Throughout the exercises, unless stated otherwise, S is a countable space, (X_n) is the canonical Markov chain with transition kernel p, and Green function G. Notation for natural numbers: $\mathbb{N} := \{0, 1, 2, \cdots\}$, and $\mathbb{N}^* := \{1, 2, \cdots\}$.

Exercise 1. Let ξ be a real-valued random variable with $E(|\xi|) < \infty$. Let X be an S-valued random variable. Let $h : S \to \mathbb{R}$ be a mapping. Prove that

$$E(\xi \,|\, X) = h(X), \quad \text{a.s.} \quad \Leftrightarrow \quad E(\xi \,|\, X = x) = h(x), \;\; \forall x \in S \text{ such that } P(X = x) > 0.$$

Solution. "$\Rightarrow$" Let x be such that $P(X = x) > 0$. Since $E[\xi\, \mathbf{1}_{\{X=x\}} \,|\, X] = \mathbf{1}_{\{X=x\}}\, E[\xi \,|\, X] = \mathbf{1}_{\{X=x\}}\, h(X)$, we have $E[\xi\, \mathbf{1}_{\{X=x\}}] = E[\mathbf{1}_{\{X=x\}}\, h(X)] = h(x)\, P(X = x)$. Therefore,

$$E(\xi \,|\, X = x) = \frac{E[\xi\, \mathbf{1}_{\{X=x\}}]}{P(X = x)} = \frac{h(x)\, P(X = x)}{P(X = x)} = h(x).$$

"$\Leftarrow$" For any $x \in S$, $|h(x)| \le E[\,|\xi| \,|\, X = x]$, so that $E(|h(X)|) = \sum_{x \in S} |h(x)|\, P(X = x) \le E(|\xi|) < \infty$. Since $h(X)$ is $\sigma(X)$-measurable, it remains to check that for any $A \in \sigma(X)$, we have $E(h(X)\, \mathbf{1}_A) = E(\xi\, \mathbf{1}_A)$.

By definition, there exists $B \subset S$ such that $A = X^{-1}(B)$. So $\mathbf{1}_A = \mathbf{1}_B(X)$. Thus $E(h(X)\, \mathbf{1}_A) = E(h(X)\, \mathbf{1}_B(X)) = \sum_{x \in B} h(x) P(X = x) = \sum_{x \in B} E(\xi\, \mathbf{1}_{\{X=x\}})$, which is $= E(\xi\, \mathbf{1}_B(X)) = E(\xi\, \mathbf{1}_A)$. $\square$

Exercise 2. Let $n \ge 1$ and $f : S \to \mathbb{R}_+ := [0, \infty)$. Let $\{i_1, \cdots, i_k\} \subset \{0, 1, \cdots, n-1\}$. Prove that

$$E[f(X_{n+1}) \,|\, X_{i_1}, \cdots, X_{i_k}, X_n] = E[f(X_{n+1}) \,|\, X_n].$$

Solution. For any $x \in S$,

$$P[X_{n+1} = y \,|\, X_0, X_1, \cdots, X_n] = p(X_n, y) = P[X_{n+1} = y \,|\, X_n].$$

Therefore,

$$E[f(X_{n+1}) \,|\, X_0, X_1, \cdots, X_n] = \sum_{y \in S} f(y)\, p(X_n, y) = E[f(X_{n+1}) \,|\, X_n].$$

If $\{i_1, \cdots, i_k\} \subset \{0, 1, \cdots, n-1\}$, then

$$\begin{aligned} E[f(X_{n+1}) \,|\, X_{i_1}, \cdots, X_{i_k}, X_n] &= E\{E[f(X_{n+1}) \,|\, X_0, X_1, \cdots, X_n] \,|\, X_{i_1}, \cdots, X_{i_k}, X_n\} \\ &= E\{E[f(X_{n+1}) \,|\, X_n] \,|\, X_{i_1}, \cdots, X_{i_k}, X_n\}, \end{aligned}$$

which is $E[f(X_{n+1}) \,|\, X_n]$, being itself $\sigma(X_{i_1}, \cdots, X_{i_k}, X_n)$-measurable. $\square$

Exercise 3. Prove that there exists a probability space $(\Omega, \mathscr{F}, P)$ on which one can define $(U_n, n \geq 1)$, a sequence of i.i.d. uniform $(0, 1)$ random variables.

Solution. Let $\Omega := [0, 1)$, endowed with the Borel σ-field and with the Lebesgue measure. For any $\omega \in \Omega = [0, 1)$ and integer $n \geq 1$, let

$$Y_n(\omega) := \lfloor 2^n \omega \rfloor - 2 \lfloor 2^{n-1} \omega \rfloor \in \{0, 1\}.$$

Then we have the (proper) dyadic development

$$\omega = \sum_{n=1}^{\infty} \frac{Y_n(\omega)}{2^n}, \qquad \forall \omega \in \Omega,$$

as $\omega - \sum_{k=1}^n \frac{Y_k(\omega)}{2^k} = \frac{2^n\omega - \lfloor 2^n \omega \rfloor}{2^n} \in [0, \frac{1}{2^n}[$, $\forall n$.

For any p and any $i_1, \cdots, i_p \in \{0, 1\}$,

$$\{Y_1 = i_1, \cdots, Y_p = i_p\} = \Big[\sum_{j=1}^p \frac{i_j}{2^j}, \sum_{j=1}^p \frac{i_j}{2^j} + \frac{1}{2^p} \Big),$$

so that

$$P(Y_1 = i_1, \cdots, Y_p = i_p) = \frac{1}{2^p}.$$

Summing over possible values of Y_1, we get $P(Y_2 = i_2, \cdots, Y_p = i_p) = \frac{1}{2^{p-1}}$, and by induction, $P(Y_p = i_p) = \frac{1}{2}$. So

$$P(Y_1 = i_1, \cdots, Y_p = i_p) = P(Y_1 = i_1) \cdots P(Y_p = i_p).$$

In other wods, $(Y_n, \ n \geq 1)$ is a sequence of i.i.d. random variables such that $P(Y_n = 0) = \frac{1}{2} = P(Y_n = 1)$.

Let $\varphi : \mathbb{N}^* \times \mathbb{N}^* \to \mathbb{N}^*$ be injective. Then $(\xi_{i,j} := Y_{\varphi(i,j)}, \, i, \, j \geq 1)$ is again a collection of i.i.d. random variables. We set

$$U_i := \sum_{j=1}^{\infty} \frac{\xi_{i,j}}{2^j}, \qquad i \geq 1.$$

It is clear that $(U_i, \, i \geq 0)$ is a sequence of i.i.d. random variables. For any $m \geq 1$, $\sum_{j=1}^m \frac{\xi_{i,j}}{2^j}$ has the same distribution as $\sum_{n=1}^m \frac{Y_n}{2^n}$; letting $m \to \infty$ yields that each U_i has the uniform distribution on $(0, 1)$. $\square$

Exercise 4. Prove that in the proof of the Markov property, we only need to check that

$$E_x[(\mathbf{1}_A \circ \theta_n)\, \mathbf{1}_B] = E_x[\mathbf{1}_B\, P_{X_n}(A)],$$

for $B := \{\omega \in \Omega : X_0(\omega) = y_0, X_1(\omega) = y_1, \cdots, X_n(\omega) = y_n\}$ (for $y_0, y_1, \cdots, y_n \in S$) and any cylinder set A.

Solution. The Markov property is equivalent to the following: if $Y : \Omega \to \mathbb{R}_+$ is measurable and $n \geq 0$, then $E_x[(Y \circ \theta_n)\, \mathbf{1}_B] = E_x[\mathbf{1}_B\, E_{X_n}(Y)]$ for any $B \in \mathscr{F}_n$.

By a usual argument (from indicator functions to simple functions, and then to non-negative functions by means of the monotone convergence theorem), it suffices to prove the theorem for $Y = \mathbf{1}_A$, where $A \in \mathscr{F}$.

By the π-λ theorem, it suffice to prove the theorem for A of the form (for some $m \in \mathbb{N}$ and $x_0, x_1, \cdots, x_m \in S$)

$$\begin{aligned} A &= \{x_0\} \times \{x_1\} \times \cdots \times \{x_m\} \times S \times S \times \cdots \\ &= \{\omega \in \Omega : X_0(\omega) = x_0, X_1(\omega) = x_1, \cdots, X_m(\omega) = x_m\}, \end{aligned}$$

because the family of all such sets A is a π-system (i.e., stable by finite intersections) and generates $\mathscr{F}$, whereas the family of all sets A such that $Y = \mathbf{1}_A$ satisfies the desired identity is a λ-system.

