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Decay of correlations for normally hyperbolic trapping

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Abstract We prove that for evolution problems with normally hyperbolic trapping in phase space, correlations decay exponentially in time. Normally hyperbolic trapping means that the trapped set is smooth and symplectic and that the flow is hyperbolic in directions transversal to it. Flows with this structure include contact Anosov flows, classical flows in molecular dynamics, and null geodesic flows for black holes metrics. The decay of correlations is a consequence of the existence of resonance free strips for Green's functions (cut-off resolvents) and polynomial bounds on the growth of those functions in the semiclassical parameter.

1 Statement of results

1.1 Introduction

We prove the existence of resonance free strips for general semiclassical problems with normally hyperbolic trapped sets. The width of the strip is related to certain Lyapunov exponents and, for the spectral parameter in that strip, the Green's function (cut-off resolvent) is polynomially bounded. Such estimates

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are closely related to exponential decay of correlations in classical dynamics and in scattering problems. The framework to which our result applies covers both settings.

To illustrate the results consider

$$P = -h^2\Delta + V(x), \quad V \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathbb{R}). \tag{1.1}$$

The classical flow $\varphi_t : (x(0), \xi(0)) \mapsto (x(t), \xi(t))$ is obtained by solving Newton's equations $x'(t)(t) = 2\xi(t)$, $\xi'(t) = -\nabla V(x(t))$. The trapped set at energy E, K_E, is defined as the set of (x, ξ) such that $p(x, \xi) \stackrel{\text{def}}{=} \xi^2 + V(x) = E$ and $\varphi_t(x, \xi) \not\to \infty$, as $t \to \infty$ and as $t \to -\infty$.

The flow φ_t is said to be *normally hyperbolic near energy* E, if for some $\delta > 0$,

$$K^\delta \stackrel{\text{def}}{=} \bigcup_{|E-E'|<\delta} K_{E'} \text{ is a smooth symplectic manifold, and}$$
$$\text{the flow } \varphi_t \text{ is hyperbolic in the directions transversal to} K^\delta, \tag{1.2}$$

see (1.17) below for a precise definition, and [29] for physical motivation for considering such dynamical setting. A simplest consequence of Theorems 2 and 6 is the following result about decay of correlations.

Theorem 1 *Suppose that* P *is given by* (1.1) *and that* (1.2) *holds, that is the classical flow is normally hyperbolic near energy* E. *Then for* $\psi \in \mathcal{C}_c^\infty((E - \delta/2, E + \delta/2))$, *and any* $f, g \in L^2(\mathbb{R}^n)$, *with* $\|f\|_{L^2} = \|g\|_{L^2} = 1$, supp f, supp $g \subset B(0, R)$,

$$\left|\langle e^{-itP/h}\psi(P)f, g\rangle_{L^2(\mathbb{R}^n)}\right| \leq \frac{C_R \log(1/h)}{h^{1+\gamma c_0}} e^{-\gamma t} + C_{R,N}h^N, \quad t > 0, \tag{1.3}$$

for any $\gamma < \lambda_0/2$ *and for all* N. *Here* λ_0 *and* c_0 *are the same as in* (1.18) *and* C_R, $C_{R,N}$ *are constants depending on* R *and on* R *and* N, *respectively.*

This means that the correlations decay rapidly in the semiclassical limit: we start with a state localized in space (the support condition) and energy, $\psi(P)f$, propagate it, and test it against another spatially localized state g. The estimate (1.3) is a consequence of the existence of a band without scattering resonances and estimates on cut-off resolvent given in Theorem 2. When there is no trapping, that is when $K_E = \emptyset$, then the right hand side in (1.3) can be replaced by $\mathcal{O}((h/t)^\infty)$, provided that $t > T_E$, for some T_E—see for instance [36, Lemma 4.2]. On the other hand when strong trapping is present, for instance when the potential has an interaction region separated from infinity by a barrier, then the correlation does not decay—see [36] and references given there.

More interesting quantitative results can be obtained for the wave equation or for decay of classical correlations: see Sect. 1.2 for motivation and [54, Theorem 3] and Corollary 5 below for examples. When the outgoing and incoming sets at energy E,

$$\Gamma_E^{\pm} \stackrel{\text{def}}{=} \{(x,\xi) : p(x,\xi) = E,\ \varphi_t(x,\xi) \not\to \infty, t \to \mp\infty\},$$

are sufficiently regular and of codimension one, Theorem 1 and Theorem 2 below (without the specific constant λ_0) are already a consequence of earlier work by Wunsch–Zworski [54, Theorem 2][1] and, in the case of closed trajectories, Christianson [10,11]. For a survey of other recent results on resolvent estimates in the presence of weak trapping we refer to [52].

When normal hyperbolicity is strengthened to r-normal hyperbolicity for large r (which implies that $\Gamma_E^{\pm}$ are C^r manifolds) and provided a certain pinching condition on Lyapunov exponents is satisfied, much stronger results have been obtained by Dyatlov [19]. In particular, [19] provides an asymptotic counting law for scattering resonances *below* the band without resonances given in Theorems 2, 4 and 6. It shows the optimality of the size of the band in a large range of settings, for instance, for perturbations of Kerr–de Sitter black holes.

Similar results on asymptotic counting laws in strips have been proved by Faure–Tsujii in the case of Anosov diffeomorphisms [24], and recently announced in the case of contact Anosov flows [25]. In the latter situation, described in Theorem 4 below, the trapped set is a normally hyperbolic smooth symplectic manifold, but the dependence of the stable and unstable subspaces on points on the trapped set is typically nonsmooth, but C^1 or Hölder continuous (see Remark 1.2 below). For compact manifolds of constant negative curvature Dyatlov–Faure–Guillarmou [21] have provided a precise description of Pollicott–Ruelle resonances in terms of eigenvalues of the Riemannian Laplacian acting on section of certain natural vector bundles.

In this paper we do *not* assume any regularity on $\Gamma_E^{\pm}$ and provide a quantitative estimate on the resonance free strip. For operators with analytic coefficients this result was already obtained by Gérard–Sjöstrand [27] with even weaker assumptions on K^δ. A new component here, aside from dropping the analyticity assumption, is the polynomial bound on the Green's function/resolvent that allows applications to the decay of correlations.

The proof is given first for an operator with a complex absorbing potential. This allows very general assumptions which can then be specialized to scattering and dynamical applications.

[1] Recently Dyatlov [20] provided a much simpler proof of that result, including the optimal size of the gap established in this paper and the optimal resolvent bound $o(h^{-2})$, for smooth and orientable stable and unstable manifolds.

Finally we comment on the comparison between the resonance free regions in this paper and the results of [38,39] where the existence of a resonance free strip was given for *hyperbolic* trapped sets, provided a certain *pressure condition* was satisfied. In the setting of [38] the trapped set is typically very irregular but, the assumptions of [38] also include the situation where K^δ is a smooth symplectic submanifold, and the flow is hyperbolic both transversely to K^δ and along each K_E. In that case the resonance gap obtained in [38] involves a topological pressure associated with the full (that is, longitudinal and transverse) unstable Jacobian, namely

$$\mathcal{P}\left(-\frac{1}{2}(\log J_{\parallel}^{+}+\log J_{\perp}^{+})\right) = \sup_{\mu}\left(H(\mu) - \frac{1}{2}\int(\log J_{\parallel}^{+}+\log J_{\perp}^{+})\,d\mu\right), \tag{1.4}$$

where the supremum is taken over all flow-invariant probability measures on K^δ and $H(\mu)$ is the Kolmogorov–Sinai entropy of the measure μ with respect to the flow. The bound is nontrivial only if this pressure is negative. In the case of mixing Anosov flows discussed in Sect. 9 the transverse and longitudinal unstable Jacobians are equal to each other; the above pressure is then equal to the pressure $\mathcal{P}(-\log J_{\parallel}^{+})$, equivalent with the pressure $\mathcal{P}(-\log J^{u})$ of the Anosov flow, which is known to vanish [7, Proposition 4.4], and hence gives only a trivial bound. For this situation, our spectral bound (Theorem 4) is thus sharper than the pressure bound. On the other hand, one can construct examples where the longitudinal and transverse unstable Jacobians are independent of one another, and such that the pressure (1.4) is more negative - hence sharper - than the value $-\lambda_0$ given in (1.19), which may be expressed as $-\lambda_0 = \sup_\mu\Big(-\frac{1}{2}\int\log J_{\perp}^{+}\,d\mu\Big)$.

Notation. We use the following notation $g = \mathcal{O}_k(f)_V$ means that $\|g\|_V \leq C_k f$ where the norm (or any seminorm) is in the space V, an the C_k depends on k. When either k or V are dropped then the constant is universal or the estimate is scalar, respectively. When $F = \mathcal{O}_k(f)_{V\to W}$ then the operator $F : V \to W$ has its norm bounded by $C_k f$.

1.2 Motivation

To motivate the problem we consider the following elementary example. Let $X = \mathbb{R}$ and $P = -\partial_x^2$. A wave evolution is given by $U(t) \stackrel{\text{def}}{=} \sin(\sqrt{P}t)/\sqrt{P}$. Then for $f, g \in \mathcal{C}_c^\infty(\mathbb{R})$ and any time $t \in \mathbb{R}$ we define the wave *correlation function* as

$$C(f,g)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} [U(t)f](x)\, g(x)\, dx \tag{1.5}$$

In this 1-dimensional setting, the correlation function becomes very simple for large times. Indeed, for a certain $T > 0$ depending on the support of f and g, it satisfies

$$\forall t \geq T, \qquad C(f,g)(t) = \frac{1}{2} \int_{\mathbb{R}} f(x) \int_{\mathbb{R}} g(x)\, dx$$

This particular behaviour is due to the fact that the resolvent of P,

$$R(\lambda) \stackrel{\text{def}}{=} (P - \lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \ \ \operatorname{Im} \lambda > 0,$$

continues meromorphically to $\mathbb{C}$ in λ as an operator $L^2_{\text{comp}} \to L^2_{\text{loc}}$ and has a pole at $\lambda = 0$. In this basic case we see this from an explicit formula,

$$[R(\lambda) f](x) = \frac{i}{2\lambda} \int_{\mathbb{R}} e^{i\lambda|x-y|} f(y) dy.$$

More generally, we can consider $P = -\partial_x^2 + V(x)$, $V \in L^\infty_{\text{c}}(\mathbb{R})$, with $V \geq 0$, for simplicity. With the same definition of $U(t)$ we now have the Lax–Phillips expansion generalizing (1.5):

$$\begin{aligned} C(f,g)(t) &= \int_{\mathbb{R}} U(t) f\, g\, dx \\ &= \sum_{\operatorname{Im} \lambda_j > -A} e^{-i\lambda_j t} \int_{\mathbb{R}} f\, u_j\, dx \int_{\mathbb{R}} g\, u_j\, dx + \mathcal{O}(e^{-At}), \end{aligned} \tag{1.6}$$

where λ_j are the poles of the meromorphic continuation of $R(\lambda) = (P - \lambda^2)^{-1}$ (for simplicity assumed to be simple), and u_j are solutions to $(P - \lambda_j^2) u_j = 0$ satisfying $u_j(x) = a_{\operatorname{sgn} x} e^{i\lambda|x|}$ for $|x| \gg 1$. Since u_j are not in L^2 their normalization is a bit subtle: they appear in the residues of $R(\lambda)$ at λ_j.

The expansion (1.6) makes sense since the number of poles of $R(\lambda)$ with $\operatorname{Im} \lambda > -A$ is finite for any A. If we define $C(f,g)$ to be 0 for $t \leq 0$, the Fourier transform of (1.6) gives (provided 0 is not a pole of $R(\lambda)$),

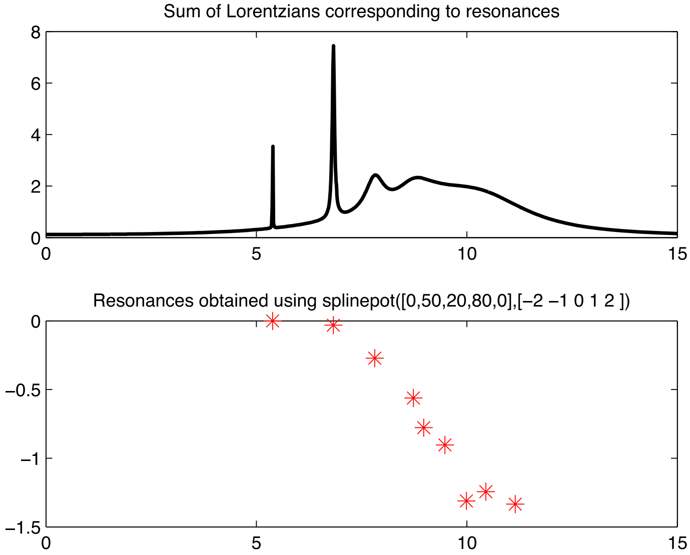

Fig. 1 The effect of resonances on the Fourier transform of correlations as described in (1.7). The resonances are computed using the code `scatpot.m` [4]

$$\widehat{C(f,g)}(-\lambda) = \sum_{\operatorname{Im}\lambda_j > -A} \frac{c_j}{\lambda_j - \lambda} + \mathcal{O}\left(\frac{1}{A}\right),$$
$$c_j \stackrel{\text{def}}{=} -i \int_{\mathbb{R}} f u_j \, dx \int_{\mathbb{R}} g u_j \, dx. \tag{1.7}$$

The Lorentzians

$$\frac{|\operatorname{Im}\lambda_j|}{|\lambda - \lambda_j|^2} = -2 \operatorname{Im} \frac{1}{\lambda_j - \lambda},$$

peak at $\lambda = \Re\lambda_j$ and are more pronounced for $\operatorname{Im}\lambda_j$ small. This stronger response in the spectrum of correlations is one of the reasons for calling λ_j (or λ_j^2) *scattering resonances* (Fig. 1).

In more general situations, to have a finite expansion of type (1.6), modulo some exponentially decaying error $\mathcal{O}(e^{-\gamma t})$, we need to know that the number of poles of $R(\lambda)$ is finite in a strip $\operatorname{Im}\lambda > -\gamma$. Hence exponential decay of correlations is closely related to *resonance free strips*.

This elementary example is related through our approach to recent results of Dolgopyat [16], Liverani [35], and Tsujii [46,47] on the decay of correlations in classical dynamics.

Let X be a compact contact manifold of (odd) dimension n, and let γ_t be an Anosov flow on X preserving the contact structure—see Sect. 9 for details. The standard example is the geodesic flow on the cosphere bundle $X = S^*M$, where (M, g) is a smooth negatively curved Riemannian manifold. Let $U(t) : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ be defined by $U(t)f = \gamma_t^* f = f \circ \gamma_t$ and let dx be the measure on X induced by the contact structure and normalized so that $\mathrm{vol}(X) = 1$. The results of [16,35] show that, for any test functions $f, g \in C^\infty(X)$, the correlation function satisfies the following asymptotical behavior for large times:

$$C(f, g)(t) \stackrel{\mathrm{def}}{=} \int_X [U(t)f](x)\, g(x)\, dx = \int_X f\, dx \int_X g\, dx + \mathcal{O}(e^{-\Gamma t}),\ t \to \infty, \tag{1.8}$$

and the exponent Γ is independent of f, g. In other words, the Anosov flow is exponentially mixing with respect to the invariant measure dx.

From the microlocal point of view of Faure–Sjöstrand [23], this result is related to a resonance free strip for the generator of the flow γ_t. The resonances in this setting are called *Pollicott–Ruelle resonances*.

In this paper we consider general semiclassical operators modeled on P given in (1.1), for which the classical flow has a normally hyperbolic trapped set. Schrödinger operators for which (1.2) holds appear in molecular dynamics—see the recent review [29] for an introduction and references. In particular, [29, Chapter 5] discusses the resonances in some model cases and the relation between the size of the resonance free strip and the transverse Lyapounov exponents. As reviewed in Sect. 9, the setting can be extended such as to include the generator of the Anosov flow of (1.8), namely the operator $P(h)$ on X such that $U(t) = \gamma_t^* = \exp(-itP/h)$.

1.3 Assumptions and the result

The general result, Theorem 2, is proved for operators modified using a *complex absorbing potential* (CAP). Results about such operators can then be used for different problems using *resolvent gluing techniques* of Datchev–Vasy [14]—see Theorems 3 and 4. The assumptions on the manifold X, operator P, and the complex absorbing potential may seem unduly general, they are justified by the broad range of applications.

Let X be a smooth compact manifold with a density dx and let

$$P = P(x, hD) \in \Psi^m(X), \quad m > 0,$$

be an unbounded self-adjoint semiclassical pseudodifferential operator on $L^2(X, dx)$ (see Sect. 3.1 and [55, §14.2] for background and notations), with principal symbol $p(x,\xi)$ independent of h. Let

$$W = W(x, hD) \in \Psi^k(X), \quad 0 \le k \le m, \quad W \ge 0,$$

be another operator, also self-adjoint and with h-independent principal symbol $w(x,\xi)$, which we call a (generalized) complex absorbing potential (CAP). We should stress that W plays a purely *auxiliary* role and can be chosen quite freely.

If the principal symbols $p(x,\xi) \in S^m(T^*X)$ and $w(x,\xi) \in S^k(T^*X)$, we assume that, for some fixed $C_0 > 0$ and for any phase space point $(x,\xi) \in T^*X$,

$$\begin{aligned} &|p(x,\xi) - iw(x,\xi)| \ge \langle\xi\rangle^m/C_0 - C_0, \qquad 1 + w(x,\xi) \ge \langle\xi\rangle^k/C_0, \\ &\quad \exp(tH_p)(x,\xi) \text{ is defined for all } t \in \mathbb{R}. \end{aligned} \tag{1.9}$$

Here, for $\xi \in T_x^*X$ we have denoted $\langle\xi\rangle^2 = 1 + \|\xi\|_x^2$ for some smoothly varying metric on X, $x \mapsto \|\bullet\|_x^2$, and by H_p the Hamilton vector field of p. The map $\exp(tH_p) : T^*X \to T^*X$ is the corresponding flow at time t. This flow will often be denoted by φ_t, the Hamiltonian $p(x,\xi)$ being clear from the context.

For technical reasons (see Lemma 10.4) we will need an additional smoothness assumption on w:

$$|\partial^\alpha w(x,\xi)| \le C_\alpha w(x,\xi)^{1-\gamma}, \quad 0 < \gamma < \frac{1}{2}, \tag{1.10}$$

when $w(x,\xi) \le 1$. This can be easily arranged and is invariant under changes of variables.

We call the operator

$$\widetilde{P} = P - iW \in \Psi^m(X), \tag{1.11}$$

the CAP-modified P. The condition (1.9) means that the CAP-modified P is classically elliptic and that for any fixed $z \in \mathbb{C}$

$$\{(x,\xi) \ : \ \widetilde{p}(x,\xi) - z = p(x,\xi) - iw(x,\xi) - z = 0\} \Subset T^*X.$$

We define the *trapped set* at energy E as

$$K_E \stackrel{\text{def}}{=} \{\rho = (x,\xi) \ : \ \rho \in p^{-1}(E), \ \ \varphi_{\mathbb{R}}(\rho) \subset w^{-1}(0)\}. \tag{1.12}$$

K_E is compact and consists of points in $p^{-1}(E)$ which never reach the *damping region* $\{\rho \in T^*X \ : \ w(\rho) > 0\}$ in backward or forward propagation by the flow φ_t.

We illustrate this setup with two simple examples:

Example 1 Suppose that $P_0 = -h^2\Delta + V$, $V \in \mathcal{C}^\infty_c(\mathbb{R}^n;\mathbb{R})$, $\operatorname{supp} V \Subset B(0, R_0)$. Define the torus $X = \mathbb{R}^n/(6R_0\mathbb{Z})^n$, and $W \in \mathcal{C}^\infty(X;[0,\infty))$, satisfying

$$W(x) = 0, \quad x \in B(0, R_0), \qquad W(x) = 1, \quad x \in X\backslash B(0, 2R_0),$$
$$\partial^\alpha W = \mathcal{O}_\alpha(W^{2/3}),$$

(here we identified the balls in $\mathbb{R}^n$ with subsets of the torus). The last condition can be arranged by taking $W(x) = \chi(|x|^2 - R_0^2)\psi(x)$ where $\chi(x) = \exp(-x^{-1})\mathbb{1}_{\mathbb{R}_+}(x)$, and $\psi \in \mathcal{C}^\infty(X,(0,\infty))$ is suitably chosen. The power of W on the right hand side can be any number greater than $\frac{1}{2}$.

Because of the support properties of V, $P \stackrel{\text{def}}{=} -h^2\Delta + V \in \Psi^2(X)$ and $P - iW$ satisfy all the properties above. The trapped set K_E can be identified with a subset of $T^*_{B(0,R_0)}\mathbb{R}^n$ and is then equal to the trapped set of scattering theory:

$$K_E = \{(x,\xi) \in T^*\mathbb{R}^n \ : \ \xi^2 + V(x) = E, \ \ x(t) \not\to \infty, \ \ t \to \pm\infty\}.$$

Remark 1.1 Normally hyperbolic trapped sets occur in the semiclassical theory of chemical reaction dynamics, where they are usually called Normally Hyperbolic Invariant Manifolds (NHIM). They are of fundamental importance to quantitatively understand the kinetics of the chemical reaction. See for instance [48] for a description of the classical phase space structure, and [29] and references given there for the adaptation to the quantum framework. The focus there is on examples for which the Hamiltonian flow exhibits a

$$\text{saddle} \times \text{saddle} \times \cdots \times \text{center} \cdots \times \text{center}$$

fixed point: after an appropriate linear symplectic change of coordinates, the quadratic expansion of the Hamiltonian $p(x,\xi)$ near the fixed point (set at the origin) reads as:

$$p_{\text{quad}}(x,\xi) = \frac{1}{2}\sum_{i=1}^{d-d_\perp}\left(\xi_i^2 + \omega_i^2 x_i^2\right) + \sum_{i=d-d_\perp+1}^{d}\frac{1}{2}\left(\xi_i^2 - \lambda_i^2 x_i^2\right).$$

For this quadratic model the NHIM at a positive energy $E > 0$[2], is given by

$$p^{-1}(E) \cap \left\{\xi_{d-d_\perp+1} = x_{d-d_\perp+1} = \ldots = x_d = \xi_d = 0\right\}$$

[2] For the distribution of resonances at the fixed point energy $E = 0$ see [34,41].

which is a $2d - 2d_\perp - 1$-dimensional sphere. The stable/unstable distributions are $d_\perp$-dimensional (see (1.17) below), and are generated by the vectors $\{\partial/\partial\xi_i \pm \lambda_i \partial/\partial x_i\}_{i=d-d_\perp+1}^{d}$. For this quadratic model the flow along the NHIM is completely integrable. This implies that the latter is structurally stable to perturbations (it is then r-normally hyperbolic for any $r \in \mathbb{N}$), meaning that for any given regularity $r > 0$, a small enough perturbation of p_{quad} will still lead to the presence of a NHIM of regularity C^r [31]. However, the flow on the perturbed NHIM is generally not integrable. This situation occurs if one considers the full Hamiltonian p with quadratic expansion p_{quad}: for small positive energies p will still exhibit a NHIM, which is a deformed sphere.

Physical systems featuring this type of fixed point are presented in the literature: for instance the isomerization of hydrogen cyanide [51] or the quantum dynamics of the nitrogen-nitrogen exchange [29]. Strictly speaking the potentials appearing in these physical models are more complicated than the ones allowed here. However, the behaviour near the NHIM determines the phenomena which are studied here and which are relevant in physics.

We conclude this remark by recalling that when $d_\perp = 1$ (most relevant from the point of view of [29]) and when the system is r-normally hyperbolic for sufficiently large r very precise results on the distribution of resonances have been obtained by Dyatlov [19,20].

Example 2 Suppose that X is a compact manifold with a volume form dx and a vector field Ξ generating a volume preserving flow ($\mathcal{L}_\Xi dx = 0$). Then $P = -ih\Xi$ is a selfadjoint operator on $L^2(X, dx)$, and the corresponding propagator $\exp(-itP/h)$ is the push-forward of the flow $\gamma_t = \exp(t\Xi)$ generated by Ξ on functions $f \in L^2(X, dx)$: $\exp(-itP/h) f = f \circ \gamma_{-t}$.

To define the CAP in this setting we choose a Riemannian metric g on X, and a function

$$\begin{aligned} &f \in C^\infty(\mathbb{R}, [0, \infty)), \quad |f^{(k)}(s)| \le C_k f(s)^{1-\gamma}, \quad \text{for some } \gamma \in (0, 1/2), \\ &\qquad f^{-1}(0) = [-\infty, M] \text{ for some } M > 0, \quad f(s) = \sqrt{s}, \quad s > 2M. \end{aligned} \tag{1.13}$$

If Δ_g is the corresponding Laplacian on X, we set $W(x, hD) = f(-h^2\Delta_g)$.

Then the operator $P - iW \in \Psi^1(X)$ satisfies the assumptions above. The principal symbols read $p(x, \xi) = \xi(\Xi_x)$, $w(x, \xi) = f(\|\xi\|_x^2)$, where the norm $\| \bullet \|_x$ is associated with the metric g.

At a given energy $E \in \mathbb{R}$, the trapped set is given by the points which never enter the absorbing region:

$$K_E = \{(x, \xi) \in T^*X \;:\; \xi(\Xi_x) = E, \;\; \|(\gamma_{-t})_*\xi\|_g \le M, \;\; \forall t \in \mathbb{R}\}.$$

At this stage the trapped set seems to depend on the choice of M. Below we will be concerned with $\exp(t\,\Xi)$ being an Anosov flow, in which case this explicit dependence will disappear, as long as we choose M large enough compared with the energy E (see the second assumption (1.15) below).

Returning to general considerations we also define

$$K^\delta \stackrel{\text{def}}{=} \bigcup_{|E|\le\delta} K_E, \tag{1.14}$$

which is a compact subset to T^*X and assume that

$$dp\restriction_{K^\delta} \neq 0, \qquad K^\delta \cap \mathrm{WF}_h(W) = \emptyset. \tag{1.15}$$

The first assumption implies that for $|E| \le \delta$, the energy shell $p^{-1}(E)$ is a smooth hypersurface close to $w^{-1}(0)$. The second assumption is consistent with the definition (1.12) of K_E. It implies that the latter is contained in the interior of the region $w^{-1}(0)$, a property which is stable when enlarging K_E to K^δ, or when slightly modifying the support of w.

We now make the following *normal hyperbolicity* assumption on K^δ:

$$K^\delta \text{ is a smooth symplectic submanifold of } T^*X, \tag{1.16}$$

and there exists a continuous distribution of linear subspaces

$$K^\delta \ni \rho \longmapsto E^\pm_\rho \subset T_\rho(T^*X),$$

invariant under the flow,

$$\forall t \in \mathbb{R}, \quad (\varphi_t)_* E^\pm_\rho = E^\pm_{\varphi_t(\rho)},$$

and satisfying, for some $\lambda > 0$, $C > 0$ and any point $\rho \in K^\delta$,

$$\begin{gathered} T_\rho K^\delta \cap E^\pm_\rho = E^+_\rho \cap E^-_\rho = \{0\}, \quad \dim E^\pm_\rho = d_\perp, \\ T_\rho(T^*X) = T_\rho K^\delta \oplus E^+_\rho \oplus E^-_\rho, \\ \forall v \in E^\pm_\rho, \ \ \forall t > 0, \quad \|d\varphi_{\mp t}(\rho)v\|_{\varphi_{\mp t}(\rho)} \le C e^{-\lambda t}\|v\|_\rho. \end{gathered} \tag{1.17}$$

Here $\rho \mapsto \|\bullet\|_\rho$ is any smoothly varying norm on $T_\rho(T^*X)$, $\rho \in K^\delta$. The choice of norm may affect C but not λ.

Remark 1.2 A large class of examples for which the distributions $\rho \mapsto E^\pm_\rho$ are not smooth is provided by considering contact Anosov flows on compact manifolds—see [23,47] and Sect. 9.1 below for the natural appearance of normally hyperbolic trapping for the flow lifted to the cotangent bundle of the manifold. The regularity is inherited from the regularity of the stable and unstable distributions tangent to the manifold, which in general are only known to be Hölder continuous [3]. More is known on the regularity of these distributions when the manifold is 3-dimensional (and preserves a contact structure). In this situation, Hurder-Katok showed [32] that there is a dichotomy (or "rigidity"): either the stable/unstable distributions are $C^{2-\epsilon}$ for any $\epsilon > 0$ but not C^2 (this is due to a certain obstruction, namely the *Anosov cocycle* is not cohomologous to zero), or the distributions are as smooth as the flow. If that 3-dimensional flow is the geodesic flow on a surface of negative curvature, then following Ghys [28] they show (Corollary. 3.7) that the latter case imposes a metric of *constant* negative curvature. Hence, for the geodesic flow on a surface of *non-constant* negative curvature, the stable/unstable distributions, and hence their lifts $E^{\pm}_{\rho}$, are not C^2.

We do not know of an example of a Schrödingier operator (that is of a classical Hamiltonian of the form $p(x,\xi) = |\xi|^2 + V(x)$) for which the trapped set is smooth—or sufficiently regular: as with all microlocal results a certain high level of regularity, depending on the dimension, is sufficient—and the distributions $\rho \mapsto E^{\pm}_{\rho}$ are irregular. However there is no general result which prevents that possibility. Interesting regular examples of $E^{\pm}_{\rho}$ of any dimension $1 \leq d_{\perp} \leq d-1$ were discussed in Remark 1.1.

We also remark that higher dimensional distributions can lead to complicated topological issues, which would make the global approach of [19,20,53] difficult. This is visible already for flows on constant curvature manifolds for which smooth foliations may have nontrivial topology [21, §2.2].

Except for the construction of the escape function, for which we need to use [37,43], the analysis in Sects. 5 and 6 would not be simplified by a smoothness assumption on the distributions.

We can now state our main result.

Theorem 2 *Suppose that X is a smooth compact manifold and that P and W satisfy the assumptions above. If the trapped set K^{δ} given by* (1.12),(1.14) *is normally hyperbolic, in the sense that* (1.16) *and* (1.17) *hold, then for any $\epsilon_0 > 0$ there exists h_0, c_0, C_1, such that for $0 < h < h_0$,*

$$\begin{aligned} &\|(P - iW - z)^{-1}\|_{L^2\to L^2} \leq C_1 h^{-1+c_0\, Im\, z/h} \log(1/h), \\ &\quad \textit{for} \quad z \in [-\delta + \epsilon_0, \delta - \epsilon_0] - ih[0, \lambda_0/2 - \epsilon_0], \end{aligned} \tag{1.18}$$

where $\lambda_0 > 0$ is the minimal transverse unstable expanding rate:

$$\lambda_0 \stackrel{\text{def}}{=} \liminf_{t\to\infty} \frac{1}{t} \inf_{\rho \in K^{\delta}} \log\det\left(d\varphi_t \restriction_{E^+_{\rho}} \right). \tag{1.19}$$

Here det *is taken using any fixed volume form on* E_ρ^+, *the value of* $\lambda_0 > 0$ *being independent of the choice of volume forms.*

This theorem will be proved in Sect. 6 after preparation in Sects. 4, 5. The bound $\log(1/h)/h$ on the real axis is optimal as shown in [5]. Using the methods of [14] the estimate (1.18) almost immediately applies to the setting of scattering theory. As an example we present an application to scattering on asymptotically hyperbolic manifolds, which will be proved in Sect. 8:

Theorem 3 *Suppose* (Y, g) *is a conformally compact* n*-manifold with even power metric:* Y *is compact,* $\partial Y = \{x = 0\}$, $dx\restriction_{\partial Y} \neq 0$, $g = (dx^2 + h)/x^2$ *where* h *is a smooth 2-tensor on* Y *with only even powers of* x *appearing in its Taylor expansion at* $x = 0$. *If the trapped set for the geodesic flow on* Y *is normally hyperbolic, then the following resolvent estimate holds:*

$$\|x^{k_0}\big(-\Delta_g - (n-1)^2/4 - \lambda^2 \pm i0\big)^{-1}x^{k_0}\|_{L^2\to L^2} \leq C_0\frac{\log\lambda}{\lambda}, \quad \lambda > 1.$$

The next application is a rephrasing of a recent theorem of Tsujii [46,47]; it will be proved in Sects. 9. We take the point of view of Faure–Sjöstrand [23], see also [13].

Theorem 4 *Suppose* X *is a compact manifold and* $\gamma_t : X \to X$ *a contact Anosov flow on* X. *Let* Ξ *be the vector field generating* γ_t, *and* $P = -ih\Xi$ *the corresponding semiclassical operator, self-adjoint on* $L^2(X, dx)$ *for* dx *the volume form derived from the contact structure.*

Define the minimal asymptotic unstable expansion rate

$$\lambda_0 \stackrel{\text{def}}{=} \mathop{lim\,inf}_{t\to\infty}\frac{1}{t}\inf_{x\in X}\log\det\big(d\gamma_t\restriction_{E_u(x)}\big), \tag{1.20}$$

with $E_u(x) \subset T_xX$ *the unstable subspace of the flow at* x.

For any $t > 0$ *there exists a Hilbert space,* $H_{t\mathcal{G}}$ *(see (9.10)),*

$$\mathcal{C}^\infty(X) \subset H_{t\mathcal{G}}(X) \subset \mathcal{D}'(X),$$

such that $(P - z)^{-1} : H_{t\mathcal{G}} \to H_{t\mathcal{G}}$ *is meromorphic in the half-space* $\{Im\, z > -th\}$.

Then for any small $\epsilon_0, \delta > 0$, *there exist* $h_0, c_0 > 0$ *and* $C_1 > 0$ *such that, taking any* $t > \lambda_0/2$ *and any* $0 < h < h_0$,

$$\|(P-z)^{-1}\|_{H_{t\mathcal{G}}\to H_{t\mathcal{G}}} \leq C_1 h^{-1+c_0\, Im\, z/h}\log(1/h),$$
$$z \in [\delta, \delta^{-1}] - ih[0, \lambda_0/2 - \epsilon_0]. \tag{1.21}$$

The Hilbert space $H_{t\mathcal{G}}$ in the above theorem is not optimal as far as sharp resolvent estimates are concerned[3]. It is obtained by applying a microlocal weight $e^{t\mathcal{G}^w}$ on L^2, with a function $\mathcal{G}(x,\xi)$ vanishing in a fixed neighbourhood of the trapped set. In [46] Tsujii constructed Hilbert spaces B^β leading to resolvent estimates $\|(P-z)^{-1}\|_{B^\beta} \leq C_1 h^{-1}$ in the same region. A similar resolvent estimate could be obtained in our framework, by further modifying $H_{t\mathcal{G}}$ using the "sharp" escape function G presented in Sect. 2 [see the estimate (2.4)].

Under a pinching condition on the Lyapunov exponents, the recent results announced by Faure–Tsujii [25] provide a much more precise description of the spectrum of $P = -ih\Xi$ on $H_{t\mathcal{G}}$: the Ruelle–Pollicott resonances are localized in horizontal strips below the real axis, and the number of resonances in each strip satisfies a Weyl's law asymptotics. That is analogous to the result proved by Dyatlov [19], which was motivated by quasinormal modes for black holes.

Theorems 3 and 4 have applications to the decay of correlations, respectively for the wave equation and for contact Anosov flows. As an example we state a refinement of the decay of correlation result (1.8) of Dolgopyat [16] and Liverani [35].

