that is write the action S_Z as

$$-S_Z[n] = [\phi_{L/R}^{(\pm\mu, \pm\nu)}]^\dagger \mathcal{M}(n) [\phi_{L/R}^{(\pm\mu, \pm\nu)}], \quad (34)$$

where the column vector $[\phi_{L/R}^{(\pm\mu, \pm\nu)}]$ contains the $2N_b = 4d(d-1)$ fields at the site n . The coupling matrix $\mathcal{M}(n)$ is therefore of dimension $4d(d-1)$; it contains components of the matrices Z and Z^\dagger carried by the links touching n . Our task is to write down the matrix \mathcal{M} explicitly, using a judicious grouping of the field components. As was already the case in two dimensions, the matrix \mathcal{M} has many null entries, so that its determinant may be simplified.

We will group the auxiliary fields in *site quadruplets* associated with the planes in the d -dimensional lattice. Each plane, indexed by a couple $(\mu\nu)$ with $1 \leq \mu < \nu \leq d$, contains 4 plaquettes touching n . To each plane we associate two site quadruplets at n :

$$\Phi^{(\mu\nu)}(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_R^{(\mu, \nu)}(n) \\ \phi_L^{(\mu, -\nu)}(n) \\ \phi_L^{(-\mu, \nu)}(n) \\ \phi_R^{(-\mu, -\nu)}(n) \end{pmatrix}; \quad \Psi^{(\mu\nu)}(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_L^{(\mu, \nu)}(n) \\ \phi_R^{(\mu, -\nu)}(n) \\ \phi_R^{(-\mu, \nu)}(n) \\ \phi_L^{(-\mu, -\nu)}(n) \end{pmatrix}. \quad (35)$$

These quadruplets generalize the ones defined in Eq. (??) to any plane in the d -dimensional lattice. The total number of these planes is $\frac{d(d-1)}{2}$, so that the ‘concatenation’ of all the above site quadruplets yields the correct number of field components. To perform this concatenation, we need to *order* the different planes, that is, to order the couples $(\mu\nu)$.

These couples are in one-to-one correspondence with the positive roots of the Lie algebra $\mathfrak{gl}(d)$: each plane $(\mu\nu)$ can indeed be associated with the generator $e_{\mu\nu}$ of the algebra, satisfying the relations

$$[e_{\mu\nu}, e_{\rho\eta}] = \delta_{\nu\rho} e_{\mu\eta} - \delta_{\mu\eta} e_{\rho\nu}. \quad (36)$$

There is no canonical ordering of the positive generators (or the positive roots), on the other hand it seems natural to require that $(\mu\nu) < (\rho\eta)$ if $\nu \leq \rho$; this condition is satisfied by the following ordering:

$$(12) < (13) < (14) < \dots < (1d) < (23) < (24) < \dots < (2d) < (34) < \dots < (d-1d). \quad (37)$$

¹A particular case of $d=3$ effective action was considered in [?].

The site quadruplets will be ordered according to the above convention, starting with all quadruplets $\Phi^{(\mu\nu)}$ and finishing with the quadruplets $\Psi^{(\mu\nu)}$.

Now that we ordered the vector $[\phi_{L/R}^{(\pm\mu, \pm\nu)}]$, we need to compute the matrix \mathcal{M} , and for this to derive which quadruplets $\Phi^{(\mu\nu)\dagger}$ or $\Psi^{(\mu\nu)\dagger}$ are coupled with which quadruplets $\Phi^{(\rho\eta)}$ or $\Psi^{(\rho\eta)\dagger}$, and through which matrices Z or Z^\dagger . For this aim, we have to compare the components of, on one side, the quadruplets $\Phi^{(\mu\nu)}$, $\Psi^{(\mu\nu)}$; on the other side, the site-link multiplets $\Phi(n; \pm\alpha)$, $\Psi(n; \pm\alpha)$ which were used to write the action (??). For instance, the first component $\phi_R^{(\mu, \nu)}$ of $\Phi^{(\mu\nu)}$ is contained in the site-link multiplet $\Psi(n; \mu)$ (because $\mu < \nu$), so its complex conjugate is coupled to the matrix $Z^\dagger(n + \frac{\hat{\mu}}{2})$ on the left; on the other hand, $\phi_R^{(\mu, \nu)}$ is also contained in the multiplet $\Phi(n; \nu)$ (because $\nu > \mu$), so it is coupled to the matrix $Z^\dagger(n + \frac{\hat{\nu}}{2})$ on the right. Below we schematically represent the couplings of the quadruplets with the Z matrices by taking all components in the quadruplets into account:

$$\begin{aligned} \Phi^{(\mu\nu)\dagger} &\longrightarrow Z^\dagger(n + \frac{\hat{\mu}}{2}), Z(n - \frac{\hat{\mu}}{2}) & Z^\dagger(n + \frac{\hat{\eta}}{2}), Z(n - \frac{\hat{\eta}}{2}) &\longleftarrow \Phi^{(\rho\eta)} \\ \Psi^{(\mu\nu)\dagger} &\longrightarrow Z^\dagger(n + \frac{\hat{\nu}}{2}), Z(n - \frac{\hat{\nu}}{2}) & Z^\dagger(n + \frac{\hat{\rho}}{2}), Z(n - \frac{\hat{\rho}}{2}) &\longleftarrow \Psi^{(\rho\eta)}. \end{aligned} \quad (38)$$

In general, one of the two indices in the couple $(\mu\nu)$ specifies the direction of the link carrying the matrix Z or Z^\dagger , while the other index shows which entries of the matrix are concerned: for instance, $\bar{\phi}_R^{(\mu, \nu)}$ couples to the entries $Z^\dagger(n + \frac{\hat{\mu}}{2})_{\nu, \cdot}$, while $\phi_R^{(\rho\eta)}$ couples to the entries $Z(n - \frac{\hat{\eta}}{2})_{\cdot, \rho}$.

As a result, the couplings between the site quadruplets satisfy ‘selection rules’, which mean that the matrix \mathcal{M} contains many 4×4 empty blocks. The non-empty blocks connect the following pairs of quadruplets:

$$\begin{aligned} \Phi^{(\mu\nu)\dagger} &\longleftrightarrow \Phi^{(\rho\eta)} \text{ iff } \mu = \eta \\ \Phi^{(\mu\nu)\dagger} &\longleftrightarrow \Psi^{(\rho\eta)} \text{ iff } \mu = \rho \\ \Psi^{(\mu\nu)\dagger} &\longleftrightarrow \Phi^{(\rho\eta)} \text{ iff } \nu = \eta \\ \Psi^{(\mu\nu)\dagger} &\longleftrightarrow \Psi^{(\rho\eta)} \text{ iff } \nu = \rho. \end{aligned} \quad (39)$$

By analogy with the 2-dimensional case, we will call $V_\mu^{\nu, \eta}$ the matrix coupling $\Phi^{(\mu\nu)\dagger}$ with $\Psi^{(\rho\eta)}$, and $W_\nu^{\mu, \rho}$ the matrix coupling $\Psi^{(\mu\nu)\dagger}$ with $\Phi^{(\rho\eta)}$. The structure of these matrices is similar to the ones in Eq. (??), except that the matrices Z , Z^\dagger are replaced by 2×2 submatrices:

$$\begin{aligned} V_\mu^{\nu, \eta}(n) &= \begin{pmatrix} Z_{\nu, \eta}^\dagger(n + \frac{\hat{\mu}}{2}) & Z_{\nu, -\eta}^\dagger(n + \frac{\hat{\mu}}{2}) & 0 & 0 \\ Z_{-\nu, \eta}^\dagger(n + \frac{\hat{\mu}}{2}) & Z_{-\nu, -\eta}^\dagger(n + \frac{\hat{\mu}}{2}) & 0 & 0 \\ 0 & 0 & Z_{\nu, \eta}(n - \frac{\hat{\mu}}{2}) & Z_{\nu, -\eta}(n - \frac{\hat{\mu}}{2}) \\ 0 & 0 & Z_{-\nu, \eta}(n - \frac{\hat{\mu}}{2}) & Z_{-\nu, -\eta}(n - \frac{\hat{\mu}}{2}) \end{pmatrix}, \\ W_\nu^{\mu, \rho}(n) &= \begin{pmatrix} Z_{\mu, \rho}^\dagger(n + \frac{\hat{\nu}}{2}) & 0 & Z_{\mu, -\rho}^\dagger(n + \frac{\hat{\nu}}{2}) & 0 \\ 0 & Z_{\mu, \rho}(n - \frac{\hat{\nu}}{2}) & 0 & Z_{\mu, -\rho}(n - \frac{\hat{\nu}}{2}) \\ Z_{-\mu, \rho}^\dagger(n + \frac{\hat{\nu}}{2}) & 0 & Z_{-\mu, -\rho}^\dagger(n + \frac{\hat{\nu}}{2}) & 0 \\ 0 & Z_{-\mu, \rho}(n - \frac{\hat{\nu}}{2}) & 0 & Z_{-\mu, -\rho}(n - \frac{\hat{\nu}}{2}) \end{pmatrix}. \end{aligned} \quad (40)$$

In $d \geq 3$, the fields $\Phi^{(\mu\nu)\dagger}$ and $\Phi^{(\rho\mu)}$ are also coupled, through a matrix $X_\mu^{\nu\rho}$; similarly, $\Psi^{(\mu\nu)\dagger}$ and $\Psi^{(\nu\eta)}$ are coupled through a matrix $Y_\nu^{\mu\eta}$. As for V and W , the lower index refers to the direction of the links carrying the elements of Z , Z^\dagger which appear in X (or Y). These matrices

have similar forms as the matrices V , W above (we won't need their explicit expression in the following). These four sets of matrices can be grouped separately into matrices of size $N_b \times N_b$, which we call $\mathcal{V}(n)$, $\mathcal{W}(n)$, $\mathcal{X}(n)$, $\mathcal{Y}(n)$. These four matrices make up the complete coupling matrix $\mathcal{M}(n)$: the action reads

$$-S_Z[n] = \begin{pmatrix} \Phi^{(\mu\nu)} \\ \Psi^{(\mu\nu)} \end{pmatrix}^\dagger \begin{pmatrix} \mathcal{X} & \mathcal{V} \\ \mathcal{W} & \mathcal{Y} \end{pmatrix} \begin{pmatrix} \Phi^{(\mu\nu)} \\ \Psi^{(\mu\nu)} \end{pmatrix}. \quad (41)$$

Taking the mass term into account, the integral over the auxiliary fields yields

$$\text{Det}(m_b \mathbb{I} - \mathcal{M})^{-1} \propto \exp \left\{ -\text{Tr} \ln(1 - m_b^{-1} \mathcal{M}) \right\} \approx \exp \left\{ m_b^{-1} \text{Tr} \mathcal{M} + \frac{m_b^{-2}}{2} \text{Tr} \mathcal{M}^2 \right\}, \quad (42)$$

where we performed the large- m_b expansion up to second order. To analyze the traces, we use the 'selection rules' given by (??). $X_\mu^{\nu\rho}$ connects planes $(\mu\nu) > (\rho\mu)$, therefore its block appears under the diagonal in the matrix \mathcal{X} ; on the opposite, $Y_\nu^{\mu\eta}$ connects planes $(\mu\nu) < (\nu\eta)$, so its block is over the diagonal in \mathcal{Y} . As a result, $\text{Tr} \mathcal{M} = 0$, and $\text{Tr} \mathcal{X}^2 = \text{Tr} \mathcal{Y}^2 = 0$. Therefore, the first nontrivial term appears at the order $1/m_b^2$, and takes the value $\text{Tr} \mathcal{M}^2 = 2\text{Tr}(\mathcal{V}\mathcal{W})$.