Another application of the π-λ theorem then tells us that we only need to check the identity for the set B given in the exercise. $\square$

Exercise 5. Let $(Z_n, n \geq 1)$ be a sequence of i.i.d. random variables defined on a certain probability space, taking values in a measurable space $(E, \mathscr{E})$. Let $\Phi : S \times E \to S$ be a measurable mapping. Let $y \in S$. We define $(Y_n, n \in \mathbb{N})$ by $Y_0 := y$ and $Y_{n+1} := \Phi(Y_n, Z_{n+1})$ (for $n \in \mathbb{N}$). Prove that $(Y_n, n \in \mathbb{N})$ is a Markov chain, and determine its transition probability.

Solution. Let $n \in \mathbb{N}$ and $x_0, x_1, \cdots, x_n \in S$. We have

$$\begin{aligned} & P\{Y_0 = x_0, \cdots, Y_n = x_n\} \\ = \;& P\{Y_0 = x_0, \Phi(x_0, Z_1) = x_1, \Phi(x_1, Z_2) = x_2, \cdots, \Phi(x_{n-1}, Z_n) = x_n\} \\ = \;& \mathbf{1}_{\{x_0 = y\}} P\{\Phi(x_0, Z_1) = x_1\}\, P\{\Phi(x_1, Z_1) = x_2\} \cdots P\{\Phi(x_{n-1}, Z_1) = x_n\}. \end{aligned}$$

By taking $q(u, v) := P\{\Phi(u, Z_1) = v\}$ for $u, v \in S$, we have

$$P\{Y_0 = x_0, \cdots, Y_n = x_n\} = \mathbf{1}_{\{x_0 = y\}}\, q(x_0, x_1)\, q(x_1, x_2) \cdots q(x_{n-1}, x_n),$$

which means $(Y_n,\, n \in \mathbb{N})$ is a Markov chain with transition probability q. $\square$

Exercise 6. Let $A \subset S$. Let $\tau_A := \inf\{n \geq 0 :\, X_n \in A\}$ ($\inf \varnothing := \infty$). Let $Y_n := X_{n \wedge \tau_A}$, $n \in \mathbb{N}$. Prove that $(Y_n,\, n \in \mathbb{N})$ is a Markov chain, and determine its transition kernel.

Solution. By admitting that $(Y_n,\, n \in \mathbb{N})$ is a Markov chain, let us determine its transition probability q: for $x,\, y \in S$,

$$q(x,\, y) = P_x\{Y_1 = y\} = P_x\{X_{1 \wedge \tau_A} = y\}.$$

If $x \in A$, then $\tau_A = 0$, so $P_x\{X_{1 \wedge \tau_A} = y\} = P_x\{X_0 = y\} = \mathbf{1}_{\{y=x\}}$. If $x \in A^c$, then $\tau_A \geq 1$, so $P_x\{X_{1 \wedge \tau_A} = y\} = P_x\{X_1 = y\} = p(x,\, y)$. Therefore, the transition kernel of $(Y_n,\, n \in \mathbb{N})$ is

$$q(x,\, y) = \mathbf{1}_{\{x \in A\}}\, \mathbf{1}_{\{y=x\}} + \mathbf{1}_{\{x \in A^c\}}\, p(x,\, y).$$

It remains to check that $(Y_n,\, n \in \mathbb{N})$ is indeed a Markov chain. We have, for $x,\, y_0,\, \cdots,\, y_n,\, y \in S$ such that $P_x(B_n) > 0$ with $B_n := \{Y_0 = y_0,\, \cdots,\, Y_n = y_n\}$,

$$P_x\{Y_{n+1} = y \,|\, B_n\} = P_x\{Y_{n+1} = y,\, Y_n \in A \,|\, B_n\} + P_x\{Y_{n+1} = y,\, Y_n \in A^c \,|\, B_n\}.$$

On the set $\{Y_n \in A\}$, we have $\tau_A \leq n$, so that $Y_{n+1} = Y_n$, and thus $P_x\{Y_{n+1} = y,\, Y_n \in A \,|\, B_n\} = \mathbf{1}_{\{y=y_n\}}\, \mathbf{1}_{\{y_n \in A\}}$. On the set $\{Y_n \in A^c\}$, we have $\tau_A \geq n+1$, so $Y_n = X_n$ and $Y_{n+1} = X_{n+1}$, which implies $P_x\{Y_{n+1} = y,\, Y_n \in A^c \,|\, \mathscr{F}_n\} = \mathbf{1}_{\{y_n \in A^c\}}\, p(y_n,\, y)$. As a consequence,

$$P_x\{Y_{n+1} = y \,|\, Y_0 = y_0,\, \cdots,\, Y_n = y_n\} = \mathbf{1}_{\{y=y_n\}}\, \mathbf{1}_{\{y_n \in A\}} + \mathbf{1}_{\{y_n \in A^c\}}\, p(y_n,\, y) = q(y_n,\, y),$$

proving that $(Y_n,\, n \in \mathbb{N})$ is a Markov chain with transition kernel q. $\square$

Exercise 7. To each probability α on $\{2,\, 3,\, \cdots\}$, we associate a Markov chain $(X_n,\, n \geq 0)$ taking values in $\{0,\, 1,\, 2,\, \cdots\}$, with initial distribution δ_0 (Dirac measure on 0) and transition probability p defined by

$$p(0,\, i) = \alpha(i+1)\,, \qquad p(i,\, i-1) = 1\,, \qquad i \geq 1\,.$$

Let $T := \inf\{n \geq 1 :\, X_n = 0\}$ ($\inf \varnothing := \infty$). What is the law of T?

Solution. The only possible way for the chain (X_n) to come back to 0 in k steps is to move from 0 to $k-1$ at the first step, and to move from $k-1$ to $k-2$, and from $k-2$ to $k-3$,

$\cdots$, and from 1 to 0 in the next $k-1$ steps. As such, the event $\{T=k\}$ is identical to $\{X_1 = k-1\}$. We have $P(T=k) = P(X_1 = k-1) = \alpha(k)$.

[So, all probability measure on $\{2, 3, \cdots\}$ can be considered as the law of the first return time to the initial state for a certain Markov chain.] $\square$

Exercise 8 (Transition kernel and Doob's h-transform).

Question A. Let $h : S \to \mathbb{R}_+$ be a bounded mapping such that for any $x \in S$, $(h(X_n), n \in \mathbb{N})$ under P_x is a martingale with respect to the canonical filtration. For any x, $y \in S^h := \{z \in S : h(z) > 0\}$, we define

$$p^h(x, y) := \frac{h(y)}{h(x)} p(x, y).$$

Prove that p^h is a transition kernel on S^h, and is called the h-transform of p.

Question B. We consider the example of simple random walk on $\mathbb{Z}$, i.e., under P_x, $(X_n, n \in \mathbb{N})$ is a simple random walk on $\mathbb{Z}$ with $P_x(X_0 = x) = 1$. Let $T_i := \inf\{n \in \mathbb{N} : X_n = i\}$ for $i \in \mathbb{Z}$. For $N \geq 1$ and $1 \leq k \leq N-1$, we write

$$P_k^{(N)} := P_k(\,\cdot\,|\, T_N < T_0).$$

(B1) Prove that for $N \geq 1$ and $1 \leq k \leq N-1$,

$$P_k(T_N < T_0) = \frac{k}{N}.$$

(B2) Prove that under $P_k^{(N)}$, $(X_{n\wedge T_N}, n \in \mathbb{N})$ is a Markov chain taking values in $\{1, 2, \cdots, N\}$. Determine its transition kernel.

(B3) Find a function $h : \{0, 1, 2, \cdots, N\} \to \mathbb{R}_+$ such that the transition kernel of $(X_{n\wedge T_N}, n \in \mathbb{N})$ under $P_k^{(N)}$ is the h-transform of the transition kernel of $(X_{n\wedge T_0 \wedge T_N}, n \in \mathbb{N})$ under P_k.

Solution. (A) Let $x \in S^h$. Under P_x, $(h(X_n), n \in \mathbb{N})$ is a martingale, thus

$$h(x) = E_x[h(X_1)] = \sum_{y\in S} h(y)\, p(x, y) = \sum_{y \in S^h} h(y)\, p(x, y),$$

which is equivalent to saying that $\sum_{y\in S^h} p^h(x, y) = 1$. Since $p^h(x, y) \geq 0$, this shows that p^h is a transition kernel on S^h.

(B1) Write $a_k := P_k(T_N < T_0)$, for $0 \leq k \leq N$. Of course, $a_0 = 0$ and $a_N = 1$. By the Markov property, $a_k = \frac{1}{2}\, a_{k+1} + \frac{1}{2}\, a_{k-1}$ for $1 \leq k \leq N-1$.

It is easy to solve this system of linear equations: writing $b_k := a_k - a_{k-1}$, $1 \le k \le N$, we have $b_{k+1} = b_k$, $\forall 1 \le k \le N-1$. Since $\sum_{k=1}^N b_k = a_N - a_0 = 1$, this yields $b_k = \frac{1}{N}$, $\forall 1 \le k \le N$, i.e., $a_k - a_{k-1} = \frac{1}{N}$, $\forall 1 \le k \le N$. Consequently, $a_k = \frac{k}{N}$, $0 \le k \le N$.