Corollary 5 *Suppose that $\gamma_t : X \to X$ is a contact Anosov flow on a compact manifold X (see Sect. 9.1 for the definitions) and that λ_0 is given by (1.20).*

Then there exist a sequence of complex numbers, μ_j,

$$0 > Im\, \mu_j \geq Im\, \mu_{j+1},$$

and of distributions $u_{j,k}, v_{j,k} \in \mathscr{D}'(X)$, $0 \leq k \leq K_j$, such that, for any $\epsilon_0 > 0$, there exists $J(\epsilon_0) \in \mathbb{N}$ such that for any $f, g \in \mathcal{C}^\infty(X)$,

$$\int_X f(x)\, \gamma_t^* g(x)\, dx = \int_X f dx \int_X g dx + \sum_{j=1}^{J(\epsilon_0)} \sum_{k=1}^{K_j} t^k e^{-it\mu_j} u_{j,k}(f) v_{j,k}(g) + \mathcal{O}_{f,g}(e^{-t(\lambda_0-\epsilon_0)/2}), \tag{1.22}$$

for $t > 0$. Here dx is the measure on X induced by the contact form and normalized so that $\mathrm{vol}(X) = 1$, and $u(f)$, $u \in \mathcal{D}'(X)$, $f \in \mathcal{C}^\infty(X)$ denotes the distributional pairing.

The exponential mixing estimate (1.22) has been obtained by Tsujii [46, Corollary 1.2] in the more general case of contact Anosov flows of regularity C^r. We restate it here to stress its analogy with resonance expansions in wave scattering, see for instance [45].

[3] We are grateful to Frédéric Faure for this remark.

For information about microlocal structure of the distributions $u_{j,k}$ and $v_{j,k}$ the reader should consult [23]. Here we only mention that (with the standard wave front set of [30])

$$\mathrm{WF}(u_{j,k}) \subset E_s^*, \quad \mathrm{WF}(v_{j,k}) \subset E_u^*,$$

where $E_\bullet^* = \bigcup_{x\in X} E_\bullet^*(x)$, and $E_\bullet^*(x) \subset T_x^*X$ is the annihilator of $\mathbb{R}\Xi_x + E_\bullet(x) \subset T_xX$, $\bullet = u, s$. The spaces $E_\bullet(x)$ appear in the Anosov decomposition of the tangent space (9.2)

2 Outline of the proof of Theorem 2

The proof proceeds via the analysis of the propagator for the operator

$$\widetilde{P}_G \stackrel{\text{def}}{=} e^{-G^w(x,hD)}(P - iW)e^{G^w(x,hD)},$$

where the function $G(x, \xi; h)$ belongs to a certain exotic class of symbols. Our G is closely related to the escape function constructed in [37], it depends on an additional small parameter, $\tilde{h}$, which will be chosen independently of h.

For a large t_0, any fixed $\Gamma > 0$ and $\epsilon > 0$, we can construct G so that, for some constant C_0, the following holds uniformly in $0 < h < h_0$, $0 < \tilde{h} < \tilde{h}_0$:

$$\begin{aligned}&G(\rho) = \mathcal{O}(\log(1/h)), \quad G(\rho) - G(\varphi_{-t_0}(\rho)) \geq -C_0, \quad \rho \in T^*X,\\ &G(\rho) - G(\varphi_{-t_0}(\rho)) \geq 2\Gamma, \ \ \rho \in p^{-1}([-\delta, \delta]), \ \ d(\rho, K^\delta) > (h/\tilde{h})^{\frac{1}{2}}, \ w(\rho) < \epsilon,\end{aligned} \tag{2.1}$$

where $d(\bullet, \bullet)$ is any given distance function in T^*X.

The proof of Theorem 2 is based on the following estimate. For some $\epsilon_1 > 0$, take an operator $A \in \Psi^0(X)$ such that $\mathrm{WF}_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_1))$. We will prove the following norm estimate: for any ϵ_0 and M there exists M_{ϵ_0} and $\tilde{h}_0 > 0$, $h_0 > 0$ such that for any $\tilde{h} < \tilde{h}_0$, $h < h_0$, we have the estimate

$$\begin{gathered}\| \exp(-it\widetilde{P}_G/h)A\|_{L^2(X)\to L^2(X)} \leq e^{-t(\lambda_0-\epsilon_0)/2},\\ \text{uniformly for times } \ M_{\epsilon_0} \log\frac{1}{\tilde{h}} \leq t \leq \max(M, M_{\epsilon_0}) \log\frac{1}{\tilde{h}}.\end{gathered} \tag{2.2}$$

As a result, for $\operatorname{Im} z > -(\lambda_0 - 2\epsilon_0)/2$,

$$(\widetilde{P}_G - z)\frac{i}{h}\int_0^T e^{-it(\widetilde{P}_G - z)/h} A dt = (I - e^{-iT(\widetilde{P}_G - z)/h})A$$
$$= A - \mathcal{O}(e^{-T\epsilon_0})_{L^2 \to L^2}. \tag{2.3}$$

Hence, by taking T large enough and using the ellipticity of $\widetilde{P}_G - z$ away from $p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_1))$, we obtain

$$(\widetilde{P}_G - z)^{-1} = \mathcal{O}(h^{-1})_{L^2 \to L^2}, \quad \operatorname{Im} z > -(\lambda_0 - 2\epsilon_0)/2. \tag{2.4}$$

Since $e^{\pm G^w} = \mathcal{O}(h^{-M_0})_{L^2 \to L^2}$ from the growth condition on G, a polynomial bound for $(P - iW - z)^{-1}$ follows. The more precise bound (1.18) follows from a semiclassical maximum principle.

To prove the estimate (2.2) we proceed in a number of steps:

Step 1. The most delicate part of the argument concerns the evolution near the trapped set. For some fixed $R > 1$, we introduce a cut-off function $\chi \in \widetilde{S}_{\frac{1}{2}}$ supported in the set

$$\{\rho \in p^{-1}((-\delta, \delta)) \; : \; d(\rho, K^\delta) \leq 2R(h/\tilde{h})^{\frac{1}{2}}\}.$$

This cut-off is quantized into an operator $\chi^w \stackrel{\text{def}}{=} \chi^w(x, hD)$.

We then claim that for any $\epsilon_0 > 0$ and $M > 0$, there exists $C > 0$ such that, for $\tilde{h} < \tilde{h}_0$ and $h < h_0(\tilde{h})$,

$$\|\chi^w e^{-itP/h} \chi^w\|_{L^2 \to L^2} \leq C\tilde{h}^{-d_\perp/2} e^{-t(\lambda_0 - \epsilon_0/2)/2},$$
$$\text{uniformly for } 0 \leq t \leq M \log \frac{1}{\tilde{h}}. \tag{2.5}$$

The proof of this bound is provided in Sect. 5.

Step 2. For the weighted operator we obtain an improved estimate, now with a fixed large time t_0 related to the construction of G, and for χ which in addition satisfies

$$\chi(\rho) = 1 \;\text{ for } d(\rho, K^\delta) \leq R(h/\tilde{h})^{\frac{1}{2}}, |p(\rho)| \leq \delta/2.$$

Using Egorov's theorem and (from (2.1)) the positivity of $G - G \circ \varphi_{-t_0}$ on the set $\operatorname{supp}(1 - \chi) \cap \mathrm{WF}_h(A)$, we get following the weighted estimate:

$$\|(1 - \chi^w)e^{-it_0 \widetilde{P}_G/h} A\| \leq e^{-\Gamma}, \tag{2.6}$$

When constructing the function G it is essential to choose Γ such that

$$\Gamma > \frac{t_0\lambda_0}{2}.$$

We also show that

$$\|e^{-it_0\widetilde{P}_G/h}A\| \leq e^{2C_0}, \tag{2.7}$$

for a constant C_0 independent of $h, \tilde{h}$. Formally, these results follow from Egorov's theorem but care is needed as G is a symbol in an exotic class. To obtain (2.6) and (2.7) we proceed as in the proof of [37, Proposition 3.11]. This is done in Sect. 6.

Step 3. The last step combines the two previous estimates, by decomposing

$$e^{-int_0\widetilde{P}_G/h} = (U_{G,+} + U_{G,-})^n,$$
$$U_{G,+} \stackrel{\text{def}}{=} e^{-it_0\widetilde{P}_G/h}\chi^w, \quad U_{G,-} \stackrel{\text{def}}{=} e^{-it_0\widetilde{P}_G/h}(1-\chi^w).$$

In order to apply (2.5) we use the fact that

$$\chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w}\chi^w = \chi^w_{G,1} e^{-itP/h}\chi^w_{G,2} + \mathcal{O}(\tilde{h}^\infty) + \mathcal{O}(h^{\frac{1}{2}}),$$

where the symbols $\chi_{G,i}$ have the properties required in (2.5). A clever expansion of $(e^{int_0\widetilde{P}_G/h})^n$ into terms involving $U_{G,\pm}$ and an application of Steps 1 and 2 lead to the estimate (2.2) for $t = nt_0$. The argument is presented in Sect. 7.

3 Preliminaries

In this section we will briefly recall basic concepts of semiclassical quantization on manifolds with detailed references to previous papers.

3.1 Semiclassical quantization

The semiclassical pseudodifferential operators on a compact manifold X are quantizations of functions belonging to the symbol classes S^m modeled on symbol classes for $\mathbb{R}^n$:

$$S^m(T^*\mathbb{R}^n) = \Big\{a \in \mathcal{C}^\infty(T^*\mathbb{R}^n \times (0,1]_h) :$$
$$\forall \alpha, \beta \in \mathbb{N}^n, |\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\beta|}\Big\},$$

see [55, §14.2.3]. The Weyl quantization, which we informally write as

$$S^m(T^*X) \ni a(x,\xi) \longmapsto a^w(x,hD) \in \Psi^m(X),$$

maps symbols to pseudodifferential operators. It is modeled on the quantization on $\mathbb{R}^n$:

$$\begin{aligned}[a^w u](x) &= a^w(x,hD)u(x) = [\mathrm{Op}_h^w(a)u](x) \\ &\stackrel{\text{def}}{=} \frac{1}{(2\pi h)^d}\int\int a\Big(\frac{x+y}{2},\xi\Big)e^{i\langle x-y,\xi\rangle/h}u(y)dyd\xi, \quad u\in\mathscr{S}(\mathbb{R}^n). \end{aligned} \tag{3.1}$$

The symbol map

$$\sigma : \Psi^m(X) \to S^m(T^*X)/hS^{m-1}(T^*X),$$

is well defined as an equivalence class and its kernel is $h\Psi^{m-1}(X)$—see [55, Theorem 14.3]. If $\sigma(A)$ has a representative independent of h we call that invariantly defined element of $S^m(T^*X)$ the *principal symbol* of A.

Following [12] we define the class of *compactly microlocalized operators*

$$\Psi^{\mathrm{comp}}(X) \stackrel{\text{def}}{=} \{a^w(x,hD) : a\in(S^0\cap\mathcal{C}_{\mathrm{c}}^\infty)(T^*X)\} + h^\infty\Psi^{-\infty}(X).$$

These operators have well defined semiclassical wave front sets:

$$\Psi^{\mathrm{comp}}(X) \ni A \longmapsto \mathrm{WF}_h(A) \Subset T^*X,$$

see [12, §3.1] and [55, §8.4].

Let $u=u(h)$, $\|u(h)\|_{L^2}=\mathcal{O}(h^{-N})$ (for some fixed N) be a wavefunction microlocalized in a compact set in T^*X, in the sense that for some $A\in\Psi^{\mathrm{comp}}$, one has $u = Au + \mathcal{O}_{\mathcal{C}^\infty}(h^\infty)$. The semiclassical wavefront set of u is then defined as:

$$\mathrm{WF}_h(u) = \complement\big\{\rho\in T^*X : \exists\, a\in S^0(T^*X),\ a(x,\xi)=1,\ \|a^w u\|_{L^2}=\mathcal{O}(h^\infty)\big\}. \tag{3.2}$$

When $A\in\Psi^{\mathrm{comp}}(X)$ we also define

$$\mathrm{WF}_h(I-A) := \bigcup_{B\in\Psi^{\mathrm{comp}}(X)} \mathrm{WF}_h(B(I-A)),$$

and note that $\mathrm{WF}_h(B) \cap \mathrm{WF}_h(A)$ is defined for any $B \in \Psi^m(X)$ as

$$\mathrm{WF}_h(B) \cap \mathrm{WF}_h(A) := \mathrm{WF}_h(CB) \cap \mathrm{WF}_h(A), \quad C \in \Psi^{\mathrm{comp}},$$
$$\mathrm{WF}_h(I - C) \cap \mathrm{WF}_h(A) = \emptyset.$$

Semiclassical Sobolev spaces, $H_h^s(X)$ are defined using the norms

$$\|u\|_{H_h^s(X)} = \|(I - h^2\Delta_g)^{s/2}u\|_{L^2(X)}, \tag{3.3}$$

for some choice of Riemannian metric g on X (notice that $H_h^s(X)$ represents the same vector space as the usual Sobolev space $H^s(X)$).

3.2 $S_{\frac{1}{2}}$ calculus with two parameter

Another standard space of symbols $S_\delta(\mathbb{R}^{2n})$, $0 < \delta \leq 1/2$, is defined by demanding that $\partial^\alpha a = \mathcal{O}(h^{-|\alpha|\delta})$. The quantization procedure $a \mapsto \mathrm{Op}_h^w a$ gives well defined operators and $\mathrm{Op}_h^w a \circ \mathrm{Op}_h^w b = \mathrm{Op}_h^w c$ with $c \in S_\delta$.

For $0 < \delta < 1/2$ we still have a pseudodifferential calculus, with asymptotic expansions in powers of h. However, for $\delta = 1/2$ we are at the border of the uncertaintly principle, and there is no asymptotic calculus—see [55, §4.4.1]. To obtain an asymptotic calculus the standard $S_{\frac{1}{2}}$ spaces is replaced by a symbol space where a second asymptotic parameter is introduced:

$$\widetilde{S}_{\frac{1}{2}}(\mathbb{R}^{2n}) \stackrel{\mathrm{def}}{=} \Big\{ a = a(\rho, h, \tilde{h}) \in \mathcal{C}^\infty(\mathbb{R}^{2n}_\rho \times (0,1]_h$$
$$\times (0,1]_{\tilde{h}}) : |\partial_\rho^\alpha a| \leq C_\alpha (h/\tilde{h})^{-|\alpha|/2} \Big\}.$$

Then the quantization $a \mapsto a^w(x, hD) \in \widetilde{\Psi}_{\frac{1}{2}}(\mathbb{R}^n)$ is unitarily equivalent to

$$\tilde{a} \mapsto \tilde{a}^w(\tilde{x}, \tilde{h}D) = \mathrm{Op}_{\tilde{h}}^w(\widetilde{a}), \quad \tilde{a}(\rho) = a((h/\tilde{h})^{\frac{1}{2}}\rho). \quad \tilde{a} \in S(\mathbb{R}^{2n}), \tag{3.4}$$

—see [55, §§4.1.1,4.7.2]. Hence, we now have expansions in powers of $\tilde{h}$, as in the standard calculus, with better properties (powers of $(h\tilde{h})^{\frac{1}{2}}$) when operators in $\widetilde{\Psi}_{\frac{1}{2}}$ and Ψ are composed—see [43, Lemma 3.6].

For the case of manifolds we refer to [12, §5.1] which generalizes and clarifies the presentations in [43, §3.3] and [54, §3.2]. The basic space of symbols, and the only one needed here, is

$$\widetilde{S}^{\mathrm{comp}}_{\frac{1}{2}}(T^*X) = \Big\{ a \in \mathcal{C}^\infty_{\mathrm{c}}(T^*X) : V_1 \cdots V_k a = \mathcal{O}((h/\tilde{h})^{-\frac{k}{2}}), \ \ \forall\, k, \\ V_j \in \mathcal{C}^\infty(T^*X, T(T^*X)) \Big\} + h^\infty S^{-\infty}(T^*X).$$

The quantization procedure

$$\widetilde{S}^{\mathrm{comp}}_{\frac{1}{2}}(T^*X) \ni a \to \mathrm{Op}^w_h(a) \in \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X)$$

defines the class of operators $\widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X)$ modulo $h^\infty \Psi^{-\infty}(X)$, and the symbol map:

$$\widetilde{\sigma} : \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X) \longrightarrow \widetilde{S}^{\mathrm{comp}}_{\frac{1}{2}}(T^*X)/h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}\widetilde{S}^{\mathrm{comp}}_{\frac{1}{2}}(T^*X). \tag{3.5}$$

The properties of the resulting calculus are listed in [12, Lemma 5.1] and we will refer to those results later on.

When $\tilde{h} = 1$ we use the notation $S^{\mathrm{comp}}_{\frac{1}{2}}(T^*X)$ for symbols and denote by $\Psi^{\mathrm{comp}}_{\frac{1}{2}}(X)$ the corresponding class of pseudodifferential operators. The symbol map

$$\sigma : \Psi^{\mathrm{comp}}_{\frac{1}{2}}(X) \longrightarrow S^{\mathrm{comp}}_{\frac{1}{2}}(T^*X)/h^{\frac{1}{2}} S^{\mathrm{comp}}_{\frac{1}{2}}(T^*X),$$

is still well defined but the operators in this class do not enjoy a proper symbol calculus in the sense that $\sigma(AB)$ cannot be related to $\sigma(A)\sigma(B)$. However, when $A \in \Psi^{\mathrm{comp}}_{\frac{1}{2}}(X)$ and $B \in \Psi(X)$ then $\sigma(AB) = \sigma(A)\sigma(B) + \mathcal{O}(h^{\frac{1}{2}})_{S_{\frac{1}{2}}(T^*X)}$—see [43, Lemma 3.6] or [12, Lemma 5.1].

3.3 Fourier integral operators

In this paper we will consider Fourier integral operators associated to canonical transformations. It will also be sufficient to consider operators which are compactly microlocalized as we will always work near $p^{-1}([-2\delta, 2\delta]) \cap w^{-1}(0)$ which by assumption (1.9) is a compact subset of T^*X.

Suppose that Y_1, Y_2 are two compact smooth manifolds ($Y_j = X$ or $Y_j = \mathbb{T}^n$ in what follows) and that, $U_j \subset T^*Y_j$ are open subsets. Let

$$\kappa : U_1 \to U_2, \\ \Gamma'_\kappa \stackrel{\mathrm{def}}{=} \{(x, \xi, y, -\eta) : (x, \xi) = \kappa(y, \eta), (y, \eta) \in U_1\} \subset T^*Y_2 \times T^*Y_1,$$

be a symplectic transformation, for instance $\kappa = \varphi_t$, $U_1 = U_2 = T^*X$. Here Γ_κ is the graph of κ and $'$ denotes the twisting $\eta \mapsto -\eta$. This follows the standard convention [30, Chapter 25].

Following [12, §5.2] we introduce the class of *compactly microlocalized h-Fourier integral operator quantizing* κ, $I_h^{\mathrm{comp}}(Y_2 \times Y_1, \Gamma'_\kappa)$. If $T \in I_h^{\mathrm{comp}}(Y_2 \times Y_1, \Gamma'_\kappa)$ then it has the following properties: $T = \mathcal{O}(1)_{L^2(Y_1)\to L^2(Y_2)}$; there exist $A_j \in \Psi^{\mathrm{comp}}(Y_j)$, $\mathrm{WF}_h(A_j) \Subset U_j$ such that

$$A_2 T = T + \mathcal{O}(h^\infty)_{\mathcal{D}'(Y_1)\to\mathcal{C}^\infty(Y_2)}, \quad T A_1 = T + \mathcal{O}(h^\infty)_{\mathcal{D}'(Y_1)\to\mathcal{C}^\infty(Y_2)};$$

for any $B_j \in \Psi^m(Y_j)$,

$$\begin{aligned} &T B_1 = C_1 T + h T_1, \quad \sigma(C_1) = \sigma(B_1)\circ\kappa^{-1}, \\ &B_2 T = T C_2 + h T_2, \quad \sigma(C_2) = \sigma(B_2)\circ\kappa, \quad T_j \in I_h^{\mathrm{comp}}(Y_2 \times Y_1, \Gamma'_\kappa). \end{aligned} \tag{3.6}$$

The last statement is a form of Egorov theorem.

When $B_j \in \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X)$ then an analogue of (3.6) still holds in a modified form

$$\begin{aligned} T B_1 &= C_1 T + h^{\frac12}\tilde{h}^{\frac12} D_1 T_1, \quad \sigma(C_1) = \sigma(B_1)\circ\kappa^{-1}, \\ B_2 T &= T C_2 + h^{\frac12}\tilde{h}^{\frac12} T_2 D_2, \quad \sigma(C_2) = \sigma(B_2)\circ\kappa, \\ T_j &\in I_h^{\mathrm{comp}}(Y_2\times Y_1, \Gamma'_\kappa), \quad C_j, D_j \in \widetilde{\Psi}^{\mathrm{comp}}_{\frac12}(X), \end{aligned} \tag{3.7}$$

see Proposition 6.3 (applied with $g \equiv 0$).

An example is given by the operators

$$A\,e^{-itP/h}, \quad e^{-itP/h}A \in I^{\mathrm{comp}}(X\times X, \Gamma'_{\varphi_t}), \quad \text{if} A \in \Psi^{\mathrm{comp}}(X). \tag{3.8}$$

In Sect. 5 we will also need a local representation of elements of I^{comp} as oscillatory integrals—see [1],[22, §3.2] and references given there. If $T \in I^{\mathrm{comp}}(\mathbb{R}^n \times \mathbb{R}^n, \Gamma'_\kappa)$ is microlocalized to a sufficiently small neighbourhood $\kappa(U)\times U \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n$ ([55, 8.4.5]) then

$$Tu(x) = (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta) u(y)\,dy\,d\theta + \mathcal{O}(h^\infty)_{\mathcal{S}} \|u\|_{H^{-M}}, \tag{3.9}$$

for any M. Here $a \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n}\times\mathbb{R}^k)$, $\psi \in \mathcal{C}^\infty(\mathbb{R}^{2n}\times\mathbb{R}^k)$, and near $\kappa(U)\times U$, the graph of κ is given by

$$\Gamma_\kappa = \{((x, d_x\psi(x,y,\theta)), (y, -d_y\psi(x,y,\theta)) \;:\; (x,y,\theta) \in C_\psi\},$$
$$C_\psi \stackrel{\text{def}}{=} \{(x,y,\theta) \;:\; d_\theta\psi(x,y,\theta) = 0, \},$$
$$d_{x,y,\theta}(\partial_{\theta_j}\psi), \quad j = 1, \dots, k, \quad \text{are linearly independent,} \tag{3.10}$$

For given symplectic coordinates (x, ξ) and (y, η) in neighbourhoods of $\kappa(U)$ and U respectively, such a representation exists with an extra variable of dimension k, where $0 \le k \le n$, and $n + k$ is equal to the rank of the projection

$$\Gamma_\kappa \ni ((x,\xi),(y,\eta)) \longmapsto (x,\eta),$$

assumed to be constant in the neighbourhood of $\kappa(U) \times U$—see for instance [55, Theorem 2.14]. Since Γ_κ in (3.10) is the graph of a symplectomorphism it follows that for some $y' = (y_{j_1}, \dots, y_{j_{n-k}}) \in \mathbb{R}^{n-k}$,

$$D_\psi(x,y,\theta) \stackrel{\text{def}}{=} \det\left(\frac{\partial^2\psi}{\partial x_i \partial y'_{j'}}, \frac{\partial^2\psi}{\partial x_i \partial\theta_j}\right) \neq 0. \tag{3.11}$$

For the use in Sect. 5 we record the following lemma, proved using standard arguments (see for instance [1]):

Lemma 3.1 *Suppose that T is given by (3.9) and that $B \in \widetilde{\Psi}_{\frac{1}{2}}(\mathbb{R}^n)$. Then for any $u \in L^2$ with $\|u\|_{L^2} = 1$,*

$$\begin{aligned} BTu(x) &= (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta) \\ &\qquad \times b(x, d_x\psi(x,y,\theta)) u(y)\, dy\, d\theta + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}})_{L^2}, \\ TBu(x) &= (2\pi h)^{-\frac{n+k}{2}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta) \\ &\qquad \times b(y, -d_y\psi(x,y,\theta)) u(y)\, dy\, d\theta + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}})_{L^2}, \end{aligned}$$

where $b = \sigma(B)$.

3.4 Fourier integral operators with operator valued symbols

In Sect. 5 we will also use a class of Fourier integral operators with *operator valued* symbols. We present what we need in an abstract form in this section. Only local aspects of the theory will be relevant to us and we opt for a direct presentation.

Suppose that $\mathcal{H}$ is a separable Hilbert space and Q is an (unbounded) self-adjoint operator with domain $\mathcal{D} \subset \mathcal{H}$. We assume that $Q : \mathcal{D} \to \mathcal{H}$ is invertible and we put $\mathcal{D}^\ell \stackrel{\text{def}}{=} Q^{-\ell}\mathcal{H}$, for $\ell \geq 0$. For $\ell < 0$, we define $\mathcal{D}^\ell$ as the completion of $\mathcal{H}$ with respect to the norm $\|Q^\ell u\|_{\mathcal{H}}$.

We define the following class of operator valued symbols:

$$\mathcal{S}_\delta(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathcal{H}, \mathcal{D}), \tag{3.12}$$

to consist of operator valued functions

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^k \ni (x, y, \theta) \longmapsto N(x, y, \theta) : \mathcal{D}^\infty \longrightarrow \mathcal{H}$$

which satisfy the following estimates:

$$\partial^\alpha_{x,y,\theta} N(x, y, \theta) = \mathcal{O}_{\alpha,\ell}(1) : \mathcal{D}^{\ell+\delta|\alpha|} \longrightarrow \mathcal{D}^\ell, \tag{3.13}$$

for any multiindex α and $\ell \in \mathbb{Z}$, uniformly in (x, y, θ). We note that this class is closed under pointwise composition of the operators: if $N_j \in \mathcal{S}_\delta$ then N_j defines a family of operators $\mathcal{D}^\ell \to \mathcal{D}^\ell$, hence so does their product $N_1 N_2$; the estimate (3.13) follows for the composition, since for $|\beta| + |\gamma| = |\alpha|$,

$$\partial^\beta N_1 \partial^\gamma N_2 = \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta|\alpha|} \to \mathcal{D}^{\ell+\delta(|\alpha|-|\beta|)}} \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta(|\alpha|-|\beta|)} \to \mathcal{D}^\ell} = \mathcal{O}(1)_{\mathcal{D}^{\ell+\delta|\alpha|} \longrightarrow \mathcal{D}^\ell}.$$

Proposition 3.5 at the end of this section describes a class which will be used in Sect. 5.

Suppose that ψ satisfies (3.10) and (3.11). We can assume that ψ is defined on $\mathbb{R}^{2n} \times \mathbb{R}^k$. For $N \in \mathcal{S}_\delta$ and $a \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^{2n} \times \mathbb{R}^k)$ we define the operator

$$T : L^2(\mathbb{R}^n) \otimes \mathcal{H} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{H}, \quad L^2(\mathbb{R}^n) \otimes \mathcal{H} \simeq L^2(\mathbb{R}^n, \mathcal{H}),$$

(the second identification is valid as $\mathcal{H}$ is separable [40, Theorem II.10] but it is convenient in definitions to use the tensor product notation) by

$$T(u \otimes v) \stackrel{\text{def}}{=} (2\pi h)^{-\frac{n+k}{2}} \int\limits_{\mathbb{R}^k}\int\limits_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x, y, \theta)\, (u(y) \otimes N(x, y, \theta)v)\, dy d\theta. \tag{3.14}$$

This operator is well-defined since a is compactly supported, but to obtain a norm estimate which is uniform in h we need to assume that $N \in \mathcal{S}_0$:

Lemma 3.2 *Suppose that $N \in \mathcal{S}_0(\mathbb{R}^{2n+k}, \mathcal{H}, \mathcal{D})$ and that T is given by (3.14). Then*

$$\|T\|_{L^2(\mathbb{R}^n)\otimes\mathcal{H} \to L^2(\mathbb{R}^n)\otimes\mathcal{H}} = \max_{C_\psi} \frac{|a| \|N\|_{\mathcal{H}\to\mathcal{H}}}{\sqrt{|D_\psi|}} + \mathcal{O}(h), \tag{3.15}$$

where $C_\psi \stackrel{\text{def}}{=} \{(x, y, \theta) : \partial_\theta \psi = 0\}$, *and* D_ψ *is given by* (3.11).

If $N \in \mathcal{S}_\delta(\mathbb{R}^{2n+k}, \mathcal{H}, \mathcal{D})$ *then*

$$T = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\delta m_n + \ell} \longrightarrow L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell, \tag{3.16}$$

where m_n *depends only on the dimension* n.

Proof The estimate (3.15) follows from a standard argument based on considering T^*T and from [55, Theorem 13.13]. The estimates (3.13) with $\delta = 0$ and $\ell = 0$ show that the operators can be treated just as scalar symbols.

To obtain (3.16) we note that

$$\partial^\alpha_{x,y,\theta}\left(Q^{-L}N(x, y, \theta)\right) = \mathcal{O}(1) : \mathcal{H} \to \mathcal{H}, \quad \text{for } |\alpha|\delta \le L.$$

To obtain the norm estimate (3.15) we only need a finite number of derivatives, M_n, depending only on the dimension. Taking $m_n\delta \ge L \ge M_n\delta$, we can then apply (3.15) to the operator $Q^{-L}T$, which gives the bound (3.16) for T. □

A special case of is given by $\kappa = id$. In that case we deal with pseudodifferential operators with operator valued symbols. The following lemma summarizes their basic properties:

Lemma 3.3 *Suppose that* $N_j \in \mathcal{S}_{\delta_j}(\mathbb{R}^{2n})$, $j = 1, 2$. *For* $u \in \mathscr{S}(\mathbb{R}^n)$ *and* $v \in \mathcal{D}^\infty$ *we define*

$$\mathrm{Op}^w_h(N_j)(u \otimes v) \stackrel{\text{def}}{=} \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} \left[N_j\left(\frac{x+y}{2}, \xi\right) v\right] u(y) dy d\xi.$$

These operators extend to

$$\mathrm{Op}^w_h(N_j) = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\ell + m_n\delta_j} \to L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell, \tag{3.17}$$

and satisfy the following product formula:

$$\begin{gathered} \mathrm{Op}^w_h(N_1)\mathrm{Op}^w_h(N_2) = \mathrm{Op}^w_h(N_1N_2) + hR, \\ R = \mathcal{O}(1) : L^2(\mathbb{R}^n) \otimes \mathcal{D}^{\ell + m_n(\delta_1+\delta_2)} \to L^2(\mathbb{R}^n) \otimes \mathcal{D}^\ell. \end{gathered} \tag{3.18}$$

Here and in (3.17), ℓ *is arbitrary and* m_n *depends only on the dimension* n.

Proof When $\delta_1 = \delta_2 = 0$ the proof is an immediate vector valued adaptation of the standard arguments presented in [55, §§4.4,4.5] where we note that only a finite number (depending on the dimension) of seminorms of symbols is needed. In general, (3.13) gives

$$\partial^\alpha_{x,\xi} Q^{-L} N_j Q^{-M} = \mathcal{O}(1) : \mathcal{D}^\ell \to \mathcal{D}^\ell, \ \ |\alpha|\delta_j \le L + M, \tag{3.19}$$

and the norm estimates (3.17) follows. To obtain the product formula we note that, using (3.19), it applies to $Q^{-M}N_1$ and $N_2 Q^{-M}$ for M sufficiently large depending on n. Hence

$$
\begin{aligned}
\mathrm{Op}_h^w(N_1)\mathrm{Op}_h^w(N_2) &= Q^M \mathrm{Op}_h^w(Q^{-M}N_1)\mathrm{Op}_h^w(N_2 Q^{-M})Q^M \\
&= Q^M \mathrm{Op}_h^w(Q^{-M}N_1 N_2 Q^{-M})Q^M \\
&\quad + Q^M \mathcal{O}(h)_{L^2\otimes\mathcal{D}^p \to L^2\otimes\mathcal{D}^p} Q^M \\
&= \mathrm{Op}_h^w(N_1N_2) + \mathcal{O}(h)_{L^2\otimes\mathcal{D}^{p+M}\to L^2\otimes\mathcal{D}^{p-M}},
\end{aligned}
$$

which gives (3.18) provided $m_n(\delta_1+\delta_2) \geq 2M$. □

We can also factorize the operator T using the pseudodifferential operators described in Lemma 3.3, the proof being an adaptation of the standard argument. When $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ we also write S for $S \otimes I_{\mathcal{H}} : L^2(\mathbb{R}^n)\otimes\mathcal{H} \to L^2(\mathbb{R}^n)\otimes\mathcal{H}$.

Lemma 3.4 *Suppose that T is given by (3.14) with $N \in \mathcal{S}_\delta$. Then*

$$
T = T^{\|}\mathrm{Op}_h^w(N_1) + hR_1 = \mathrm{Op}_h^w(N_2)T^{\|} + hR_2,
$$

where

$$
\begin{aligned}
& T^{\|} \in I^{\mathrm{comp}}(\mathbb{R}^n\times\mathbb{R}^n, \Gamma'_\kappa), \\
& T^{\|}u(x) = (2\pi h)^{-\frac{k+n}{2}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^n} e^{\frac{i}{h}\psi(x,y,\theta)} a(x,y,\theta)u(y)dyd\theta, \\
& N_2(x, d_x\psi(x,y,\theta)) = N_1(y, -d_y\psi(x,y,\theta)) = N(x,y,\theta), \\
& \quad (x,y,\theta)\in C_\psi \\
& R_j = \mathcal{O}(1) : L^2(\mathbb{R}^n)\otimes\mathcal{D}^{\delta m_n+\ell} \longrightarrow L^2(\mathbb{R}^n)\otimes\mathcal{D}^\ell.
\end{aligned}
\tag{3.20}
$$

Here, $N_j \in \mathcal{S}_\delta(\mathbb{R}^n\times\mathbb{R}^n, \mathcal{H}, \mathcal{D})$, and

$$
\mathrm{Op}_h^w(N_j) = \mathcal{O}(1) : L^2(\mathbb{R}^n)\otimes\mathcal{D}^{\delta m_n+\ell} \longrightarrow L^2(\mathbb{R}^n)\otimes\mathcal{D}^\ell.
$$

In our applications we will have

$$
\mathcal{H} = L^2(\mathbb{R}^{d_\perp}, d\tilde{y}), \quad Q = -\tilde{h}^2\Delta_{\tilde{y}} + \tilde{y}^2 + 1, \tag{3.21}
$$

so that $\mathcal{D}^\ell$ are analogous to Sobolev spaces (see [55, §8.3]). In the rest of this section (as well in Sect. 5), we will use the shorthand notations $\rho_\| = (x,y,\theta)$ in order to shorten the expressions, and to differentiate between these variables and the "transversal variables" $(\tilde{y}, \tilde{\eta})$.

We consider a specific class of metaplectic operators:

$$N(\rho_\parallel)u(\tilde{y}) = (2\pi\tilde{h})^{-d_\perp} \int\limits_{\mathbb{R}^{d_\perp}} \int\limits_{\mathbb{R}^{d_\perp}} (\det \partial^2_{\tilde{y},\tilde{\eta}} q_{\rho_\parallel})^{\frac{1}{2}} e^{\frac{i}{\tilde{h}}(q_{\rho_\parallel}(\tilde{y},\tilde{\eta}) - \langle \tilde{\eta}, \tilde{y}' \rangle)} u(\tilde{y}') d\tilde{y}', \tag{3.22}$$

where $q_{\rho_\parallel}(\tilde{y}, \tilde{\eta})$ is a real quadratic form in the variables $\tilde{y}$, $\tilde{\eta}$, with coefficients depending on $\rho_\parallel$, being in the class $S(\mathbb{R}^{2n+k})$, and the matrix of coefficients $\partial^2_{\tilde{y},\tilde{\eta}} q_{\rho_\parallel}$ is assumed to be uniformly non-degenerate for all $\rho_\parallel$. The definition involves a *choice* of the branch of the square root—see Remark 5.8 for further discussion of that. For any fixed $\rho_\parallel$ these operators are unitary on $\mathcal{H}$ (see for instance [55, Theorem 11.10]).

