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# QUANTUM DECAY RATES IN CHAOTIC SCATTERING

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In this talk we consider a simplified model of chaotic scattering by studying the semi-classical operator

$$P_0(h) = -h^2\Delta_g + V(x)\,, \quad x \in \mathbb{R}^2$$

modified by a complex absorbing potential

$$P(h) = -h^2\Delta_g + V(x) - iW(x)\,.$$

Here $W \geq 0$ is large outside of the *interaction region*. We show that, if the corresponding classical flow is hyperbolic, and if the dimension of the trapped set is small enough, then there is a gap between the eigenvalues of $P(h)$ and the real axis. In other words, the quantum decay rate is bounded from below if the classical repeller is sufficiently *filamentary*. The proof and also the more natural and invariant result about quantum resonances will be presented in [14]. The higher dimensional statement also follows from the proof but is more complicated to present.

More precisely, we work on a Riemannian manifold $(X, g)$ which coincides with $(\mathbb{R}^2, g_0)$, where $g_0$ is the Euclidean metric outside a compact set, say, $B(0, R_0) \subset \mathbb{R}^2$. We let $V \in \mathcal{C}^\infty_{\rm c}(X;\mathbb{R})$ be supported in $B(0, R_0)$ The complex absorbing barrier is given by $W \in \mathcal{C}^\infty(\mathbb{R}^2)$, satisfying $W \geq 0$, $W = 0$ in $B(0, R_0)$, and $W > 1$ in $B(0, R_1)$, $R_1 > R_0$.

The absorbing barrier created by $W$ is a model of infinity since it produces no reflection in semiclassical propagation. We use the notation

$$\Phi^t(\rho) = \exp(tH_{p_0})(\rho)\,, \quad \rho = (x, \xi) \in T^*X\,,$$

where $H_{p_0}$ is the Hamilton vector field of $p_0 = |\xi|^2_g + V(x)$. The incoming and outgoing sets are defined as

$$\Gamma^{\pm}_E \stackrel{\rm def}{=} \left\{ \rho \in T^*_{B(0,R_0)}X \ : \ p_0(\rho) = E\,, \ \ \Phi^t(\rho) \not\to \infty\,, \ t \to \mp\infty \right\}\,. \tag{1}$$

The trapped set,

$$K_E \stackrel{\rm def}{=} \Gamma^+_E \cap \Gamma^-_E \tag{2}$$

is a compact, locally maximal invariant set, contained inside $T^*_{B(0,R_0)}X$. We say that the flow $\Phi^t$ is *hyperbolic near* $K_{E_0}$, if for any energy $E$ near $E_0$, $\Phi^t$ has no fixed point on $p_0^{-1}(E)$, and for any $\rho \in K_E$, the tangent space to $p_0^{-1}(E)$ at $\rho$ splits into flow, unstable and stable subspaces:

$$(3) \quad \begin{aligned} &i)\ T_\rho(p_0^{-1}(E)) = \mathbb{R}H_{p_0}(\rho) \oplus E_\rho^+ \oplus E_\rho^- \quad , \quad \dim E_\rho^\pm = 1\,, \\ &ii)\ d\Phi^t_\rho(E^\pm_\rho) = E^\pm_{\Phi^t(\rho)}\,, \quad \forall t \in \mathbb{R} \\ &iii)\ \exists\, \lambda > 0\,, \quad \|d\Phi^t_\rho(v)\| \leq Ce^{-\lambda|t|}\|v\|\,, \text{ for all } v \in E^\mp_\rho,\ \pm t \geq 0. \end{aligned}$$

The following properties are then satisfied:

$$(4) \quad \begin{aligned} &iv)\ K_E \ni \rho \longmapsto E^\pm_\rho \subset T_\rho(p_0^{-1}(E)) \text{ is Hölder-continuous,} \\ &v)\ \rho \text{ admits local (un)stable manifolds } W^\pm_{\rm loc}(\rho) \text{ tangent to } E^\pm_\rho. \end{aligned}$$

If periodic orbits are dense in $K_E$, then the flow is said to be *Axiom A* on $K_E$ [3].

Classes of potentials satisfying this assumption at a range of non-zero energies are given in [11] and [17, Appendix c]. The dimension of the trapped set appears in the fractal upper