(B2) Let $Y_n := X_{n \wedge T_N}$, which, under $P_k^{(N)}$, takes values in $\{1, 2, \cdots, N\}$. Let us compute

$$P_k^{(N)}\{Y_0 = y_0, \cdots, Y_n = y_n\} = \frac{N}{k} P_k\{Y_0 = y_0, \cdots, Y_n = y_n, T_N < T_0\},$$

for $y_i \in \{1, 2, \cdots, N\}$ satisfying:(a) $y_0 = k$, (b) $|y_{i+1} - y_i| = 1$ if $y_i < N$, (c) if $y_i = N$, then $y_j = N$, $\forall j \ge i$. [If either of the conditions are violated, the probability in question vanishes.]

We distinguish two situations.

First situation: $y_i < N$, $\forall i \le n$. Then on the event $\{\{Y_0 = y_0, \cdots, Y_n = y_n\}$, we have $T_N > n$, so $\mathbf{1}_{\{T_N < T_0\}} = \mathbf{1}_{\{T_N < T_0\}} \circ \theta_n$. By the Markov property,

$$\begin{aligned} P_k\{Y_0 = y_0, \cdots, Y_n = y_n, T_N < T_0\} &= E_k[\mathbf{1}_{\{X_0 = y_0, \cdots, X_n = y_n\}} P_{y_n}(T_N < T_0)] \\ &= \frac{y_n}{N} P_k\{X_0 = y_0, \cdots, X_n = y_n\} \\ &= \frac{y_n}{N} \frac{1}{2^n}. \end{aligned}$$

Therefore, in this case,

$$P_k^{(N)}\{Y_0 = y_0, \cdots, Y_n = y_n\} = \frac{y_n}{k} 12^n = \frac{y_1}{2k} \frac{y_2}{2y_1} \cdots \frac{y_n}{2y_{n-1}}.$$

Second situation: $y_\ell = N$ for some ℓ (while $y_{\ell-1} < N$) and thus $y_i = N$, $\forall i \ge \ell$. On the event $\{\{Y_0 = y_0, \cdots, Y_n = y_n\}$, we have $T_N = \ell$, which is indeed smaller than T_0. Accordingly,

$$P_k\{Y_0 = y_0, \cdots, Y_n = y_n, T_N < T_0\} = P_k\{X_0 = y_0, \cdots, X_\ell = y_\ell\} = \frac{1}{2^\ell},$$

so that in this situation,

$$P_k^{(N)}\{Y_0 = y_0, \cdots, Y_n = y_n\} = \frac{N}{2^\ell k} = \frac{y_1}{2k} \frac{y_2}{2y_1} \cdots \frac{y_\ell}{2y_{\ell-1}} \times 1 \times \cdots \times 1.$$

Conclusion: under $P_k^{(N)}$, $(Y_n, n \in \mathbb{N})$ is a Markov chain with transition probability

$$\begin{aligned} q(x, x+1) &= \frac{x+1}{2x}, \qquad 1 \le x \le N-1, \\ q(x, x-1) &= \frac{x-1}{2x}, \qquad 2 \le x \le N-1, \\ q(N, N) &= 1, \\ q(x, y) &= 0, \qquad \text{otherwise}. \end{aligned}$$

(B3) It is seen that $(X_{n\wedge T_0\wedge T_N},\, n\in\mathbb{N})$ is a Markov chain, taking values in $\{0,\,1,\,\cdots,\,N\}$, with transition kernel

$$\begin{aligned} p(x,\,x+1) &= \frac{1}{2}, && 1\le x\le N-1,\\ p(x,\,x-1) &= \frac{1}{2}, && 1\le x\le N-1,\\ p(N,\,N) &= 1, &&\\ p(0,\,0) &= 1, &&\\ p(x,\,y) &= 0, && \text{otherwise.} \end{aligned}$$

In view of the kernels p on $\{0,\,1,\,\cdots,\,N\}$ and q on $\{1,\,2,\,\cdots,\,N\}$, let us define the function $h:\,\{0,\,1,\,\cdots,\,N\}\to\mathbb{R}_+$ by $h(x):=x$ for $x\in\{0,\,1,\,\cdots,\,N\}$. Since $(X_{n\wedge T_0\wedge T_N},\, n\in\mathbb{N})$ under P_k is a martingale (being a martingale stopped at a stopping time), it is immediate that q is the h-transform of p. $\square$

Exercise 9 (Markov chains and martingales). Let $\mathscr{H}:=\{f:\,S\to\mathbb{R}\text{ bounded}\}$. Prove that there exists a mapping $A:\,\mathscr{H}\to\mathscr{H}$ such that for any $f\in\mathscr{H}$,

$$f(X_n)-\sum_{i=0}^{n-1}Af(X_i),\qquad (\sum_{i=0}^{-1}:=0)$$

is a martingale with respect to the canonical filtration.

Solution. Assuming that such a mapping exists, what should it look like?

Let $Y_n:=f(X_n)-\sum_{i=0}^{n-1}Af(X_i)$, $n\in\mathbb{N}$. Saying that $(Y_n,\,n\in N)$ is an $(\mathscr{F}_n)$-martingale means that Y_n is integrable, $\mathscr{F}_n$-measurable, an $E(Y_{n+1}\,|\,\mathscr{F}_n)=Y_n$. Since

$$\begin{aligned} E(Y_{n+1}\,|\,\mathscr{F}_n) &= E[f(X_{n+1})\,|\,\mathscr{F}_n]-\sum_{i=0}^{n}Af(X_i)\\ &= E[f(X_{n+1})\,|\,X_n]-\sum_{i=0}^{n}Af(X_i)\\ &= \sum_{y\in S}p(X_n,\,y)f(y)-\sum_{i=0}^{n}Af(X_i)\\ &= Y_n+\sum_{y\in S}p(X_n,\,y)f(y)-f(X_n)-Af(X_n), \end{aligned}$$

this gives $Af(X_n)=\sum_{y\in S}p(X_n,\,y)f(y)-f(X_n)$.

So, by defining $A : \mathscr{H} \to \mathscr{H}$ by $Af(x) := \sum_{y\in S} p(x, y)f(y) - f(x)$, $\forall x \in S$, we immediately see that Y_n is integrable and $\mathscr{F}_n$-measurable, and by the computations done above, $E(Y_{n+1} \,|\, \mathscr{F}_n) = Y_n$. In other words, $(Y_n,\, n \in \mathbb{N})$ is a martingale. $\square$

Exercise 10 (Simple random walk on $\mathbb{Z}$). For any $x \in \mathbb{Z}$, let P_x denote the probability on the canonical space, under which $(X_n,\, n \in \mathbb{N})$ is a simple random walk on $\mathbb{Z}$ with $P_x(X_0 = x) = 1$.

For $a \in \mathbb{N}$, we define $\tau_a := \inf\{n \in \mathbb{N} : X_n - X_0 = a\}$ (with $\inf \varnothing := \infty$).

(i) Prove that for any $x \in \mathbb{Z}$, $P_x\{\tau_a < \infty,\, \forall a \in \mathbb{N}\} = 1$.

(ii) Prove that $(\tau_{a+1} - \tau_a,\, a \in \mathbb{N})$ under P_0 is a sequence of i.i.d. random variables.

Solution. (i) Under P_0, simple random walk on $\mathbb{Z}$ is sum of i.i.d. Bernoulli($\frac{1}{2}$) random variables. By central limit theorem (why?), $\limsup_{n\to\infty} \frac{X_n}{n^{1/2}} = \infty$ P_0-a.s., and a fortiori, $\limsup_{n\to\infty} X_n = \infty$ P_0-a.s. By symmetry, $\liminf_{n\to\infty} X_n = -\infty$ P_0-a.s. Thus P_0-a.s., $\tau_a < \infty$ for any $a \in \mathbb{N}$.

For $x \in \mathbb{Z}$, we note that $P_x\{\tau_a < \infty,\, \forall a \in \mathbb{N}\} = P_0\{\tau_a < \infty,\, \forall a \in \mathbb{N}\} = 1$.

(ii) We work under P_0. For any $a \geq 1$, $\tau_{a+1} - \tau_a = \tau_1 \circ \theta_{\tau_a}$. An application of the strong Markov property at stopping time τ_a tells us that $\tau_{a+1} - \tau_a$ is independent of $\mathscr{F}_{\tau_a}$, and has the law of τ_1.

For any $n \geq 1$, by an induction argument in n, we see that $\tau_{a+1} - \tau_a$, $0 \leq a \leq n$, are i.i.d. random variables. $\square$

Exercise 11. Let $f : S \to \mathbb{R}_+$.