The next proposition shows that this class fits nicely into our framework:

Proposition 3.5 *The operators $N(\rho_\parallel)$ given by (3.22) satisfy*

$$\partial^\alpha_{\rho_\parallel} N(\rho_\parallel) = \mathcal{O}_{\alpha,\ell}(\tilde{h}^{-|\alpha|}) : \mathcal{D}^{|\alpha|+\ell} \longrightarrow \mathcal{D}^\ell, \tag{3.23}$$

for all ℓ, That means that (3.13) holds with $\delta = 1$ (the loss in $\tilde{h}$ is considered as dependence on α).

If $\tilde{\chi} \in \mathscr{S}(\mathbb{R}^{2d_\perp})$ is fixed, $\Lambda > 1$, and $\tilde{\chi}_\Lambda(\bullet) \stackrel{\text{def}}{=} \tilde{\chi}(\Lambda^{-1}\bullet)$, then for any ℓ and $k \geq 0$,

$$\tilde{\chi}^w_\Lambda(\tilde{y}, \tilde{h}D_{\tilde{y}}) = \mathcal{O}_\ell(\Lambda^{2k}) : \mathcal{D}^\ell \to \mathcal{D}^{\ell+k}, \tag{3.24}$$

so that

$$\begin{aligned} \partial^\alpha_{\rho_\parallel} \left(\tilde{\chi}^w_\Lambda(\tilde{y}, \tilde{h}D_{\tilde{y}}) N(\rho_\parallel) \right) &= \mathcal{O}_{\alpha,\ell}(\Lambda^{2|\alpha|}\tilde{h}^{-|\alpha|}) : \mathcal{D}^\ell \longrightarrow \mathcal{D}^\ell, \\ \partial^\alpha_{\rho_\parallel} \left(N(\rho_\parallel) \tilde{\chi}^w_\Lambda(\tilde{y}, \tilde{h}D_{\tilde{y}}) \right) &= \mathcal{O}_{\alpha,\ell}(\Lambda^{2|\alpha|}\tilde{h}^{-|\alpha|}) : \mathcal{D}^\ell \longrightarrow \mathcal{D}^\ell. \end{aligned} \tag{3.25}$$

Proof We see that $\partial^\alpha_{\rho_\parallel} N(\rho_\parallel)$ is an operator of the same form as (3.22) but with the amplitude multiplied by

$$\sum_{|\beta| \leq 2|\alpha|} \tilde{h}^{-m_\beta} \tilde{y}^{\beta_1} (\tilde{y}')^{\beta_2} \tilde{\eta}^{\beta_3} q_\beta(\rho_\parallel),\, q_\beta \in S(\mathbb{R}^{2n+k}),$$
$$\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^{3d_\perp},\, \beta_j \in \mathbb{N}^{d_\perp},$$

where $m_\beta \leq |\alpha|$. Hence to obtain (3.23), it is enough to prove that

$$Q^\ell \tilde{y}^{\beta_1} N(\rho_\parallel) \left((\tilde{y}')^{\beta_2} (\tilde{h}D_{\tilde{y}'})^{\beta_3} Q^{-\ell-|\alpha|} v(\tilde{y}') \right) = \mathcal{O}(\|v\|_\mathcal{H})_\mathcal{H}.$$

Using the exact Egorov's theorem for metaplectic operators (see for instance [55, Theorem 11.9]) we see that the left hand side is equal to

$$N(\rho_{\|})\left(p_\beta^w(\tilde{y}', \tilde{h} D_{\tilde{y}'})\left(K_q^* Q\right)^\ell Q^{-\ell-|\alpha|} v(\tilde{y}')\right), \quad K_q : (\partial_{\tilde{\eta}} q, \tilde{\eta}) \mapsto (\tilde{y}, \partial_{\tilde{y}} q),$$

$q = q_{\rho_{\|}}$ and where p_β is a polynomial of degree less than or equal to $|\beta|$. Since $|\beta| \leq 2|\alpha|$, the operator $p_\beta^w (K_q^* Q)^\ell Q^{-\ell-|\alpha|}$ is bounded on $\mathcal{H}$ (see for instance [55, Theorem 8.10]) so the unitarity of N gives the boundedness in $\mathcal{H}$.

To obtain (3.24) we first note that $\widetilde{\chi}_\Lambda \in S(\mathbb{R}^{2d_\perp})$ uniformly in $\Lambda > 1$. Hence $Q^{-\ell} \widetilde{\chi}_\Lambda^w Q^\ell = \mathcal{O}(1)_{\mathcal{H} \to \mathcal{H}}$, uniformly in Λ (again, see [55, Theorem 8.10]). This gives (3.24) for $k = 0$. For the general case we put $Q_\Lambda = 1 + \Lambda^{-2}((\tilde{h} D_{\tilde{y}})^2 + \tilde{y}^2)$, and note that for any M, $Q_\Lambda^M \widetilde{\chi}_\Lambda^w = \widetilde{\chi}_{\Lambda,M}^w$, where $\widetilde{\chi}_{\Lambda,M} \in S(\mathbb{R}^{2d_\perp})$ uniformly in Λ. Hence it is bounded on $L^2(\mathbb{R}^{d_\perp})$ uniformly in Λ and $\tilde{h}$. We then write

$$\begin{aligned}
&Q^k \widetilde{\chi}_\Lambda^w(\tilde{y}, \tilde{h} D_{\tilde{y}}) = Q^k Q_\Lambda^{-k} Q_\Lambda^k \widetilde{\chi}_\Lambda^w(\tilde{y}, \tilde{h} D_{\tilde{y}}) \\
&\quad = (1 + (\tilde{h} D_{\tilde{y}})^2 + \tilde{y}^2)^k (1 + \Lambda^{-2}(\tilde{h} D_{\tilde{y}})^2 + \Lambda^{-2} \tilde{y}^2)^{-k} \widetilde{\chi}_{\Lambda,k}^w(\tilde{y}, \tilde{h} D_{\tilde{y}}) \\
&\quad = \Lambda^{2k}(1 + (\tilde{h} D_{\tilde{y}})^2 + \tilde{y}^2)^k (\Lambda^2 + (\tilde{h} D_{\tilde{y}})^2 + \tilde{y}^2)^{-k} \widetilde{\chi}_{\Lambda,k}^w(\tilde{y}, \tilde{h} D_{\tilde{y}}) \\
&\quad = \mathcal{O}(\Lambda^{2k})_{L^2(\mathbb{R}^{d_\perp}) \to L^2(\mathbb{R}^{d_\perp})},
\end{aligned}$$

completing the proof of (3.24). □

4 Classical dynamics

In this section we will describe the consequences of the normal hyperbolicity assumption (1.16), (1.17) needed in the proof of Theorem 2.

4.1 Stable and unstable distributions

Let K^δ be the trapped set (1.14) and $E_\rho^\pm \subset T_\rho X$, $\rho \in K^\delta$, the distributions in (1.17). We recall our notation $\varphi_t \stackrel{\text{def}}{=} \exp t H_p$ for the Hamiltonian flow generated by the function $p(x, \xi)$.

We start with a simple

Lemma 4.1 *If ω is the canonical symplectic form on T^*X then*

$$\omega_\rho \restriction_{E_\rho^\pm} = 0, \tag{4.1}$$

that is $E_\rho^\pm$ are isotropic.

Without loss of generality we can assume that the distributions $E^\pm_\rho$ satisfy

$$E^+_\rho \oplus E^-_\rho = (T_\rho K^\delta)^\perp, \tag{4.2}$$

where $V^\perp$ denotes the symplectic orthogonal of V.

Proof The property (4.1) follows from the fact that φ_t preserves the symplectic structure ($\varphi_t^*\omega = \omega$). For $X, Y \in E^\pm_\rho$,

$$\omega_\rho(X, Y) = \omega_{\varphi_{\mp t}(\rho)}((d\varphi_{\mp t})(\rho)X, d\varphi_{\mp t}(\rho)Y) \to 0, \quad t \to +\infty.$$

To see that we can assume (4.2) we note that the distribution $\{(T_\rho K^\delta)^\perp,\ \rho \in K^\delta\}$ is invariant by the flow: $d\varphi_t(\rho) : T_\rho K \to T_{\varphi_t(\rho)}K$, and $d\varphi_t(\rho)$ is a symplectic transformation. If $\pi_\rho : T_\rho(T^*X) \to (T_\rho K^\delta)^\perp$ is the symplectic projection, then $\pi_{d\varphi_t(\rho)} \circ d\varphi_t(\rho) = d\varphi_t(\rho) \circ \pi_\rho$. This means that we may safely replace $E^\pm_\rho$ with $\pi_\rho(E^\pm_\rho)$, without altering the properties (1.17). □

4.2 Construction of the escape function

To construct the escape function near the trapped set we need a lemma concerning invariant cones near K^δ. To define them we introduce a Riemannian metric on T^*X and use the tubular neighbourhood theorem (see for instance [30, Appendix C.5]) to make the identifications

$$\begin{aligned} \mathrm{neigh}(K^\delta) &\simeq N^*K^\delta \cap \{(\rho, \zeta) \in T^*(T^*X) : \|\zeta\|_\rho \le \epsilon_1\} \\ &\simeq (TK^\delta)^\perp \cap \{(\rho, z) \in T(T^*X) : \|z\|_\rho \le \epsilon_1\} \\ &\simeq \{(m, z) : m \in K^\delta,\ z \in \mathbb{R}^{2d_\perp},\ \|z\|_\rho \le \epsilon_1\}. \end{aligned} \tag{4.3}$$

Here $(TK^\delta)^\perp$ denotes the symplectic orthogonal of $TK^\delta \subset T_{K^\delta}(T^*X) \subset T(T^*X)$. Since K^δ is symplectic, the symplectic form identifies $(TK^\delta)^\perp$ with the conormal bundle N^*K^δ. The norm $\|\bullet\|_\rho$ is a smoothly varying norm on $T_\rho(T^*X)$. We write $d_\rho(z, z') = \|z - z'\|_\rho$ and introduce a distance function $d : \mathrm{neigh}(K^\delta) \times \mathrm{neigh}(K^\delta) \to [0, \infty)$ obtained by choosing a Riemannian metric on $\mathrm{neigh}(K^\delta)$. We have $d((m, z), (m, z')) \sim d_m(z, z')$ and the notation $a \sim b$, here and below, means that there exists a constant $C \ge 1$ (independendent of other parameters) such that $b/C \le a \le Cb$.

Assuming that $E_\rho^\pm$ are chosen so that (4.2) holds we can define (closed) cone fields by putting

$$C_\rho^\pm \stackrel{\text{def}}{=} \{z \in (T_\rho K^\delta)^\perp : d_\rho(z, E_\pm^\rho) \leq \epsilon_2 \|z\|_\rho,\ \|z\|_\rho \leq \epsilon_1\},$$
$$C^\pm \stackrel{\text{def}}{=} \bigcup_{\rho \in K^\delta} C_\rho^\pm \subset \text{neigh}(K^\delta), \tag{4.4}$$

where we used the identification (4.3). Since the maps $\rho \mapsto E_\rho^\pm$ are continuous, $C^\pm$ are closed.

The basic properties $C^\pm$ are given in the following

Lemma 4.2 *There exists $t_0 > 0$ and $\epsilon_2^0 > 0$ such that, for every $t > t_0$ there exists ϵ_1^0 such that if one chooses $\epsilon_j < \epsilon_j^0$, $j = 1, 2$ in the definition of* neigh(K^δ) *and $C^\pm$, then*

$$\rho \in C_\pm,\ \varphi_{\pm t}(\rho) \in \text{neigh}(K^\delta) \implies \varphi_{\pm t}(\rho) \in C_\pm. \tag{4.5}$$

In fact a stronger statement is true: for some constant $\lambda_1 > 0$ and any $t \geq t_0$,

$$\rho, \varphi_{\pm t}(\rho) \in \text{neigh}(K^\delta) \implies d(\varphi_{\pm t}(\rho), C^\pm) \leq e^{-\lambda_1 t} d(\rho, C^\pm). \tag{4.6}$$

Finally,

$$d(\rho, C^+)^2 + d(\rho, C^-)^2 \sim d(\rho, K^\delta)^2. \tag{4.7}$$

The conclusions (4.6) and (4.7) are similar to [37, Lemma 4.3] and [42, Lemma 5.2] but the proof does not use foliations by stable and unstable manifolds which seem different under our assumptions.

Proof For $\rho \in$ neigh(K^δ) let (m, z), $m \in K^\delta$ and $z \in \mathbb{R}^{2d_\perp} \simeq (T_m K^\delta)^\perp$ be local coordinates near ρ. Similarly let $(\tilde{m}, \tilde{z})$ be local coordinates near $\varphi_t(\rho) \in$ neigh(K^δ) (by assumption in (4.5)). Then if for each m we put $d_\perp \varphi_t(m) \stackrel{\text{def}}{=} d\varphi_t(m)\restriction_{(T_m K^\delta)^\perp}$, the map φ_t can be written as,

$$\begin{aligned}\varphi_t(m, z) &= \big(\varphi_t(m) + \mathcal{O}_t(\|z\|^2), d_\perp \varphi_t(m) z + \mathcal{O}_t(\|z\|^2)\big)\\ &= \big(\varphi_t(m_1), d_\perp \varphi_t(m_1) z + \mathcal{O}_t(\|z\|^2)\big), \quad m_1 = m + \mathcal{O}_t(\|z\|^2).\end{aligned} \tag{4.8}$$

(Here we identify $(T_m K^\delta)^\perp$ with $\mathbb{R}^{2d_\perp}$ and consider $d_\perp \varphi_t(m) : \mathbb{R}^{2d_\perp} \to \mathbb{R}^{2d_\perp}$, with similar identification near $\varphi_t(\rho)$. The norm $\| \bullet \|$ is now fixed in that neighbourhood.)

Let $z = z_+ + z_-$ be the decomposition of z corresponding to $(T_{m_1} K^\delta)^\perp = E_{m_1}^+ \oplus E_{m_1}^-$ (we assumed without loss of generality that (4.2) holds). The continuity of $\rho \mapsto E^{\pm}_{\rho}$ and the definition of C^{+}_{ρ} show that if ϵ_1 is small enough depending on t (so that $d(m, m_1) = \mathcal{O}_t(\|z\|^2)$ is small)

$$z \in C^{+}_{m} \implies \|z_-\|_{m_1} \leq 2\epsilon_2 \|z_+\|_{m_1}. \tag{4.9}$$

Since

$$d_{\perp}\varphi_t(m_1)z = \sum_{\pm} d_{\perp}\varphi_t(m_1)z_{\pm}, \quad d_{\perp}\varphi_t(m_1)z_{\pm} \in E^{\pm}_{\varphi_t(m_1)},$$

normal hyperbolicity implies that for some $C > 0$ and $\lambda_1 > 0$

$$\begin{aligned} \|d_{\perp}\varphi_t(m_1)z_+\| &\geq \frac{1}{C} e^{2\lambda_1 t}\|z_+\|, \\ \|d_{\perp}\varphi_t(m_1)z_-\| &\leq C e^{-2\lambda_1 t}\|z_-\|, \end{aligned} \tag{4.10}$$

for all positive times t.

If $z \in C^{+}_{m}$, then this and (4.9) show

$$\|d_{\perp}\varphi_t(m_1)z_-\| \leq 2C^2\, e^{-4\lambda_1 t}\, \epsilon_2 \|d_{\perp}\varphi_t(m_1)z_+\|.$$

Let us take t_0 such that $2C^2\, e^{-4\lambda_1 t_0} < 1/2$. For $t \geq t_0$ and ϵ_1 small enough depending on t this shows that

$$\begin{aligned} &z \in C^{+}_{m} \text{ and } \|z\| \leq \epsilon_1,\ \|d_{\perp}\varphi_t(m_1)z\| \leq \epsilon_1 \\ &\quad \implies d_{\perp}\varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2) \in C^{+}_{\varphi_t(m_1)}, \end{aligned} \tag{4.11}$$

which in view of (4.8) proves (4.5) in the $+$ case with the $-$ case being essentially the same.

To obtain (4.6) we note that for $(m, z) \in \text{neigh}(\rho, K^{\delta})$,

$$d((m,z), C^{+}) \sim d_m(z, C^{+}_{m}) \sim \|z_-\|(1 - \mathbb{1}_{C^{+}_{m}}(z)), \quad z = z_+ + z_-, \quad z_{\pm} \in E^{\pm}_{m},$$

where $\mathbb{1}_A$ is the characteristic function of a set A. (To see the first $\sim$ we need to show that $d((m,z), C^{+}) \leq c_0 d_m(z, C^{+}_{m})$ for some c_0, which follows from an argument by contradiction using pre-compactness of K^{δ}.)

We also observe that if $d\varphi_t(m_1)z \in \text{neigh}(K^{\delta})$ then (4.11) gives, for ϵ_1 small enough depending on t,

$$1 - \mathbb{1}_{C^{+}_{\varphi_t(m_1)}}\big(d_{\perp}\varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2)\big) \leq 1 - \mathbb{1}_{C^{+}_{m}}(z).$$

Hence, using (4.8) and (4.10), writing $z = z_- + z_+$ as before, and taking ϵ_1 sufficiently small depending on $t \geq t_0$,

$$
\begin{aligned}
d(\varphi_t(m,z), C^+) &\sim d_{\varphi_t(m_1)}\big(d_\perp\varphi_t(m_1)z + \mathcal{O}_t(\|z\|^2), C^+_{\varphi_t(m_1)}\big)\\
&\sim \|d_\perp\varphi_t(m_1)z_-\|\,(1+\mathcal{O}_t(\|z_-\|))\\
&\quad\times\big(1 - \mathbb{1}_{C^+_{\varphi_t(m_1)}}(d_\perp\varphi_t(m_1)z + \mathcal{O}_t(\|z\|)^2)\big)\\
&\le Ce^{-2\lambda_1 t}\|z_-\|\,(1+\mathcal{O}_t(\|z_-\|))\,(1-\mathbb{1}_{C^+_{m_1}}(z))\\
&\le C'e^{-2\lambda_1 t}d_{m_1}(z, C^+_{m_1}) \sim C'e^{-2\lambda_1 t}d_m(z, C^+_m)\\
&\le e^{-\lambda_1 t}d((m,z), C^+).
\end{aligned}
$$

Here in the second line we used the fact that $\|z\| \le C\|z_-\|$ if the distance is non zero (with C depending on ϵ_2). In the fourth line we used the continuity of the cone field, $m \mapsto C^+_m$.

This proves (4.6). The last claim (4.7) is immediate from the construction of $C^\pm$ and the fact that $E^+_\rho \cap E^-_\rho = \{0\}$. ◻

We now regularize $d(\rho, C^\pm)^2$ uniformly with respect to a parameter ϵ. It will eventually be taken to be $h/\tilde{h}$, where $\tilde{h}$ is a small constant independent of h. Lemma 4.2 and the arguments of [37, §4] and [43, §7] immediately give

Lemma 4.3 *There exists $t_0 > 0$ such that for any $t > t_0$, there exists a neighbourhood $\mathcal{V}_t$ of $K^{2\delta}$ and a constant $C_0 > 0$ such that the following holds.*

For any small $\epsilon > 0$ there exist functions $\gamma_\pm \in \mathcal{C}^\infty(\mathcal{V}_t \cup \varphi_t(\mathcal{V}_t))$ such that for $\rho \in \mathcal{V}_t \cap p^{-1}([-\delta, \delta])$,

$$
\begin{aligned}
&\gamma_\pm(\rho) \sim d(\rho, C_\pm)^2 + \epsilon, \quad \gamma_\pm(\rho) \ge \epsilon,\\
&\quad \pm(\gamma_\pm(\rho) - \gamma_\pm(\varphi_t(\rho))) + C_0\epsilon \sim \gamma_\pm(\rho),\\
&\partial^\alpha\gamma_\pm(\rho) = \mathcal{O}(\gamma_\pm(\rho)^{1-|\alpha|/2}),\\
&\gamma_+(\rho) + \gamma_-(\rho) \sim d(\rho, K^\delta)^2 + \epsilon.
\end{aligned}
\tag{4.12}
$$

Following [37, §4] and [43, §7] again this gives us an escape function for a small neighbourhood of the trapped set. We record this in

Proposition 4.4 *Let $\gamma_\pm$ be the functions given in Lemma 4.3. For $L \gg 1$ independent of ϵ, define*

$$
G_0 \stackrel{\text{def}}{=} \log(L\epsilon + \gamma_-) - \log(L\epsilon + \gamma_+) \tag{4.13}
$$

on a neighbourhood $\mathcal{V}$ of the trapped set $K^{2\delta}$.

For any t_0 large enough, and L depending on t_0, we can find a neighbourhood of $U_1 \Subset \mathcal{V}$ of $K^{2\delta}$ and $c_1, c_2, C_1, C_2, > 0$, independent of L, such that

$$
G_0 = \mathcal{O}(\log(1/\epsilon)), \quad \partial^\alpha_\rho G_0 = \mathcal{O}(\min(\gamma_+, \gamma_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \ge 1,
$$

and such that for $\rho \in U_1 \cap p^{-1}([-\delta, \delta])$,

$$\begin{aligned} \partial_\rho^\alpha (G_0(\varphi_{t_0}(\rho)) - G_0(\rho)) &= \mathcal{O}(\min(\gamma_+, \gamma_-)^{-\frac{|\alpha|}{2}}) = \mathcal{O}(\epsilon^{-\frac{|\alpha|}{2}}), \quad |\alpha| \geq 0, \\ d(\rho, K^\delta)^2 \geq C_1 \epsilon &\implies G_0(\varphi_{t_0}(\rho)) - G_0(\rho) \geq c_1/L, \\ d(\rho, K^\delta)^2 \leq c_2 L \epsilon &\implies |G(\rho)| \leq C_2. \end{aligned} \tag{4.14}$$

Remark 4.5 For the reader's convenience we make some comments on the constants in Proposition 4.4 referring to the proof of [37, Lemma 4.4] for details. The constant L has to be large enough depending on the implicit constants in (4.12). The constants C_1, C_2 have to be large enough, and constants c_1, c_2 small enough, depending on the implicit constants in (4.12). In Sect. 6.2 it matters that we can take $c_2 L > C_1$ which is certainly possible.

In the intermediate region between U_1 and $\{x : w(x) > 0\}$ we need an escape function similar to the one constructed in [15, §4] and [26, Appendix]. We work here under the general assumptions of Sect. 1.3 and present a slightly modified argument.

Lemma 4.6 *Suppose that* X *is a compact smooth manifold,* $p \in S^m(T^*X; \mathbb{R})$, $w \in S^k(T^*X; [0, \infty))$, $k \leq m$, *and that* (1.9) *holds. For any open neighbourhood* V_1 *of* $K^{3\delta}$, *there exists* $\epsilon_1 > 0$ *and a function* $G_1 \in \mathcal{C}_c^\infty\big(p^{-1}((-2\delta, 2\delta))\big)$ *such that*

$$\begin{aligned} &G_1(\rho) = 0 \ \textit{for } \rho \textit{ in some neighbourhood of } K^{3\delta}, \\ &H_p G_1(\rho) \geq 0 \ \textit{for } \rho \notin w^{-1}((\epsilon_1, \infty)), \\ &H_p G_1(\rho) > 0 \ \textit{for } \rho \in p^{-1}([-\delta, \delta]) \backslash \big(V_1 \cup w^{-1}((\epsilon_1, \infty))\big). \end{aligned} \tag{4.15}$$

Proof Call $U_0 \stackrel{\text{def}}{=} w^{-1}((0, \infty))$ and suppose $\rho \in p^{-1}([-2\delta, 2\delta]) \backslash (V_1 \cup U_0)$. We first claim that there exist $T_\pm = T_\pm(\rho)$, $T_- < 0 < T_+$, such that

$$\varphi_{T_+}(\rho) \in U_0 \ \text{ or } \ \varphi_{T_-}(\rho) \in U_0, \tag{4.16}$$

$$\varphi_{T_\pm}(\rho) \in V_1 \cup U_0. \tag{4.17}$$

(Here and below we use the notation $\varphi_A(\rho) = \{\varphi_t(\rho) : t \in A\}$.)

To justify these claims we first note that since $\rho \notin K^{2\delta}$, $\varphi_{\mathbb{R}}(\rho) \cap U_0 \neq \emptyset$ which implies that

$$\exists\, T_1, \quad \varphi_{T_1}(\rho) \in U_0. \tag{4.18}$$

Assuming that $T_1 < 0$ we want to show that $\varphi_{T_2}(\rho) \in V_1 \cup U_0$ for some $T_2 > 0$. Suppose that this is not true, that is

$$\varphi_{(0,\infty)}(\rho) \cap (V_1 \cup U_0) = \emptyset. \tag{4.19}$$

Then for any $t_j \to \infty$,

$$\rho_j \stackrel{\text{def}}{=} \varphi_{t_j}(\rho) \in p^{-1}([-2\delta, 2\delta])\backslash(V_1 \cup U_0), \quad \varphi_{[0,\infty)}(\rho_j) \cap (V_1 \cup U_0) = \emptyset.$$

By (1.9) the set $p^{-1}([-2\delta, 2\delta])\backslash(V_1 \cup U_0)$ is compact and hence, by passing to a subsequence, we can assume that $\rho_j \to \bar{\rho} \notin V_1 \cup U_0$. We have $\varphi_t(\rho_j) \to \varphi_t(\bar{\rho})$, as $j \to \infty$, uniformly for $|t| \leq T$, and it follows that $\varphi_{[0,\infty)}(\bar{\rho}) \cap (V_1 \cup U_0) = \emptyset$. For $t \geq -t_j$,

$$\varphi_t(\rho_j) = \varphi_{t+t_j}(\rho) \subset \varphi_{[0,\infty)}(\rho) \subset p^{-1}([-2\delta, 2\delta])\backslash(V_1 \cup U_0),$$

which means that $\varphi_t(\bar{\rho}) \notin V_1 \cup U_0$ for $t > -t_j \to -\infty$. We conclude that

$$\varphi_{\mathbb{R}}(\bar{\rho}) \cap V_1 \cup U_0 = \emptyset \implies \varphi_{\mathbb{R}}(\bar{\rho}) \in K^{3\delta}.$$

This contradicts the property $\bar{\rho} \notin V_1$, and proves the existence of $T_2 > 0$ such that $\varphi_{T_2}(\rho) \in V_1 \cup U_0$. We call $T_-(\rho) = T_1$, $T_+(\rho) = T_2$.

In the case T_1 in (4.18) is positive, a similar argument shows the existence of $T_2 < 0$ such that $\varphi_{T_2}(\rho) \in (V_1 \cup U_0) \neq \emptyset$. In this case we call $T_-(\rho) = T_2$, $T_+(\rho) = T_1$.

For each $\rho \in p^{-1}([-2\delta, 2\delta])$ we can find an open hypersuface Γ_ρ, transversal to H_p at ρ, such that, if $\varphi_{T_\pm}(\rho) \in U_0$, then for $\rho' \in \Gamma_\rho$,

$$\varphi_{T_\pm}(\rho') \in U_0, \quad \varphi_{T_\mp}(\rho') \in V_1 \cup U_0.$$

Notice that the closure of the tube $\Omega_\rho \stackrel{\text{def}}{=} \varphi_{(T_-,T_+)}(\Gamma_\rho)$ does not intersect $K^{3\delta}$. Using this tube, we construct a local escape functions $g_\rho \in \mathcal{C}_c^\infty(\Omega_\rho)$, with the following properties: for some $\epsilon_\rho > 0$, and an slightly smaller tube $\Omega'_\rho \subset \Omega_\rho$ containing $\varphi_{(T_-,T_+)}(\rho)$,

$$\begin{aligned} &H_p g_\rho(\rho') \geq 0, \quad \rho' \notin w^{-1}((\epsilon_\rho, \infty)), \\ &H_p g_\rho(\rho') > 0, \rho' \in \Omega'_\rho\backslash(w^{-1}((\epsilon_\rho, \infty)) \cup V_1). \end{aligned} \tag{4.20}$$

Here ϵ_ρ is chosen so that if $\varphi_{T_\pm}(\rho) \in U_0$ then $\varphi_{T_\pm}(\Gamma_\rho) \subset w^{-1}((2\epsilon_\rho, \infty))$.

To construct g_ρ we take $(t, m) \in (T_-, T_+) \times \Gamma_\rho$ as local coordinates: $(t, m) \mapsto \varphi_t(m) \in \Omega_\rho$. Suppose that $\varphi_{T_-}(\rho) \in U_0$, and that $\varphi_{(T_-,T_-+\gamma)}(\Gamma_\rho) \subset w^{-1}((\epsilon_\rho, \infty))$ and $\varphi_{(T_+-\gamma,T_+)}(\Gamma_\rho) \subset V_1 \cup U_0$. Choose $\chi_\rho \in \mathcal{C}_c^\infty((T_-, T_+))$ which is strictly increasing on $(T_- + \gamma, T_+ - \gamma)$ and non-decreasing on $(T_+ - \gamma, T_+)$. Also, choose $\psi_\rho \in \mathcal{C}_c^\infty(\Gamma_\rho)$ with $\psi_\rho(\rho) = 1$. Then put $g_\rho(\varphi_t(m)) \stackrel{\text{def}}{=} \chi_\rho(t)\psi_\rho(m)$. Since $H_p g_\rho = \chi'_\rho(t)\psi_\rho(m)$, (4.20) holds. A similar construction can be applied in the case where $\varphi_{T_-}(\rho) \in V_1$, $\varphi_{T_+}(\rho) \in U_0$.

From the open cover

$$p^{-1}([-\delta,\delta]) \setminus (V_1 \cup U_0) \subset \bigcup \left\{ \Omega_\rho : \rho \in p^{-1}([-\delta,\delta]) \setminus (V_1 \cup U_0) \right\},$$

one may extract a finite subcover $\bigcup_{j=1}^{L} \Omega_{\rho_j}$. The closure of this cover does not intersect $K^{3\delta}$, so that the function $G_1(\rho) \stackrel{\text{def}}{=} \sum_{l=1}^{L} g_{\rho_L}(\rho)$ satisfies (4.15), for $\epsilon_0 = \min_j \epsilon_{\rho_j}$. □

We conclude this section with a global escape function which combines the ones in Proposition 4.4 and Lemma 4.6. The estimates will be needed to justify the quantization of the escape function in Sect. 6. The proof is an immediate adaptation of the proof of [37, Proposition 4.6] and is omitted.

Proposition 4.7 *Let $\mathcal{V}$, U_1, G_0 and t_0 be as in Proposition 4.4, and let W_1 be a neighbourhood of $K^{2\delta}$ such that $W_1 \Subset U_1$, $W_1 \cup \varphi_{t_0}(W_1) \Subset \mathcal{V}$.*

Take $\chi \in \mathcal{C}_{\rm c}^\infty(\mathcal{V})$ equal to 1 in $W_1 \cup \varphi_{t_0}(W_1)$, and let G_1 be the escape function constructed in Lemma 4.6 for $V_1 = W_1$. Then for any $\Gamma > 1$, $G \in \mathcal{C}_{\rm c}^\infty(T^*X;\mathbb{R})$ defined by

$$G \stackrel{\text{def}}{=} \chi C_3 \Gamma G_0 + C_4 \log(1/\epsilon)\, G_1 \tag{4.21}$$

where C_3 and C_4 are sufficiently large, satisfies the following estimates

$$\begin{aligned}
&|G(\rho)| \leq C_6 \log(1/\epsilon), \quad \partial^\alpha G = \mathcal{O}(\epsilon^{-|\alpha|/2}), \quad |\alpha| \geq 1,\\
&\rho \in W_1 \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq -C_7,\\
&\rho \in W_1 \cap p^{-1}([-\delta,\delta]), \quad d(\rho, K^\delta)^2 \geq C_1\epsilon \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq 2\Gamma,\\
&\rho \in p^{-1}([-\delta,\delta]) \setminus \left(W_1 \cup w^{-1}((\epsilon_1,\infty))\right) \implies G(\varphi_{t_0}(\rho)) - G(\rho) \geq C_8 \log(1/\epsilon),
\end{aligned} \tag{4.22}$$

with $C_8 > 0$.

In addition we have

$$\frac{\exp G(\rho)}{\exp G(\rho')} \leq C_9 \left(1 + \frac{d(\rho,\rho')}{\sqrt{\epsilon}}\right)^{N_1}, \tag{4.23}$$

for some constants C_9 and N_1.

5 Analysis near the trapped set

In this section we will analyse the cut-off propagator

$$\chi^w \exp(-itP/h)\chi^w, \tag{5.1}$$

where $\chi^w = \mathrm{Op}_h^w(\chi)$, $\chi \in \mathcal{C}_c^\infty \cap \widetilde{S}_{\frac{1}{2}}$ and $\operatorname{supp} \chi \subset \{\rho : d(\rho, K^\delta) < R(h/\tilde{h})^{\frac{1}{2}}\}$ for some $R > 1$ independent of $\tilde{h}$, h. We could take two different cut-offs on both sides, as long as they share the above properties.

Our objective is to prove the following bound (announced in (2.5)):

Proposition 5.1 *For any $\epsilon_0 > 0$ and $M > 0$, there exist $C_0 > 0$, $\tilde{h}_0$, and a function $\tilde{h} \mapsto h_0(\tilde{h}) > 0$, such that for $0 < \tilde{h} < \tilde{h}_0$ and $0 < h < h_0(\tilde{h})$,*

$$\begin{aligned} &\|\chi^w e^{-itP/h} \chi^w\|_{L^2 \to L^2} \leq C_0\, \tilde{h}^{-d_\perp/2} \exp\left(-\frac{1}{2} t(\lambda_0 - \epsilon_0)\right), \\ &0 \leq t \leq M \log 1/\tilde{h}, \end{aligned} \tag{5.2}$$

where λ_0 is given by (1.19).

Since $e^{-itP/h}$ is unitary, the above bound is nontrivial only for

$$0 \leq \frac{d_\perp}{\lambda_0} \log \frac{1}{\tilde{h}} \leq t \leq M \log \frac{1}{\tilde{h}}.$$

5.1 Darboux coordinate charts

We start by setting up an adapted atlas of Darboux coordinate charts near K^δ, that is take a finite open cover

$$K^\delta \subset \bigcup_{j \in J} U_j,$$

and symplectomorphisms $\kappa_j : U_j \to V_j = \operatorname{neigh}(0, \mathbb{R}^{2d})$. The standard symplectic coordinates on V_j then appear as a local symplectic coordinate frame on U_j. We may choose the coordinates such that they split into

$$X = (x, y), \quad \Xi = (\xi, \eta), \quad y, \eta \in \mathbb{R}^{d_\perp}, \quad x, \xi \in \mathbb{R}^{d - d_\perp},$$

such that the symplectic submanifold $K^\delta \cap U_j$ is identified with $\mathcal{K} \cap V_j \subset \mathbb{R}^{2d}$, where

$$\mathcal{K} \stackrel{\text{def}}{=} \{y = \eta = 0\} \subset \mathbb{R}^{2d}.$$

That is, (x, ξ) is a local coordinate frame on K^δ, while (y, η) describes the transversal directions.

We also assume that for each $\rho \in K^\delta \cap U_j$, identified with some $(x, 0, \xi, 0) \in V_j$, the subspace $\{(x, y, \xi, 0)\,,\ y \in \mathbb{R}^{d_\perp}\}$ is ϵ-close to the transversal unstable space $d\kappa_j(E_\rho^+)$, while the subspace $\{(x_0, 0, \xi_0, \eta)\,,$ $\eta \in \mathbb{R}^{d_\perp}\}$ is ϵ-close to the transversal stable space $d\kappa_j(E_\rho^-)$.

We want to describe the flow in the vicinity of K^δ, using these local coordinates. We choose a (large) time $t_0 > 0$, and express the time-t_0 flow $\varphi_{t_0} : U_{j_0} \to U_{j_1}$ in the local coordinate frames, through the maps

$$\kappa_{j_1 j_0} \stackrel{\text{def}}{=} \kappa_{j_1} \circ \varphi_{t_0} \circ \kappa_{j_0}^{-1} : D_{j_1 j_0} \to A_{j_1 j_0}, \tag{5.3}$$

where $D_{j_1 j_0} \subset V_{j_0}$ is the *departure set*, while $A_{j_1 j_0} \subset V_{j_1}$ is the *arrival set*. This is defined when $\varphi_{t_0}(U_{j_0}) \cap U_{j_1} \neq \emptyset$ and such a pair $j_1 j_0$ for which this holds will be called *physical*.