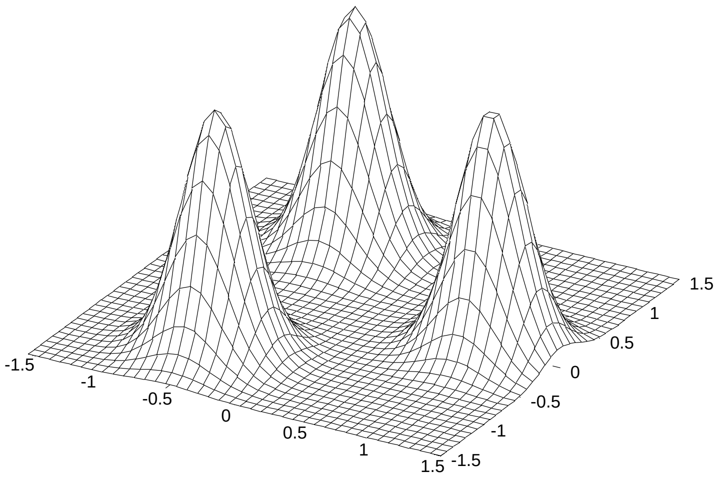


Figure 1. A three bump potential exhibiting hyperbolic dynamics for a range of energies. When the intersections of the graph of $V$ with the plane $V = E$ have radii $a$, and the centers of the bumps are at equilateral distance $R$, $d_H$ in (6) is approximately $\log 2/\log(R/a)$, $R \gg a$.

bounds on the number of resonances. We recall the following result [18] (see [17] for the first result of this type):

**Theorem 1.** *Suppose that the flow of $H_{p_0}$ is hyperbolic near $K_E$. Then*

$$(5) \qquad |\operatorname{Spec}(P(h)) \cap D(E, Ch)| = \mathcal{O}(h^{-d_H})\,,$$

*where*

$$(6) \qquad 2d_H + 1 = \text{ Hausdorff dimension of } K_E.$$

We note that using [15, Theorem 4.1], and in dimension $n = 2$, we strengthened the formulation of the result in [18] by replacing upper Minkowski (or box) dimension by the Hausdorff dimension.

In this talk we address a different question which has been present in the physics literature at least since the seminal paper by Gaspard and Rice [5]. In the same setting of scattering by several convex obstacles, it has also been considered around the same time by Ikawa [6] (see also the careful analysis by Burq in [4]).

**Question:** What properties of the flow $\Phi_t$, or of $K_E$ alone, imply the existence of a *gap* $\gamma > 0$ such that, for $h > 0$ sufficiently small,

$$z \in \mathrm{Spec}(P(h)) \,, \quad \mathrm{Re}\, z \sim E \implies \mathrm{Im}\, z < -\gamma h\,?$$

In other words, what dynamical conditions guarantee a lower bound on the quantum decay rate?

Numerical investigations in different settings of semiclassical three bump potentials [8, 9] three disc scattering [5, 10, 20], Cantor-like Julia sets for $z \mapsto z^2 + c$, $c < -2$ [19], and quantum maps [13, 16], all indicate that a trapped set $K_E$ of low dimension (a "filamentary" fractal set) guarantees the existence of a *resonance gap* $\gamma > 0$.

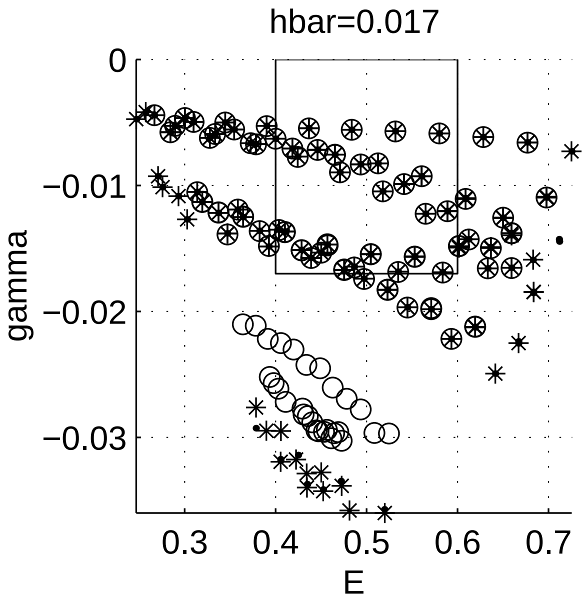


FIGURE 2. A sample of numerical results of [8]: the plot shows resonances for the potential of Fig. 1 ($h = 0.017$). For the energies inside the box, the fractal dimension is approximately $d_H \simeq 0.288 < 0.5$ (see [8, Table 2]), and resonances are separated from the real axis in agreement with Theorem 2.

We remark that some of these works also confirm the fractal Weyl law of Theorem 1 which, unlike Theorem 2 below, was first conjectured in the mathematical works on counting resonances.