(i) Let $u(x) := \sum_{y\in S} G(x, y)f(y)$, $x \in S$. Prove that

$$u(x) = f(x) + \sum_{y\in S} p(x, y)u(y).$$

(ii) Let $v : S \to \mathbb{R}_+ \cup \{\infty\}$ be such that

$$v(x) = f(x) + \sum_{y\in S} p(x, y)v(y), \qquad \forall x \in S.$$

Prove that $u(x) \leq v(x)$, $\forall x \in S$.

Solution. (i) It suffices to use $G(x, y) = \sum_{n=0}^{\infty} p^n(x, y)$ and Fubini–Tonelli.

(ii) Let us check that $v(x) \ge \sum_{i=0}^n \sum_{y\in S} p^i(x, y)f(y)$, $\forall n \ge 0$. It is holds trivially for $n = 0$ because $v(x) \ge f(x)$ by definition. Assume this holds for n. Then

$$\begin{aligned}
v(x) &= f(x) + \sum_{y\in S} p(x, y)v(y) \\
&\ge f(x) + \sum_{y\in S} p(x, y) \sum_{i=0}^{n} \sum_{z\in S} p^i(y, z)f(z) \\
&= f(x) + \sum_{z\in S} p^{n+1}(x, z)f(z),
\end{aligned}$$

so the inequality holds for $n + 1$ as well. By induction, it holds for all $n \ge 0$. By letting $n \to \infty$, we obtain $v(x) \ge \sum_{i=0}^{\infty} \sum_{y\in S} p^i(x, y)f(y) = \sum_{y\in S} G(x, y)f(y) = u(x)$. $\square$

Exercise 12. **(i)** Let $x \in S$. Let y be a recurrent state such that $G(x, y) > 0$. Is it true that $P_x\{N(y) = \infty\} = 1$?

(ii) Give an example of $x, y \in S$ such that $G(x, y) > 0$, but $G(y, x) = 0$.

(iii) If $G(x, y) = \infty$, is y necessarily recurrent?

(iv) If y is recurrent, is $G(x, y)$ necessarily infinite?

(v) Is it possible to have $0 < G(x, y) < \infty$ for a recurrent state y?

(vi) If $G(x, y) = \infty$, what are possible values for $G(y, x)$?

(vii) Assume that for all $x \in S$, the set $B_x := \{y \in S : G(x, y) > 0\}$ is finite. Prove that there are recurrent states.

Solution. (i) Not necessarily. Let $y \ne x$. We only have $P_x\{T_y < \infty\} > 0$; so, on the set $\{T_y = \infty\}$ which can have positive P_x-probability, we have $N(y) = 0$.

(ii) Any chain such that y is absorbing and that $G(x, y) > 0$. For example, a branching process such that $\mu(0) > 0$, with $x = 1$ and $y = 0$.

(iii) We know that $G(x, y) \le G(y, y)$. So $G(x, y) = \infty$ implies $G(y, y) = \infty$, which means y is recurrent.

(iv) If $P_x\{T_y = \infty\} = 1$ (which is possible if x is transient), then $G(x, y) = 0$.

(v) If $x = y$, then saying that y is recurrent is equivalent to saying that $G(y, y) = \infty$, so it is not possible to have $0 < G(x, y) < \infty$ in this case.

If $x \ne y$, then $G(x, y) = G(y, y)\, P_x\{T_y < \infty\}$, with $G(y, y) = \infty$, which implies that $G(x, y)$ is either 0 (if $P_x\{T_y < \infty\} = 0$) or ∞ (if $P_x\{T_y < \infty\} > 0$). In any case, it is not possible to have $0 < G(x, y) < \infty$.

(vi) By (iii), $G(x, y) = \infty$ implies that y is recurrent, so $G(y, x) = \infty$ if x is recurrent and in the same recurrence class as y and $G(y, x) = 0$ otherwise.

(vii) Let $x \in S$. Under P_x, the chain lies a.s. in B_x, so that for any $n \in \mathbb{N}$,

$$\sum_{y \in B_x} p^n(x, y) = \sum_{y \in S} p^n(x, y) = 1.$$

By summing over all $n \in \mathbb{N}$, we get

$$\sum_{n \in \mathbb{N}} \sum_{y \in B_x} p^n(x, y) = \infty.$$

However, on the left-hand side, $\sum_{n\in\mathbb{N}} \sum_{y \in B_x} p^n(x, y) = \sum_{y\in B_x} \sum_{n\in\mathbb{N}} p^n(x, y)$, which is $\sum_{y \in B_x} G(x, y)$. Since $\# B_x < \infty$, there exists $y \in B_x$ such that $G(x, y) = \infty$, which implies (see (iii)) that y is recurrent. $\square$

Exercise 13. Let $x, y, z \in S$.

(i) Prove that $E_x(\mathbf{1}_{\{T_y<\infty\}} \sum_{n=T_y}^{\infty} \mathbf{1}_{\{X_n=z\}}) = P_x(T_y < \infty)\, G(y, z)$.

(ii) Assume y is a transient state. Prove that $E_x(\mathbf{1}_{\{T_y<\infty\}} \sum_{n=T_y}^{\infty} \mathbf{1}_{\{X_n=z\}}) = \frac{G(x,y)}{G(y,y)}\, G(y, z)$.

Solution. (i) By the strong Markov property,

$$\begin{aligned} E_x\Big(\mathbf{1}_{\{T_y<\infty\}} \sum_{n=T_y}^{\infty} \mathbf{1}_{\{X_n=z\}}\Big) &= E_x\Big(\mathbf{1}_{\{T_y<\infty\}} \Big(\sum_{n=0}^{\infty} \mathbf{1}_{\{X_n=z\}}\Big) \circ \theta_{T_y}\Big) \\ &= E_x\Big(\mathbf{1}_{\{T_y<\infty\}} E_{X_{T_y}}(N_z)\Big) \\ &= P_x(T_y < \infty)\, G(y, z)\,, \end{aligned}$$

as desired.

(ii) Follows from (i) by recalling that in the class, we have proved that $G(x, y) = P_x(T_y < \infty)\, G(y, y)$. $\square$

Exercise 14 (A criterion for recurrence). Assume there exists $w \in S$ such that for all $x \in S$, $G(w, x) > 0$ and $P_x\{T_w < \infty\} = 1$. Prove that the chain is irreducible and recurrent.

Solution. Since $P_w\{T_w < \infty\} = 1$, w is by definition recurrent. For any $x \in S$, since $G(w, x) > 0$, x is also recurrent, and is in the same recurrence class as w: there is thus only one recurrence class. $\square$

Exercise 15. Let $(Y_n,\, n \ge 1)$ be an i.i.d. sequence of Bernoulli random variables of parameter $0 < p < 1$. Let $X_0 := 0$ and $X_n := Y_1 + \cdots + Y_n$ for $n \ge 1$. We observe that $X_{n+1} \ge X_n$ a.s. for all n. Let $T_y := \inf\{n \ge 0 : X_n = y\}$ ($\inf \varnothing := \infty$) for all $y \in \{0, 1, 2, \cdots\}$.

(i) Prove that $\lim_{n\to\infty} X_n = \infty$ a.s. and that $P_0(T_y < \infty) = 1$ for all $y \in \{0, 1, 2, \cdots\}$.

(ii) Prove that $M_n := X_n - np$, $n \geq 0$, is an $(\mathscr{F}_n)$-martingale, where for any n, $\mathscr{F}_n := \sigma(X_i,\ 0 \leq i \leq n)$.

(iii) By considering $(M_{n\wedge T_y})$, compute $E_0(T_y)$.

(iv) Let $N(y) := \sum_{k=0}^{\infty} \mathbf{1}_{\{X_k=y\}}$. Compute $\mathbf{1}_{\{X_k=y\}}$ for $k < T_y$, $T_y \leq k < T_{y+1}$ and $k \geq T_{y+1}$, respectively.

(v) Prove that $N(y) = T_{y+1} - T_y$ a.s. Compute $E_0[N(y)]$.

We notice that (X_n) is a random walk with transition probability p given by $p(x, x) = 1 - p$, $p(x, x+1) = p$, $x \in \{0, 1, 2, \cdots\}$ (you are not asked to prove this).

(vi) Compute the law of X_n (for given n). Compute the law of T_1.

(vii) Prove that $N(y)$ has the same law as T_1.

(viii) Compute the law of T_y.

Solution. (i) The sequence of random variables (Y_n) being i.i.d. and $Y_1 \in L^1$ with $E_0(Y_1) = p$, it follows from Kolmogorov's law of large numbers that $\lim_{n\to\infty} \frac{X_n}{n} = p > 0$ a.s.; a fortiori, $\lim_{n\to\infty} X_n = \infty$ a.s. Since $X_n - X_{n-1} \in \{0,1\}$ for all n, $X_0 = 0$ and $\lim_{n\to\infty} X_n = \infty$ a.s., we see that (X_n) visits all sites $y \in \{0, 1, 2, \cdots\}$ and thus $T_y < \infty$ a.s.