To compute this term, we notice that the block $V_\mu^{\nu,\nu}$ connects planes $(\mu\nu)$, $(\mu\eta)$ sharing the same lower index μ (that is, positive generators $e_{\mu\nu}$, $e_{\mu\eta}$ situated on the same row); on the opposite, a block $W_\nu^{\mu,\rho}$ connects generators situated on the same column. Therefore, through \mathcal{V} we can jump along a row, and through \mathcal{W} we jump along a column. When computing $2\text{Tr}(\mathcal{V}\mathcal{W})$ we want to be back at the initial position after two jumps, so that only 'immobile jumps' are allowed:

$$\frac{1}{2} \text{Tr} \mathcal{M}^2 = \text{Tr}(\mathcal{V}\mathcal{W}) = \sum_{\mu < \nu} \text{Tr}(V_\mu^{\nu,\nu} W_\nu^{\mu,\mu}). \quad (43)$$

Thus, to this order the planes are 'decoupled' from one another, and the contribution of each plane is identical to what we had found in the two-dimensional framework (Eq. (??)):

$$\begin{aligned} \text{Tr}(\mathcal{V}(n)\mathcal{W}(n)) = & \sum_{1 \leq \mu < \nu \leq d} Z_{\nu,\nu}^\dagger(n+\frac{\hat{\mu}}{2}) Z_{\mu,\mu}^\dagger(n+\frac{\hat{\nu}}{2}) + Z_{-\nu,-\nu}^\dagger(n+\frac{\hat{\mu}}{2}) Z_{\mu,\mu}(n-\frac{\hat{\nu}}{2}) \\ & + Z_{\nu,\nu}(n-\frac{\hat{\mu}}{2}) Z_{-\mu,-\mu}^\dagger(n+\frac{\hat{\nu}}{2}) + Z_{-\nu,-\nu}(n-\frac{\hat{\mu}}{2}) Z_{-\mu,-\mu}(n-\frac{\hat{\nu}}{2}). \end{aligned} \quad (44)$$

As in two dimensions, only the diagonal elements of the Z -fields contribute to the action up to order $1/m_b^2$. Each of the above terms is the product of two diagonal matrix elements which self-coupled auxiliary fields carried by the same plaquette, so each term can be associated with a well-defined plaquette (we come back to this property in next section). The higher-order terms in $1/m_b$ are more complicated, since the matrices X , Y and non-diagonal blocks V , W start contributing.

To summarize, the effective action to second order in $1/m_b$ has the same structure in any dimension:

$$-S[Z] = \sum_n m_b^{-2} \sum_{\mu < \nu} \text{Tr}(V_\mu^{\nu,\nu} W_\nu^{\mu,\mu})(n) + \sum_{\alpha=1}^d \text{Tr} \ln \left(1 - Z(n+\frac{\hat{\alpha}}{2}) Z^\dagger(n+\frac{\hat{\alpha}}{2}) \right). \quad (45)$$

3.3 Stationary point of the large- m_b effective action

The factor N_c in front of the action $S[Z]$ suggests to study the large- N_c limit of the theory, that is look for stationary points of this action with respect to variations of the matrix fields Z , Z^\dagger . Since we computed the action $S[Z]$ up to second order in $1/m_b$, we will keep this approximation (??) and compute its saddle-point equations.