Here we provide the following

**Theorem 2.** *Suppose that the dimension $d_H$ defined in (6) satisfies*

$$d_H < \frac{1}{2}\,. \tag{7}$$

*Then there exists $\delta,\ \gamma > 0$, and $h_{\delta,\gamma} > 0$ such that*

$$0 < h < h_{\delta,\gamma} \implies \operatorname{Spec}(P(h)) \cap ([E_0 - \delta, E_0 + \delta] - i[0, h\gamma]) = \emptyset\,. \tag{8}$$

The statement of the theorem can be made more precise using a more sophisticated dynamical object, namely the *topological pressure* of the flow on $K_E$. By taking $\delta$ small enough in (8), we can take any $\gamma$ satisfying

$$0 < \gamma < \min_{|E_0 - E| \leq \delta} (-P_E(1/2))\,, \quad P_E(s) = \text{ pressure of the flow on } K_E. \tag{9}$$

The a priori existence of a resonance gap depends on the *sign* of $P_E(1/2)$. For $n = 2$, one has the equivalence $d_H < 1/2 \iff P_E(1/2) < 0$, which explains the statement of our theorem in terms of the Hausdorff dimension. The connection between $P_E(1/2)$ and a resonance gap also holds in dimension $n \geq 3$; however, for $n \geq 3$ there is generally no simple link between $P_E(1/2)$ and the value of $d_H$ (except when the flow is "conformal" in the unstable, resp. stable directions [15]).

Since the topological pressure will play a crucial rôle, we recall its definition in our context (see [7, Definition 20.2.1] or [15, Appendix A]). If $d$ is a Riemannian distance function on $p_0^{-1}(E)$, we say that a set $\mathcal{E} \subset K_E$ is $(\epsilon, t)$-*separated* if for $\rho_1, \rho_2 \in \mathcal{E}$, $\rho_1 \neq \rho_2$, we have $d(\Phi^{t'}(\rho_1), \Phi^{t'}(\rho_2)) > \epsilon$ for some $0 \leq t' \leq t$. Obviously, such a set must be finite.

We first define the unstable Jacobian:

$$\exp \lambda_t^+(\rho) \stackrel{\text{def}}{=} \det \left( d\Phi^t(\rho)|_{E_\rho^+} \right)\,, \tag{10}$$

and then define, for any $s \in \mathbb{R}$,

$$Z_t(\epsilon, s) \stackrel{\text{def}}{=} \sup_{\mathcal{E}} \sum_{\rho \in \mathcal{E}} \exp\left(-s\, \lambda_t^+(\rho)\right)\,, \tag{11}$$

where the supremum is taken over all $(\epsilon, t)$-separated sets. From there the pressure of the flow $\Phi^t$ on $K_E$ is defined as

$$\forall s \in \mathbb{R}, \qquad P_E(s) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \limsup_{t \to \infty} \frac{1}{t} \log Z_t(\epsilon, s)\,. \tag{12}$$

The proof of Theorem 2 is based on the ideas developed in the recent work of Anantharaman and the first author [1, 2] on semiclassical defect measures for eigenfuctions of the Laplacian on manifolds with Anosov geodesic flows. Although we do not use semiclassical defect measures in the proof of Theorem 2, the following simple result provides a general connection:

**Theorem 3.** *Consider a sequence of values* $(h_k \to 0)$ *and corresponding eigenstates*

$$P(h_k)u(h_k) = z(h_k)u(h_k)\,, \quad \|u(h_k)\|_{L^2} = 1\,,$$

*satisfying* $\operatorname{Re} z(h_k) = E + o(1)$ *and* $\operatorname{Im} z(h) \geq -Ch$*. Suppose that a semiclassical defect measure* $d\mu$ *on* $T^*X$ *is associated to the sequence* $(u(h_k))$*:*

$$\langle a^w(x, h_k D)u(h_k), u(h_k)\rangle \longrightarrow \int_{T^*X} a(\rho)\, d\mu(\rho)\,, \quad k \to \infty\,.$$

*Then*

$$\operatorname{supp} \mu \cap T^*_{B(0,R_0)}X \subset \Gamma_E^+\,, \tag{13}$$

*and there exists* $\lambda > 0$ *such that*

$$\lim_{k\to\infty} \operatorname{Im} z(h_k)/h_k = -\lambda/2\,, \quad \text{and} \quad \mathcal{L}_{H_{p_0}}\mu = \lambda\mu \quad \text{in} \quad T^*_{B(0,R_0)}X\,. \tag{14}$$

Connecting this theorem with Theorem 2, the semiclassical defect measures associated with sequences of resonant states have decay rates $\lambda$ bounded from below by $2\gamma$ if the dimension of the trapped set is small enough.

Finally, we comment on the optimality of Theorem 2. Except in some very special cases (for instance when $K_E$ consists of one hyperbolic orbit) we do not expect the estimate on the size of the resonance free region in terms of the pressure to be optimal. In fact, in the analogous case of scattering on convex co-compact hyperbolic surfaces the results of Naud (see [12] and references given there) show that the resonance free strip is larger at high energies than the strip predicted by the pressure. That relies on delicate zeta function analysis following the work of Dolgopyat: at zero energy there exists a Patterson-Sullivan resonance with the imaginary part (width) given by the pressure but all other resonances have more negative imaginary parts.

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