(ii) The sequence $M_n := X_n - np = \sum_{i=1}^{n}(Y_i - p)$, $n \geq 1$, is a martingale, because $(Y_i - p,\ i \geq 1)$ is a sequence of i.i.d. mean-zero random variables.

(iii) Since $n \wedge T_y$ is a bounded stopping time, we have $E_0(M_{n\wedge T_y}) = E(M_0) = 0$, thus $E_0(n\wedge T_y) = E_0(X_{n\wedge T_y})/p$. Since $T_y < \infty$ and $|X_{n\wedge T_y}| \leq y$ a.s., it follows from the dominated convergence theorem that $\lim_{n\to\infty} E_0(X_{n\wedge T_y}) = E_0(X_{T_y}) = y$. By monotone convergence, this yields $\lim_{n\to\infty} E_0(n \wedge T_y) = E_0(T_y)$. So $E_0(T_y) = \frac{y}{p}$.

(iv) Since $X_{n+1} \geq X_n$ a.s. for all n, we have

$$\mathbf{1}_{\{X_k=y\}} = 0, \qquad \text{if} \quad k < T_y \quad \text{or} \quad k \geq T_{y+1},$$

and

$$\mathbf{1}_{\{X_k=y\}} = 1, \qquad \text{if} \quad T_y \leq k < T_{y+1}.$$

(v) It follows from (iv) that $N(y) = \sum_{i=T_y}^{T_{y+1}-1} 1 = T_{y+1} - T_y$, which yields

$$E_0[N(y)] = E_0(T_{y+1} - T_y) = \frac{y+1}{p} - \frac{y}{p} = \frac{1}{p}\,.$$

(vi) It is clear that for any n, X_n is a binomial random variable of parameter (n, p). To determine the law of T_1, we note that for $n \geq 1$,

$$\begin{aligned} P_0(T_1 = k) = P_0(X_1 = \cdots = X_{k-1} = 0,\ X_k = 1) &= P_0(Y_1 = \cdots = Y_{k-1} = 0,\ Y_k = 1) \\ &= (1-p)^{k-1}\,p\,. \end{aligned}$$

In words, T_1 has the geometric distribution of parameter p.

(vii) Let us compute $P_0(N(y) = k)$. By (iv),

$$N(y) = T_{y+1} - T_y = T_{y+1} \circ \theta_{T_y}.$$

Applying the strong Markov property gives that

$$\begin{aligned} P_0(N(y) = k) &= P_0(T_{y+1} \circ \theta_{T_y} = k) \\ &= E_0(P_{X_{T_y}}(T_{y+1} = k)) \\ &= P_y(T_{y+1} = k) = P_0(T_1 = k)\,. \end{aligned}$$

(viii) Let $y \geq 2$ and $k \geq y$. We have

$$P_0(T_y = k) = P_0(X_{k-1} = y-1,\ X_k = y) = P_0(X_{k-1} = y-1,\ Y_k = 1)\,,$$

which is $\binom{k-1}{y-1}\ p^y(1-p)^{k-y}$. □

Exercise 16 (Second moment for hitting numbers). The aim of this exercise is to find an explicit formula for the second moment of $N(y)$ in terms of the Green function. Assume that the chain is irreducible and that $G(x,\,y) < \infty$, $\forall x,\, y \in S$.

(i) For any mapping $f : S \to \mathbb{R}_+$, we define $Gf : S \to \mathbb{R}_+$ by

$$Gf(x) := \sum_{y \in S} G(x,\,y) f(y), \qquad (\infty \times 0 := 0)$$

Prove that for any $x \in S$ and any $f : S \to \mathbb{R}_+$,

$$E_x\Big[\sum_{n=0}^{\infty}\sum_{m=n}^{\infty} f(X_n)\, f(X_m)\Big] = Gg(x),$$

where $g(y) := f(y)Gf(y)$, $y \in S$.

(ii) Prove that for any $x \in S$ and any $f : S \to \mathbb{R}_+$,

$$E_x\Big[\Big(\sum_{n=0}^{\infty} f(X_n)\Big)^2\Big] = 2Gg(x) - Gh(x),$$

where $h(y) := f(y)^2$, $y \in S$.

(iii) Prove that for any $x,\, y \in S$,

$$E_x[N(y)^2] = G(x,\,y)\,[2G(y,\,y) - 1].$$

Solution. (i) It follows from the Fubini–Tonelli theorem that $E_x[\sum_{n=0}^\infty \sum_{m=n}^\infty f(X_n)\, f(X_m)] = \sum_{n=0}^\infty \sum_{m=n}^\infty E_x[f(X_n)\, f(X_m)]$. For $m \ge n$, the Markov property gives

$$E_x[f(X_n)\, f(X_m)] = E_x\{f(X_n)\, E_{X_n}[f(X_{m-n})]\}.$$

Therefore, by Fubini–Tonelli again,

$$\begin{aligned} E_x\Big[\sum_{n=0}^\infty \sum_{m=n}^\infty f(X_n)\, f(X_m)\Big] &= \sum_{n=0}^\infty E_x\Big\{f(X_n)\, E_{X_n}\Big[\sum_{m=n}^\infty f(X_{m-n})\Big]\Big\} \\ &= \sum_{n=0}^\infty E_x\{f(X_n)\, Gf(X_n)\}, \end{aligned}$$

which is $= \sum_{n=0}^\infty E_x\{g(X_n)\} = Gg(x)$.

(ii) We have

$$\begin{aligned} E_x\Big[\Big(\sum_{n=0}^\infty f(X_n)\Big)^2\Big] &= 2E_x\Big[\sum_{n=0}^\infty \sum_{m=n}^\infty f(X_n)\, f(X_m)\Big] - E_x\Big[\sum_{n=0}^\infty f(X_n)^2\Big] \\ &= 2Gg(x) - E_x\Big[\sum_{n=0}^\infty h(X_n)\Big] \\ &= 2Gg(x) - Gh(x). \end{aligned}$$

(iii) Fix $x, y \in S$. By definition, $N(y) = \sum_{n=0}^\infty f(X_n)$, where $f := \mathbf{1}_{\{y\}}$. For this choice of f, we have $h = f^2 = f$, so $Gh(x) = G(x, y)$, whereas $g(z) := f(z)\, Gf(z) = \mathbf{1}_{z=y}\, G(z, y) = \mathbf{1}_{z=y}\, G(y, y)$, which yields $Gg(x) = \sum_{z\in S} G(x, z)g(z) = G(x, y)\, G(y, y)$. As a consequence, $E_x[N(y)^2] = 2G(x, y)\, G(y, y) - G(x, y)$, as desired. $\square$

Exercise 17 (Irreducible chains). Prove that $(X_n,\, n \in \mathbb{N})$ is irreducible if and only if there exists no $A \subset S$, with $A \neq \varnothing$ and $A \neq S$, such that $p(x, y) = 0$, $\forall x \in A$, $\forall y \in A^c$.

Solution. "$\Rightarrow$" Suppose there exists $A \subset S$, with $A \neq \varnothing$ and $A \neq S$, such that $p(x, y) = 0$, $\forall x \in A$, $\forall y \in A^c$.

Let $x \in A$ and $y \in A^c$. Since $p(x, y) = 0$ while the chain is irreducible, there exists $n \ge 2$ such that $p^n(x, y) > 0$, and thus $x =: x_1, x_2, \cdots, x_n := y \in S$ such that $p(x_i, x_{i+1}) > 0$ for any $1 \le i \le n-1$. Let

$$k := \max\{i :\ 1 \le i \le n-1,\ x_i \in A\}.$$

Then $x_k \in A$, $x_{k+1} \in A^c$, and $p(x_k, x_{k+1}) > 0$, which is impossible.

"$\Leftarrow$" By assumption, for any $A \subset S$ with $A \neq \varnothing$ and $A \neq S$, there exist $a \in A$ and $b \in A^c$ such that $p(a, b) > 0$.

For any $x \in S$, let $B_x := \{y \in S : G(x, y) > 0\} \supset \{x\}$. We need to show that $B_x = S$.

Suppose $B_x \neq S$. Then by assumption, there exist $a \in B_x$ and $b \in B_x^c$ such that $p(a, b) > 0$. Since $a \in B_x$, we have $G(x, a) > 0$; so there exists $n \in \mathbb{N}$ such that $p^n(x, a) > 0$. This implies $p^{n+1}(x, b) \geq p^n(x, a)\, p(a, b) > 0$; thus $G(x, b) > 0$, i.e., $b \in B_x$. Contradiction. $\square$

Exercise 18 (Wright–Fisher reproduction model). Let $S := \{0, 1, 2, \cdots, N\}$, and let

$$p(i, j) := C_N^j \, (\frac{i}{N})^j \, (1 - \frac{i}{N})^{N-j}.$$

The chain describes, in an idealized model, the process of genetic drift in a population of fixed size.

(i) Classify the states of $(X_n, n \in \mathbb{N})$.