Below we will also consider the maps $\kappa^n_{j_1 j_0}$ representing the time-nt_0 flow in the charts $V_{j_0} \to V_{j_1}$—see Sect. 5.6.

5.2 Splitting $e^{-it_0 P/h}$ into pieces

We want to use the fact that the propagator $e^{-it_0 P/h}$ is a Fourier integral operator on M associated with φ_{t_0}. To make this remark precise, we will use a smooth partition of unity $\big(\pi_j \in \mathcal{C}^\infty_{\mathrm{c}}(U_j, [0, 1])\big)$ such that each cut-off π_j is equal to unity near some $\widetilde{U}_j \Subset U_j$, and the quantized cut-offs $\Pi_i \stackrel{\text{def}}{=} \mathrm{Op}^w_h(\pi_i)$ satisfy the following quantum partition of unity:

$$\Pi \stackrel{\text{def}}{=} \sum_{j=1}^{J} \Pi_j\, \Pi_j^* \equiv I \quad \text{microlocally in a neighbourhood of } K^\delta. \tag{5.4}$$

We will then split $e^{-it_0 P/h}$ into the local propagators

$$T^\flat_{j_1 j_0} \stackrel{\text{def}}{=} \Pi^*_{j_1} e^{-it_0 P/h} \Pi_{j_0}, \tag{5.5}$$

which can be represented by operators on $L^2(\mathbb{R}^d)$ as follows. We define Fourier integral operators $\mathcal{U}_j : L^2(X) \to L^2(\mathbb{R}^d)$ quantizing the coordinate changes κ_j, and microlocally unitary in some subset of $V_j \times U_j$ containing $\kappa_j(\operatorname{supp} \pi_j) \times \operatorname{supp} \pi_j$, so that

$$\forall j, \quad \Pi_j \Pi_j^* = \Pi_j \mathcal{U}_j^* \mathcal{U}_j \Pi_j^* + \mathcal{O}(h^\infty), \tag{5.6}$$

The local propagators $T^\flat_{j_1 j_0}$ are then represented by

$$T_{j_1 j_0} \stackrel{\text{def}}{=} \mathcal{U}_{j_1} \Pi^*_{j_1} e^{-it_0 P/h} \Pi_{j_0} \mathcal{U}^*_{j_0}. \tag{5.7}$$

Notice that for an unphysical pair $j_1 j_0$, $T_{j_1 j_0} = \mathcal{O}(h^\infty)_{L_2 \to L_2}$. For a physical pair $j_1 j_0$, $T_{j_1 j_0}$ is a Fourier integral operator associated with the local symplectomorphism $\kappa_{j_1 j_0}$.

From the unitarity of $e^{-it_0 P/h}$ we draw the following property of the operators $T_{j' j}$.

Lemma 5.2 *The operator-valued matrix* $\boldsymbol{T} \stackrel{\text{def}}{=} (T_{ij})_{i,j=1,\dots,J}$, *acting on the space* $L^2(\mathbb{R}^d)^J$ *with the Hilbert norm* $\|\boldsymbol{u}\|^2 = \sum_{j=1}^J \|u_j\|^2_{L^2}$, *satisfies*

$$\|\boldsymbol{T}\|_{L^2(\mathbb{R}^d)^J \to L^2(\mathbb{R}^d)^J} = 1 + \mathcal{O}(h).$$

Proof From (5.6), the action of $T_{j_1 j_0}$ on $L^2(\mathbb{R}^d)$ is (up to an error $\mathcal{O}(h^\infty)_{L^2 \to L^2}$) unitarily equivalent with the action of $T^\flat_{j_1 j_0}$ on $L^2(X)$. Hence, the action of $\boldsymbol{T}$ on $L^2(\mathbb{R}^d)^J$ is equivalent to the action of $\boldsymbol{T}^\flat$ on $L^2(X)^J$, where $\boldsymbol{T}^\flat$ is the matrix of operators (5.5).

To obtain the norm estimate we follow [2, Lemma 6.5], put $\mathcal{H} \stackrel{\text{def}}{=} L^2(X)$, $U = e^{-it_0 P/h}$, and define the row vector of cut-off operators $C = (\Pi_i)_{i=1,\dots,J}$. The operator valued matrix $\boldsymbol{T}^\flat$ can be written as $\boldsymbol{T}^\flat = C^*(U \otimes I_J)C$. Its operator norm on $\mathcal{L}(\mathcal{H}^J)$ satisfies

$$\begin{aligned}\|\boldsymbol{T}^\flat\|^2_{\mathcal{L}(\mathcal{H}^J)} &= \|(\boldsymbol{T}^\flat)^* \boldsymbol{T}^\flat\|_{\mathcal{L}(\mathcal{H}^J)} = \|C^*(U \otimes I_J)CC^*(U^* \otimes I_J)C\|_{\mathcal{L}(\mathcal{H}^J)} \\ &= \|C^*(U \Pi U^* \otimes I_J)C\|_{\mathcal{L}(\mathcal{H}^J)}.\end{aligned}$$

Egorov's theorem (see (3.6)) and [55, Theorem 13.13] imply that $\Pi^1 \stackrel{\text{def}}{=} U \Pi U^*$ is a positive semidefinite operator of norm $1 + \mathcal{O}(h)$, with symbol equal to $1 + \mathcal{O}(h)$ near K^δ, and its square root $\sqrt{\Pi^1}$, as well as the product $\sqrt{\Pi^1}\, \Pi \sqrt{\Pi^1}$, have the same properties. Hence,

$$\begin{aligned}\|\boldsymbol{T}^\flat\|^2_{\mathcal{L}(\mathcal{H}^J)} &= \Big\| \Big((\sqrt{\Pi^1} \otimes I_J)\, C \Big)^* (\sqrt{\Pi^1} \otimes I_J)\, C \Big\|_{\mathcal{L}(\mathcal{H}^J)} \\ &= \Big\| (\sqrt{\Pi^1} \otimes I_J)\, C \Big((\sqrt{\Pi^1} \otimes I_J)\, C \Big)^* \Big\|_{\mathcal{L}(\mathcal{H})} \\ &= \|\sqrt{\Pi^1}\, \Pi \sqrt{\Pi^1}\|_{\mathcal{L}(\mathcal{H})} = 1 + \mathcal{O}(h).\end{aligned}$$

$\square$

5.3 Iterated propagator

In this section we explain how to use the $T_{j'j}$ to study our cut-off propagator (5.1).

First of all, Egorov's theorem (3.7) applied to $T = \mathcal{U}_j \Pi_j^*$, $B_2 = \chi^w$ allows us to write

$$\mathcal{U}_j \, \Pi_j^* \, \chi^w = \chi_j^w \, \mathcal{U}_j \, \Pi_j^* + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \to L^2}, \quad j = 1, \dots, J, \tag{5.8}$$

where the symbol $\chi_j = \chi \circ \kappa_j^{-1} \in \widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d)$.

We start from a arbitrary normalized state $u \in L^2(X)$, and represent the part of u microlocalized near K^δ through the (column) vector of states

$$\begin{aligned} \boldsymbol{u} &\stackrel{\text{def}}{=} (u_j)_{j=1,\dots,J}, \quad u_j \stackrel{\text{def}}{=} \mathcal{U}_j \Pi_j^* u, \\ \|\boldsymbol{u}\|^2 &\stackrel{\text{def}}{=} \sum_j \|u_j\|^2 = \langle u, \Pi u \rangle + \mathcal{O}(h^\infty) \|u\|_{L^2}^2. \end{aligned}$$

The Eqs. (5.7) and (5.8) show that

$$\begin{aligned} \Pi e^{-it_0 P/h} \chi^w u &= \sum_j \Pi_j \Pi_j^* e^{-it_0 P/h} \chi^w u = \sum_{j_1, j_0} \Pi_{j_1} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0} \Pi_{j_0}^* \chi^w u \\ &= \sum_{j_1, j_0} \Pi_j \mathcal{U}_{j_1}^* \mathcal{U}_{j_1} \Pi_{j_1}^* e^{-it_0 P/h} \Pi_{j_0} \mathcal{U}_{j_0}^* \mathcal{U}_{j_0} \Pi_{j_0}^* \chi^w u + \mathcal{O}(h^\infty)_{L^2 \to L^2} \\ &= \sum_{j_1, j_0} \Pi_{j_1} \mathcal{U}_{j_1}^* \, T_{j_1 j_0} \chi_{j_0}^w u_{j_0} + \mathcal{O}(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \to L^2}. \end{aligned}$$

Similarly, for $n \geq 2$ the propagator $e^{-int_0 P/h}$ can be represented by iteratively applying the operator valued matrix $\boldsymbol{T}$ to the vector $\boldsymbol{u}$. By inserting the identities (5.4), (5.6) n times in the expression $\pi e^{-int_0 P/h} \chi^w u$, we get the following

Lemma 5.3 *For any $n \in \mathbb{N}$ (independent of h), we have*

$$\begin{aligned} \Pi e^{-int_0 P/h} \chi^w u &= \sum_{j_n, \dots, j_0} \Pi_{j_n} \mathcal{U}_{j_n}^* \, T_{j_n j_{n-1}} \cdots T_{j_1 j_0} \, \chi_{j_0}^w \, u_{j_0} + \mathcal{O}_n(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \to L^2} \\ &= \sum_{j_n, j_0} \Pi_{j_n} \mathcal{U}_{j_n}^* \, [(\boldsymbol{T})^n]_{j_n j_0} \, \chi_{j_0}^w \, u_{j_0} + \mathcal{O}_n(h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}})_{L^2 \to L^2}, \end{aligned} \tag{5.9}$$

where the matrix of operators, $\boldsymbol{T}$, was defined in Lemma 5.2.

5.3.1 Inserting nested cut-offs

In this section we modify the Fourier integral operators $T_{j'j}$, taking into account that in the above expression their products are multiplied by narrow cut-offs χ_j^w.

By construction of χ_j, there exists $R_0 > 0$ (independent of h, $\tilde{h}$) such that for any index j the cut-off $\chi_j \in \widetilde{S}_{\frac{1}{2}}$ is supported inside the microscopic cylinder

$$B_{R_0(h/\tilde{h})^{1/2}} \stackrel{\text{def}}{=} \{(x, y, \xi, \eta) : |y|, |\eta| \leq R_0(h/\tilde{h})^{1/2}\} \subset T^*\mathbb{R}^d. \quad (5.10)$$

Fix some $R_1 \geq 2R_0$, and choose a function $\widetilde{\chi}^0 \in C_0^\infty(\mathbb{R}^{2d_\perp}, [0, 1])$ equal to unity in the ball $\{|\tilde{y}|, |\tilde{\eta}| \leq R_1\}$, and supported in $\{|\tilde{y}|, |\tilde{\eta}| \leq 2R_1\}$. Normal hyperbolicity implies that there exists $\Lambda > 2$ such that the cylinders $B_\bullet$ (see (5.10)) satisfy

$$\kappa_{j'j}(B_{2R}(h/\tilde{h})^{1/2}) \Subset B_{R\Lambda}(h/\tilde{h})^{1/2}, \quad (5.11)$$

for all $0 < R < 1$ and any physical pair $j'j$.

We then define the families of nested[4] cut-offs $\{\chi^n\}_{n\in\mathbb{N}}, \{\widetilde{\chi}^n\}_{n\in\mathbb{N}}$ as follows:

$$\forall n \in \mathbb{N}, \quad \widetilde{\chi}^n(y, \eta) \stackrel{\text{def}}{=} \widetilde{\chi}^0(y\Lambda^{-n}, \eta\Lambda^{-n}), \quad (5.12)$$

$$\chi^n(x, y, \xi, \eta) \stackrel{\text{def}}{=} \widetilde{\chi}^n\big(y(\tilde{h}/h)^{1/2}, \eta(\tilde{h}/h)^{1/2}\big) \in \widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d). \quad (5.13)$$

We stress that the $\widetilde{S}_{\frac{1}{2}}(T^*\mathbb{R}^d)$ seminorms of χ^n hold uniformly in n: the smoothness of χ^n actually improves when n grows. From the assumption $R_1 > R_0$ we draw the nesting $\chi^0 \succ \chi_j$ for any $j = 1, \ldots, J$. Furthermore, the property (5.11) implies that

$$\text{for any physical pair } j'j, \qquad \chi^{n+1} \succ \chi^n \circ \kappa_{j'j}. \quad (5.14)$$

From these nesting properties and from Egorov's property (3.7) we easily obtain the following

Lemma 5.4 *For any $j = 1, \ldots, J$ we have*

$$(\chi^0)^w \chi_j^w = \chi_j^w + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}, \quad \chi_j^w (\chi^0)^w = \chi_j^w + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}. \quad (5.15)$$

[4] Below we use the notation $\chi^0 \succ \chi$ for nested cut-offs, meaning that $\chi^0 \equiv 1$ near $\mathrm{supp}(\chi)$.

In addition, we have the estimate

$$T_{j'j}(\chi^n)^w = (\chi^{n+1})^w\, T_{j'j}(\chi^n)^w + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}, \tag{5.16}$$

uniformly for all $j, j' = 1, \dots, J$ *and for all* n *independent of* h.

We will actually only use n smaller than $M \log 1/\tilde{h}$ for some $M > 0$ independent of $\tilde{h},\ h$, so our cut-offs χ^n will all be localized in microscopic neighbourhoods of $\mathcal{K}$ when $h \to 0$. Furthermore, for such a logarithmic time the number of terms in the sum in the middle expression in (5.9) is bounded above by $J^{n+1} \le \tilde{h}^{-N}$ for some $N > 0$. As a result, taking into account the above cut-off insertions, this sum can be rewritten as

$$\begin{aligned}
&\Pi e^{-int_0 P/h}\chi^w u\\
&\quad = \sum_{j_n,\dots,j_0} \Pi_{j_n}\mathcal{U}^*_{j_n}\, T_{j_n j_{n-1}}(\chi^{n-1})^w \cdots T_{j_2 j_1}(\chi^1)^w T_{j_1 j_0}(\chi^0)^w \chi_{j_0} u_{j_0}\\
&\qquad + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}.
\end{aligned} \tag{5.17}$$

In the next section we will carefully analyze the kernels of the operators $T_{j'j}\,(\chi^k)^w$.

5.4 Structure of the local phase function

To analyze the Fourier integral operators we will examine the structure of the generating function for the symplectomorphism $\kappa_{j_1 j_0}$.

We start by studying the transverse linearization $d_\perp\kappa(\rho)$ of the map $\kappa = \kappa_{j_1 j_0}$, for a point $\rho \in \mathcal{K} \cap D_{j_1 j_0}$. In our symplectic coordinate frames, this transverse map is represented by the symplectic matrix $S_{j_1 j_0}(\rho) = S(\rho) \in \mathrm{Sp}(2d_\perp, \mathbb{R})$ given by

$$S(\rho) \stackrel{\text{def}}{=} \frac{\partial(y^1, \eta^1)}{\partial(y^0, \eta^0)}(\rho), \qquad \rho \in \mathcal{K}. \tag{5.18}$$

The linear symplectomorphism $S(\rho)$ admits a quadratic generating function $Q_\rho(y^1, y^0, \theta')$, where $\theta' \in \mathbb{R}^{d_\perp}$ is an auxiliary variable: the graph of the map ${}^T(y^0, \eta^0) \mapsto {}^T(y^1, \eta^1) = S(\rho)^T(y^0, \eta^0)$ can be obtained by identifying the critical set

$$C_{Q_\rho} = \left\{(y^1, y^0, \theta')\,:\, \partial_{\theta'} Q(y^1, y^0, \theta') = 0\right\} \subset \mathbb{R}^{3d_\perp}.$$

This critical set is in bijection with the graph of $S(\rho)$ through the rules

$$\eta^1 = \partial_{y^1} Q_\rho(y^1, y^0, \theta'), \quad \eta^0 = -\partial_{y^0} Q_\rho(y^1, y^0, \theta'), \quad (y^1, y^0, \theta') \in C_{Q_\rho}.$$

More structure comes from taking the normal hyperbolicity into account. Recall that our coordinates are chosen so that E^+ and E^- are ϵ-close to $\{\eta = 0\}$ and $\{y = 0\}$, respectively. (Here we identified $E^\pm$ with their images under $d\kappa_j$—see Sect. 5.1.) This implies the existence of a continuous family of symplectic transformation

$$\mathcal{K} \cap D_{j_1 j_0} \ni \rho \longmapsto R(\rho) \in \mathrm{Sp}(2d_\perp, \mathbb{R}),$$

such that

$$R(\rho)(\{\eta=0\}) = E^+_\rho, \quad R(\rho)(\{y=0\}) = E^-_\rho, \quad R(\rho) = I + \mathcal{O}(\epsilon). \tag{5.19}$$

Since $d_\perp \kappa(\rho) \equiv S(\rho)$ maps $E^\pm_\rho$ to $E^\pm_{\kappa(\rho)}$, the matrix

$$\widetilde{S}(\rho) \stackrel{\text{def}}{=} R(\kappa(\rho))^{-1}\, S(\rho)\, R(\rho), \qquad \rho \in \mathcal{K} \cap D_{j_1 j_0}, \tag{5.20}$$

is block-diagonal:

$$\widetilde{S}(\rho) = \begin{pmatrix} \Lambda(\rho) & 0 \\ 0 & {}^T\Lambda(\rho)^{-1} \end{pmatrix}. \tag{5.21}$$

The normal hyperbolicity (1.17) implies that, provided t_0 has been chosen large enough[5], the matrix $\Lambda(\rho)$ is expanding, uniformly with respect to ρ:

$$\exists\, \nu > 0, \quad \forall \rho \in \mathcal{K}, \qquad \|\Lambda^{-1}(\rho)\| \le e^{-\nu} < 1. \tag{5.22}$$

More precisely, for any small $\varepsilon > 0$, if t_0 is chosen large enough the coefficient ν can be taken of the form $\nu = t_0(\lambda_{\min} - \epsilon_0)$, where $\lambda_{\min} > 0$ is the smallest positive transverse Lyapunov exponent of φ_t near K^δ.

Combining (5.19), (5.20) and (5.21) gives

$$S(\rho) = \begin{pmatrix} \Lambda(\rho) + \mathcal{O}(\epsilon\Lambda) & \mathcal{O}(\epsilon\Lambda) \\ \mathcal{O}(\epsilon\Lambda) & \mathcal{O}(\epsilon^2\Lambda + {}^T\Lambda(\rho)^{-1}) \end{pmatrix}, \qquad \rho \in \mathcal{K}. \tag{5.23}$$

This explicit form, more precisely the fact that the upper left block is invertible, allows to use a special type of quadratic generating function:

Lemma 5.5 *If t_0 is chosen large enough, for each ρ the generating function $Q_\rho(y^1, y^0, \theta')$ can be chosen in the following form:*

$$Q_\rho(y^1, y^0, \theta') = q_\rho(y^1, \theta') - \langle y^0, \theta' \rangle. \tag{5.24}$$

[5] Recall that κ represents φ_{t_0}.

For any point (y^1, y^0, θ') on the critical set C_{Q_ρ}, the auxiliary variable θ' is identified with η^0 of the corresponding phase space point.

The specific form of the generating function corresponds to the geometric fact that the graph of $S(\rho)$ admits (y^1, η^0) as coordinates (that is, the graph of $S(\rho)$ projects bijectively onto the (y^1, η^0)-plane).

The function $q_\rho(y^1, \eta^0)$ can be written in terms of a symmetric matrix $H(\rho)$:

$$q_\rho(y^1, \eta^0) = \frac{1}{2}\langle (y^1, \eta^0), {}^T H(\rho)(y^1, \eta^0)\rangle, \quad H(\rho) = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}, \\ H_{12} \text{ invertible.} \tag{5.25}$$

The matrix $S(\rho)$ is related to $H(\rho)$ in the following way:

$$S(\rho) = \begin{pmatrix} H_{21}^{-1} & -H_{21}^{-1} H_{22} \\ H_{11} H_{21}^{-1} & H_{12} - H_{11} H_{21}^{-1} H_{22} \end{pmatrix}. \tag{5.26}$$

Comparing with (5.23) we see that

$$H_{12}^T = H_{21} = \Lambda(\rho)^{-1} + \mathcal{O}(\epsilon \Lambda(\rho)^{-1}), \quad H_{11} = \mathcal{O}(\epsilon), \quad H_{22} = \mathcal{O}(\epsilon), \tag{5.27}$$

uniformly with respect to ρ. The quadratic phase function Q_ρ will be relevant when we consider the metaplectic operator $M(\rho)$ quantizing $S(\rho)$ in Sect. 5.5.4 [see also (3.22)].

From the study of the linearized flow in the transverse direction, we now consider the dynamics of

$$\widetilde{\kappa} = \widetilde{\kappa}_{j_1 j_0} : D_{j_1 j_0} \cap \mathcal{K} \longrightarrow A_{j_1 j_0} \cap \mathcal{K}. \tag{5.28}$$

along the trapped set—see Fig. 2 in Sect. 5.6. When no confusion is likely to arise we use the notation $D_\bullet$ and $A_\bullet$ for the corresponding subsets of $\mathcal{K}$. There we have *no* assumptions on the flow, except for it being symplectic.

Possibly after refining the covers U_j, each map $\widetilde{\kappa}$ can be generated by a non-degenerate phase function $\psi = \psi_{j_1 j_0}(x^1, x^0, \theta)$ defined in a neighbourhood of the origin in $\mathbb{R}^{d-d_\perp} \times \mathbb{R}^{d-d_\perp} \times \mathbb{R}^k$, where $0 \le k \le n$—see Sect. 3.3.

Since the U_j have been chosen small, the map $C_\psi \to \Gamma_{\widetilde{\kappa}}$ can be assumed to be injective. Notice that the values of ψ away from C_ψ are irrelevant.

We now want to extend ψ into a generating function of the map κ, at least in a small neighbourhood of $\mathcal{K}$. The intuitive idea is to "glue together" the generating function ψ for $\widetilde{\kappa}$, with the quadratic generating functions Q_ρ for the transverse dynamics $d_\perp \kappa(\rho)$.

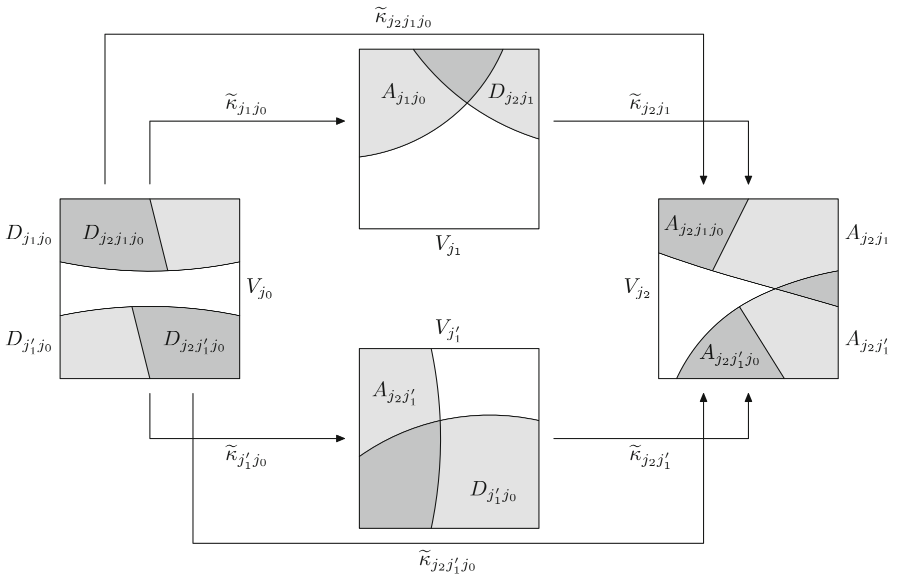

Fig. 2 Schematic representation of the departure and arrival sets for $\mathbf{j}$ of length 1 and 2. We show two *physical* sequences $j_2 j_1 j_0$ and $j_2 j_1' j_0$ and the corresponding maps (5.28). As remarked there we use the same notation for the departure and arrival sets on $\mathcal{K}$

Let us consider the following Ansatz for a generating function Ψ for κ:

$$\Psi(x^1, x^0, \theta; y^1, y^0, \theta') = \psi(x^1, x^0, \theta) + \delta\Psi(x^1, x^0, \theta; y^1, y^0, \theta'), \tag{5.29}$$

with an additional auxiliary variable $\theta' \in \mathbb{R}^{d_\perp}$. To simplify notation we split the variables into longitudinal and transversal ones:

$$\rho_\parallel = (x^1, x^0, \theta), \quad \rho_\perp = (y^1, y^0, \theta'). \tag{5.30}$$

Lemma 5.6 *Near any point $\rho \in \mathcal{K}$, κ is generated by Ψ of the form (5.29) with the transversal correction, $\delta\Psi(\rho_\parallel, \rho_\perp)$, satisfying*

$$\delta\Psi(\rho_\parallel, \rho_\perp) = Q_{\rho_\parallel}(\rho_\perp) + \mathcal{O}((y^1, \theta')^3),$$

where $Q_{\rho_\parallel}(\bullet)$ is a quadratic form of the same type as (5.24, 5.25), which depends smoothly on $\rho_\parallel$. If $\rho_\parallel \in C_\psi$ corresponds to the point $(\rho^1; \rho^0) \in \Gamma_{\widetilde{\kappa}}$, then $Q_{\rho_\parallel} = Q_{\rho^0}$.

In other words, the quadratic forms $Q_{\rho_\parallel}$ extend the forms Q_ρ to a neighbourhood of C_ψ.

Proof Since $\mathcal{K}$ is preserved by κ and carries the map $\widetilde{\kappa}$, we may assume that for any $\rho_\parallel$, the function $\delta\Psi(\rho_\parallel, \bullet)$ has no linear part in the variables $\rho_\perp$. At each point $\rho_\parallel \in C_\psi$ (identified with some $\rho^0 \in \mathcal{K}$), the quadratic part $Q_{\rho_\parallel}(\rho_\perp)$ generates the linear transverse deviation from $\widetilde{\kappa}$ near the point ρ^0, namely $d_\perp\kappa(\rho^0)$. This means that $Q_{\rho_\parallel} = Q_{\rho^0}$, which has the form (5.24). This form corresponds to the geometric fact that the graph of $d_\perp\kappa(\rho^0)$ admits (y^1, η^0) as coordinates.

This projection property locally extends to the graph of κ: in some neighbourhood of $\mathcal{K}$, the points of Γ_κ can be represented by the coordinates $(\rho^0 = (x^0, \xi^0) \in \mathcal{K}; y^1, \eta^0)$, where $y^1, \eta^0 \in \mathrm{neigh}(0)$. This property shows that $\delta\Psi$ can be written in the form

$$\delta\Psi(\rho_\parallel, \rho_\perp) = \delta\widetilde{\Psi}(\rho_\parallel, y^1, \theta') - \langle y^0, \theta'\rangle. \tag{5.31}$$

As explained above, the quadratic part $q_{\rho_\parallel}(\bullet)$ of $\delta\widetilde{\Psi}(\rho_\parallel; \bullet)$ must be equal, for $\rho_\parallel \in C_\psi$, to the corresponding q_{ρ^0} generating $S(\rho^0)$. The equations for C_Ψ show that, if we fix small values $(y^1, \theta' = \eta^0)$, then value $\rho_\parallel$ such that $(\rho_\parallel, y^1, y^0, \eta^0) \in C_\Psi$ is $\mathcal{O}((y^1, \eta^0)^2)$-close to C_ψ. □

5.5 Structure of the propagators $T_{j'j}$

From the above informations about the phase function $\Psi = \Psi_{j'j}$, we can write the integral kernel of $T = T_{j'j}$ defined in (5.7) and quantizing the map $\kappa_{j'j}$, as an oscillatory integral. The general theory of Fourier integral operators (see Sect. 3.3) tells us that its kernel takes the form

$$T(x^1, y^1; x^0, y^0) = \int_{\mathbb{R}^{L+d_\perp}} \frac{d\theta\, d\theta'}{(2\pi h)^{(k+d_\perp+d)/2}}\, a(\rho_\parallel, \rho_\perp)\, e^{\frac{i}{h}\Psi(\rho_\parallel, \rho_\perp)}$$
$$+\mathcal{O}(h^\infty)_{L^2\to L^2}, \tag{5.32}$$

where we use the notation (5.30). Let us group the variables $(x, y) = X$, $(\xi, \eta) = \Xi$, $(\theta, \theta') = \Theta$. We may assume that the symbol $a(X^1, X^0, \Theta)$ is supported in a small neighbourhood of the critical set C_Ψ. In particular, for small values of the transversal variables $\rho_\perp$, $a(\bullet, \rho_\perp)$ is supported near C_ψ. From (5.7), this Fourier integral operator is microlocally subunitary in $V_{j'} \times V_j$.

5.5.1 Using the cut-off near $\mathcal{K}$

We now take into account the cut-offs $(\chi^k)^w$, and study the truncated propagator $T\,(\chi^k)^w$ appearing in (5.17).

Lemma 5.7 *For any $k \geq 0$ we have*

$$T\,(\chi^k)^w = T^{\chi^k} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \to L^2}, \tag{5.33}$$

where the Schwartz kernel of the operator T^{χ^k} is given by

$$T^{\chi^k}(x^1, y^1; x^0, y^0) \stackrel{\text{def}}{=} \int \frac{d\theta\, d\eta^0}{(2\pi h)^{(k+d_\perp+d)/2}}\, a(\rho_\parallel, \rho_\perp)\, \chi^{\sharp(k+1)}(y^1) \times \chi^k(y^0, \eta^0)\, e^{\frac{i}{h}\Psi(\rho_\parallel, \rho_\perp)}, \tag{5.34}$$

where $\chi^{\sharp k} \stackrel{\text{def}}{=} \chi^k|_{\eta=0}$, with χ^k given in (5.12).

Proof As in (5.16), the nesting property $\chi^{\sharp(k+1)} \succ \chi^k \circ \kappa_{j'j}$ and the uniformity (in k) of the symbol estimates on χ^k imply that

$$(\chi^{\sharp(k+1)})^w\, T\,(\chi^k)^w = T\,(\chi^k)^w + \mathcal{O}(\tilde{h}^\infty), \tag{5.35}$$

uniformly for all $k \geq 0$. (We recall that uniformity in k is due to (5.12) and (5.13) and the uniform error estimate comes from (3.7).) The Fourier integral operator calculus in the class $\widetilde{S}_{\frac{1}{2}}$ presented in Lemma 3.1 has the following consequence:

$$(\chi^{\sharp(k+1)})^w\, T\,(\chi^k)^w = T^{\chi^k} + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}),$$

which combined with (5.35) gives (5.33). □

5.5.2 Rescaling the transversal coordinates

Since we work at distances $\sim (h/\tilde{h})^{\frac{1}{2}}$ from the trapped set, it will be convenient to use the rescaled transversal variables

$$\tilde{y} = (\tilde{h}/h)^{\frac{1}{2}} y, \qquad \tilde{\eta} = (\tilde{h}/h)^{\frac{1}{2}} \eta. \tag{5.36}$$

Our cut-offs χ^k, $\widetilde{\chi}^k$ defined in (5.12, 5.13) are then related by $\widetilde{\chi}^\bullet(\tilde{y}, \tilde{\eta}) = \chi^\bullet(y, \eta)$. This change of variables induces the following unitary rescaling $\mathcal{T} : L^2(dx\, dy) \to L^2(dx\, d\tilde{y})$:

$$\mathcal{T}u(x, \tilde{y}) \stackrel{\text{def}}{=} (h/\tilde{h})^{d_\perp/2}\, u(x, (h/\tilde{h})^{\frac{1}{2}}\tilde{y}) = (h/\tilde{h})^{d_\perp/2}\, u(x, y). \tag{5.37}$$

We recall (see for instance [55, (4.7.16)]) that

$$\begin{aligned}&\mathcal{T} a^w(x,y,hD_x,hD_y)\mathcal{T}^* = \tilde{a}^w(x,\tilde{y},hD_x\tilde{h}D_{\tilde{y}}),\\ &\quad \tilde{a}(x,\tilde{y},\xi,\tilde{\eta}) \stackrel{\text{def}}{=} a(x,y,\xi,\eta).\end{aligned}$$

Through this rescaling, the operator T^{χ^k} is transformed into

$$\widetilde{T}^{\chi^k} \stackrel{\text{def}}{=} \mathcal{T} T^{\chi^k} \mathcal{T}^* \ :\ L^2(dxd\tilde{y}) \longrightarrow L^2(dxd\tilde{y}),$$

with Schwartz kernel

$$\begin{aligned}\widetilde{T}^{\chi^k}(x^0,\tilde{y}^0,x^1,\tilde{y}^1) = \int\limits_{\mathbb{R}^k}\int\limits_{\mathbb{R}^{d_\perp}} \frac{d\theta}{(2\pi h)^{\frac{k+d-d_\perp}{2}}}\frac{d\tilde{\eta}^0}{(2\pi\tilde{h})^{d_\perp}}\, a(\rho_\parallel,(h/\tilde{h})^{\frac12}\tilde{\rho}_\perp)\\ \times\, \widetilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\,\widetilde{\chi}^k(\tilde{y}^0,\tilde{\eta}^0)\, e^{\frac{i}{h}\psi(\rho_\parallel)+\delta\Psi(\rho_\parallel;(h/\tilde{h})^{\frac12}\tilde{\rho}_\perp)}\end{aligned} \tag{5.38}$$

5.5.3 Transversal linearization

The factor $\widetilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\widetilde{\chi}^k(\tilde{y}^0,\tilde{\eta}^0)$ appearing in the integrand (5.38) allows us to simplify the above kernel. Indeed, it implies that the variables $\tilde{\rho}_\perp = (\tilde{y}^1,\tilde{y}^0,\tilde{\eta}^0)$ are integrated over a set of diameter $\sim R_1\Lambda^k$. One can then Taylor expand the amplitude and phase function $\delta\Psi$ in (5.38):

$$\begin{aligned}&a(\rho_\parallel,(h/\tilde{h})^{\frac12}\tilde{\rho}_\perp)\,e^{\frac{i}{h}\delta\Psi(\rho_\parallel;(h/\tilde{h})^{\frac12}\tilde{\rho}_\perp)}\widetilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\widetilde{\chi}^k(\tilde{y}^0,\tilde{\eta}^0)\\ &\quad = \big(a(\rho_\parallel,0)+\mathcal{O}_{\tilde{h},k}(h^{\frac12})_{S(T^*\mathbb{R}^d)}\big)\,e^{\frac{i}{\tilde{h}}Q_{\rho_\parallel}(\tilde{\rho}_\perp)}\,\widetilde{\chi}^{\sharp(k+1)}(y^1)\widetilde{\chi}^k(\tilde{y}^0,\tilde{\eta}^0).\end{aligned}$$

Since we will restrict ourselves to values $k\le M\log 1/\tilde{h}$, uniformly bounded with respect to h, we may omit to indicate the k-dependence in the remainder. As a result, up to a small error we may keep only the quadratic part of $\delta\Psi$, namely consider the operator with the Schwartz kernel

$$\int\limits_{\mathbb{R}^k}\int\limits_{\mathbb{R}^{d_\perp}} \frac{d\theta}{(2\pi h)^{\frac{k+d-d_\perp}{2}}}\frac{d\tilde{\eta}^0}{(2\pi\tilde{h})^{d_\perp}}\, a(\rho_\parallel,0)\,\widetilde{\chi}^{\sharp(k+1)}(\tilde{y}^1)\widetilde{\chi}^k(\tilde{y}^0,\tilde{\eta}^0)\,e^{\frac{i}{h}\psi(\rho_\parallel)}\,e^{\frac{i}{\tilde{h}}Q_{\rho_\parallel}(\tilde{\rho}_\perp)}.$$

Combining the above pointwise estimates with the fact that $a \in S(T^*\mathbb{R}^{3d})$, and with (5.35) and Lemma 5.7, gives

$$\widetilde{T}^{\chi^k} = \widetilde{T}(\tilde{\chi}^k)^w(\tilde{y}, \tilde{h} D_{\tilde{y}}) + \mathcal{O}(\tilde{h}^\infty)_{L^2 \to L^2},$$
$$\widetilde{T}(x^1, y^1; x^0, y^0) = \int \frac{d\theta}{(2\pi h)^{(k+d-d_\perp)/2}} \frac{d\tilde{\eta}^0}{(2\pi \tilde{h})^{d_\perp}}\, a(\rho_\parallel, 0)\, e^{\frac{i}{h}\psi(\rho_\parallel)} e^{\frac{i}{\tilde{h}} Q_{\rho_\parallel}(\tilde{\rho}^\perp)} \tag{5.39}$$

uniformly for $k \leq M \log 1/\tilde{h}|$.