The variation of the quadratic terms $\text{Tr}(\mathcal{V}\mathcal{W})$ is easy to compute from (??): it only involves variations of diagonal elements of the matrices Z or Z^\dagger . On the opposite, the variation of the second term in Eq. (??) involves all matrix elements:

$$\begin{aligned} \delta \text{Tr} \ln(1 - ZZ^\dagger) &= -\text{Tr} \left[\delta Z Z^\dagger (1 - ZZ^\dagger)^{-1} + \delta Z^\dagger Z (1 - Z^\dagger Z)^{-1} \right] \\ &= -\sum_{a,b} \delta Z_{ab} \left(Z^\dagger (1 - ZZ^\dagger)^{-1} \right)_{ba} + \delta Z_{ab}^\dagger \left(Z (1 - Z^\dagger Z)^{-1} \right)_{ba}. \end{aligned} \quad (46)$$

Therefore, setting $\frac{\delta S[Z]}{\delta Z_{ab}} = 0$ for all $a \neq b$ implies that the matrix $Z(1 - Z^\dagger Z)^{-1}$ is diagonal; this implies that Z is itself a diagonal matrix. We then compute the saddle point equations with respect to variations of the diagonal elements Z_{aa} . For any site n and $\mu \neq \nu$, Eq. (??) yields the following variations:

$$\frac{\delta S[Z]}{\delta Z_{\nu,\nu}^\dagger(n+\frac{\hat{a}}{2})} = \frac{Z_{\nu,\nu}(n+\frac{\hat{a}}{2})}{1 - |Z_{\nu,\nu}(n+\frac{\hat{a}}{2})|^2} - m_b^{-2} Z_{\mu,\mu}^\dagger(n+\frac{\hat{a}}{2}), \quad (47)$$

and similar expressions for the variation of $S[Z]$ with respect to the components $Z_{-\nu,-\nu}^\dagger(n+\frac{\hat{a}}{2})$, $Z_{\nu,\nu}(n-\frac{\hat{a}}{2})$ and $Z_{-\nu,-\nu}(n-\frac{\hat{a}}{2})$. Setting these variations to zero, we get the full set of saddle-points equations. These equations obviously admit the trivial configuration $Z \equiv 0$ as solution. Taking into account the condition $m_b \gg 1$, one can show that this solution is the unique one.

It therefore makes sense to expand the action (??) to quadratic order in Z , Z^\dagger :

$$-S[Z]_{\text{quad}} = m_b^{-2} \sum_n \left(\sum_{\mu < \nu} \text{Tr}(V_\mu^{\nu,\nu} W_\nu^{\mu,\mu})(n) - \sum_{\alpha=1}^d \sum_{a,b=1}^{d'} m_b^2 |Z_{ab}(n+\frac{\hat{a}}{2})|^2 \right). \quad (48)$$

From the expression (??), the above action seems to describe free bosonic fields. The nondiagonal elements Z_{ab} with $a \neq b$ are nondynamical, since they only appear in the mass term. On the opposite, the diagonal terms Z_{aa} appear both in the mass term and the 'kinetic energy term' (??), so they seem to correspond to propagating modes. This is actually not the case: as we already noticed, the terms Eq. (??) only couple matrix elements related to the same plaquette, so that these fields can only propagate around one plaquette. The diagonal fields are therefore non-propagating modes as well, so the above quadratic action is non-dynamical. This is not so surprising, since the auxiliary bosonic fields $\phi(n)$ were from the beginning also confined to one plaquette. The same phenomenon persists if one includes several generations $n_b > 1$.

Propagation can be induced by including higher-order terms in ZZ^\dagger when expanding the logarithm. This way, one obtains a quartic contribution $\text{Tr}(ZZ^\dagger)^2$, which allows to couple together different diagonal elements through non-diagonal ones. This contribution includes for instance terms of the form $Z_{2,2} Z_{2,-2}^\dagger Z_{-2,-2} Z_{-2,2}^\dagger$ (all elements on the link $(n + \hat{1}/2)$), which couple fields carried by two adjacent plaquettes, namely the plaquette $(n, 1, 2)$ carrying $Z_{2,2}(n+\frac{\hat{1}}{2})$, and the plaquette $(n, 1, -2)$ carrying $Z_{-2,-2}(n+\frac{\hat{1}}{2})$ (see Fig. (??)).