(ii) Prove, for any $k \in S$, that $(X_n, n \in \mathbb{N})$ under P_k is a martingale with respect to the canonical filtration, and that the limit

$$X_\infty := \lim_{n \to \infty} X_n$$

exists P_k-a.s. What is the law of X_∞ under P_k?

Solution. (i) Since $p(0, 0) = p(N, N) = 1$, 0 and N are absorbing, thus recurrent. For $i \in \{1, 2, \cdots, N-1\}$, we have $p(i, 0) > 0$; therefore, with positive P_i-probability, the chain, starting at i, never returns to i. Therefore, $1, 2, \cdots, N-1$ are transient states.

(ii) For any $n \in \mathbb{N}$, X_n is $\mathscr{F}_n$-measurable, and P_k-integrable (being bounded). Moreover, by the Markov property,

$$E_k(X_{n+1} \,|\, \mathscr{F}_n) = E_k(X_{n+1} \,|\, X_n),$$

and given X_n, X_{n+1} has the binomial distribution of parameter $(N, \frac{X_n}{N})$ (thus of mean $N \times \frac{X_n}{N} = X_n$), so that $E_k(X_{n+1} \,|\, X_n) = X_n$. Therefore, $(X_n, n \in \mathbb{N})$ is a martingale.

A non-negative martingale converges a.s., so $X_\infty := \lim_{n \to \infty} X_n$ exists P_k-a.s.

To determine the law of X_∞, we observe that, the number of visits at a transient state being finite, X_N takes values in $\{0, N\}$. Since (X_n) is a uniformly integrable martingale (being bounded), it also converges to X_∞ in L^1: thus $E_k(X_\infty) = E_k(X_0) = k$. This means $P_k(X_\infty = N) = \frac{k}{N}$; so the law of X_∞ under P_k is $\frac{k}{N}\, \delta_N + (1 - \frac{k}{N})\delta_0$. $\square$

Exercise 19 (A random walk on $\mathbb{Z}^d$ resembling simple random walk). Consider the transition probability on $\mathbb{Z}^d$:

$$p(x,\,y) = \frac{1}{2^d}\prod_{i=1}^{d} \mathbf{1}_{\{|y_i - x_i| = 1\}}, \qquad x := (x_1,\,\cdots,\,x_d),\ y := (y_1,\,\cdots,\,y_d) \in \mathbb{Z}^d.$$

Classify the states of the chain.

Solution. We know that for random walks, since Green's function depends only on $x-y$, there are only two possible situations: either all states are recurrent, or all states are transient.

Our Markov chain under P_0 has the law of $(Y_n^1,\,\cdots,\,Y_n^d)_{n\in\mathbb{N}}$, where the processes $(Y_n^1)_{n\in\mathbb{N}}$, $\cdots$, $(Y_n^d)_{n\in\mathbb{N}}$ are independent copies of the simple symmetric random walk on $\mathbb{Z}$ starting from $0 \in \mathbb{Z}$. Therefore,

$$p^n(0,\,0) = P(Y_n^1 = 0,\,\cdots,\,Y_n^d = 0) = P(Y_n^1 = 0)^d.$$

It is easy to compute $P(Y_n^1 = 0)$: the probability is 0 if n is odd, and if $n = 2k$, then

$$P(Y_{2k}^1 = 0) = \frac{C_{2k}^k}{2^{2k}}.$$

Therefore, for $x \in \mathbb{Z}^d$,

$$G(x,\,x) = G(0,\,0) = \sum_{k=0}^{\infty} p^{2k}(0,\,0) = \sum_{k=0}^{\infty} (\frac{C_{2k}^k}{2^{2k}})^d.$$

By the Stirling formula, when $k \to \infty$,

$$\frac{C_{2k}^k}{2^{2k}} = \frac{(2k)!}{2^{2k}\,(k!)^2} \sim \frac{(\frac{2k}{\mathrm{e}})^{2k}(4\pi k)^{1/2}}{2^{2k}\,[(\frac{k}{\mathrm{e}})^k(2\pi k)^{1/2}]^2} = \frac{1}{(\pi k)^{1/2}},$$

So that $G(x,\,x) = \infty$ if $d = 1$ or 2, and $G(x,\,x) < \infty$ if $d \geq 3$. We conclude that in dimensions $d = 1$ and 2, all states are recurrent, whereas in dimensions $d \geq 3$, all states are transient. □

Exercise 20 (Birth-and-death chain). Let $S = \mathbb{N}$ and consider the transition matrix

$$q := \begin{pmatrix} r_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $p_i > 0$ and $r_i \geq 0$ (for $i \geq 0$) and $q_j > 0$ (for $j \geq 1$) are such that $p_0 + r_0 = 1$ and $p_j + r_j + q_j = 1$ (for $j \geq 1$).

(i) Prove that $(X_n,\, n \in \mathbb{N})$ is irreducible.

(ii) Prove that the chain is recurrent if and only if

$$\sum_{i=1}^{\infty} \frac{q_1 \cdots q_i}{p_1 \cdots p_i} = \infty.$$

Solution. (i) Let $x,\, y \in \mathbb{N}$. If $x < y$, then $G(x,\, y) > 0$ (because $p_i > 0$, $\forall i \geq 0$). If $x > y$, then $G(x,\, y) > 0$ (because $p_j > 0$, $\forall j \geq 1$). If $x = y$, then $G(x,\, x) \geq p^2(x,\, x) \geq p_x\, q_{x+1} > 0$. As a consequence, the chain is irreducible.

(ii) Since the chain is irreducible, either all states are recurrent or they are all transient. It suffices to check for state 0. Since $P_0\{T_0 < \infty\} = r_0 + p_0\, P_1\{T_0 < \infty\}$, we need to know whether $P_1\{T_0 < \infty\} = 1$.

Let M be an integer (at the end, $M \to \infty$), and let $a_i := P_i\{\tau_0 < \tau_M\}$ for $0 \leq i \leq M$, where $\tau_x := \inf\{n \in \mathbb{N} :\, X_n = x\}$. So $a_0 = 1$ and $a_M = 0$ while by the Markov property, $a_i = q_i\, a_{i-1} + r_i\, a_i + p_i\, a_{i+1}$ for $1 \leq i \leq M-1$, which becomes $p_i(a_{i+1} - a_i) = q_i(a_i - a_{i-1})$. Write $b_i := a_{i+1} - a_i$, $0 \leq i \leq M-1$, then $b_i = \frac{q_i}{p_i}\, b_{i-1}$, $1 \leq i \leq M-1$, so that $b_i = \frac{q_1 \cdots q_i}{p_1 \cdots p_i}\, b_0$. Summing over $i \in [1,\, M-1] \cap \mathbb{Z}$, we get $\sum_{i=1}^{M-1} b_i = b_0 \sum_{i=1}^{M-1} \frac{q_1 \cdots q_i}{p_1 \cdots p_i}$. But $\sum_{i=1}^{M-1} b_i = a_M - a_0 = -1$, whereas $b_0 = a_1 - 1$, we get

$$P_1\{\tau_0 < \tau_M\} = a_1 = 1 - \frac{1}{\sum_{i=1}^{M-1} \frac{q_1 \cdots q_i}{p_1 \cdots p_i}}.$$

We let $M \uparrow \infty$. By the monotone convergence theorem, this gives

$$P_1\{T_0 < \infty\} = 1 - \frac{1}{\sum_{i=1}^{M-1} \frac{q_1 \cdots q_i}{p_1 \cdots p_i}},$$

which equals 1 if and only if $\sum_{i=1}^{\infty} \frac{q_1 \cdots q_i}{p_1 \cdots p_i} = \infty$. □

Exercise 21 (Kolmogorov's condition for reversibility). Assume that $(X_n,\, n \in \mathbb{N})$ is irreducible. Prove that the chain has a reversible measure if and only if the following two conditions are satisfied:

(i) $\forall x,\, y \in S, \quad p(x,\, y) > 0 \;\Rightarrow\; p(y,\, x) > 0$;

(ii) for any n and any $x_0,\, x_1,\, \cdots,\, x_n := x_0 \in S$, we have

$$p(x_0,\, x_1)\, p(x_1,\, x_2) \cdots p(x_{n-1},\, x_n) = p(x_n,\, x_{n-1}) \cdots p(x_2,\, x_1)\, p(x_1,\, x_0).$$

Solution. "$\Rightarrow$" Let μ be a reversible measure for the chain. Let us first prove that $\mu(x) > 0$, $\forall x \in S$.

Suppose there exists $x \in S$ such that $\mu(x) = 0$. Let $y \in S\backslash\{x\}$. Since the chain is irreducible, there exist $x_0 = x,\, x_1,\, \cdots,\, x_n := y$ such that $p(x_{i-1},\, x_i) > 0$, $\forall 1 \le i \le n$. By the definition of reversibility,

$$\mu(x_{n-1})\, p(x_{n-1},\, x) = \mu(x)\, p(x,\, x_{n-1}) = 0,$$

so $\mu(x_{n-1}) = 0$. By induction, this yields $\mu(x_i) = 0$, $0 \le i \le n$. In particular, $\mu(x) = 0$, contradicting the choice of x.