5.5.4 *Factoring out the transversal contribution*

For each $\rho_\parallel \in \operatorname{supp} a(\bullet, 0)$, the quadratic phase $Q_{\rho_\parallel}(\bullet)$ generates a symplectic transformation $S(\rho_\parallel)$ (which, in the case $\rho_\parallel \in C_\psi$ corresponds coincides with the transformation $S(\rho^0)$ of (5.18)). As already shown in (3.22), this phase allows to represent the metaplectic operator $M(\rho_\parallel) : L^2(d\tilde{y}) \to L^2(d\tilde{y})$ which $\tilde{h}$-quantizes this symplectomorphism:

$$M(\rho_\parallel)(\tilde{y}^1, \tilde{y}^0) \stackrel{\text{def}}{=} (2\pi\tilde{h})^{-d_\perp} \int_{\mathbb{R}^{d_\perp}} \det(H_{12}(\rho_\parallel))^{1/2}\, e^{\frac{i}{\tilde{h}} Q_{\rho_\parallel}(\tilde{\rho}_\perp)} d\tilde{\eta}^0, \tag{5.40}$$

where $H_{12}(\rho_\parallel)$ is the block matrix appearing in $Q_{\rho_\parallel}$, similarly as in (5.24, 5.25).

Remark 5.8 In the expression (5.40) we implicitly chose a sign for the square root of $\det(H_{12}(\rho_\parallel))$. Indeed, the metaplectic representation of the symplectic group is 1-to-2, a given symplectic matrix S being quantized into two possible operators $\pm M$. The relations (5.27) and the uniform expansion property (5.22) show that $\det(H_{12}(\rho_\parallel))$ does not vanish on the support of the amplitude $a(\bullet, 0)$ (which is a small neighbourhood of $C_\psi \times \{\tilde{y}^0 = \tilde{y}^1 = \tilde{\eta}^0 = 0\}$), so we may fix the sign in each connected component of this support. This remark will be relevant in Sect. 5.6.

Defining the symbol

$$\widetilde{a}(\rho_\parallel) \stackrel{\text{def}}{=} \frac{a(\rho_\parallel, 0)}{\det(H_{12}(\rho_\parallel))^{\frac{1}{2}}},$$

we interpret the operator $\widetilde{T}$ in (5.39) as a Fourier integral operator with an operator valued symbol, $M(\rho_\parallel)$, where M is given by (5.40). That fits exactly in the framework presented in Proposition 3.5:

$$\widetilde{T}(u \otimes v)(x^1, \tilde{y}^1)$$
$$= (2\pi h)^{-(k+d-d_\perp)/2} \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} \widetilde{a}(\rho_\parallel)\, [M(\rho_\parallel)v](\tilde{y}^1)\, e^{\frac{i}{h}\psi(\rho_\parallel)} u(x^0) dx^0 d\theta.$$

We now apply Lemma 3.4 to see that

$$\widetilde{T} = \mathrm{Op}_h^w(M) T^\parallel + \mathcal{O}_{\tilde{h}}(h)_{\mathcal{D}^{m+\ell} \to \mathcal{D}^\ell}, \tag{5.41}$$

where $m = m_{d-d_\perp}$ is defined in (3.7) and where the Schwartz kernel of $T^\parallel$ is given by

$$T^\parallel(x^0, x^1) = (2\pi h)^{-k} \int_{\mathbb{R}^k} \widetilde{a}(\rho_\parallel)\, e^{\frac{i}{h}\psi(\rho_\parallel)} d\theta. \tag{5.42}$$

The operator valued symbol $M(\rho^1)$ is the metaplectic operator $\tilde{h}$-quantizing $S(\rho^0)$, where $\rho^1 = \widetilde{\kappa}(\rho^0)$ and $S(\rho^0)$ is given in (5.18). We summarize these findings in the following

Proposition 5.9 *Suppose that the Schwartz kernel of T is given by (5.32), χ^k, $\widetilde{\chi}^k$ are given in (5.12), and $\mathcal{T}$ is the unitary rescaling defined in (5.37).*

Then for $k \leq K(\tilde{h})$, where $K(\tilde{h})$ may depend on $\tilde{h}$ but not on h,

$$\mathcal{T}\left(T(\chi^k)^w\right)\mathcal{T}^* = \mathrm{Op}_h^w(M) T^\parallel (\widetilde{\chi}^k)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2 \to L^2} + \mathcal{O}_{\tilde{h}}(h)_{L^2 \to L^2}, \tag{5.43}$$

where $T^\parallel$ is given by (5.42) and $M(x^1, \xi^1)$ given by (5.40) with $\rho_\parallel \in C_\psi$ determined by $(x^1, \xi^1) = (x^1, \partial_{x^1}\psi(\rho_\parallel))$. Here and below we use the abbreviation $(\widetilde{\chi}^k)^{\tilde{w}} \overset{\text{def}}{=} (\widetilde{\chi}^k)^w(\tilde{y}, \tilde{h} D_{\tilde{y}})$.

Proof Lemma 5.7, (5.39), and (5.41) give (5.43) with the remainder

$$\mathcal{O}(\tilde{h}^\infty)_{L^2(dxd\tilde{y}) \to L^2(dxd\tilde{y})} + \mathcal{O}_{\tilde{h}}(h)_{L^2(dx)\otimes\mathcal{D}^m \to L^2(dxd\tilde{y})} (\widetilde{\chi}^k)^{\tilde{w}},$$

where $m = m_{d-d_\perp}$ is given in (3.7). The definition of $\widetilde{\chi}^k$ in (5.12) and (3.24) show that

$$(\widetilde{\chi}^k)^{\tilde{w}} = \mathcal{O}(\Lambda^{2mk}) : L^2(d\tilde{y}) \longrightarrow \mathcal{D}^m,$$

and that gives the remainder in (5.43). $\square$

5.6 Back to the iterated propagator

We can now come back to (5.9) and (5.17), re-establishing the subscripts $j_{k+1}j_k$ on the relevant objects. We rescale all the operators by conjugating them through $\mathcal{T}$. Fixing the limit indices j_0, j_n, we want to study the sum of operators obtained by conjugation of terms in (5.17) by $\mathcal{T}$:

$$\begin{aligned}\mathcal{T}[\boldsymbol{T}^n]_{j_n j_0}(\chi^0)^w\mathcal{T}^* &= \mathcal{T}\left(\sum_{\boldsymbol{j}}\prod_{k=n-1}^{0} T_{j_{k+1}j_k}(\chi^k)^w\right)\mathcal{T}^* + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}\\ &= \sum_{\boldsymbol{j}} \widetilde{T}_{j_n j_{n-1}}(\widetilde{\chi}^{n-1})^{\tilde{w}}\cdots(\widetilde{\chi}^1)^{\tilde{w}}\widetilde{T}_{j_1 j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}\end{aligned} \tag{5.44}$$

where the sum runs over all possible sequences $\boldsymbol{j} = j_{n-1}\dots j_1$. A sequence (which could be thought of geometrically as a path) $j_n\boldsymbol{j}\,j_0$ will be relevant only if it is *physical*, meaning that there exists points $\rho\in K^\delta$ such that $\varphi_{kt_0}(\rho)\in U_{j_k}$ for all times $k=0,\dots,n$ (we say that the path $j_n\boldsymbol{j}\,j_0$ contains the trajectory of ρ). Any unphysical sequence leads to a term of order $\mathcal{O}(h^\infty)$. On the other hand, for a given point $\rho\in K^\delta$ there are usually many sequences $\boldsymbol{j}$ containing its trajectory, since the neighbourhoods (U_j)'s overlap, and so do the cut-offs (π_j).

For physical sequences $j_n\boldsymbol{j}\,j_0$ we define the departure set $D_{j_n\boldsymbol{j}\,j_0}$ as the set of points $\kappa_{j_0}(\rho)$, $\rho\in U_{j_0}=\kappa_{j_0}^{-1}(V_{j_0})$ such that $\varphi_{\ell t_0}(\rho)\in U_{j_\ell}$ for $0\le\ell\le n$. We then put

$$D^n_{j_n j_0} = \bigcup_{\boldsymbol{j}} D_{j_n\boldsymbol{j}\,j_0} = \kappa_{j_0}\left(\{\rho\in U_{j_0}\cap K^\delta,\ \varphi_{nt_0}(\rho)\in U_{j_n}\}\right). \tag{5.45}$$

We now simplify the expression (5.44), in the following way.

Lemma 5.10 *In the notation of* (5.9) *and* (5.44), *and for* $n\le M\log 1/\tilde{h}$,

$$\mathcal{T}[\boldsymbol{T}^n]_{j_n j_0}(\chi^0)^w\mathcal{T}^* = \mathrm{Op}_h^w(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}. \tag{5.46}$$

Here $T^{n\|}_{j_n j_0}$ *is a Fourier integral operator on* $L^2(dx)$ *quantizing the map* $\widetilde{\kappa}^n : V_{j_0}\to V_{j_n}$, *defined on the departure set* $D^n_{j_0 j_n}$. *For each* $\rho\in A^n_{j_n j_0} = \widetilde{\kappa}^n(D^n_{j_n j_0})$ *(the arrival set) the operator valued symbol* $M^n_{j_n j_0}(\rho)$ *is a metaplectic operator quantizing the symplectic map*

$$S^n_{j_n j_0}((\widetilde{\kappa}^n)^{-1}(\rho)) = d_\perp\kappa^n((\widetilde{\kappa}^n)^{-1}(\rho)).$$

Proof If we insert the approximate factorizations (5.43) in a term $\boldsymbol{j}$ of the sum in the left hand side of (5.44), this term becomes

$$\mathrm{Op}_h^w(M_{j_n j_{n-1}})T^{\parallel}_{j_n j_{n-1}}(\widetilde{\chi}^{n-1})^{\tilde{w}} \cdots \mathrm{Op}_h^w(M_{j_1 j_0})T^{\parallel}_{j_1 j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}. \tag{5.47}$$

We now observe that just as we inserted the cut-offs χ^k to obtain (5.17) from (5.9) we can remove them so that each term becomes

$$\mathrm{Op}_h^w(M_{j_n j_{n-1}})T^{\parallel}_{j_n j_{n-1}} \cdots \mathrm{Op}_h^w(M_{j_1 j_0})T^{\parallel}_{j_1 j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}. \tag{5.48}$$

We can now apply Lemmas 3.3, 3.4 and Proposition 3.5 to see that

$$\begin{aligned}&\mathrm{Op}_h^w(M_{j_n j_{n-1}})T^{\parallel}_{j_n j_{n-1}} \cdots \mathrm{Op}_h^w(M_{j_1 j_0})T^{\parallel}_{j_1 j_0}\\ &\quad = Op(M_{j_n j_{n-1}\cdots j_0})T^{\parallel}_{j_n \boldsymbol{j} j_0} + \mathcal{O}(\tilde{h}^{-2m_{d-d_\perp}n}h)_{L^2(dx)\otimes\mathcal{D}^{2nm_{d-d_\perp}}\to L^2},\end{aligned} \tag{5.49}$$

where we use the shorthands

$$\begin{aligned}
T^{\parallel}_{j_n j_{n-1}\cdots j_0} &\stackrel{\text{def}}{=} T^{\parallel}_{j_n j_{n-1}}\, T^{\parallel}_{j_{n-1} j_{n-2}} \cdots T^{\parallel}_{j_1 j_0},\\
M_{j_n j_{n-1}\cdots j_0} &\stackrel{\text{def}}{=} (M_{j_n j_{n-1}})(M_{j_{n-1} j_{n-2}} \circ \widetilde{\kappa}_{j_{n-1} j_n}) \cdots (M_{j_2 j_1} \circ \widetilde{\kappa}_{j_2\cdots j_n})\\
&\qquad (M_{j_1 j_0} \circ \widetilde{\kappa}_{j_1\cdots j_n}),\\
\widetilde{\kappa}_{j_k j_{k-1}\cdots j_0} &\stackrel{\text{def}}{=} \widetilde{\kappa}_{j_k j_{k-1}} \circ \widetilde{\kappa}_{j_{k-1} j_{k-2}} \cdots \circ \widetilde{\kappa}_{j_1 j_0}.
\end{aligned}$$

These expressions only make sense for physical sequences $j_n \boldsymbol{j} j_0$. The map $\widetilde{\kappa}_{j_n \boldsymbol{j} j_0}$ is defined on the departure set $D_{j_n \boldsymbol{j} j_0}$.

The metaplectic operator $M_{j_n \boldsymbol{j} j_0}(\rho)$ quantizes the symplectomorphism $S_{j_n \boldsymbol{j} j_0}(\rho^0)$, with $\rho = \widetilde{\kappa}^n(\rho^0) \in A_{j_n \boldsymbol{j} j_0}$. This symplectomorphism represents, in the charts $V_{j_0} \to V_{j_n}$, the transverse linearization of the flow φ_{nt_0} at the point $\kappa_{j_0}^{-1}(\rho^0)$. As a consequence, the symplectic matrix $S_{j_n \boldsymbol{j} j_0}(\rho^0)$ is identical for all sequences $j_n \boldsymbol{j} j_0$ containing the trajectory of ρ^0, and we call this matrix $S^n_{j_n j_0}(\rho^0)$. Hence, two metaplectic operators $M_{j_n \boldsymbol{j} j_0}(\rho)$, $M_{j_n \boldsymbol{j}' j_0}(\rho)$ corresponding to two different allowed sequences can at most differ by a global sign.

For all ρ in the arrival set

$$A^n_{j_n j_0} = \bigcup_{\boldsymbol{j}} A_{j_n \boldsymbol{j} j_0} = \kappa_{j_n}\big(\big\{\rho \in U_{j_n} \cap K^\delta,\ \varphi_{-nt_0}(\rho) \in U_{j_0}\big\}\big),$$

we choose the sign of the metaplectic operator $M^n_{j_n j_0}(\rho)$ quantizing $S^n_{j_n j_0}(\rho^0)$, such that $M^n_{j_n j_0}(\rho)$ depends smoothly on ρ on each connected component of $A^n_{j_n j_0}$ (there is no obstruction to this fact, due to the property mentioned in the Remark 5.8: the symplectomorphisms $S^n_{j_n j_0}(\rho)$ also have the form (5.23)). Hence, for each physical sequence $j_n \boldsymbol{j} j_0$ we have

$$M_{j_n \boldsymbol{j} j_0}(\rho) = \varepsilon_{j_n \boldsymbol{j} j_0}(\rho) M^n_{j_n j_0}(\rho), \quad \rho \in D_{j_n \boldsymbol{j} j_0}, \tag{5.50}$$

for some sign $\varepsilon_{j_n \boldsymbol{j} j_0}(\rho) \in \{\pm\}$ constant on each connected component of $A_{j_n \boldsymbol{j} j_0}$. As before, the functions $\rho \mapsto \varepsilon_{j_n \boldsymbol{j} j_0}(\rho)$, $\rho \mapsto M_{j_n \boldsymbol{j} j_0}(\rho)$ can be smoothly extended outside $A_{j_n \boldsymbol{j} j_0}$, into compactly supported symbols. Lemma 3.3 and the identity (5.50) give

$$\begin{aligned}\mathrm{Op}^w_h(M_{j_n j_{n-1}\cdots j_0}) T^{\parallel}_{j_n \boldsymbol{j} j_0} = \mathrm{Op}^w_h(M^n_{j_n j_0})\, (\varepsilon_{j_n \boldsymbol{j} j_0})^w,\, T^{\parallel}_{j_n \boldsymbol{j} j_0} \\ + \mathcal{O}_{\tilde{h}}(h)_{L^2(dx)\otimes\mathcal{D}^{m d_\perp}\to L^2}.\end{aligned} \tag{5.51}$$

When $(\widetilde{\chi}^0)^{\tilde{w}}$ is inserted in (5.49) and (5.51) we apply (3.24) to see that

$$\begin{aligned}\mathcal{O}(\tilde{h}^{-2nm_{d-d_\perp}} h)_{L^2(dx)\otimes\mathcal{D}^{2nm_{d-d_\perp}}\to L^2} (\chi^0)^{\tilde{w}} &= \mathcal{O}(\tilde{h}^{-2nm_{d-d_\perp}} h)_{L^2\to L^2} \\ &= \mathcal{O}_{\tilde{h}}(h)_{L^2\to L^2},\end{aligned}$$

and hence that error term can be absorbed into $\mathcal{O}(\tilde{h}^\infty)$.

Returning to (5.47) we see that the sum in the right hand side of (5.44) can be factorized in the following way:

$$\begin{aligned}&\sum_{\boldsymbol{j}} \widetilde{T}_{j_n j_{n-1}} (\widetilde{\chi}^{n-1})^{\tilde{w}} \cdots \widetilde{T}_{j_1 j_0} (\widetilde{\chi}^0)^{\tilde{w}} \\ &\quad = \mathrm{Op}^w_h(M^n_{j_n j_0}) \left(\sum_{\boldsymbol{j}} T^{\parallel}_{j_n \boldsymbol{j} j_0}\, (\varepsilon_{j_n \boldsymbol{j} j_0})^w \right) (\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2},\end{aligned} \tag{5.52}$$

with a uniform remainder for $n \le M \log 1/\tilde{h}$. Let us put $T^{n\parallel}_{j_n j_0} \stackrel{\text{def}}{=} \sum_{\boldsymbol{j}} T^{\parallel}_{j_n \boldsymbol{j} j_0} (\varepsilon_{j_n \boldsymbol{j} j_0})^w$, so that the above identity reads exactly like in the statement of the Lemma. The operator $T^{n\parallel}_{j_n j_0}$ is sum of Fourier integral operators $T^{\parallel}_{j_n \boldsymbol{j} j_0}$ defined with different phase functions $\psi_{j_n \boldsymbol{j} j_0}$, yet these phases generate (on different parts of phase space) the same map $\widetilde{\kappa}^n : D^n_{j_n j_0} \to A^n_{j_n j_0}$. Hence, $T^{n\parallel}_{j_n j_0}$ is a Fourier integral operator quantizing $\widetilde{\kappa}^n$. This completes the proof of (5.46). □

The next lemma shows that the Fourier integral operator $T^{n\|}_{j_n j_0}$ is essentially subunitary.

Lemma 5.11 *Let $M > 0$. For any small $\tilde{h} > 0$, there exists $h_0 = h_0(\tilde{h})$ such that, for any sequence $\boldsymbol{j}$ of length $n \leq M \log 1/\tilde{h}$ and any $h \leq h_0(\tilde{h})$, the operator $T^{n\|}_{j_n j_0}$ satisfies the following norm estimate:*

$$\|T^{n\|}_{j_n j_0}\|_{L^2(dx)\to L^2(dx)} \leq 1 + \mathcal{O}(\tilde{h}). \tag{5.53}$$

Proof We first note that we can bound the left hand side of (5.53) by $\tilde{h}^{-CM}$, for some C—that follows from a trivial estimate of the terms $T^{\|}_{j_n \boldsymbol{j} j_0}$ in (5.52).

To prove (5.53) it is clearly enough to prove the bound $\|T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}}\|_{L^2(dx d\tilde{y})\to L^2(dx d\tilde{y})} \leq 1 + \mathcal{O}(\tilde{h}^\infty)$. From Lemma 5.2 we know that $\|\mathbf{T}^n\|_{(L^2)^J\to(L^2)^J} \leq 1 + \mathcal{O}(h)$, which implies that $\|[\mathbf{T}^n]_{j_0 j_n}\|_{L^2\to L^2} \leq 1 + \mathcal{O}(h)$. Lemma 5.10 then shows that

$$\|\mathrm{Op}^w_h(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}}\|_{L^2\to L^2} \leq 1 + \mathcal{O}(\tilde{h}^\infty). \tag{5.54}$$

The family of unitary metaplectic operators $\rho \mapsto M^n_{j_n j_0}(\rho)^{-1}$ is well defined for ρ in the neighbourhood of the arrival set $A^n_{j_n j_0}$, and $T^{n\|}_{j_n j_0}$ is microlocalized in any small neighbourhood of $A^n_{j_n j_0} \times D^n_{j_n j_0} \subset V_{j_n} \times V_{j_0}$. Lemma 3.3 and (3.24) then show that

$$\begin{aligned}
T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}} &= \mathrm{Op}^w_h((M^n_{j_n j_0})^{-1})\mathrm{Op}^w_h(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}} \\
&\quad + \mathcal{O}_{\tilde{h}}(h\|T^{n\|}_{j_n j_0}\|)_{L^2(dx)\otimes\mathcal{D}^{2md-d_\perp}\to L^2}(\widetilde{\chi}^0)^w \\
&= \mathrm{Op}^w_h((M^n_{j_n j_0})^{-1})\mathrm{Op}^w_h(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}_{\tilde{h}}(h)_{L^2\to L^2}.
\end{aligned}$$

where we used the above a priori bound on $\|T^{n\|}_{j_n j_0}\|$.

Just as before we can insert the cut-off $\widetilde{\chi}^n$ (see (5.12)) with a $\mathcal{O}(\tilde{h}^\infty)$ loss. We also introduce a cut-off $\psi = \psi(x, \xi)$ to a small neighbourhood of $A_{j_n j_0}$. (It was not necessary before as $T^{n\|}_{j_n j_0}$ provided the needed localization.) This and (5.54) give the bound

$$\begin{aligned}
\|T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}}\| &\leq \|\mathrm{Op}^w_h((M^n_{j_n j_0})^{-1}\psi)(\widetilde{\chi}^n)^{\tilde{w}}\|\|\mathrm{Op}^w_h(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}}\| \\
&\quad + \mathcal{O}(\tilde{h}^\infty) \\
&\leq \|\mathrm{Op}^w_h((M^n_{j_n j_0})^{-1}\psi)(\widetilde{\chi}^n)^{\tilde{w}}\|(1 + \mathcal{O}(\tilde{h}^\infty)) \\
&\quad + \mathcal{O}(\tilde{h}^\infty).
\end{aligned}$$

Since by Lemma 3.3 and (3.24)

$$[\mathrm{Op}_h^w((M^n_{j_n j_0})^{-1}\psi)(\widetilde{\chi}^n)^{\tilde{w}}]^*\mathrm{Op}_h^w((M^n_{j_n j_0})^{-1}\psi)(\widetilde{\chi}^n)^{\tilde{w}} \\ = [\psi^w]^*\psi^w[(\widetilde{\chi}^n)^{\tilde{w}}]^*(\widetilde{\chi}^n)^{\tilde{w}} + \mathcal{O}_{\tilde{h}}(h)_{L^2\to L^2},$$

we have

$$\|\mathrm{Op}_h^w((M^n_{j_n j_0})^{-1}\psi)(\widetilde{\chi}^n)^{\tilde{w}}\| \le \|\psi^w\|\|(\widetilde{\chi}^n)^{\tilde{w}}\| + \mathcal{O}_{\tilde{h}}(h) \le 1 + \mathcal{O}(\tilde{h}),$$

and the bound (5.53) follows. □

5.7 Inserting the final cut-off

We now return to the operator $\chi^w e^{-itn_0P/h}\chi^w$. From Lemma 5.3 we easily obtain

$$\chi^w e^{-int_0P/h}\chi^w u = \sum_{j_n, j_0} \Pi_{j_n}\mathcal{U}^*_{j_n}\chi^w_{j_n}[(\boldsymbol{T})^n]_{j_n j_0}\chi^w_{j_0}u_{j_0} + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}}) \\ = \sum_{j_n, j_0} \Pi_{j_n}\mathcal{U}^*_{j_n}\chi^w_{j_n}(\chi^0)^w[(\boldsymbol{T})^n]_{j_n j_0}(\chi^0)^w\chi^w_{j_0}u_{j_0} + \mathcal{O}(\tilde{h}^\infty), \tag{5.55}$$

where in the first line we used (5.8), while in the second line we used (5.15). Hence our last step will consist in estimating the norm of the operator $(\chi^0)^w[\boldsymbol{T}^n]_{j_n j_0}(\chi^0)^w$ (or its conjugate through $\mathcal{T}$). To this aim we will use Lemma 3.4, Proposition 3.5 and the factorization (5.46) to obtain

$$(\widetilde{\chi}^0)^{\tilde{w}}\mathcal{T}[\boldsymbol{T}^n]_{j_n j_0}\mathcal{T}^*(\widetilde{\chi}^0)^{\tilde{w}} = (\widetilde{\chi}^0)^{\tilde{w}}\mathrm{Op}_h^w(M^n_{j_n j_0})T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2} \\ = T^{n\|}_{j_n j_0}(\widetilde{\chi}^0)^{\tilde{w}}\mathrm{Op}_h^w(N^n_{j_n j_0})(\widetilde{\chi}^0)^{\tilde{w}} + \mathcal{O}(\tilde{h}^\infty)_{L^2\to L^2}. \tag{5.56}$$

Here the operator valued symbol $N^n_{j_n j_0}(\rho) = M^n_{j_n j_0}((\widetilde{\kappa}^n)^{-1}(\rho))$, $\rho \in D^n_{j_n j_0}$, is a metaplectic operator quantizing the symplectic map $S^n_{j_n j_0}(\rho) = d_\perp\kappa^n(\rho)$. (Having it on the right now makes the notation slightly less cumbersome.)

In Lemma 5.11 we control the norm of $T^{n\|}_{j_n j_0}$. There remains to control the norm of the factor $(\widetilde{\chi}^0)^{\tilde{w}}\mathrm{Op}_h^w(N^n_{j_n j_0})(\widetilde{\chi}^0)^{\tilde{w}}$. For that it is enough to control the operator-valued symbol $\mathrm{Op}^w_{\tilde{h},\tilde{y}}(\widetilde{\chi}^0)\,N^n_{j_n j_0}(\rho)\,\mathrm{Op}^w_{\tilde{h},\tilde{y}}(\widetilde{\chi}^0)$.

5.7.1 Controlling the symbol

In (5.19) we defined, for each point $\rho \in \mathcal{K} \cap D_{j_1 j_0}$, a symplectic transformation $R(\rho) \in \mathrm{Sp}(2d_\perp, \mathbb{R})$ which maps the y-space to E^+_ρ and the $\tilde{\eta}$-space to E^-_ρ. This transformation is ϵ-close to the identity and in particular it is uniformly bounded with respect to ρ.

By iteration of this property, for any $\rho_0 \in D^n_{j_n j_0}$, the map

$$\widetilde{S}^n_{j_n j_0}(\rho_0) \stackrel{\text{def}}{=} R(\rho_n)^{-1} S^n_{j_n j_0}(\rho_0) R(\rho_0)$$

is block-diagonal in the basis (y, η):

$$\widetilde{S}^n_{j_n j_0}(\rho_0) = \begin{pmatrix} \Lambda^n(\rho_0) & 0 \\ 0 & {}^T\Lambda^n(\rho_0)^{-1} \end{pmatrix}, \tag{5.57}$$

where $\Lambda^n(\rho_0)$ is expanding. We may quantize $R(\rho)$ into metaplectic operators $A(\rho)$, and define

$$\widetilde{N}^n_{j_n j_0}(\rho_0) \stackrel{\text{def}}{=} A(\rho_n)^{-1}\, N^n_{j_n j_0}(\rho_0)\, A(\rho_0)$$

which quantizes $\widetilde{S}^n_{j_n j_0}(\rho_0)$.

We can then rewrite

$$(\widetilde{\chi}^0)^{\tilde{w}}\, N^n_{j_n j_0}(\rho)\, (\widetilde{\chi}^0)^{\tilde{w}} = (\widetilde{\chi}^0)^{\tilde{w}}\, A(\rho_n)\, \widetilde{N}^n_{j_n j_0}(\rho_0)\, A(\rho_0)^{-1}\, (\widetilde{\chi}^0)^{\tilde{w}}. \tag{5.58}$$

We are interested in the $L^2 \to L^2$ norm of this operator. Since metaplectic operators are unitary, and using the covariance of the Weyl quantization with respect to metaplectic operators, this norm is equal to that of

$$(\widetilde{\chi}^0_{\rho_n})^{\tilde{w}}\, (\widetilde{\chi}^0_{\rho_0} \circ \widetilde{S}^n_{j_n j_0}(\rho_0)^{-1})^{\tilde{w}}, \qquad \widetilde{\chi}^0_{\rho_n} \stackrel{\text{def}}{=} \widetilde{\chi}^0 \circ R(\rho_n), \quad \widetilde{\chi}^0_{\rho_0} \stackrel{\text{def}}{=} \widetilde{\chi}^0 \circ R(\rho_0).$$

The block diagonal form of $\widetilde{S}^n_{j_n j_0}(\rho_0)$ shows that

$$\left[\widetilde{\chi}^0_{\rho_0} \circ (\widetilde{S}^n_{j_n j_0}(\rho_0))^{-1}\right](\tilde{y}, \tilde{\eta}) = \widetilde{\chi}^0_{\rho_0}(\Lambda^n(\rho_0)^{-1}\tilde{y}, {}^T\Lambda^n(\rho_0)\tilde{\eta}). \tag{5.59}$$

We may now invoke the following simple

Lemma 5.12 *Suppose that A is a $m \times m$ real invertible matrix and that $\chi_1, \chi_2 \in \mathscr{S}(\mathbb{R}^{2m})$. Then*

$$\|\chi_1^w(x, \tilde{h}D_x)\chi_2^w(Ax, {}^TA^{-1}\tilde{h}D_x)\|_{L^2(\mathbb{R}^m)\to L^2(\mathbb{R}^m)} \leq C|\det A|^{\frac{1}{2}}\tilde{h}^{-\frac{m}{2}}, \tag{5.60}$$

where C depends on certain seminorms of χ_1 and χ_2, but not on A.

We remark that the upper bound becomes nontrivial only if $|\det A| \ll \tilde{h}^{m/2}$. When that holds one cannot apply the $\tilde{h}$-symbol calculus any longer because the second factor is not the quantization of a symbol in the class $S(\mathbb{R}^{2m})$, uniformly in $\tilde{h}$ and A. When applicable, the symbol calculus would give the norm equal to $\max_{x,\xi} |\chi_1(x,\xi)\, \chi_2(Ax, {}^T A^{-1}\xi)| + \mathcal{O}(\tilde{h})$—see [55, Theorem 13.13].

Proof If we put $\hat{\chi}_j(x, Z) \stackrel{\text{def}}{=} \int_{\mathbb{R}^m} \chi_j(x,\xi) e^{i\langle Z,\xi\rangle} d\xi$, then the kernel of the operator in the lemma is given by

$$K(x,y) = \frac{1}{(2\pi\tilde{h})^{2m}} \int_{\mathbb{R}^{3m}} \chi_1\left(\frac{x+z}{2}, \xi\right) \chi_2\left(\frac{Az+Ay}{2}, {}^T A^{-1}\eta\right)$$
$$\times e^{i\langle x-z,\xi\rangle/\tilde{h} + i\langle z-y,\eta\rangle/\tilde{h}} d\xi\, d\eta\, dz$$
$$= \frac{|\det A|}{(2\pi\tilde{h})^{2m}} \int_{\mathbb{R}^m} \hat{\chi}_1\left(\frac{x+z}{2}, \frac{x-z}{\tilde{h}}\right) \hat{\chi}_2\left(\frac{Az+Ay}{2}, \frac{Az-Ay}{\tilde{h}}\right) dz.$$

We will estimate the norm using Schur's Lemma and hence we need to show that

$$\left(\max_{x\in\mathbb{R}^m} \int |K(x,y)| dy\right) \left(\max_{y\in\mathbb{R}^m} \int |K(x,y)| dx\right) \leq C^2 |\det A|\, \tilde{h}^{-m}. \tag{5.61}$$

Making a change of variables $Z = (x-z)/\tilde{h}$ and $X = (x+z)/\tilde{h}$ we obtain

$$\int |K(x,y)| dx \leq C_1 (\max_{\mathbb{R}^{2m}} |\hat{\chi}_2|)\, |\det A| \tilde{h}^{-m} \iint |\hat{\chi}_1(X,Z)| dZ dX$$
$$\leq C |\det A|\, \tilde{h}^{-m}.$$

To estimate the integral in y let

$$F(Z) = \max_{\mathbb{R}^m} |\hat{\chi}_1(\bullet, Z)|, \quad G(Y) = \max_{\mathbb{R}^m} |\hat{\chi}_2(\bullet, Y)|,$$

noting that our assumptions give $F(Z) = \mathcal{O}(\langle Z\rangle^{-\infty})$, $G(Y) = \mathcal{O}(\langle Y\rangle^{-\infty})$. Changing variables to $Z = (x-z)/\tilde{h}$ and $Y = (Az - Ay)/\tilde{h}$ we obtain,

$$\int |K(x,y)| dy \leq C_3 \iint F(Z) G(Y) dZ dY \leq C.$$

This proves the upper bound (5.60). □

Applying Lemma 5.12 to the product on the right hand side of (5.58) we get the bound

$$\|(\widetilde{\chi}^0)^{\tilde{w}}\, N^n_{j_n j_0}(\rho_0)\, (\widetilde{\chi}^0)^{\tilde{w}}\|_{L^2(d\tilde{y})\to L^2(d\tilde{y})} \leq C(\widetilde{\chi}^0_{\rho_0}, \widetilde{\chi}^0_{\rho_n})|\det \Lambda^n(\rho_0)|^{-1/2}\, \tilde{h}^{-d_\perp/2}.$$

Since the transformations $R(\rho)$ are uniformly bounded, the prefactor $C(\widetilde{\chi}^0_{\rho_0}, \widetilde{\chi}^0_{\rho_n})$ is uniformly bounded with respect to ρ_0. On the other hand, the determinant of $\Lambda^n(\rho_0)^{-1}$ can be bounded as follows.

Lemma 5.13 *Take $\epsilon_0 > 0$ arbitrary small. Then there exists $C_{\epsilon_0} > 0$ such that,*

$$\forall n \geq 1,\ \forall \rho_0 \in D^n_{j_n j_0}, \quad |\det \Lambda^n(\rho_0)^{-1}| \leq C_{\epsilon_0} e^{-(\lambda_0-\epsilon_0)n t_0},$$

where λ_0 was defined by (1.19), and $t_0 > 0$ is chosen large enough, as explained in the comment following (5.22).

Proof This follows from writing the definition of λ_0 using the local coordinate frames. □

We have thus obtained the following upper bound:

$$\|(\widetilde{\chi}^0)^{\tilde{w}}\, N^n_{j_n j_0}(\rho_0)\, (\widetilde{\chi}^0)^{\tilde{w}}\|_{L^2(d\tilde{y})\to L^2(d\tilde{y})} \leq C_\epsilon\, \tilde{h}^{-d_\perp/2}\, e^{-(\lambda_0-\epsilon_0)n t_0}, \quad (5.62)$$

valid for any $n \geq 1$ and any $\rho_0 \in D^n_{j_n j_0}$. In particular, the time n may arbitrarily depend on $\tilde{h}$.

When $n \leq M \log 1/\tilde{h}$, for $M > 0$ arbitrary large but independent of $\tilde{h}$ or h, we combine this bound with (5.17), Lemma 5.11 and Lemma 5.10 to obtain the estimate (5.2), which was the goal of this section.