4 Concluding remarks

We have considered the dual formulation of the lattice theory which induces Wilson's pure gauge action after integrating over auxiliary bosonic fields, in the limit of large mass and many 'generations'. In our model the 'flavor' degrees of freedom are associated not only with the number of 'generations' of the inducing field but also with a particular plaquette; besides, we need fields with left resp. right 'chirality', which doubles the number of flavor degrees of freedom.

We investigate the properties of the lattice model in the simpler case of one generation. The structure of the inducing theory allows us to apply the color-flavor transformation to obtain a 'dual' effective theory in terms of colorless matrices Z carried by the lattice links. After integrating over the auxiliary bosons, we obtain an effective action uniquely in terms of the Z fields, which is computed explicitly in the limit of large auxiliary mass, leading to a trivial non-propagating theory in the large- N_c limit.

The color-flavor transformation for the $SU(N_c)$ gauge group yields some differences, related with the decomposition of the colorless sector into disconnected subsectors labeled by the baryonic charge Q [?, ?]. Our above derivations correspond to the sectors $Q = 0$ with no contribution of closed baryon loops [?]. The investigation of the effect of these loops is in progress. Note that the choice of $U(N_c)$ instead of the realistic $SU(N_c)$ is irrelevant in the large- N_c limit.

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References

- [1] K. Wilson, Phys. Rev. **D 10** (1974) 2445.
- [2] D. Gross and E. Witten, Phys. Rev. **D 21** (1980) 446.
- [3] T. Banks, R. Myerson and J. Kogut, Nucl. Phys. **B129** (1977) 493.
- [4] M.E. Peskin, Ann. Phys. **113** (1978) 122.
- [5] L. Polley and U.J. Wiese, Nucl. Phys. **B356** (1991) 629.
- [6] M.B. Halpern, Phys. Rev. **D19** (1979) 517.
- [7] B. Rusakov, Phys. Lett. **B398** (1997) 331.
- [8] R. Anishety, S. Chelubaraja and H.S. Sharatchandra, Phys. Lett. **B314** (1993) 387.
- [9] D. Diakonov and V. Petrov, hep-th/9912268.
- [10] M.R. Zirnbauer, J. Phys. **A29** (1996) 7113; *ibid.*, Contribution to the Proceedings of the XIIth International Congress of Mathematical Physics, Brisbane, 1997.

- [11] M. Bander, Phys. Lett. **B126** (1983) 463.
- [12] H. Hamber. Phys. Lett. **B126** (1983) 471.
- [13] I.I. Kogan, G.W. Semenoff and N. Weiss, Phys. Rev. Lett. **69** (1992) 3435.
- [14] V.A. Kazakov and A.A. Migdal, Nucl. Phys. **B397** (1993) 214.
- [15] Yu. Makeenko and K. Zarembo, Nucl. Phys. **B422** (1994) 237.
- [16] A. Hasenfratz and P. Hasenfratz, Phys. Lett. **B297** (1992) 166.
- [17] J. Budzgies, S. Nonnenmacher, Ya. Shnir and M.R. Zirnbauer, Nucl. Phys. **B635** (2002) 309.
- [18] P. Suranyi, Phys. Rev. **D57** (1998) 5084.
- [19] B. Schlittgen and T. Wettig, Nucl. Phys. **B632** (2002) 155.
- [20] B. Schlittgen and T. Wettig, hep-th/0208044.
- [21] J. Budzgies, *The Color-Flavor Transformation and its application to Quantum Chromodynamics*, PhD Thesis, University of Cologne, 2002.
- [22] Ya. Shnir, ICTP preprint IC/2002/124, Trieste, 2002.