Now that we know $\mu(x) > 0$, $\forall x \in S$, we have, by reversibility,

$$p(y,\, x) = \frac{\mu(x)}{\mu(y)}\, p(x,\, y), \qquad \forall x,\, y \in S.$$

In particular, if $p(x,\, y) > 0$, then $p(y,\, x) > 0$.

To check (ii), we see that

$$\prod_{i=1}^{n} p(x_{i-1},\, x_i) = \Big(\prod_{i=1}^{n} \frac{\mu(x_{i-1})}{\mu(x_i)} \Big) \prod_{i=1}^{n} p(x_i,\, x_{i-1}),$$

which is $\prod_{i=1}^{n} p(x_i,\, x_{i-1})$ since $\prod_{i=1}^{n} \mu(x_{i-1}) = \prod_{i=1}^{n} \mu(x_i)$ (recalling that $x_0 = x_n$).

"$\Leftarrow$" Let us construct a reversible measure. Fix an arbitrary $x \in S$, and let $\mu(x) := 1$. Let $y \in S\backslash\{x\}$.

By irreducibility, there exist $x_0 = x,\, x_1,\, \cdots,\, x_n := y$ such that $p(x_{i-1},\, x_i) > 0$, $\forall 1 \le i \le n$. Since we also have $p(x_i,\, x_{i-1}) > 0$, we can define

$$\mu(y) := \prod_{i=1}^{n} \frac{p(x_{i-1},\, x_i)}{p(x_i,\, x_{i-1})}.$$

An elementary calculation tells us that μ is well-defined, in the sense that it does not depend on the choice of the path $x_0 = x,\, x_1,\, \cdots,\, x_n := y$.

Let us check that μ is reversible. Let $z \in S$, and we need to verify $\mu(y)\, p(y,\, z) = \mu(z)\, p(z,\, y)$.

If $p(y,\, z) = 0$, then $p(z,\, y) = 0$, and the identity holds trivially.

Assume $p(y,\, z) > 0$. Then by using the path $x_0 = x,\, x_1,\, \cdots,\, x_n := y$ connecting x to y, and add a last element $x_{n+1} := z$, we have, by definition of μ,

$$\mu(z) = \prod_{i=1}^{n+1} \frac{p(x_{i-1},\, x_i)}{p(x_i,\, x_{i-1})} = \mu(y)\, \frac{p(y,\, z)}{p(z,\, y)},$$

giving again the desired equality. $\square$

Exercise 22. Let π be an invariant probability for p such that $\pi(x) > 0$ for all $x \in S$. Define

$$p^*(x,\, y) := \frac{p(y,\, x)\pi(y)}{\pi(x)}, \qquad x,\, y \in S.$$

(i) Prove that p^* is a transition kernel on S and that π is an invariant probability for p^*. Give a necessary and sufficient condition for $p^* = p$.

(ii) Let $(X_n,\, n \geq 0)$ be the canonical Markov chain with transition kernel p. Fix $N \in \mathbb{N}$, and define $X_n^* := X_{N-n}$, $0 \leq n \leq N$. Compute $P_\pi(X_0^* = x_0, \cdots, X_N^* = x_N)$. Prove that $(X_n^*,\, 0 \leq n \leq N)$ is, under P_π, a Markov chain with initial law π and transition kernel p^*.

(iii) Let $0 < \alpha < 1$, and let p be the transition kernel on $\mathbb{N}$ defined by

$$p(x,\, y) = \alpha\, \mathbf{1}_{\{y=x+1\}} + (1-\alpha)\, \mathbf{1}_{\{y=0\}}, \qquad x,\, y \in \mathbb{N}\,.$$

Find an invariant probability π. Is π unique?

(iv) Compute p^* and prove that

$$p^*(x,\, y) = \mathbf{1}_{\{y=x-1\}} + \pi(y)\, \mathbf{1}_{\{x=0\}}, \qquad x,\, y \in \mathbb{N}\,.$$

Solution. (i) By definition, $p^*(x,\, y) \geq 0$ for all $x,\, y \in S$, and

$$\sum_y p^*(x,\, y) = \sum_y \frac{p(y,\, x)\pi(y)}{\pi(x)},$$

which is (because π is an invariant probability) $= \frac{\pi(x)}{\pi(x)} = 1$. So p^* is a transition kernel. Furthermore, for any $y \in S$,

$$\sum_{x\in S} \pi(x)\, p^*(x,\, y) = \sum_{x\in S} \pi(x)\, \frac{p(y,\, x)\pi(y)}{\pi(x)} = \sum_{x\in S} p(y,\, x)\pi(y) = \pi(y)\,,$$

which implies that π is invariant for p^*.

We have $p^* = p$ if and only if $\pi(x)\, p(x,\, y) = p(y,\, x)\, \pi(y)$ for all $x,\, y \in S$, i.e., if and only if π is reversible for p.

(ii) We have

$$\begin{aligned} P_\pi(X_0^* = x_0, \cdots, X_N^* = x_N) &= P_\pi(X_0 = x_N, \cdots, X_N = x_0) \\ &= \pi(x_N)\, p(x_N,\, x_{N-1}) \cdots p(x_1, x_0)\,. \end{aligned}$$

Since $\pi(x_i)\, p(x_i,\, x_{i-1}) = p^*(x_{i-1},\, x_i)\, \pi(x_{i-1})$, this yields

$$\begin{aligned}
& P_\pi(X_0^* = x_0,\, \cdots,\, X_N^* = x_N) \\
= \ & p^*(x_{N-1},\, x_N)\, \pi(x_{N-1})\, p(x_{N-1},\, x_{N-2}) \cdots p(x_1,\, x_0) \\
= \ & p^*(x_{N-1},\, x_N)\, p^*(x_{N-2},\, x_{N-1})\, \pi(x_{N-2})\, p(x_{N-2},\, x_{N-3}) \cdots p(x_1,\, x_0) \\
= \ & \cdots = p^*(x_{N-1},\, x_N) \cdots p^*(x_0,\, x_1)\, \pi(x_0) \\
= \ & \pi(x_0)\, p^*(x_0,\, x_1) \cdots p^*(x_{N-1},\, x_N).
\end{aligned}$$

Therefore, $(X_n^*,\, n \in [0,\, N])$ is, under P_π, a Markov chain with initial law π and transition matrix p^*.

(iii) Consider the system of equations $\pi(y) = \sum_{x\in\mathbb{N}} \pi(x)\, p(x,\, y)$. For $y = 0$, this gives

$$\pi(0) = \sum_{x=0}^{\infty} \pi(x)\, p(x,\, 0) = \sum_{x=0}^{\infty} \pi(x)(1-\alpha) = 1-\alpha\,.$$

For $y > 0$,

$$\pi(y) = \sum_{x=0}^{\infty} \pi(x)\, p(x,\, y) = \alpha\, \pi(y-1)\,,$$

and thus by induction, $\pi(y) = \alpha^y\, \pi(0) = (1-\alpha)\, \alpha^y$, $y \geq 1$. In other words, π is the geometric law of parameter α, and is clearly unique.

(iv) We have

$$p^*(x,\, y) = \frac{\pi(y)}{\pi(x)}\, p(y,\, x) = \alpha^{y-x} \Big(\alpha\, \mathbf{1}_{\{x=y+1\}} + (1-\alpha)\, \mathbf{1}_{\{x=0\}}\Big) = \mathbf{1}_{\{y=x-1\}} + \pi(y)\, \mathbf{1}_{\{x=0\}}\,,$$

for all $x,\, y \in \mathbb{N}$. □

Exercise 23. Let $(X_n,\, n \geq 0)$ be a Markov chain taking values in $S := \{1,\, 2,\, 3\}$ with transition matrix

$$p := \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

(i) Determine the recurrent classes, as well as the transient states. Determiner all states x such that $G(x,\, x) = \infty$. Determiner all states x and y such that $G(x,\, y) = 0$.

(ii) Prove that $G = I + pG$, where I is the 3×3 identity matrix. Compute G. Compute $E_x(N_y)$ all all $x,\, y \in S$, where $N_y := \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = y\}}$.

(iii) Let $T_1 := \inf\{n \geq 0 : X_n = 1\}$ (with $\inf \varnothing := \infty$), and let $v(x) = E_x(T_1)$. Prove that

$$v(x) = 1 + \sum_{y \in S} p(x, y) v(y), \quad x \in \{2, 3\}, \qquad v(1) = 0.$$

Compute $E_x(T_1)$ for all $x \in S$.

(iv) Compute $E_x(T_3)$ where $T_3 := \inf\{n \geq 0 : X_n = 3\}$ ($\inf \varnothing := \infty$).

(v) Give an invariant probability, and determine whether it is unique.