6 Microlocal weights and estimates away from the trapped set

In this section we will justify the estimates described as Step 2 of the proof in Sect. 2. That will involve a quantization of the escape function G given in Proposition 4.7 with $\epsilon = (h/\tilde{h})^{\frac{1}{2}}$. That means that we will use the calculus described in Sect. 3.2.

6.1 Exponential weights

Suppose that $g \in \mathcal{C}^\infty_c(T^*X;\mathbb{R})$ satisfies the following estimates:

$$\frac{\exp g(\rho)}{\exp g(\rho')} \leq C\big(1+(\tilde{h}/h)^{\frac{1}{2}} d(\rho,\rho')\big)^N, \quad \partial^\alpha_\rho g = \mathcal{O}\big((h/\tilde{h})^{-|\alpha|/2}\big),\ |\alpha|>0, \quad (6.1)$$

for some N and C, and for some distance function $d(\rho, \rho')$ on $T^*X \times T^*X$ (since g is compactly supported, the estimate is independent of the choice of d—we can d to be the distance function given by a Riemannian metric). We note that G defined in Proposition 4.7 with $\epsilon = (h/\tilde{h})^{\frac{1}{2}}$ satisfies these assumptions.

We first recall a variant of the Bony-Chemin theorem [6, Théorème 6.4], [55, Theorem 8.6] in the form presented in [37, Proposition 3.5, (3.21), (3.22)] (as usual $g^w = \mathrm{Op}_h^w(g)$):

Proposition 6.1 *Suppose that $g \in \mathcal{C}_c^\infty(T^*X)$ satisfies (6.1). Then*

$$\exp(g^w) = b^w, \tag{6.2}$$

where the symbol $b(x, \xi)$ satisfies the bounds

$$|\partial^\alpha b(\rho)| \leq C_\alpha \, e^{g(\rho)} \big(h/\tilde{h}\big)^{-|\alpha|/2}, \tag{6.3}$$

in any local coordinates near the support of g.

If supp $g \Subset U$*, for an open $U \Subset T^*X$, then*

$$\partial_x^\alpha \partial_\xi^\beta (b(x, \xi) - 1) = \mathcal{O}(h^\infty \langle \xi \rangle^{-\infty}), \quad (x, \xi) \in \complement U. \tag{6.4}$$

Also, if $A \in \Psi^{\mathrm{comp}}(X)$, $B \in \widetilde{\Psi}_{\frac{1}{2}}^{\mathrm{comp}}(X)$ and $C \in \Psi_{\frac{1}{2}}^{\mathrm{comp}}(X)$ then

$$\begin{aligned} e^{g^w} A e^{-g^w} &= A + i(h\tilde{h})^{\frac{1}{2}} A_1, \quad A_1 \in \widetilde{\Psi}_{\frac{1}{2}}^{\mathrm{comp}}(X), \quad \mathrm{WF}_h(A_1) \subset \mathrm{WF}_h(A), \\ e^{g^w} B e^{-g^w} &= B + i\tilde{h} B_1, \quad B_1 \in \widetilde{\Psi}_{\frac{1}{2}}^{\mathrm{comp}}(X), \quad WF_h(B_1) \subset WF_h(B), \\ e^{g^w} C e^{-g^w} &= C + i\tilde{h}^{\frac{1}{2}} C_1, \quad C_1 \in \Psi_{\frac{1}{2}}^{\mathrm{comp}}(X), \quad WF_h(C_1) \subset WF_h(C). \end{aligned} \tag{6.5}$$

The assumptions in (6.1) show that $\exp g$ is an order function for the $\widetilde{S}_{\frac{1}{2}}$ calculus—see [37, §3.3, (3.17), (3.18)]. Hence we can apply composition formulae. In particular if g_j, $j = 1, 2$ satisfy (6.1) then

$$\exp(g_1^w) \exp(g_2^w) = c^w, \quad |\partial^\alpha c(\rho)| \leq C_\alpha \exp(g_1 + g_2) \big(h/\tilde{h}\big)^{-|\alpha|/2}. \tag{6.6}$$

Because of the compact supports of g_j's and because of (6.3) derivatives can be taken in any local coordinates.

The consequence of (6.6) useful to us here is given in the following Lemma.

Lemma 6.2 *Suppose that $A \in \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X)$ and that*

$$\widetilde{\sigma}(A) = a + \mathcal{O}\big((h\tilde{h})^{\frac{1}{2}}\big)_{\widetilde{S}_{\frac{1}{2}}}, \quad a \in \mathcal{C}^{\infty}_{\mathrm{c}}(T^*X) \cap \widetilde{S}_{\frac{1}{2}}(T^*X).$$

If $U_{h,\tilde{h}} \stackrel{\mathrm{def}}{=} \{\rho \in T^*X : d(\rho, \operatorname{supp} a) < (h/\tilde{h})^{\frac{1}{2}}\}$, then

$$\|A\, e^{g_1^w} e^{g_2^w}\|_{L^2 \to L^2} = \sup_{T^*X}(|a| e^{g_1+g_2}) + \mathcal{O}\left(\tilde{h} \sup_{U_{h,\tilde{h}}} e^{g_1+g_2}\right) + \mathcal{O}\left(h^{\frac{1}{2}} \log(1/h)\right). \tag{6.7}$$

Proof We first consider this statement in $\mathbb{R}^n$. We apply the standard rescaling (3.4) noting that (6.1) imply that $\tilde{m}_j = \exp \tilde{g}_j$ are order functions. If d is the Euclidean distance and if we put

$$n_N(\tilde{\rho}) \stackrel{\mathrm{def}}{=} (1 + d(\tilde{\rho}, \widetilde{U}))^{-N}, \quad \widetilde{U} \stackrel{\mathrm{def}}{=} (\tilde{h}/h)^{\frac{1}{2}} U_{h,\tilde{h}},$$

then n_N is an order function for any N, and $\tilde{a} \in S(n_N)$ for all N. We have

$$A = \mathrm{Op}_h^w(a + (h\tilde{h})^{\frac{1}{2}} a_1), \text{ for some } a_1 \in \widetilde{S}_{\frac{1}{2}},$$

and hence, after rescaling,

$$\begin{gathered}\tilde{A}\, e^{\mathrm{Op}_{\tilde{h}}^w(\tilde{g}_1)} e^{\mathrm{Op}_{\tilde{h}}^w(\tilde{g}_2)} = \mathrm{Op}_{\tilde{h}}^w(\tilde{b}) + (h\tilde{h})^{\frac{1}{2}} \mathrm{Op}_{\tilde{h}}^w(\tilde{b}_1), \\ \tilde{b} \in S(n_N \tilde{m}_1 \tilde{m}_2), \quad \tilde{b} - \tilde{a} e^{\tilde{g}_1 + \tilde{g}_2} \in \tilde{h} S(n_N \tilde{m}_1 \tilde{m}_2), \quad \tilde{b}_1 \in S(\tilde{m}_1 \tilde{m}_2).\end{gathered}$$

Put

$$\begin{aligned}M = M(h, \tilde{h}) &\stackrel{\mathrm{def}}{=} \sup_{\mathbb{R}^{2n}} n_N \tilde{m}_1 \tilde{m}_2 \leq \sup_{\tilde{\rho}} \left(\big(1 + d(\tilde{\rho}, \widetilde{U})\big)^{-N} e^{\tilde{g}_1(\tilde{\rho}) + \tilde{g}_2(\tilde{\rho})} \right) \\ &\leq \left(\sup_{\widetilde{U}} e^{\tilde{g}_1 + \tilde{g}_2} \right) \left(1 + \sup_{\tilde{\rho}} (1 + C_1 C_2 d(\tilde{\rho}, \widetilde{U}))^{-N + N_1 + N_2} \right) \\ &\leq C \sup_{U_{h,\tilde{h}}} e^{g_1 + g_2},\end{aligned}$$

where we took $N \geq N_1 + N_2$, with N_j, C_j appearing in (6.1) for g_j.

We now apply [55, Theorem 13.13] (with h replaced by $\tilde{h}$) to $\tilde{b}/M \in S$. That gives

$$\|\mathrm{Op}_h^w(\tilde{b})\| = \sup |a| e^{g_1 + g_2} + \mathcal{O}(\tilde{h}) \sup_{U_{h,\tilde{h}}} e^{g_1 + g_2}.$$

Since $\tilde{m}_1\tilde{m}_2 = \mathcal{O}(\log(1/h))$, applying the same argument to $\tilde{b}_1/\log(1/h)$ gives (6.7).

The calculus is invariant modulo $\mathcal{O}((h\tilde{h})^{\frac{1}{2}})$ terms (see (3.5) and [12, §5.1],[54, §3.2]), so these local estimates on $\mathbb{R}^n$ imply similar estimates on manifolds. $\square$

The next result is a version of (3.6) for exponentiated weights g. It is a special case of [37, Proposition 3.14] which follows from globalization of the local result [37, Proposition 3.11]. We state it using concepts recalled in Sect. 3.3.

Proposition 6.3 *Suppose that $T \in I^{\mathrm{comp}}(X \times X, \Gamma'_\kappa)$ where $\kappa : U_1 \to U_2$, $U_j \subset T^*X$, is a symplectomorphism, that $g \in \mathcal{C}^\infty_{\mathrm{c}}(T^*X)$ satisfies (6.1), and that $A \in \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}$. Then*

$$\begin{gathered} e^{g^w} A T = T e^{(\kappa^* g)^w} B + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} T_1 e^{(\kappa^* g)^w} C, \\ T_1 \in I^{\mathrm{comp}}_h(X \times X, \Gamma'_\kappa), \quad B, C \in \widetilde{\Psi}_{\frac{1}{2}}(X), \quad \sigma(B) = \kappa^* \sigma(A). \end{gathered} \tag{6.8}$$

6.2 Estimates away from the trapped set

We now provide precise versions of the estimates (2.6) and (2.7) described in the Step 2 of the proof in Sect. 2.

For the escape function G constructed in Proposition 4.7 we define the operator

$$G^w \stackrel{\text{def}}{=} \mathrm{Op}^w_h(G) \in \log(\tilde{h}/h) \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}(X), \quad \widetilde{\sigma}(G) = G + \mathcal{O}\big((h\tilde{h})^{\frac{1}{2}-}\big)_{\widetilde{S}_{\frac{1}{2}}}. \tag{6.9}$$

Since G satisfies (6.1), Proposition 6.1 describes the exponentiated operator $e^{G^w} = e^{\mathrm{Op}^w_h(G)}$. We refer to Remark 4.5 for the requirements on the constants in the definition of G. Intuitively, G is bounded (independently of h and $\tilde{h}$) in a $(h/\tilde{h})^{\frac{1}{2}}$-neighbourhood of $\mathcal{K}$, and satisfies the growth condition $G(\varphi_{t_0}(\rho)) - G(\rho) \geq 2\Gamma$ outside of a *smaller* $(h/\tilde{h})^{\frac{1}{2}}$-neighbourhood of $\mathcal{K}$.

The first lemma shows that the weights are bounded near the trapped set:

Lemma 6.4 *Suppose that $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(T^*X) \cap \widetilde{S}_{\frac{1}{2}}(T^*X)$ has the property*

$$\operatorname{supp} \chi \subset \left\{ \rho \in T^*X : d(\rho, K^{2\delta}) < C_0 (h/\tilde{h})^{\frac{1}{2}} \right\}, \tag{6.10}$$

for some constant C_0 satisfying $0 < (C_0 + 1)^2 < c_2 L$, in the notation of (4.14).

Then for some constants $h_0, \tilde{h}_0, C_1 > 0$ *we have for* $0 < h < h_0$, $0 < \tilde{h} < \tilde{h}_0$,

$$\|\chi^w e^{G^w}\| \leq C_1, \quad \|e^{G^w}\chi^w\| \leq C_1. \tag{6.11}$$

Proof Since $\widetilde{\sigma}(\chi^w) = \chi + \mathcal{O}(h^{\frac{1}{2}}\tilde{h}^{\frac{1}{2}})_{\widetilde{S}_{\frac{1}{2}}}$, and $|G(\rho)| \leq C_3\Gamma C_2$ for $d(\rho, K^{2\delta}) < (C_0+1)(h/\tilde{h})^{\frac{1}{2}}$ (see (4.14) and (4.21)), the estimates in (6.11) follow directly from Lemma 6.2. □

The main result of this section provides bounds for the conjugated propagator. It relies heavily on the material about the propagator for the complex absorbing potential (CAP) modified Hamiltonian, $\exp(-it(P-iW)/h)$, presented in the Appendix.

Proposition 6.5 *Suppose that* G^w *is given by* (6.9) *and that* $A \in \Psi^{\mathrm{comp}}(X)$ *satisfies*

$$WF_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_1)), \tag{6.12}$$

for some $\epsilon_1 > 0$.

Then for some constants $h_0, \tilde{h}_0, C_1 > 0$ *we have for* $0 < h < h_0$, $0 < \tilde{h} < \tilde{h}_0$,

$$\|e^{-G^w}e^{-it_0(P-iW)/h}e^{G^w}A\| \leq e^{2C_1}. \tag{6.13}$$

If χ *satisfies* (6.10) *and in addition*

$$\chi(\rho) \equiv 1 \quad \text{for } d(\rho, K^{2\delta}) < \frac{1}{2}C_0(h/\tilde{h})^{\frac{1}{2}}, |p(\rho)| \leq \delta, \tag{6.14}$$

where C_0 *is a large constant dependending on* t_0, *then, if* $\|A\| \leq 1$,

$$\|(1-\chi^w)e^{-G^w}e^{-it_0(P-iW)/h}e^{G^w}A\| < e^{-\Gamma}, \tag{6.15}$$

where Γ *is the constant appearing in the definition* (4.21) *of* G.

Proof Let $A_{-G} \stackrel{\text{def}}{=} e^{G^w}Ae^{-G^w}$. Then (6.5) in Proposition 6.1 shows that

$$A_{-G} = A + \mathcal{O}_{L^2\to L^2}(h^{\frac{1}{2}}) = \mathcal{O}(1)_{L^2\to L^2} \text{ and } A_{-G} = \widetilde{A}A_{-G} + \mathcal{O}(h^{\infty}), \tag{6.16}$$

where $\widetilde{A}$ satisfies (10.10). To prove (6.13) we use the notation of Proposition 10.3, and rewrite the operator on the right hand side as

$$\begin{aligned} e^{-G^w} e^{-it_0(P-iW)/h} e^{G^w} A &= e^{-G^w} e^{-it_0 P/h} e^{G^w} e^{-G^w} V_{\widetilde{A}}(t_0) A_{-G} e^{G^w} \\ &\quad + \mathcal{O}(h^{\frac{1}{2}})_{L^2\to L^2} \\ &= e^{-G^w} e^{-it_0 P/h} e^{G^w} C(t_0) + \mathcal{O}(h^{\frac{1}{2}})_{L^2\to L^2}, \end{aligned} \tag{6.17}$$

where using (6.5) and Proposition 10.3,

$$C(t_0) \in \Psi^{\mathrm{comp}}_{\frac{1}{2}}(X), \quad \mathrm{WF}_h(C(t_0)) \subset \mathrm{WF}_h(A) \cap w^{-1}(0).$$

Since

$$e^{\pm G^w} = B e^{\pm G^w} + (I - B) + \mathcal{O}(h^\infty)_{L^2\to L^2}, \quad \text{for some } B \in \Psi^{\mathrm{comp}}(X),$$

Proposition 6.3 (applied with $A \equiv I$) and (3.8) show that for some $B_0 \in \widetilde{\Psi}_{\frac{1}{2}}(X)$,

$$e^{-G^w} e^{-it_0 P/h} e^{G^w} = e^{-it_0 P/h} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} \left(I + h^{\frac{1}{2}} \tilde{h}^{\frac{1}{2}} B_0 \right) + \mathcal{O}(h^\infty)_{L^2\to L^2}.$$

From this and (6.17) we see that to prove (6.13) it is enough to show that

$$\begin{gathered} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} B_1 = \mathcal{O}(1)_{L^2\to L^2}, \quad B_1 \in \Psi^{\mathrm{comp}}(X), \\ \mathrm{WF}_h(B_1) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_1)). \end{gathered} \tag{6.18}$$

Lemma 6.2 applied with $g_1 = -\varphi_{t_0}^* G$ and $g_2 = G$, and the property $G - \varphi_{t_0}^* G \le C_7$ in (4.22) which holds in a neighbourhood of $\mathrm{WF}_h(B_1)$, give (6.18) and hence (6.13).

To obtain (6.15) we proceed similarly but applying the property $\varphi_{t_0}^* G - G \ge 2\Gamma$ which is valid outside a $(h/\tilde{h})^{\frac{1}{2}}$ neighbourhood of K^δ—see (4.22). In more detail, Proposition 6.3 applied with $A = 1 - \chi^w$ gives[6]

$$\begin{aligned} &(1-\chi^w) e^{-G^w} e^{-it_0(P-iW)/h} e^{G^w} A \\ &= (1-\chi^w) e^{-G^w} e^{-it_0 P/h} e^{G^w} e^{-G^w} V_{\widetilde{A}}(t_0) A_{-G} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2\to L^2} \\ &= e^{-it_0 P/h} e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - (\varphi_{t_0}^* \chi)^w) e^{-G^w} V_{\widetilde{A}}(t_0) A_{-G} e^{G^w} + \mathcal{O}(h^{\frac{1}{2}})_{L^2\to L^2}, \end{aligned}$$

[6] Strictly speaking $1 - \chi^w \notin \widetilde{\Psi}^{\mathrm{comp}}_{\frac{1}{2}}$ but the operator $A \in \Psi^{\mathrm{comp}}$ provides the needed localization: we can write $A = A_0 A + \mathcal{O}(h^\infty)_{L^2\to L^2}$ where $\mathrm{WF}_h(I - A_0) \cap \mathrm{WF}_h(A) = \emptyset$ and apply Proposition 6.1 to A_0.

where we used the boundedness established in (6.13) to control the lower order terms. Defining $\chi_1 \stackrel{\text{def}}{=} \varphi_{t_0}^* \chi$, we have, by the invariance of K^δ under the flow,

$$\chi_1 \equiv 1 \ \text{ for } \ d(\rho, K^\delta) \leq C_1 (h/\tilde{h})^{\frac{1}{2}}, \ |p(\rho)| \leq \delta.$$

Let $\psi \in \mathcal{C}_{\mathrm{c}}^\infty(T^*X)$ be equal to 1 in the set W_1 of Proposition 4.7, and $\operatorname{supp} \psi \subset (w^{-1}(0))^\circ$.

Since (6.5) and Proposition 10.3 give

$$\begin{aligned}\|e^{-G^w} V_{\widetilde{A}}(t_0) A_{-G} e^{G^w}\| &\leq \|e^{-G^w} V_{\widetilde{A}}(t_0) e^{G^w}\| \|A\| \leq \|A\| \left(\|\tilde{A}\| + \mathcal{O}_{L^2 \to L^2}(\tilde{h}^{\frac{1}{2}}) \right) \\ &\leq 1 + \mathcal{O}(\tilde{h}^{\frac{1}{2}}),\end{aligned}$$

it is enough to show that

$$\|e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - \chi_1^w) \psi^w\| \leq e^{-3\Gamma/2}, \tag{6.19}$$

$$\|e^{-(\varphi_{t_0}^* G)^w} e^{G^w} (1 - \chi_1^w)(1 - \psi^w) B\| \leq C h^{\frac{1}{2}} \log(1/h) \tag{6.20}$$

for $B \in \Psi^{\mathrm{comp}}(X)$ with $\mathrm{WF}_h(B) \subset w^{-1}([0, \epsilon_1/2]) \cap p^{-1}([-\delta, \delta])$, is as in Proposition 4.7. Both inequalities follow from Lemma 6.2 and properties of G in (4.22). For (6.19) we apply (6.7). For (6.20) we note that

$$\varphi_{t_0}^* G - G \geq C_8 \log(\tilde{h}/h), \quad \text{on} \quad \operatorname{supp}(1 - \psi) \cap \mathrm{WF}_h(B),$$

and (6.7) gives the estimate with the error dominating the leading term. □

7 Proof of Theorem 2

We first prove (2.2) which we rewrite as follows

$$\|U_G^n A\|_{L^2(X) \to L^2(X)} \leq C e^{-n t_0 (\lambda_0 - \epsilon_0)/2}, \quad M_{\epsilon_0} \log \frac{1}{\tilde{h}} \leq n \leq M \log \frac{1}{\tilde{h}} \tag{7.1}$$

where

$$U_G \stackrel{\text{def}}{=} \exp\left(-i t_0 \widetilde{P}_G / h\right) A = e^{-G^w} e^{-i t_0 (P - iW)/h} e^{G^w},$$

with t_0 chosen in previous sections, and

$$A \in \Psi^{\mathrm{comp}}(X), \quad \mathrm{WF}_h(A) \subset p^{-1}((-\delta, \delta)). \tag{7.2}$$

To apply the estimates of the last two sections we first observe that Proposition 10.2 implies that for any r there exist $B_j \in \Psi^{\mathrm{comp}}$, $j = 1, \dots r$, each satisfying (7.2), such that

$$U_G^r A = \prod_{j=1}^{r} U_G B_j + \mathcal{O}(h^\infty)_{L^2 \to L^2}, \quad B_r = A, \tag{7.3}$$

where the constants in the norm estimate $\mathcal{O}(h^\infty)$ depend on r. This means that, for r independent of h but depending on $\tilde{h}$, $U_G^r A$ can be replaced by the product of operators $U_G B_j$, to which estimates of the previous section are applicable.

We now want to decompose U_G^n in such a way that the estimates obtained in Sects. 5, 6 can be used. For that we define

$$U_G = U_{G,+} + U_{G,-}, \quad U_{G,+} \stackrel{\mathrm{def}}{=} U_G \chi^w, \quad U_{G,-} \stackrel{\mathrm{def}}{=} U_G (1-\chi)^w. \tag{7.4}$$

We note that Proposition 6.1 shows that

$$\begin{aligned} \chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} &= e^{-G^w} e^{G^w} \chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} \\ &= e^{-G^w} \chi^w e^{-it(P-iW)/h} e^{G^w} + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \to L^2} \\ &= e^{-G^w} \chi^w e^{-it(P-iW)/h} e^{itP/h} e^{-itP/h} e^{G^w} \\ &\quad + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \to L^2} \\ &= e^{-G^w} \chi_t^w e^{-itP/h} e^{G^w} + \mathcal{O}(\tilde{h}^{\frac{1}{2}} h^{\frac{1}{2}})_{L^2 \to L^2}, \end{aligned} \tag{7.5}$$

where $\chi_t^w \stackrel{\mathrm{def}}{=} \chi^w e^{-it(P-iW)/h} e^{itP/h}$. We now use Proposition 10.3 applied with P replaced by $-P$, $A \in \Psi^{\mathrm{comp}}$ satisfying $\mathrm{WF}_h(I-A) \cap \mathrm{WF}_h(\chi^w) = \emptyset$. In the notation of (10.12), $\chi_t^w = \chi^w V_A(t)^*$, $V_A(t)^* \in \Psi_\gamma^{\mathrm{comp}}(X)$. From (10.12)

$$\sigma(V_A(t)) = \exp\left(-\frac{1}{h} \int_0^t \varphi_{-s}^* W \right) \sigma(A),$$

with a full expansion of the symbol in any coordinate chart given in Lemma 10.4. For $\rho \in \operatorname{supp} \chi$, $d(\rho, K^\delta) = \mathcal{O}(h^{\frac{1}{2}})$, and as K^δ is invariant under the flow $d(\varphi_{-s}(\rho), K^\delta) = \mathcal{O}_s(h^{\frac{1}{2}})$. But that means that on the support χ, $\varphi_{-s}^* W \equiv 0$ for $s \leq t$, where t is independent of h, as long as h is small enough. This means that $\mathrm{WF}_h(I - V_A(t)^*) \cap \mathrm{WF}_h(\chi^w) = \emptyset$ and hence, for all t, $\chi_t = \chi + \mathcal{O}_t(h^{\frac{1}{2}})_{S_{\frac{1}{2}}}$.

Returning to (7.5) this means that for $t \leq C \log(1/\tilde{h})$ (in fact for any time bounded independently of h), we have

$$\chi^w e^{-G^w} e^{-it(P-iW)/h} e^{G^w} = \chi^w e^{-G^w} e^{-itP/h} e^{G^w} + \mathcal{O}_t(h^{\frac{1}{2}})_{L^2 \to L^2},$$
$$e^{-G^w} e^{-it(P-iW)/h} e^{G^w} \chi^w = e^{-G^w} e^{-itP/h} e^{G^w} \chi^w + \mathcal{O}_t(h^{\frac{1}{2}})_{L^2 \to L^2}. \tag{7.6}$$

Using the notation (7.4)

$$\begin{aligned} U_G^n &= \sum_{\varepsilon_i = \pm} U_{G,\epsilon_n} \cdots U_{G,\epsilon_2} U_{G,\epsilon_1} \\ &= \sum_{\boldsymbol{\epsilon} \in \Sigma(n)} U_{\boldsymbol{\epsilon}}, \qquad U_{\boldsymbol{\epsilon}} \stackrel{\text{def}}{=} U_{G,\epsilon_n} \cdots U_{G,\epsilon_2} U_{G,\epsilon_1}, \end{aligned} \tag{7.7}$$

where we used the symbolic words $\boldsymbol{\epsilon} = \epsilon_1 \cdots \epsilon_t \in \Sigma(n) = (\pm)^n$. Now, for each word $\boldsymbol{\epsilon} \neq - - \cdots - -$, call $n_L(\boldsymbol{\epsilon})$ ($n_R(\boldsymbol{\epsilon})$, respectively), the number of consecutive $(-)$ starting from the left (the right, respectively):

$$\boldsymbol{\epsilon} = \underbrace{- \cdots -}_{n_L(\boldsymbol{\epsilon})} + * * \cdots * * + \underbrace{- \cdots -}_{n_R(\boldsymbol{\epsilon})}.$$

Given integers n_L, n_R, call $\Sigma(n, n_L, n_R)$ the set of words $\boldsymbol{\epsilon} \in \Sigma(n)$ such that $n_L(\boldsymbol{\epsilon}) = n_L$ and $n_R(\boldsymbol{\epsilon}) = n_L$. The decomposition (7.7) can be split into

$$U_G^n = U_{G,-}^n + \sum_{n_L, n_R} \sum_{\boldsymbol{\epsilon} \in \Sigma(n, n_L, n_R)} U_{\boldsymbol{\epsilon}}.$$

where the sum runs over $n_L, n_R \geq 0$ such that $n_L + n_R \leq n - 1$.

We make the following observations:

$$\Sigma(n, n_L, n_R) = \left\{ (-)^{n_L} + (-)^{n_R} \right\}, \quad \text{if } n_L + n_R = n - 1,$$
$$\Sigma(n, n_L, n_R) = \left\{ (-)^{n_L} + \boldsymbol{\epsilon}' + (-)^{n_R} : \boldsymbol{\epsilon}' \in \Sigma(n - n_L - n_R - 2) \right\},$$
$$\text{if } n_L + n_R < n - 1.$$

Hence, the above sum can be recast into

$$\begin{aligned} U_G^n = U_{G,-}^n &+ \sum_{n_L=0}^{n-1} \left(U_{G,-}\right)^{n_L} U_{G,+} \left(U_{G,-}\right)^{n-n_L-1} \\ &+ \sum_{n_L, n_R} \left(U_{G,-}\right)^{n_L} U_{G,+} \left(U_G\right)^{n-n_R-n_L-2} U_{G,+} \left(U_{G,-}\right)^{n_R} \end{aligned} \tag{7.8}$$

where the last sum runs over $n_L, n_R \geq 0$ such that $n_L + n_R \leq n - 2$.

The following lemma provides the estimate for terms in the last sum on the right hand side of (7.8):

Lemma 7.1 *For $\tilde{h} > h > 0$ small enough, the following bound holds for $r_0 \leq r \leq C_0 \log(1/\tilde{h})$, $r \in \mathbb{N}$,*

$$\left\| U_{G,+}\, U_G^r\, U_{G,+} \right\|_{L^2 \to L^2} \leq C\, \tilde{h}^{-d_\perp/2} \exp\left(-\frac{1}{2} t_0 r (\lambda_0 - \epsilon) \right), \qquad (7.9)$$

where the constant C is uniform with respect to h, $\tilde{h}$ and r.

Proof Lemma 6.4 shows that $e^{G^w} \chi^w,\ \chi^w e^{-G^w} = \mathcal{O}(1)_{L^2 \to L^2}$. Also, Lemma 6.2 shows that for $\chi_1 \in \widetilde{S}_{\frac{1}{2}}$ with the same properties as χ but equal to 1 on the support of χ, we have

$$\chi^w e^{-G^w} = \chi^w e^{-G^w} \chi_1^w + \mathcal{O}(\tilde{h}^\infty)_{L^2 \to L^2}, \quad e^{G^w} \chi^w = \chi_1^w e^{G^w} \chi^w \\ + \mathcal{O}(\tilde{h}^\infty)_{L^2 \to L^2}.$$

Using (7.6) the operator on the left hand side of (7.9) can be rewritten as

$$\begin{aligned} U_G\, \chi^w\, (U_G)^{r+1}\, \chi^w &= U_G\, \chi^w\, e^{-G^w}\, e^{-i(r+1)t_0 P/h}\, e^{G^w}\, \chi^w + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \to L^2} \\ &= U_G \big(\chi^w e^{-G^w} \chi_1^w + \mathcal{O}(\tilde{h}^\infty) \big) e^{-i(r+1)t_0 P/h} \\ &\quad \times \big(\chi_1^w e^{G^w} \chi^w + \mathcal{O}(\tilde{h}^\infty) \big) + \mathcal{O}(h^{\frac{1}{2}})_{L^2 \to L^2} \end{aligned}$$

Hence,

$$\begin{aligned} &\left\| U_{G,+} U_G^r U_{G,+} \right\|_{L^2 \to L^2} \\ &\quad \leq \|U_G\| \left\| \chi^w e^{-G^w} \right\| \left\| \chi_1^w e^{-i(r+1)t_0 P/h} \chi_1^w \right\| \left\| e^{G^w} \chi^w \right\| + \mathcal{O}(\tilde{h}^\infty) \\ &\quad \leq C \left\| \chi_1^w e^{-i(r+1)t_0 P/h} \chi_1^w \right\|, \end{aligned}$$

where we used the fact that the operators U_G, $e^{G^w} \chi^w$ and $\chi^w e^{-G^w}$ are uniformly bounded on L^2. We can now apply Proposition 5.1, replacing t by $(r+1)t_0$ and χ by χ_1. □

Let us now take $n = C_0 \log 1/\tilde{h}$, with $C_0 \gg 1$. We recall that Γ in (2.6) was assumed to satisfy $\Gamma > t_0 \lambda_0 / 2$. We will use the bounds (7.9), and Proposition 6.5: $\left\| U_{G,-} A \right\| < e^{-\Gamma}$.

Returning to the estimate for $U_G^n A$ we first observe that (7.3) and the estimates (6.15) in Proposition 6.5 give

$$\| U_{G,-}^m A \| \leq e^{-m\Gamma} + \mathcal{O}_r(h^{\frac{1}{2}}). \qquad (7.10)$$

In (7.8), for each $\ell = 1, \dots, n-2$, we group together terms with $n_L + n_R = \ell$, and apply Lemma 7.1 and (7.10):

$$\begin{aligned}\|U_G^n A\| &\lesssim e^{-n\Gamma} + n\, e^{-(n-1)\Gamma} + \tilde{h}^{-d_\perp/2} \sum_{\ell=1}^{n-2} (\ell+1)\, e^{-\ell\Gamma}\, e^{-t_0(n-\ell)\frac{\lambda_0-\epsilon}{2}} + \mathcal{O}(h^{\frac{1}{2}}) \\ &\lesssim n\, e^{-n\Gamma} + \tilde{h}^{-d_\perp/2}\, e^{-t_0 n\frac{\lambda_0-\epsilon}{2}} \sum_{\ell=1}^{n-2} (\ell+1)\, e^{-\ell\left(\Gamma - t_0\frac{\lambda_0-\epsilon}{2}\right)} + \mathcal{O}(h^{\frac{1}{2}}) \\ &\lesssim \tilde{h}^{-d_\perp/2}\, e^{-t_0 n\frac{\lambda_0-\epsilon}{2}}.\end{aligned}$$

By taking $C_0 = M_\epsilon \gg 1/\epsilon$ we may absorb the prefactor $\tilde{h}^{-d_\perp/2}$ and obtain, for $\tilde{h} > 0$ small enough,

$$\|U_G^n A\| \le C\, \exp\big(-n t_0 \frac{\lambda_0 - 2\epsilon}{2}\big), \quad n \approx M_\epsilon \log 1/\tilde{h}. \tag{7.11}$$

We can now complete the proof of (1.18) following the outline in Sect. 2. We first note that (7.11) gives (2.2), so that (see (2.3)) for

$$z \in [-\delta/2, \delta/2] - ih[0, (\lambda_0 - 3\epsilon_0)/2] \tag{7.12}$$

and $A \in \Psi^{\mathrm{comp}}(X)$ satisfying (7.2),

$$\begin{aligned}&(\widetilde{P}_G - z) Q_A(z) = A - R(z), \quad R(z) = \mathcal{O}(\tilde{h})_{L^2 \to L^2}, \\ &Q_A(z) \stackrel{\text{def}}{=} \frac{i}{h} \int_0^{T(\tilde{h})} e^{-it(\widetilde{P}_G - z)/h} A dt = \mathcal{O}\left(\frac{T(\tilde{h})}{h}\right)_{L^2 \to L^2}, \\ &T(\tilde{h}) = M_{\epsilon_0} \log 1/\tilde{h}.\end{aligned}$$

We now apply this estimate with $A \in \Psi^{\mathrm{comp}}(X)$ such that $\sigma(A) \equiv 1$ in $p^{-1}(-3\delta/4, 3\delta/4) \cap w^{-1}([0, \epsilon_1))$. Then $\widetilde{P}_G - z \in \widetilde{\Psi}^m_{\frac{1}{2}}(X)$ is elliptic outside of $\mathrm{WF}_h(A)$. Hence, using the $\widetilde{\Psi}_{\frac{1}{2}}$ calculus of Sect. 3.2, there exists $\widetilde{Q}_A(z) \in \widetilde{\Psi}^{-m}_{\frac{1}{2}}(X)$ such that

$$(\widetilde{P}_G - z)\widetilde{Q}_A(z) = I - A + \widetilde{R}(z), \quad \widetilde{R}(z) = \mathcal{O}(\tilde{h})_{L^2 \to L^2}.$$

The Fredholm operator $\widetilde{P}_G - z$ has index 0 since $\widetilde{P}_G + i$ is invertible for small $\tilde{h}$. It follows that for $\tilde{h}$ small enough and z satisfying (7.12)

$$(\widetilde{P}_G - z)^{-1} = (Q_A(z) + \widetilde{Q}_A(x))(I + R(z) + \widetilde{R}_A(z))^{-1} = \mathcal{O}(1/h)$$

Since $e^{\pm G^w(x,hD)} = \mathcal{O}(h^{-M/2+1})_{L^2\to L^2}$ for some M, it follows that

$$(P - iW - z)^{-1} = \mathcal{O}(h^{-M})_{L^2\to L^2} \quad z\in[-\delta/2, \delta/2] - ih[0, (\lambda_0 - 3\epsilon_0)/2],$$
$$(P - iW - z)^{-1} = \mathcal{O}(1/\operatorname{Im} z)_{L^2\to L^2}, \quad \operatorname{Im} z > 0,$$

where the second is immediate from non-negativity of W as an operator.

We now use a semiclassical maximum principle [8, Lemma 4.7],[44, Lemma 2] to obtain the bound for $(P - iW - z)^{-1}$ in (1.18) (after adjusting δ and ϵ_0).

Remark 7.2 Strictly speaking we proved (1.18) for $z \in [-\delta/2, \delta/2] - ih[0, \lambda_0/2 - \epsilon_1]$, for any ϵ_1, provided that h is small enough.

8 The CAP reduction of scattering problems: Proof of Theorem 3

In this section we will prove a generalization of Theorem 3 which applies to a variety of scattering problems. Our approach of reduction to estimates for the Hamiltonian complex absorbing potential (CAP) is based on the work Datchev–Vasy [14] (see also [22, §4.1]) but as the argument is simple and elegant we reproduce it in our slightly modified setting.

Let (Y, g) be a complete Riemannian manifold and let

$$P_g = -h^2\Delta_g + V, \quad V \in \mathcal{C}^\infty(Y; \mathbb{R}). \tag{8.1}$$

We make general assumption on (Y, g) which will allow asymptotically Euclidean and asymptotically hyperbolic infinities.

We assume that Y is the interior of a compact manifold $\overline{Y}$ with a $\mathcal{C}^\infty$ boundary, $\partial Y \neq \emptyset$. We choose a defining function of ∂Y:

$$\rho \in \mathcal{C}^\infty(\overline{Y}; [0, \infty)), \quad \{\rho = 0\} = \partial Y, \quad d\rho|_{\partial Y} \neq 0. \tag{8.2}$$

Let $p_g = |\xi|_g^2 + V(x)$ be the principal symbol of P_g and let

$$(x(t), \xi(t)) = \exp tH_{p_g}(x(0), \xi(0)),$$

be the Hamiltonian flow (geodesic flow lifted to T^*Y when $V \equiv 0$. The first assumption on (Y, g) we make is a non-trapping (convexity) assumption near infinity formulated using ρ with properties (8.2):

$$\rho(x(t)) \in (0, \epsilon_1), \quad \frac{d}{dt}\rho(x(t)) = 0 \implies \frac{d^2}{dt^2}\rho(x(t)) < 0. \tag{8.3}$$

The trapped set at energy $E \in [-\delta, \delta]$ is defined as

$$(x, \xi) \in K_E \iff p_g(x, \xi) = E \text{and } \exp(\mathbb{R}H_{p_g})(x, \xi) \text{ is compact in } T^*Y.$$

We assume that the trapped set at energies $|E| \leq \delta$, (see (1.15)),

$$K^\delta \text{ is } normally\ hyperbolic \text{ in the sense of (1.17).} \tag{8.4}$$

We now make analytic assumptions on P. For that we first assume that P_g can be modified inside a compact part of Y, to obtain an operator

$$P_\infty = -h^2\Delta_g + \widetilde{V}, \quad \widetilde{V} \in \mathcal{C}^\infty(Y), \quad \widetilde{V}\restriction_{\rho<\epsilon_1} = V\restriction_{\rho<\epsilon_1},$$

with the following properties: for some $s_0 > 0$ and $C_0 > 0$,

$$\|\rho^{s_0}(P_\infty - E - i0)^{-1}\rho^{s_0}\|_{L^2(Y)\to L^2(Y)} \leq \frac{C_0}{h}, \quad |E| \leq \delta, \tag{8.5}$$

and

$$\begin{gathered} u = (P_\infty - E - i0)^{-1} f, \quad f \in \mathcal{C}_c^\infty(Y) \implies \\ \mathrm{WF}_h(u) \setminus \mathrm{WF}_h(f) \subset \exp([0,\infty)H_{\Re p_\infty})\left(\mathrm{WF}_h(f) \cap p_\infty^{-1}(E)\right), \end{gathered} \tag{8.6}$$

where $p_\infty \stackrel{\text{def}}{=} |\xi|_g^2 + \widetilde{V}$.

We note that these assumptions do not require that the resolvent of P_∞ has a meromorphic continuation from $\operatorname{Im} z > 0$ to the lower half-plane. A stronger conclusion will be possible when we make that assumption: more precisely, for $\chi \in \mathcal{C}_c^\infty(Y)$, we assume that the resolvent $(P_\infty - z)^{-1}$ continues from $\operatorname{Im} z > 0$ *analytically* to $[-\delta, \delta] - ih[0, C_0]$, for some $C_0 > 0$, and that for some N, the following resolvent estimate holds:

$$\chi(P_\infty - z)^{-1}\chi = \mathcal{O}_{L^2\to L^2}(h^{-N}), z \in [-\delta,\delta] - ih[0, C_0]. \tag{8.7}$$

When P_∞ is chosen to be selfadjoint, interpolation [8, Lemma 4.7],[44, Lemma 2] shows that (8.7) improves to

$$\begin{gathered} \chi(P_\infty - z)^{-1}\chi = \mathcal{O}_{L^2\to L^2}(h^{-1+c_1 \operatorname{Im} z/h}\log(1/h)), \\ z \in [-\delta,\delta] - ih[0, C_0]. \end{gathered} \tag{8.8}$$

We can now state a more general version of Theorem 3:

Theorem 6 *Suppose that the Riemannian manifold* (Y, g) *and the potential* V *satisfy the assumptions* (8.3), (8.4), (8.5) *and* (8.6). *In particular, the trapped set for the operator* $P = -h^2\Delta_g + V$ *is normally hyperbolic.*

Then, for some constant C_1 (and s_0 in (8.5)), we have

$$\|\rho^{s_0}(P_g - E - i0)^{-1}\rho^{s_0}\|_{L^2(Y)\to L^2(Y)} \leq C_1 \frac{\log(1/h)}{h}, \quad |E| \leq \delta. \tag{8.9}$$

If in addition (8.7) holds then, for any $\epsilon_0 > 0$, $\chi(P_g - z)^{-1}\chi$ can be continued analytically to $[-\delta/2, \delta/2] - ih[0, \min(C_0, \lambda_0/2 - \epsilon_0)]$, with λ_0 given by (1.19), and

$$\chi(P_g - z)^{-1}\chi = \mathcal{O}_{L^2\to L^2}(h^{-N}),$$
$$z \in [-\delta/2, \delta/2] - ih[0, \min(C_0, \lambda_0/2 - \epsilon_0)], \tag{8.10}$$

with the improved estimate (1.18) if (8.8) holds.

Before the proof we present two classes manifolds which satisfy our assumptions. We say (Y, g) is *asymptotically Euclidean* if

$$g = \rho^{-4}d\rho^2 + \rho^{-2}g_0(\rho), \qquad \text{near}\,\partial\text{Y},$$

where $g_0(\rho)$ is a family of metrics on ∂Y depending smoothly on ρ up to $\rho = 0$. We say (Y, g) is *evenly asymptotically hyperbolic* if

$$g = \rho^{-2}d\rho^2 + \rho^{-2}g_0(\rho), \qquad \text{near}\,\partial\text{Y},$$

where ρ is as before but this time $g_0(\rho)$ is a family of metrics on ∂Y depending smoothly on ρ^2 (hence *even*) up to $\rho = 0$.

In both cases the non-trapping assumption near infinity (8.3) is valid: see [14, Proof of Lemma 4.1] for the asymptotically hyperbolic case; the asymptotically Euclidean case follows from the same proof, with the fourth displayed equation of the proof replaced by [49, (4.3)].

For asymptotically Euclidean manifolds (8.5) and (8.6) follow from the results of [49]. The modification of V can be done in any way which produces a non-trapping classical flow: for instance we can choose $\widetilde{V} = V + V_{\text{int}}$ where V_{int} is a smooth, large non-negative potential (a barrier) supported in $\{\rho > \epsilon_1\}$.

To obtain (8.7) more care is needed but, under additional assumptions one can use an adaptation of the method of complex scaling of Aguilar-Combes, Balslev-Combes and Simon—see [53] for the case of manifolds and for references. The simplest example for which this is valid was considered in Theorem 1. For even asymptotically hyperbolic manifolds the properties (8.5), (8.6), and (8.7) all follow from the recent work of Vasy [50].

As long we are not interested in analytic continuation properties, the weaker assumptions (8.5) and (8.6) may hold in the generality considered by Cardoso-Vodev [9].

Proof of Theorem 6 To show how Theorem 6 follows from Theorem 2 we use the parametrix construction of [14, §3]. For that we first have to relate the situation in this section to the set-up in Theorem 2. It will be convenient to rescale ρ so that in (8.3) we can take $\epsilon_1 = 4$.

Let X be any compact manifold without boundary such that $\overline{Y} \subset X$ is a smooth embedding: for example, we may take X to be the double of $\overline{Y}$. We then extend ρ to $\rho \in L^\infty(X)$ to be identically 0 on $X \setminus Y$. Let $P \in \Psi^2(X)$ be any selfadjoint semiclassical differential operator satisfying

$$P|_{\rho>1} = P_g|_{\rho>1}, \quad P = -h^2\Delta_{g_X} + V_X,$$

where g_X is a Riemannian metric on X and $V_X \in \mathcal{C}^\infty(X; \mathbb{R})$.

We then take $W \in \mathcal{C}^\infty(X; [0, \infty))$ such that

$$W(x) = \begin{cases} 0 & \text{for } \rho(x) > 1; \\ 1 & \text{for } \rho(x) < \frac{1}{2}. \end{cases}$$

Let $\widetilde{V} \in \mathcal{C}^\infty(Y)$ be a potential for which (8.5) and (8.6) hold. We notice that one possibility to obtain the required properties for P_∞ is to take a complex potential $\widetilde{V} = V - iW_\infty$ where, $W_\infty \in \mathcal{C}^\infty(Y; [0, \infty))$ (see Fig. 3)

$$W_\infty(x) = \begin{cases} 0 & \text{for } \rho(x) < 4, \\ 1 & \text{for } \rho(x) > 5, \end{cases}.$$

Using the convexity property (8.3) it is easy to check that this operator satisfies (8.5) and (8.6). Then for $\operatorname{Im} z > 0, |\Re z| \le \delta$, define the following holomorphic families of operators

$$R_X(z) = (P - iW - z)^{-1}, \qquad R_\infty(z) = (P_\infty - z)^{-1}.$$

Due to the compactness of X, the family of operators $R_X(z)\colon L^2(X) \to L^2(X)$ is meromorphic for $z \in \mathbb{C}$. The resolvent $R_X(z)$ is estimated in Theorem 2. For the moment we only assume that $R_\infty(z)\colon L^2(Y) \to L^2(Y)$ is holomorphic for $\operatorname{Im} z > 0$ and satisfies (8.5), (8.6).

Now take a cutoff function $\chi_X \in \mathcal{C}^\infty(\mathbb{R}, [0, 1])$ with

$$\operatorname{supp} \chi_X \subset (2, \infty), \quad \operatorname{supp}(1 - \chi_X) \subset (-\infty, 3).$$

We put and $\chi_\infty = 1 - \chi_X$.

Our first Ansatz for the inverse of $(P_g - z)$ is the operator

$$F(z) = \chi_X(\rho(\bullet) + 1)R_X(z)\chi_X(\rho(\bullet)) + \chi_\infty(\rho(\bullet) - 1)R_\infty(z)\chi_\infty(\rho(\bullet)).$$

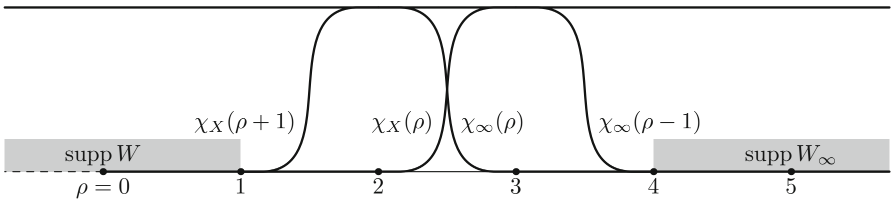

Fig. 3 Schematic representation of the cut-offs used in the proof of Theorem 6 as functions of $\rho(x)$. The spatial infinity is represented by $\rho(x)=0$ and $X\setminus Y$ corresponds to $\rho(x)\le 0$

Note that $F(z)\colon L^2(Y)\to L^2(Y)$ for $\operatorname{Im} z>0$ since all the cut-off functions are supported away from $X\setminus Y$. Also, the support properties of W, W_∞ and χ_X show that

$$(P_g-z)\chi_X(\rho(\bullet)+1)=\chi_X(\rho(\bullet)+1)(P-iW-z)+[\chi_X(\rho(\bullet)+1),h^2\Delta_g],$$
$$(P_g-z)\chi_\infty(\rho(\bullet)-1)=\chi_\infty(\rho(\bullet)-1)(P_\infty-z)+[\chi_\infty(\rho(\bullet)-1),h^2\Delta_g].$$

Hence

$$(P_g-z)F(z)=I+A_X(z)+A_\infty(z),$$

where

$$A_X(z)=[\chi_X(\rho(\bullet)+1),h^2\Delta_g]R_X(z)\chi_X(\rho(\bullet)),$$
$$A_\infty(z)=[\chi_\infty(\rho(\bullet)-1),h^2\Delta_g]R_\infty(z)\chi_\infty(\rho(\bullet)).$$

Note that $A_X(z)^2=A_\infty(z)^2=0$, due to the support properties

$$\begin{aligned}\operatorname{supp} d\,(\chi_X(\rho(\bullet)+1))\cap\operatorname{supp}\chi_X(\rho(\bullet)) &=\emptyset,\\ \operatorname{supp} d\,(\chi_\infty(\rho(\bullet)-1))\cap\operatorname{supp}\chi_\infty(\rho(\bullet)) &=\emptyset.\end{aligned}\tag{8.11}$$

Moreover, thanks to assumptions (8.3) and (8.6) [see [14, Lemma 3.1]],

$$\|A_\infty(z)A_X(z)\|_{L^2(Y)\to L^2(Y)}=\mathcal{O}(h^\infty),\ \text{uniformly for }\operatorname{Im} z>0,|\operatorname{Re} z|\le\delta.\tag{8.12}$$

Consequently

$$\begin{aligned}(P_g-z)F(z)\big((I-A_X(z)-A_\infty(z)+A_X(z)A_\infty(z)\big)&=I-E(z),\\ \text{where}\quad E(z)=A_\infty(z)A_X(z)-A_\infty(z)A_X(z)A_\infty(z)&.\end{aligned}\tag{8.13}$$

Using (8.12) we see that $E(z) = \mathcal{O}(h^\infty)_{L^2(Y)\to L^2(Y)}$, uniformly for $\operatorname{Im} z > 0$, $|\Re z| \le \delta$. This allows to write an explicit expression for $(P_g - z)^{-1}$:

$$(P_g - z)^{-1} = F(z)(I - A_X(z) - A_\infty(z) + A_X(z)A_\infty(z)) \sum_{n=0}^{\infty} E(z)^n.$$

We now want to estimate $\|\rho^{s_0}(P_g - z)^{-1}\rho^{s_0}\|_{L^2(Y)\to L^2(Y)}$. For this we expand the above identity using the expression of $F(z)$ (some terms vanish due to the support properties (8.11)). Denoting $a_X = \|R_X(z)\|$, $a_\infty = \|\rho^{s_0} R_\infty(z)\rho^{s_0}\|$, we get the bound

$$\begin{aligned}\|\rho^{s_0}(P_g - z)^{-1}\rho^{s_0}\| &\le C\big(a_\infty + a_X + h a_\infty a_X + h^2 a_\infty^2 a_X\big)\\ &\quad + \mathcal{O}(h^\infty).\end{aligned} \tag{8.14}$$

Finally, we use the bounds (8.5) for a_∞, the bound (1.18) for a_X (with $\operatorname{Im} z \ge 0$), and obtain the desired estimate (8.9).

When the assumption (8.7) holds, the construction shows that for $\chi \in \mathcal{C}_c^\infty(Y)$ equal to 1 on a sufficiently large set,

$$\chi(P_g - z)^{-1}\chi = \chi F(z)\chi\big(I - A_X(z) - A_\infty(z) + A_X A_\infty(z)\big)\chi \sum_{n=0}^{\infty}(E(z)\chi)^n,$$

continues analytically to the same region as *both* $R_X(z)$ and $\chi R_\infty(z)\chi$. The same expansion as above allows to bound from above $\|\chi(P_g - z)^{-1}\chi\|$ by the same expression as in (8.14), now using $a_X = \|\chi R_X(z)\chi\|$, $a_\infty = \|\chi R_\infty(z)\chi\|$. By using (1.18) for a_X, resp. (8.7) for a_∞ (with now $\operatorname{Im} z$ taking negative values), we obtain (8.10). □

For completeness we conclude this section with the proof of Theorem 1. The conclusion is valid under more general assumptions of Theorem 6.

Proof of Theorem 1 In the notation of Theorem 6, (1.3) is equivalent to the estimate

$$\begin{aligned}&\|\chi\psi(P_g)e^{-itP_g/h}\chi\|_{L^2(Y)\to L^2(Y)} \le C\frac{\log 1/h}{h^{1+c_0\gamma}}\, e^{-\gamma t} + \mathcal{O}(h^\infty),\\ &\gamma = \frac{1}{2}(\lambda_0 - \epsilon),\end{aligned} \tag{8.15}$$

valid (with different constants) for any $\chi \in \mathcal{C}_c^\infty(Y)$. Let $\tilde\psi \in \mathcal{C}_c^\infty(\mathbb{C})$ be an almost analytic extension of ψ, that is a function with the property that $\tilde\psi\restriction_{\mathbb{R}} = \psi$ and $\bar\partial_z\tilde\psi(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$ (see for instance [55, Theorem 3.6]). We can construct $\tilde\psi$ so that $\operatorname{supp}\tilde\psi \subset [-\delta/2, \delta/2] - i[-\delta, \delta]$). We start with

Stone's formula

$$\chi\psi(P_g)e^{-itP_g/h}\chi = \frac{1}{2\pi i}\int_{\mathbb{R}} \psi(\lambda)e^{-i\lambda t}\chi\left((P_g-\lambda-i0)^{-1}\right.$$
$$\left.-(P_g-\lambda+i0)^{-1}\right)\chi\, d\lambda.$$

We now write $R_-(z) = (P_g - z)^{-1}$, for the resolvent in $\operatorname{Im} z < 0$ (that is for the analytic continuation of $(P_g - (z - i0))^{-1}$ from $\operatorname{Im} z < 0$) and $R_+(z)$ for the meromorphic continuation of the resolvent from $\operatorname{Im} z > 0$ to the lower half-plane. We then apply Green's formula to obtain, for $0 \le \gamma < \lambda_{0/2}$,

$$\chi\psi(P_g)e^{-itP_g/h}\chi = \frac{1}{2\pi i}\int_{\operatorname{Im} z=-\gamma h} e^{-itz/h}\chi(R_+(z)-R_-(z))\chi\tilde{\psi}(z)dz$$
$$+\frac{1}{\pi}\iint_{-\gamma h\le \operatorname{Im} z\le 0} e^{-itz/h}\chi(R_+(z)-R_-(z))\chi\bar{\partial}_z\tilde{\psi}(z)dm(z), \tag{8.16}$$

where $dm(z)$ is the Lebesgue measure on $\mathbb{C}$. From (1.18) (see Theorem 6) we get

$$\|\chi R_+(z)\chi\|_{L^2\to L^2} \le Ch^{-(1+c_0\gamma)}\log(1/h),$$
$$\|\chi R_-(z)\chi\|_{L^2\to L^2} \le C/|\operatorname{Im} z|,$$

for $-\gamma h \le \operatorname{Im} z \le 0$. Inserting these bounds in (8.16) gives (8.15) and that proves (a generalized version of) Theorem 1. □

9 Decay of correlations for contact Anosov flows: Proof of Theorem 4

Most of this section is devoted to the proof of Theorem 4. This proof will be obtained by adapting the proof of Theorem 2, after reviewing the geometric point of view of Tsujii [47] and Faure–Sjöstrand [23] (see also [13]). At the end of the section we deduce Corollary 5 on the decay of correlations.

9.1 Geometric structure

Let X be a smooth compact manifold of dimension $d = 2k - 1$, $k \ge 2$. We assume that X is equipped with a contact 1-form α, that is, a form such that $(d\alpha)^{\wedge(k-1)} \wedge \alpha$ is non-degenerate. The Reeb vector field, Ξ, is defined as the unique vector field satisfying

$$\Xi_x \in \ker d\alpha_x, \quad \alpha_x(\Xi_x) = 1, \quad x \in X.$$

We assume that

$$\gamma_t \stackrel{\text{def}}{=} \exp t\,\Xi \ \text{ defines an Anosov flow on X }. \tag{9.1}$$

That means that at each point $x \in X$, the tangent space has a γ_t-invariant decomposition into neutral (one dimensional), stable and unstable subspaces (each $(k-1)$-dimensional):

$$T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x), \quad E_0(x) = \mathbb{R}\Xi_x. \tag{9.2}$$

We note that $E_u(x) \oplus E_s(x)$ span the kernel of α_x.

The dual decomposition is obtained by taking $E_0^*(x)$ to be the annihilator of $E_s(x) \oplus E_u(x)$, $E_u^*(x)$ the annihillator of $E_u(x) \oplus E_0(x)$, and similarly for $E_s^*(x)$. That makes $E_s^*(x)$ dual to $E_u(x)$, $E_u^*(x)$ dual to $E_s(x)$, and $E_0^*(x)$ dual to $E_0(x)$. The fiber of the cotangent bundle then decomposes as

$$T_x^* X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x). \tag{9.3}$$

The distributions $E_s^*(x)$ and $E_u^*(x)$ have only Hölder regularity, but $E_0^*(x)$ and $E_s^*(x) \oplus E_u^*(x)$ are smooth, and $E_0^*(x) = \mathbb{R}\alpha_x \subset T_x^* X$.

The approach of [23] highlights the analogy between this dynamical setting and the scattering theory for the Schrödinger equation. The role of the Schrödinger operator is played by the (rescaled) generator of the flow $\gamma_t = \exp t\,\Xi$:

$$\gamma_t^* u = e^{itP/h} u, \quad u \in \mathcal{C}^\infty(X), \quad P = -ih\Xi. \tag{9.4}$$

The principal symbol of P simply reads $p(x,\xi) = \xi(\Xi_x)$.

The flow γ_t can be lifted to a Hamiltonian flow φ_t on T^*X:

$$\varphi_t : (x,\xi) \longmapsto (\gamma_t(x), {}^t d\gamma_t(x)^{-1}\xi),$$

which is generated by $p(x,\xi)$: $\varphi_t = \exp t H_p$.

For each energy $E \in \mathbb{R}$, the energy shell $p^{-1}(E)$ is a union of affine subspaces:

$$p^{-1}(E) = \bigcup_{x\in X} \{(x,\xi) : \alpha_x(\xi) = E\} = \bigcup_{x\in X} (E\alpha_x + E_u^*(x) + E_s^*(x)).$$

We note that each of these energy shells has infinite volume; as opposed to the scattering theory setting, infinity occurs here in the momentum direction (the fibers), while the spatial direction is compact.

The Anosov assumption implies that for $t > 0$,

$$|\varphi_t(x,\xi)| \le Ce^{-\lambda t}|\xi|, \quad \xi \in E_s^*(x), \quad |\varphi_{-t}(x,\xi)| \le Ce^{-\lambda t}|\xi|, \\ \xi \in E_u^*(x), \tag{9.5}$$

where $|\bullet| = |\bullet|_y$ denotes a norm on T_y^*X, and we consider $\varphi_t(x,\xi) \in T^*_{\pi(\varphi_t(x,\xi))}X$. Since the action of φ_t inside each fiber T_x^*X is linear, we see that the only trapped points in T^*X must be on the line $E_0^*(x)$. More precisely, the trapped set at energy $E \in \mathbb{R}$ is given by

$$K_E = \bigcup_{x\in X} \left(E_0^*(x) \cap p^{-1}(E)\right) = \bigcup_{x\in X} E\alpha_x\,,$$

that is K_E is the image of the section $E\alpha$ in T^*X.

Stacking together energies $E \in (1-\delta, 1+\delta)$, $0 < \delta < 1$, we obtain the trapped set

$$K^\delta = K = \bigcup_{|E-1|<\delta} K_E = \{E\alpha_x,\ x \in X,\ |E-1| < \delta\} \subset T^*X.$$

This trapped set is normally hyperbolic in the sense of (1.17).

Indeed, we first note that K^δ is a symplectic submanifold of T^*X of dimension $d+1 = 2k$. Indeed, using (x, E), $x \in X$, as coordinates on K^δ, $(x,E) \mapsto E\alpha_x$, we have

$$\omega\restriction_{K^\delta} = d(E\alpha) = dE \wedge \alpha + E\,d\alpha.$$

This form is nondegenerate for $E \neq 0$ since α is a contact form.

The tangent space to K^δ is given by the image of the differential of

$$X \times \mathbb{R} \ni (x,E) \mapsto E\alpha_x \stackrel{\text{def}}{=} (x, \xi = E\beta(x)),$$

where we see $\beta(x)$ as the vector in $\mathbb{R}^d$ representing α_x. Hence,

$$T_{E\alpha_x}K^\delta = E(d\alpha)_x(T_xX, \bullet) + \mathbb{R}\alpha_x = \{(v,\, E\,d\beta(x)v + s\beta(x)) : v \in T_xX,\, s \in \mathbb{R}\} \\ \subset T_xX \oplus T_x^*X \equiv T_{E\alpha_x}(T^*X). \tag{9.6}$$

Here $d\beta(x)$ can be interpreted as the Jacobian matrix $\partial\beta/\partial x$ on $\mathbb{R}^d$.

For each $x \in X$, the symplectic orthogonal to $T_{E\alpha_x} K^\delta$, denoted $(T_{E\alpha_x} K^\delta)^\perp$, can be obtained by lifting the vectors in $\ker \alpha_x$ as follows:

$$v \in \ker \alpha_x \mapsto L_E^\perp(v) \stackrel{\text{def}}{=} (v, E\, {}^t(d\beta(x))v) \in T_x X \oplus T_x^* X \equiv T_{E\alpha_x}(T^*X),$$

where ${}^t(d\beta(x))$ denotes the transpose of $d\beta(x)$. This subspace $(T_{E\alpha_x} K^\delta)^\perp$ is symplectic and transverse to K^δ:

$$T_\rho K^\delta \oplus (T_\rho K^\delta)^\perp = T_\rho(T^*X), \quad \forall \rho = E\alpha_x \in K^\delta.$$

Since $\ker \alpha_x = E_u(x) \oplus E_s(x)$, we can naturally split the orthogonal subspace into

$$(T_\rho K^\delta)^\perp = E_\rho^+ \oplus E_\rho^-, \quad E_\rho^+ = L_E^\perp(E_u(x)), \quad E_\rho^- = L_E^\perp(E_s(x)),$$
$$\rho = E\alpha_x \in K^\delta.$$

The distributions $E^\pm_{E\alpha_x}$ are transverse to each other and flow-invariant. $E^+_{E\alpha_x}$ is a particular subspace of the global unstable subspace $E_u(x) \oplus E_u^*(x) \subset T_{E\alpha_x}(T^*X)$, and similarly for $E^-_{E\alpha_x}$. Hence, in the present setting, the subspaces $E_\rho^\pm$ exactly correspond to the subspaces described in Lemma 4.1.

9.2 Microlocally weighted spaces and the definition of resonances

Following [13] we now review the construction [23] of Hilbert spaces on which $P - z$ (with P given in (9.4)) is a Fredholm operator for $\operatorname{Im} z > -\beta h$, for some arbitrary $\beta > 0$.

The key to the definition of these Hilbert spaces is a construction of a weight function which we quote from [23, Lemma 1.2] and [13, Lemma 3.1]. We use the notation $E_\bullet^* = \bigcup_{x \in X} E_\bullet^*(x) \subset T^*X$.

Lemma 9.1 *Let U_0, U_0' be conic neighbourhoods of E_0^*, with $U_0 \Subset U_0'$ and $U_0' \cap (E_u^* \cup E_s^*) = \emptyset$. There exist real-valued functions $m \in S^0(T^*X)$, $f_0 \in S^1(T^*X)$ such that*

(1) *m is positively homogeneous of degree 0 for $|\xi| \geq 1/2$, equal to $-1, 0, 1$ near the intersection of $\{|\xi| \geq 1/2\}$ with E_u^*, E_0^*, E_s^*, respectively, and*

$$H_p m < 0 \text{ near } (U_0' \setminus U_0) \cap \{|\xi| > 1/2\}, \quad H_p m \leq 0 \text{ on } \{|\xi| > 1/2\}; \tag{9.7}$$

(2) *$\langle \xi \rangle^{-1} f_0 \geq c > 0$ for some constant c;*

(3) *the function* $\mathcal{G} \stackrel{\text{def}}{=} m \log f_0$ *satisfies*

$$H_p \mathcal{G} \leq -2 \text{ on } \{|\xi| \geq 1/2\} \setminus U_0, \quad H_p \mathcal{G} \leq 0 \text{ on } \{|\xi| \geq 1/2\}. \tag{9.8}$$

The function $\mathcal{G}$ also satisfies derivative bounds

$$\mathcal{G} = \mathcal{O}(\log\langle \xi \rangle), \quad \partial_x^\alpha \partial_\xi^\beta H_p^k \mathcal{G} = \mathcal{O}\left(\langle \xi \rangle^{-|\beta|+\epsilon}\right), \quad |\alpha| + |\beta| + k \geq 1, \tag{9.9}$$

for any $\epsilon > 0$.

As in [13, §3] we define

$$H_{t\mathcal{G}}(X) \stackrel{\text{def}}{=} e^{-t\mathcal{G}^w} L^2(X, dx), \tag{9.10}$$

where $t > 0$ is a positive parameter.

The domain of P acting on $H_{t\mathcal{G}}$ is defined as

$$\mathcal{D}_{t\mathcal{G}} \stackrel{\text{def}}{=} \{u \in \mathcal{D}'(X) \ : \ u, Pu \in H_{t\mathcal{G}}\}. \tag{9.11}$$

The action of P on $H_{t\mathcal{G}}$ is equivalent to the action of the operator $P_{t\mathcal{G}}$ on L^2:

$$\begin{aligned} P_{t\mathcal{G}} &\stackrel{\text{def}}{=} e^{t\mathcal{G}^w} P e^{-t\mathcal{G}^w} = \exp(t \operatorname{ad}_{\mathcal{G}^w}) P \\ &= \sum_{k=0}^{N} \frac{t^k}{k!} \operatorname{ad}_{\mathcal{G}^w}^k P + R_N(x, hD), \quad R_N \in h^{N+1} S^{-N+\epsilon}, \ \ \forall \epsilon > 0. \end{aligned} \tag{9.12}$$

The validity of (9.12) follows from the fact that the operators $e^{\pm t\mathcal{G}^w}$ are pseudodifferential operators [55, Theorem 8.6], hence the pseudodifferential calculus applies directly [55, Theorem 9.5, Theorem 14.1]. This expansion and the arguments in [23, §3] give

Proposition 9.2 *For $P_{t\mathcal{G}}$ defined by (9.12), we have*

i.) the operator $P_{t\mathcal{G}} - z : \mathcal{D}(P_{t\mathcal{G}}) \to L^2$ is Fredholm of index zero for $\operatorname{Im} z > -th$. Here $\mathcal{D}(P_{t\mathcal{G}})$ is the domain of $P_{t\mathcal{G}}$.

ii.) for $C > 0$ large enough, $(P_{t\mathcal{G}} - z)$ is invertible on $\{\operatorname{Im} z > Ch\}$.

In [23] the above construction was performed, replacing the h-quantization by the $h = 1$ quantization. It lead to the construction of $H_{t\mathcal{G},1}(X) = e^{-t\mathcal{G}^w(x,D)} L^2(X)$ equal, as vector space, to the above h-dependent space $H_{t\mathcal{G}}(X)$. The norms of these two spaces are equivalent with one another, but in an h-dependent way:

$$h^N \|u\|_{H_{t\mathcal{G},1}(X)}/C_0 \leq \|u\|_{H_{t\mathcal{G}}(X)} \leq C_0 h^{-N} \|u\|_{H_{t\mathcal{G},1}(X)}, \tag{9.13}$$

—see [13, §3]. As a consequence, if we call $P_1 = -i\,\Xi$, Theorem 4 translates into the fact that $P_{t\mathcal{G},1} \stackrel{\text{def}}{=} e^{t\mathcal{G}^w(x,D)} P_1 e^{-t\mathcal{G}^w(x,D)}$ is Fredholm in the strip $\{\operatorname{Im}\lambda > -t\}$, admits finitely many eigenvalues in the strip $\{\operatorname{Im}\lambda \geq -\lambda_0/2 + \epsilon_0\}$, and satisfies the resolvent estimate

$$\|(P_{t\mathcal{G},1} - \lambda)^{-1}\|_{L^2\to L^2} = \mathcal{O}(\lambda^{N_0}), \quad \operatorname{Im}\lambda \geq -\lambda_0/2 + \epsilon_0, \quad |\operatorname{Re}\lambda| \geq C. \tag{9.14}$$

9.3 Reduction to Theorem 2

In order to prove Theorem 4, we proceed as in the proof of Theorem 6 in §8, by constructing two operators which microlocally agree with $P_{t\mathcal{G}}$ (up to negligible error terms) on different subsets of T^*X.

Let $W \in \Psi^1(X)$ be as in Example 2 of Sect. 1.3. The trapped set defined in Sect. 1.3 agrees with the trapped set in Sect. 9.1 and, as shown in (9.5), it satisfies, the assumptions of normal hyperbolicity. Hence Theorem 2 applies to $\widetilde{P} = P - iW$. If $A \in \Psi^{\mathrm{comp}}(X)$ satisfies

$$\mathrm{WF}_h(A) \Subset (\mathcal{G}^{-1}(0))^\circ, \quad \mathrm{WF}_h(A) \cap \{(x,\xi) : |\xi|_{g_x} \geq M\} = \emptyset, \tag{9.15}$$

(where M is the one appearing in the definition of W—see (1.13)) then

$$A\widetilde{P} = AP_{t\mathcal{G}} + \mathcal{O}(h^\infty)_{\mathcal{D}'\to\mathcal{C}^\infty}. \tag{9.16}$$

We now introduce an operator which has better global properties and agrees with $P_{t\mathcal{G}}$ near infinity. For that we proceed as in the proof of Theorem 6, and take $W_\infty \in \Psi^{\mathrm{comp}}(X)$ such that

$$\begin{aligned} &\mathrm{WF}_h(W_\infty) \Subset (\mathcal{G}^{-1}(0))^\circ, \quad \mathrm{WF}_h(I - W_\infty) \subset \complement K^\delta, \\ &\mathrm{WF}_h(W) \cap \mathrm{WF}_h(W_\infty) = \emptyset. \end{aligned} \tag{9.17}$$

We then put

$$P_\infty = P_{t\mathcal{G}} - iW_\infty.$$

Then for any B with $\mathrm{WF}_h(I - B) \subset \complement\mathrm{WF}_h(W_\infty)$,

$$(I - B)P_\infty = (I - B)P_{t\mathcal{G}} + \mathcal{O}(h^\infty)_{\mathcal{D}'\to\mathcal{C}^\infty}. \tag{9.18}$$

Properties of the operator P_∞ are listed in the following

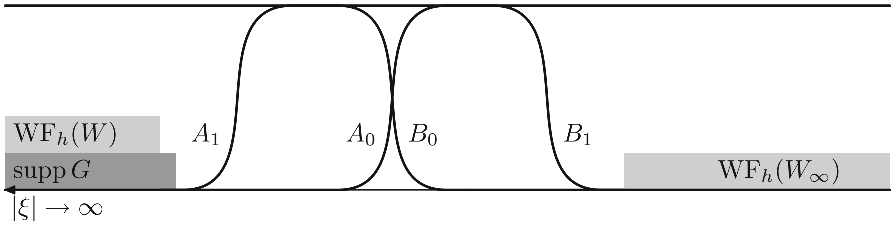

Fig. 4 Schematic representation of pseudodifferential cut-offs used in the proof of Theorem 4. The *horizontal axis* corresponds to $|\xi|$, the cotangent variable. Infinity in $|\xi|$ plays the role of $\rho = 0$ in Fig. 3. The asymmetry is intentional, to stress that there is no need for an auxiliary manifold, as opposed to the proof of Theorem 6 illustrated in Fig. 3

Lemma 9.3 *Fix $\beta > 0$ and let t be large enough so that $P_{t\mathcal{G}} - z$ is a Fredholm operator for $\operatorname{Im} z > -\beta h$. Then, there exists N_0 and h_0 such that, for $0 < h < h_0$,*

$$\|(P_\infty - z)^{-1}\|_{L^2 \to L^2} \le h^{-N_0}, \quad z \in [1 - \delta/2, 1 + \delta/2] - ih[0, \beta].$$

In addition the analogue of (8.6) holds for P_∞: in the same range of z,

$$\begin{aligned} u = (P_\infty - z)^{-1} f, \quad f \in \mathcal{C}^\infty(X) \implies \\ WF_h(u) \setminus WF_h(f) \subset \exp([0, \infty) H_p) \left(WF_h(f) \cap p^{-1}(\operatorname{Re} z) \right). \end{aligned} \tag{9.19}$$

Proof The first part follows from the now standard non-trapping estimates (see [43, §4]). In the setting of Anosov flows the details are presented in the proof of [13, Lemma 5.1] (only the escape function constructed in Lemma 4.6 above is needed).

The propagation result is a real principal type propagation result [55, Theorem 12.5] which holds when the imaginary part of the symbol is non-positive—see Lemma 10.1 below for a dynamical version. □

Proof of Theorem 4 The proof is a repetition of the proof of Theorem 3 with R_X replaced by $(\widetilde{P} - z)^{-1}$ and R_∞ by $(P_\infty - z)^{-1}$. The spatial cut-off functions are replaced by pseudodifferential operators (Fig. 4): $\chi_X(\rho(x))$ is replaced by $A_0 \in \Psi^{\mathrm{comp}}(X)$, satisfying

$$\mathrm{WF}_h(A_0) \cap \{(x, \xi) : |\xi|_{g_x} \ge M\} = \emptyset, \quad \mathrm{WF}_h(I - A_0) \cap \mathrm{WF}_h(W_\infty) = \emptyset,$$

where M is given in the definition of W, see (1.13). The function $\chi_X(\rho(x) + 1)$ is replaced by $A_1 \in \Psi^{\mathrm{comp}}(X)$, where

$$\mathrm{WF}_h(I - A_1) \cap \mathrm{WF}_h(A_0) = \emptyset, \quad \mathrm{WF}_h(A_1) \cap \{(x, \xi) : |\xi|_{g_x} \ge M\} = \emptyset,$$

$\chi_\infty(\rho(x))$ is replaced by $B_0 \stackrel{\text{def}}{=} I - A_0 \in \Psi^0(X)$, and finally $\chi_\infty(\rho(x) - 1)$ by $B_1 \in \Psi^0(X)$, where

$$\mathrm{WF}_h(W_\infty) \cap \mathrm{WF}_h(B_1) = \emptyset, \quad \mathrm{WF}_h(I - B_1) \cap \mathrm{WF}_h(B_0) = \emptyset.$$

We also require that

$$\mathrm{WF}_h(A_1), \mathrm{WF}_h(I - B_1) \subset (G^{-1}(0))^\circ.$$

The parametrix is now obtained by putting

$$F(z) = A_1(P - iW - z)^{-1} A_0 + B_1(P_{t\mathcal{G}} - iW_\infty - z)^{-1} B_0.$$

Using (9.16), (9.18) and Lemma 9.3 we obtain the theorem by proceeding as in the proof of Theorem 3 in Sect. 8. □

Proof of Corollary 5 We will use the nonsemiclassical operator $P_1 = -i\,\Xi$. It is selfadjoint on $L^2(X)$—see [23, Appendix A]—hence, by Stone's formula, we get for any $f, g \in \mathcal{C}^\infty(X)$

$$\begin{aligned}
&\int_X \gamma_{-t}^* f\, g\, dx = \langle e^{-itP_1} f, \bar{g}\rangle \\
&\quad = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} \left(\langle (P_1 - \lambda - i0)^{-1} f, \bar{g}\rangle - \langle (P_1 - \lambda + i0)^{-1} f, \bar{g}\rangle \right) d\lambda \\
&\quad = \frac{1}{2\pi i} \sum_{\pm} \mp \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda \pm i0)^{-1} (P_1 + i)^k f, \bar{g}\rangle d\lambda.
\end{aligned}$$

Here the brackets $\langle \bullet, \bullet \rangle$ represent $L^2(X)$ scalar products.

For $t > 0$ we can deform the contour in the integral corresponding to $+i0$ (λ approaching the real axis from below), where $\|(P_1 - \lambda)^{-1}\| \leq |\operatorname{Im} \lambda|^{-1}$, so that for $k > 1$ the integral is bounded as

$$\begin{aligned}
&-\frac{1}{2\pi i} \int_{\mathbb{R} - iA} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda)^{-1} (P_1 + i)^k f, \bar{g}\rangle d\lambda \\
&\quad = \mathcal{O}(e^{-tA} \|f\|_{H^k} \|g\|_{L^2}).
\end{aligned}$$

Thus,

$$\int_X \gamma_t^* f\, g\, dx = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_1 - \lambda - i0)^{-1} (P_1 + i)^k f, \bar{g} \rangle + \mathcal{O}_{f,g}(e^{-tA}),$$

for any A, with the bounds depending on seminorms of f and g in $\mathcal{C}^\infty$. We now use the nonsemiclassical weights $\mathcal{G}^w(x, D)$ constructed in Sect. 9.2 to conjugate P_1, and write

$$\begin{aligned}\int_X \gamma_t^* f\, g\, dx &= \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} (\lambda + i)^{-k} \langle (P_{t\mathcal{G},1} - \lambda - i0)^{-1} (P_{t\mathcal{G},1} + i)^k e^{t\mathcal{G}^w(x,D)} f,\\ &\quad e^{-t\mathcal{G}^w(x,D)} \bar{g} \rangle + \mathcal{O}_{f,g}(e^{-tA}).\end{aligned}$$

The nonsemiclassical analogue (9.14) of Theorem 4 shows that, by taking $k > N_0 + 1$, we may deform the contour of integration down to $\operatorname{Im} \lambda = -\lambda_0/2 + \epsilon$, collecting finitely many poles μ_j, to finally obtain the expansion (1.22). □

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Appendix: Evolution for the CAP-modified Hamiltonian

In the appendix we show some properties of the CAP-modified Hamiltonian, that is the Hamiltonian modified by adding a complex absorbing potential. At first we work under the general assumptions (1.9).

The semigroup $\exp(-it(P - iW)/h) : L^2(X) \to L^2(X)$ is defined using the Hille-Yosida theorem: for h small $P - iW - i$ is invertible as its symbol is elliptic in the semiclassical sense (see (1.11) and [55, Theorem 4.29]). Ellipticity assumption for large values of ξ also shows that $P - iW$ is a Fredholm operator, and the comment about invertibility shows that it has index 0. The estimate

$$\begin{aligned}\|(P - iW - z)u\| \|u\| &\geq -\operatorname{Im} \langle (P - iW - z)u, u\rangle\\ &\geq \operatorname{Im} z \|u\|^2, \quad u \in H_h^m(X),\end{aligned}$$

then shows invertibility for $\operatorname{Im} z > 0$, with the bound

$$\|(P - iW - z)^{-1}\|_{L^2 \to L^2} \leq \frac{1}{\operatorname{Im} z}, \quad \operatorname{Im} z > 0.$$

Since the domain of $P - iW$ is given by $H^m(X)$ which is dense in L^2, the hypotheses of the Hille-Yosida theorem are satisfied, and

$$\begin{aligned} \|e^{-it(P-iW)/h}\|_{L^2 \to L^2} \leq 1, \quad t \geq 0, \\ e^{-it(P-iW)/h} e^{-is(P-iW)/h} = e^{-i(t+s)(P-iW)/h}, \quad t, s \geq 0. \end{aligned} \tag{10.1}$$

Alternatively we can show the existence of the semigroup $\exp(-it(P-iW)/h)$ using energy estimates, just as is done in the proof of [55, Theorem 10.3]. We get that for any $T > 0$,

$$e^{-it(P-iW)/h} \in C\big([0, T]; \mathcal{L}(H^s_h(X), H^s_h(X))\big) \cap C^1\big([0, T]; \mathcal{L}(H^s_h, H^{s-m}_h)\big). \tag{10.2}$$

Our final estimates will all be given for L^2 only and that is sufficient for our purposes.

The first result we state concerns propagation of semiclassical wave front sets. We recall the notation $\varphi_t = \exp(tH_p)$ for the Hamiltonian flow generated by $p(x, \xi)$.

Lemma 10.1 *Suppose that $A \in \Psi^{\mathrm{comp}}(X)$. Then for any T independent of h there exists a smooth family of operators*

$$[0, T] \ni t \longmapsto Q(t) \in \Psi^{\mathrm{comp}}(X), \quad WF_h(I - Q(t)) \cap \varphi_t(\mathrm{WF}_h(A)) = \emptyset, \tag{10.3}$$

such that

$$(I - Q(t))\, e^{-it(P-iW)/h} A = \mathcal{O}(h^\infty)_{L^2 \to L^2}. \tag{10.4}$$

In addition if $WF_h(A) \subset w^{-1}([\epsilon_1, \infty))$, $\epsilon_1 > 0$, then for any fixed $t > 0$,

$$e^{-it(P-iW)/h} A = \mathcal{O}(h^\infty)_{L^2 \to L^2}, \quad A\, e^{-it(P-iW)/h} = \mathcal{O}(h^\infty)_{L^2 \to L^2}. \tag{10.5}$$

Proof We first construct $Q(t)$ using a semiclassical adaptation of a standard microlocal procedure—see [30, §23.1]. For that, let $Q(0) \in \Psi^{\mathrm{comp}}(X)$ be an operator satisfying $\mathrm{WF}_h(I - Q(0)) \cap \mathrm{WF}_h(A) = \emptyset$, and with the principal symbol, $q_0(0)$, independent of h. Using the fact that the flow φ_t is defined for all t we put $q_0(t) \stackrel{\text{def}}{=} \varphi_{-t}^* q_0(0)$. In terms of the Poisson bracket on the extended phase space $T^*(\mathbb{R}_t \times X) \ni (t, x, \tau, \xi)$, this means that the function $q_0(t)$ satisfies the identity $\{\tau + p, q_0(t)\} = 0$. Consequently, at the quantum level we have

$$[hD_t + P, \mathrm{Op}_h^w(q_0(t)] = hR_1(t), \quad R_1(t) \in \Psi^{\mathrm{comp}}(X),$$
$$\mathrm{Op}_h^w(q_0(0)) - Q(0) = hE_1, \quad E_1 \in \Psi^{\mathrm{comp}}(X),$$

and the principal symbols of R_1, E_1, $r_1, e_1 \in \mathcal{C}_c^\infty(T^*X)$, are independent of h. If $p_1 = \sigma((P - \mathrm{Op}_h^w(p))/h$, we then solve (in the unknown $q_1(t)$) the equation

$$\{\tau + p, q_1(t)\} = r_1 - \{p_1, q_0(t)\}, \quad q_1(0) = e_1.$$

By iteration of this procedure we obtain $q_\ell \in \mathcal{C}^\infty(T^*X)$ such that

$$[hD_t + P, \sum_{\ell=0}^{N-1} h^j \mathrm{Op}_h^w(q_\ell(t))] = h^N R_N(t), \quad R_N(t) \in \Psi^{\mathrm{comp}}(X),$$
$$\sum_{\ell=1}^{N-1} h^\ell \mathrm{Op}_h^w(q_\ell(0)) - Q(0) = h^N E_N, \quad E_N \in \Psi^{\mathrm{comp}}(X).$$

By a standard Borel resummation we may construct $Q(t) \in \Psi^{\mathrm{comp}}(X)$ such that $Q(t) \sim \sum_{\ell \geq 0} h^j \mathrm{Op}_h^w(q_\ell(t))$.

For any $N > 0$ we can iteratively construct a sequence of auxiliary operators $Q_j(t) = Q_j(t)^* \in \Psi^{\mathrm{comp}}(X)$, $0 \leq j \leq N$, satisfying

$$\begin{aligned} &\mathrm{WF}_h(I - Q_{j+1}(t)) \cap \mathrm{WF}_h(Q_j(t)) = \mathrm{WF}_h(I - Q_j(t)) \cap \varphi_t(\mathrm{WF}_h(A)) \\ &\qquad = \mathrm{WF}_h(I - Q(t)) \cap \mathrm{WF}_h(Q_j(t)) = \emptyset, \\ &[Q_j(t), hD_t + P] \in \mathcal{C}^\infty\big([0, T]; h^\infty \Psi^{\mathrm{comp}}(X)\big). \end{aligned} \tag{10.6}$$

(These assumptions imply that $\varphi_t(\mathrm{WF}_h(A)) \subset \mathrm{WF}(Q_j(t)) \subset \mathrm{WF}(Q_{j+1}(t)) \subset \mathrm{WF}(Q(t))$.)

Let $v(t) \stackrel{\text{def}}{=} e^{-it(P-iW)/h} Au$, $\|u\|_{L^2} = 1$. Our aim is to prove the following property:

$$w_j(t) \stackrel{\text{def}}{=} (I - Q_j(t))v(t) = \mathcal{O}(h^{j/2})_{L^2}, \quad \text{for } j = 0, \ldots, N, \quad 0 \leq t \leq T. \tag{10.7}$$

Since $A \in \Psi^{\mathrm{comp}}$, (10.2) shows that this property holds for $j = 0$. Let us now prove that, if true at the level j, it then holds at the level $j + 1$.

Noting that

$$w_{j+1} = (I - Q_{j+1})w_j + \mathcal{O}(h^\infty)_{C^\infty}, \tag{10.8}$$

we have

$$\begin{aligned}(hD_t + P - iW)w_{j+1} &= (I - Q_{j+1}(t))(hD_t + P - iW)w_j \\ &\quad -i[W, Q_{j+1}]w_j + \mathcal{O}(h^\infty)_{L^2}\end{aligned}$$

Dividing by h/i, taking the inner product with w_{j+1}, taking real parts and integrating gives

$$\begin{aligned}&\|w_{j+1}(t)\|_{L^2}^2 - \|w_{j+1}(0)\|_{L^2}^2 + 2\int_0^t \langle W w_{j+1}(s), w_{j+1}(s)\rangle ds \\ &= \frac{2}{h}\int_0^t \mathrm{Re}\langle [W, Q_{j+1}(s)]w_j(s), w_{j+1}(s)\rangle ds + \mathcal{O}(h^\infty),\end{aligned} \tag{10.9}$$

Now,

$$\begin{aligned}(I - Q_{j+1}(s))[W, (I - Q_{j+1}(s))] &= ihB_{j+1}(s) + h^2 C_{j+1}(s), \\ B_{j+1}(s), C_{j+1}(s) \in \Psi^{\mathrm{comp}}(X), &\quad B_{j+1}(s) = B_{j+1}(s)^*.\end{aligned}$$

Hence, using (10.8) and the induction hypothesis (10.7), the right hand side of (10.9) becomes

$$2h\int_0^t \mathrm{Re}\langle C_{j+1}(s)w_j(s), w_j(s)\rangle ds + \mathcal{O}(h^\infty) = \mathcal{O}(h^{j+1}).$$

Returning to (10.9) and using the non-negativity of W, we see that

$$\|w_{j+1}(t)\|_{L^2}^2 \leq \|w_{j+1}(0)\|_{L^2}^2 + Ch^{j+1}.$$

Since

$$w_{j+1}(0) = (I - Q_{j+1})Au = \mathcal{O}(h^\infty)_{L^2},$$

we have established (10.7) with j replaced by $j+1$.

The estimate (10.4) then follows from

$$(I - Q(t))v(t) = (I - Q(t))w_j(t) + \mathcal{O}_{L^2}(h^\infty),$$

the estimate (10.7) at the level $j = N$, and the fact that N could be taken arbitrary large.

To see (10.5) we note that if $A \in \Psi^{\mathrm{comp}}(X)$ then

$$\mathrm{WF}_h(A) \subset w^{-1}([\epsilon_1, \infty) \Longrightarrow \varphi_t(\mathrm{WF}_h(A)) \subset w^{-1}([\epsilon_1/2, \infty) \;\text{ for } 0 \le t \le \delta.$$

Hence, by (10.4),

$$\begin{aligned} &\mathrm{WF}_h(v(t)) \subset w^{-1}([\epsilon_1/2, \infty), \quad v(t) \stackrel{\text{def}}{=} e^{-it(P-iW)/h} Au, \\ &\quad \|u\|_{L^2} = 1, \quad 0 \le t \le \delta. \end{aligned}$$

This means that we can modify W into W_1, so that

$$\sigma(W_1)(x, \xi) \ge \langle \xi \rangle^k / C, \quad W_1 \ge c_0, \text{ for } 0 < h < h_0,$$

while we have

$$\begin{aligned} &0 = (hD_t + P - iW)v(t) = (hD_t + P - iW_1)v(t) + \mathcal{O}(h^\infty)_{\mathcal{C}^\infty} \\ &\quad \text{uniformly for } 0 \le t \le \delta. \end{aligned}$$

Taking the imaginary part of the inner product of the above expression with $v(t)$ gives

$$\frac{h}{2} \partial_t \|v(t)\|^2_{L^2} = -\langle W_1 v(t), v(t) \rangle + \mathcal{O}(h^\infty) \le -c_0 \|v(t)\|^2 + \mathcal{O}(h^\infty),$$

and hence

$$\|v(t)\|^2_{L^2} = \mathcal{O}(h^\infty) \quad \text{uniformly for } \delta/2 \le t \le \delta.$$

This proves the first part of (10.5). The second part follows by taking a conjugate: $A\, e^{-it(P-iW)/h} = \left(e^{-it(-P-iW)/h} A^*\right)^*$, and all the arguments remain valid for P replaced by $-P$. □

The next lemma is needed in Sect. 7 and follows immediately from Lemma 10.1:

Proposition 10.2 *Suppose that $A \in \Psi^{\mathrm{comp}}(X)$ satisfies*

$$WF_h(A) \subset p^{-1}((-\delta, \delta)) \cap w^{-1}([0, \epsilon_1)), \tag{10.10}$$

for some $\epsilon_1 > 0$ and that T is independent of h.

Then there exists $B \in \Psi^{\mathrm{comp}}(X)$ for which (10.10) holds with B in place of A, and

$$e^{-it(P-iW)/h}A = Be^{-it(P-iW)/h}A + \mathcal{O}(h^\infty)_{L^2\to L^2}, \quad 0 \le t \le T. \tag{10.11}$$

Proof Using again the operator $Q(t)$ constructed in the proof of Lemma 10.1, we take a compact set L containing $\mathrm{WF}_h(Q(t))$ for all $0 \le t \le T$. By taking $\mathrm{WF}_h(Q(0)) \subset p^{-1}((-\delta,\delta))$ (which is possible due the assumptions on A) we see that we can assume $L \subset p^{-1}((-\delta,\delta))$. We can now choose $B \in \Psi^{\mathrm{comp}}(X)$ such that

$$\mathrm{WF}_h(I-B) \cap L \cap w^{-1}([0,\epsilon_1/3]) = \emptyset,$$
$$\mathrm{WF}_h(B) \subset p^{-1}((-\delta,\delta)) \cap w^{-1}([0,\epsilon_1/2).$$

This implies that $(I-B)Q(t) = C(t)$, where $\mathrm{WF}_h(C(t)) \subset w^{-1}([\epsilon_1/3,\infty))$, and hence, by (10.4) and (10.5),

$$\begin{aligned}&(I-B)e^{-it(P-iW)/h}A = \\ &\quad \big(C(t) + (I-B)(I-Q(t)\big)e^{-it(P-iW)/h}A = \mathcal{O}(h^\infty)_{L^2\to L^2},\end{aligned}$$

proving (10.11). □

Finally we present a modification of [38, Lemma A.1]. The modification lies in slightly different assumptions on P and W, and the proof also corrects a mistake in the proof given in [38]. From now on we work under the extra assumption (1.10) on the CAP. We remark that in [38] we only needed Lemma 10.1 and hence the assumption (1.10) was not required.

Proposition 10.3 *Suppose that X is a compact manifold, P is a self-adjoint operator, $P \in \Psi^m(X)$, $W \in \Psi^k(X)$, $W \ge 0$, and that (1.9) and (1.10) hold. Then for any t independent of h, for $A \in \Psi^{\mathrm{comp}}(X)$ satisfying (10.10), we may write*

$$e^{itP/h}e^{-it(P-iW)/h}A = V_A(t) + \mathcal{O}(h^\infty)_{L^2\to L^2},$$

where

$$V_A(t) \in \Psi_\gamma^{\mathrm{comp}}(X), \quad WF_h(V_A(t)) \subset \bigcap_{0\le s\le t} (\varphi_{-s}(w^{-1}(0))) \cap WF_h(A),$$
$$\sigma(V_A(t)) = \exp\left(-\frac{1}{h}\int_0^t \varphi_s^* W ds\right)\sigma(A). \tag{10.12}$$

The class of operators $\Psi_\gamma^{\mathrm{comp}}$ was introduced in Sect. 3.2.

The proof is based on the following lemma inspired by the pseudodifferential approach to constructing parametrices for parabolic equations presented in [33].

Lemma 10.4 *Suppose that $t \mapsto p(t, z, h)$, $p(t, \bullet, h) \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}; \mathbb{R})$, is a family of functions satisfying*

$$\begin{aligned} \partial_t^k \partial_z^\alpha p(t, z, h) &= \mathcal{O}_{k,\alpha}(1), \quad p \geq -Ch, \quad 0 < h < h_0, \\ |\partial_z^\alpha p(t, z, h)| &= \mathcal{O}_\alpha(p^{1-\delta}), \quad 0 < \delta < \frac{1}{2}. \end{aligned} \tag{10.13}$$

Then, for $0 \leq s \leq t$ there exists $E(t, s) \in \Psi_\delta(\mathbb{R}^n)$ such that

$$(h\partial_t + p^w(t, x, hD_x, h))E(t, s) = 0, \quad t \geq s \geq 0, \quad E(s, s) = I.$$

Moreover, $E(t, s) = e^w(t, s, x, hD_x, h)$ where $e(t, s) \in S_\delta(\mathbb{R}^{2n})$ has an explicit expansion given in (10.27) below.

Proof Replacing p by $p + (C+1)h$, gives $p \geq h$ and $p(t, \bullet, h) \in (C+1)h + \mathcal{C}_c^\infty(\mathbb{R}_z^{2n})$. The multiplicative factor $e^{(C+1)(t-s)}$ in the evolution equation is irrelevant to our estimates.

For any $N \geq 0$ we try to approximate the symbol $e(t, s, x, \xi, h)$ by an expansion of the form

$$f_N(t, s, z, h) \stackrel{\text{def}}{=} \sum_{j=0}^{N} h^j e_j(t, s, z, h). \tag{10.14}$$

The symbol of the operator $h\partial_t f_N^w + p^w f_N^w$ can be expanded using the standard notation $a^w \circ b^w = (a\# b)^w$ and the product formula (see for instance [55, Theorem 4.12]):

$$\begin{aligned} &h\partial_t f_N(t, s) + [p(t)\# f_N(t, s)] \\ &= \sum_{j=0}^{N} h^j \Bigg(h\partial_t e_j(t, s) \\ &\quad + \sum_{k=0}^{N-j-1} \frac{1}{k!} \left(\frac{1}{2} ih\omega(D_z, D_w)\right)^k p(t, z) e_j(t, s, w)|_{z=w} + h^{N-j} r_{N,j} \Bigg) \\ &= \sum_{j=0}^{N} h^j \Bigg((h\partial_t + p(t)) e_j(t, s) \\ &\quad + \sum_{\ell=0}^{j-1} \frac{1}{(j-\ell)!} \left(\frac{1}{2} i\omega(D_z, D_w)\right)^{j-\ell} p(t, z) e_\ell(t, s, w)|_{z=w} \Bigg) + h^N r_N(t, s, z), \end{aligned}$$

$$r_N(t, s, z) \stackrel{\text{def}}{=} \sum_{j=0}^{N-1} r_{N,j}(t, s, z). \tag{10.15}$$

The remainders satisfy the following bounds (see for instance [43, (3.12)]):

$$\sup_z |\partial_z^\alpha r_{N,j}(t,s,z)| \\ \leq C_{\alpha,N,j} \sum_{\alpha_1+\alpha_2=\alpha} \sup_{z,w} \sup_{|\beta|\leq M, \beta\in\mathbb{N}^{4n}} \left| (h^{\frac{1}{2}}\partial_{z,w})^\beta (\sigma(D_z,D_w))^{N-j} \partial_z^{\alpha_1} p(z) \partial_w^{\alpha_2} e_j(w) \right|. \tag{10.16}$$

The standard strategy is now to iteratively construct the symbols e_j so that each term in the above expansion vanishes. The term $j = 0$ simply reads $(h\partial_t + p)e_0 = 0$. From the initial condition $e_0(s,s) \equiv 1$, it is solved by

$$e_0(t,s,z,h) = \exp\left(-\frac{1}{h} \int_s^t p(s',z,h) ds' \right). \tag{10.17}$$

For $j \geq 1$, the symbol e_j is obtained iteratively by solving

$$e_j(t,s,z) \stackrel{\text{def}}{=} \frac{1}{h} \int_s^t e_0(t,s',z) q_j(s',s,z) ds', \\ e_j(t,s,\bullet) \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}), \\ q_j(t,s,z) \stackrel{\text{def}}{=} \\ -\sum_{\ell=0}^{j-1} \frac{1}{(j-\ell)!} \left(\frac{1}{2} i\omega(D_z,D_w) \right)^{j-\ell} p(t,z) e_\ell(t,s,w)|_{z=w} \in \mathcal{C}_c^\infty(\mathbb{R}_z^{2n}). \tag{10.18}$$

This construction formally leads to an approximate solution:

$$h\partial_t f_N(t,s,z) + [p(t,\bullet) \# f_N(t,s,\bullet)](z) = h^N r_N(t,s,z). \tag{10.19}$$

To make the approximation effective, we now need to check that the sum (10.14) is indeed an expansion in power of h. We thus need to estimate the e_j's and thereby the remainders $r_{N,j}$'s.

We will prove the following estimate by induction:

$$|\partial_z^\alpha e_j(t,s,z)| \leq C_{\alpha,j} h^{-2\delta j - \delta|\alpha|} \left(1 + \left(\frac{1}{h} \int_s^t p(s',z) ds' \right)^{2j+|\alpha|} \right) e_0(t,s,z). \tag{10.20}$$

For that we first note that, as $p \geq h$, and $|\partial^\alpha p| \leq C_\alpha p^{1-\delta}$, we have

$$|\partial^\alpha p| \leq C_\alpha h^{-\delta} p. \tag{10.21}$$

Consequently, for $j = 0$ we have

$$\begin{aligned}
|\partial_z^\alpha e_0(t,s,z)| &\leq \sum_{\sum_{\ell=1}^k \alpha_\ell = \alpha} \prod_{\ell=1}^k \left(\frac{1}{h} \int_s^t |\partial^{\alpha_\ell} p(s',z)| ds' \right) e_0(t,s,z) \\
&\leq C_\alpha \sum_{\sum_{\ell=1}^k \alpha_\ell = \alpha} \prod_{\ell=1}^k \left(h^{-\delta} \frac{1}{h} \int_s^t p(s',z) ds' \right) e_0(t,s,z) \\
&\leq C'_\alpha h^{-\delta|\alpha|} \left(1 + \left(\frac{1}{h} \int_s^t p(s',z) ds' \right)^{|\alpha|} \right) e_0(t,s,z),
\end{aligned} \tag{10.22}$$

Here we used the fact that $k \leq |\alpha|$ and that

$$A^k \leq c_\alpha (1 + A^{|\alpha|}), \quad A = \frac{1}{h} \int_s^t p(s',z) ds' \geq 0.$$

This gives (10.20) for $j = 0$.

To proceed with the induction we put

$$\begin{aligned}
a_{j,\alpha}(t,s,z) &\stackrel{\text{def}}{=} \partial_z^\alpha e_j(t,s,z)/e_0(t,s,z), \\
b_{j,\alpha}(t,s,z) &\stackrel{\text{def}}{=} \partial_z^\alpha q_j(t,s,z)/e_0(t,s,z),
\end{aligned}$$

noting that, for some coefficients, $c_\bullet$,

$$\begin{aligned}
b_{j,\alpha}(t,s,z) &= \sum_{\ell=0}^{j-1} \sum_{\beta_1+\beta_2=\alpha} c_{\beta_1,\beta_2,\ell,j} \omega(D_z, D_w)^{j-\ell} \partial_z^{\beta_1} p(t,z) a_{\ell,\beta_2}(t,s,w)|_{z=w}, \\
a_{j,\alpha}(t,s,z) &= \frac{1}{h} \sum_{\beta_1+\beta_2=\alpha} c_{\beta_1,\beta_2,j} \int_s^t a_{0,\beta_1}(t,s',z) b_{j,\beta_2}(s',s) ds',
\end{aligned} \tag{10.23}$$

where the last equality follows from $e_0(t,s',z)e_0(s',s,z) = e_0(t,s,z)$, $s \leq s' \leq t$.

Our aim is to show

$$|b_{j,\alpha}(t,s,z)| \leq C_{\alpha,j} h^{-2\delta j-\delta|\alpha|} p(t,z) \left(1 + \left(\frac{1}{h} \int_s^t p(s',z) ds' \right)^{2j+|\alpha|-1} \right), \tag{10.24}$$

and

$$|a_{j,\alpha}(t,s,z)| \leq C'_{\alpha,j} h^{-2\delta j-\delta|\alpha|} \left(1 + \left(\frac{1}{h} \int_s^t p(s',z) ds' \right)^{2j+|\alpha|} \right), \tag{10.25}$$

assuming the statements are true for j replaced by smaller values.

We note that the case of $j = 0$ has been shown in (10.22), and since $b_{0,\alpha} \equiv 0$.

The first estimate (10.24) follows immediately from the inductive hypothesis on $a_{\ell,\alpha}$, $0 \leq \ell \leq j-1$ and the estimates on p in (10.21). The second estimate (10.25) follows from (10.22), (10.24) and the obvious fact that $\int_{s_1}^{s_2} p(s')ds' \leq \int_s^t p(s')ds'$, $s \leq s_1 \leq s_2 \leq t$.

We note that (10.20) and the definition of e_0 given in (10.17) imply that

$$\partial_z^\alpha e_j(t,s,z) = \mathcal{O}(h^{-\delta|\alpha|-2\delta j}), \quad j \geq 0.$$

so from (10.14) we see that the symbol $f_N(t,s) \in S_\delta(\mathbb{R}^{2n})$.

The bounds (10.16) then show that the remainders satisfy

$$|\partial^\alpha r_N(t,s,z)| \leq C_{N,\alpha} h^{-2\delta N-\delta|\alpha|}.$$

Going back to (10.19) we get the expression

$$E(t,s) = f_N^w(t,s,x,hD_x) + h^{N-1} \int_s^t E(t,s') r_N^w(s',s,x,hD_x). \tag{10.26}$$

(We note that, since $p^w(t,x,hD_x) \geq -Ch$ by the sharp Gårding inequality [55, Theorem 4.32], and since p^w is bounded on L^2, the operator $E(t,s)$ exists and is bounded on L^2, uniformly in h.) Since operators in Ψ_δ are uniformly bounded on L^2 [55, Theorem 4.23], it follows that

$$E(t,s) = f_N^w(t,s,x,hD_x) + \mathcal{O}(h^{(1-2\delta)N})_{L^2 \to L^2}.$$

To show that $E(t,s) - e_0^w(s,t,x,hD_x) \in \Psi_\delta^{\mathrm{comp}}(\mathbb{R}^n)$, we use (10.26) and Beals's lemma in the form given in [43, Lemma 3.5, $\tilde{h} = 1$]: ℓ_j are linear functions on $\mathbb{R}^{2n}$, $\ell_j^w = \ell_j^w(x, hD)$, then

$$\begin{aligned}
&\mathrm{ad}_{\ell_1^w} \cdots \mathrm{ad}_{\ell_J^w} E(s,t) \\
&= \mathrm{ad}_{\ell_1^w} \cdots \mathrm{ad}_{\ell_J^w} f_N^w(s,t,x,hD_x) + h^{N-1} \int_s^t \mathrm{ad}_{\ell_1^w} \cdots \mathrm{ad}_{\ell_J^w} \\
&\quad \big(E(s,s') r_N^w(s',s,x,hD_x)\big)\, ds' \\
&= \mathcal{O}(h^{(1-2\delta)J})_{L^2\to L^2} + \mathcal{O}(h^{(1-2\delta)N})_{L^2\to L^2} = \mathcal{O}(h^{(1-2\delta)J})_{L^2\to L^2},
\end{aligned}$$

if N is large enough. Here we used the fact that $f_N, r_N \in S_\delta$ and that $\mathrm{ad}_{\ell_1^w} \cdots \mathrm{ad}_{\ell_J^w} E(s,t) = \mathcal{O}(1)_{L^2\to L^2}$, which follows from considering the evolutions equation for the operator on the left hand side.

In conclusion we have shown that $E(t,s) = e^w(t,s,x,hD_x)$, where $e \in S_\delta(\mathbb{R}^n)$ admits the expansion

$$e(t,s,z,h) \sim \sum_{j\geq 0} h^j e_j(t,s,z,h), \qquad e_j(t,s) \in h^{-2\delta j} S_\delta^{\mathrm{comp}}(\mathbb{R}^{2n}), \quad j \geq 1, \tag{10.27}$$

with e_0 given by (10.17). □

Proof of Proposition 10.3 We first observe that Lemma 10.1 (applied both to propagators for $P - iW$ and for P) shows that for $B \in \Psi^{\mathrm{comp}}(X)$ satisfying $\mathrm{WF}_h(I - B) \cap \mathrm{WF}_h(A) = \emptyset$,

$$e^{itP/h} e^{-it(P-iW)/h} A = B e^{itP/h} e^{-it(P-iW)/h} A + \mathcal{O}(h^\infty)_{L^2\to L^2}.$$

We can choose $B = B^*$. Since

$$\begin{aligned}
h\partial_t \left(B e^{itP/h} e^{-it(P-iW)/h} A\right) &= -B e^{itP/h} W e^{-itP/h} e^{itP/h} e^{-it(P-iW)/h} A \\
&= -\left(B e^{itP/h} W e^{-itP/h} B\right) \\
&\quad \times \left(B e^{itP/h} e^{-it(P-iW)/h} A\right) + \mathcal{O}(h^\infty)_{L^2\to L^2},
\end{aligned}$$

it follows that

$$B\, e^{itP/h} e^{-it(P-iW)/h} A = V^B(t) + \mathcal{O}(h^\infty)_{L^2\to L^2}, \tag{10.28}$$

where

$$h\partial_t V^B(t) = -W_B(t) V^B(t), \quad W_B(t) \stackrel{\mathrm{def}}{=} B e^{itP/h} W e^{-itP/h} B. \tag{10.29}$$

We note that $W_B(t) \in \Psi^{\mathrm{comp}}(X)$, $\mathrm{WF}_h(W_B(t)) \subset \mathrm{WF}_h(B)$, and that $W_B(t) \geq 0$. Hence $V^B(t) = \mathcal{O}(1)_{L^2 \to L^2}$ and (10.28) follows from Duhamel's formula.

By decomposing A as a sum of operators, we can assume that $\mathrm{WF}_h(A)$ is supported in a neighbourhood of a fiber of a point in X. Hence, by choosing B with a sufficiently small wave front set, we only need to prove that $V^B(t) \in \Psi_\delta$ for $X = \mathbb{R}^n$; that follows from Lemma 10.4, since the symbol of $W_B(t)$ satisfies the assumptions (10.13). The second and third properties in (10.12) follows from (10.5) and (10.27). □

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