(vi) Let $T_{\{1,2\}} := \inf\{n \geq 0 : X_n \in \{1, 2\}\}$ ($\inf \varnothing := \infty$). What is the law of $T_{\{1,2\}}$ under P_3?

(vii) Prove that $E_3(T_{\{1,2\}}) = E_3(N_3)$. What is the reason?

Solution. (i) State 3 leads to states 1 and 2, but neither state 1 nor state 2 lead to state 3, so state 3 is transient.

State 1 leads to state 2 and state 2 leads to state 1. Since S is a finite set, there is at least one recurrent state. So $\{1, 2\}$ is a recurrent class, and is the only one, since 3 is transient.

Since states 1 and 2 are recurrent while state 3 is transient, we have $G(1, 1) = G(2, 2) = \infty$ and $G(3, 3) < \infty$.

Finally, since 3 cannot be reached from neither 1 nor 2, we have $G(1, 3) = G(2, 3) = 0$, while all other $G(i, j)$ are (strictly) positive.

(ii) Observe that $N_y = \mathbf{1}_{\{X_0 = y\}} + N_y \circ \theta_1$, so by the Markov property,

$$E_x(N_y \,|\, \mathscr{F}_1) = \mathbf{1}_{\{X_0 = y\}} + E_{X_1}(N_y) \, .$$

Taking expectation with respect to P_x on both sides gives that

$$G(x, y) = \mathbf{1}_{\{x=y\}} + E_x(E_{X_1}(N_y)) = \mathbf{1}_{\{x=y\}} + \sum_{z \in S} p(x, z) E_z(N_y) = \mathbf{1}_{\{x=y\}} + \sum_{z \in S} p(x, z) G(z, y) \, ,$$

i.e., $G = I + pG$. By (i), we have

$$G = \begin{pmatrix} \infty & \infty & 0 \\ \infty & \infty & 0 \\ \infty & \infty & G(3,3) \end{pmatrix},$$

whereas from the equation $G = I + pG$, we deduce that $G(3, 3) = 1 + p(3, 3)\, G(3,3) = 1 + \frac{1}{3} G(3, 3)$. Hence $G(3, 3) = \frac{3}{2}$.

By definition, $E_x(N_y) = G(x, y)$.

(iii) By defintion, $v(1) = 0$. Let $x \neq 1$. Then P_x-a.s.,

$$T_1 = \inf\{n \geq 1 : X_n = 1\} = 1 + \inf\{j \geq 0; X_{j+1} = 1\} = 1 + T_1 \circ \theta_1 \, .$$

Exactly as in (ii), we apply the Markov property to see that $v(x) = 1 + \sum_{y \in S} p(x, y)v(y)$.

Concretely, $v(2) = 1 + \frac{1}{2}(v(1) + v(2))$ and $v(3) = 1 + \frac{1}{3}(v(1) + v(2) + v(3))$; solving the system of linear equations (recalling that $v(1) = 0$) gives $v(2) = 2$ and $v(3) = \frac{5}{2}$.

Conclusion: $E_1(T_1) = 0$, $E_2(T_1) = 2$ and $E_3(T_1) = \frac{5}{2}$.

(iv) We have $T_3 = 0$ P_3-a.s., so $E_3(T_3) = 0$. On the other hand, 3 cannot be reached from 1 nor from 2, so $T_3 = \infty$ P_1-a.s. and P_2-a.s. In particular, $E_x(T_3) = \infty$ if $x \in \{1, 2\}$.

(v) Let π be an invariant probability: $\sum_{i=1}^3 \pi(i) = 1$ and $\sum_{i=1}^3 \pi(i)\, p(i, j) = \pi(j)$ for all $j \in \{1, 2, 3\}$. Solving the system of linear equations

$$(\pi(1), \pi(2), \pi(3)) \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} = (\pi(1), \pi(2), \pi(3)), \qquad \pi(1) + \pi(2) + \pi(3) = 1\,,$$

we see that $\pi(3) = \frac{1}{3}\pi(3)$, so $\pi(3) = 0$; whereas $\pi(1) = \frac{1}{2}\pi(2)$, so $\pi_(1) = \frac{1}{3}$ and $\pi(2) = \frac{2}{3}$: $\pi = (\frac{1}{3}, \frac{2}{3}, 0)$. This system has a unique solution.

(vi) From state 3, the chain stays at 3 with probability $\frac{1}{3}$ and enters $\{1, 2\}$ with probability $\frac{2}{3}$. So $T_{\{1,2\}}$ under P_3 has a geometric law of parameter $\frac{2}{3}$.

(vii) We have $E_3(T_{\{1,2\}}) = \frac{3}{2} = E_3(N_3)$. The reason is as follows: once leaving state 3, the chain never comes back; so for any k, the events $\{T_{\{1,2\}} = k\}$ and $\{X_0 = X_1 = \cdots = X_{k-1} = 3,\, X_j \neq 3,\, \forall j \geq k\}$ are P_3-a.s. identical, which leads to $T_{\{1,2\}} = N_3$ P_3-a.s. □

Exercise 24. A player visits 3 casino places, numbered as 1, 2 and 3. Every day, he chooses, with equal probability $\frac{1}{2}$, to go to one of the two casino places he has not been the day before. The initial day, day 0, the player chooses to go to one of the three casinos with probability μ on $S := \{1, 2, 3\}$. Let X_n denote the number of the casino place where the player is at day n.

(i) Prove that $(X_n,\, n \geq 0)$ is a Markov chain, and give its transition matrix p.

(ii) Compute p^n, as well as $\lim_{n\to\infty} p^n$.

(iii) Compute $\lim_{n\to\infty} P_\mu(X_n = j)$, for $j = 1$, 2 and 3.

Solution. (i) By definition, the casino where the player is at day $n+1$ depends only on the casino where he was the day before (i.e., at day n). $(X_n,\, n \geq 0)$ is a Markov chain, with transition matrix $p := (p(x, y),\, x,\, y \in S := \{1, 2, 3\})$ given by

$$p = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

(ii) Let $M := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Since $p = \frac{1}{2}(M - I)$ (with I denoting the identity matrix 3×3), we have

$$\begin{aligned} p^n &= \frac{1}{2^n}\left(\sum_{k=0}^{n} C_n^k (-1)^{n-k} M^k\right) \\ &= \frac{1}{2^n}\left[\frac{1}{3}\left(\sum_{k=1}^{n} C_n^k (-1)^{n-k} 3^k\right) M + (-1)^n I\right] \\ &= \frac{1}{2^n}\left(\frac{1}{3}(2^n - (-1)^n) M + (-1)^n I\right). \end{aligned}$$

Hence, $[p^n](i, j) = \frac{2^{-n}}{3}(2^n - (-1)^n)$, if $i \neq j$, and $\frac{2^{-n}}{3}(2^n + 2(-1)^n)$, if $i = j$. It follows that

$$\lim_{n\to\infty} p^n = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

We have $P_\mu(X_n = j) = [\mu p^n](j) = \sum_{i\in\{1,2,3\}} \mu(i)[p^n](i, j)$. So $\lim_{n\to\infty} P_\mu(X_n = j) = \frac{1}{3}$, for $j = 1$, 2 or 3. $\square$

Exercise 25 (Existence and uniqueness of invariant measures). (i) Is there always an invariant measure (not necessarily an invariant probability measure) for a transition kernel?[1]

(ii) Is there always an invariant measure for an irreducible transition kernel?[2]

(iii) If there is an invariant measure, is it necessarily unique (up to a constant multiplication)?

Solution. (i) Consider $S := \mathbb{N}$ and the kernel $p(x, x + 1) = 1$. The equation for invariant measures says $\mu(y) = \sum_{x\in\mathbb{N}} \mu(x)\, p(x, y) = \mu(y - 1)$ for $y \geq 1$ (so $\mu(y) = \mu(0)$ for all $y \in \mathbb{N}$) and $\mu(0) = \sum_{x\in\mathbb{N}} \mu(x)\, p(x, 0) = 0$, which is impossible. Conclusion: there is no invariant measure for p.

(ii) Consider $S = \mathbb{N}$ and $p(x, x + 1) = p_x$ and $p(x, 0) = 1 - p_x$ for $x \in N$, where p_x, $x \in \mathbb{N}$, are real numbers lying in $(0, 1)$. The chain is irreducible.

The equation for invariant measures says $\mu(y) = \sum_{x\in\mathbb{N}} \mu(x)\, p(x, y) = \mu(y - 1)\, p_{y-1}$ for $y \geq 1$, so $\mu(j) = (p_0 \cdots p_{j-1})\, \mu(0)$ for $j \geq 1$. On the other hand, $\mu(0) = \sum_{x\in\mathbb{N}} \mu(x)\, p(x, y) =$

[1] Hint: You can consider the chain $p(x, x + 1) = 0$ on $S := \mathbb{N}$.

[2] Hint: You can modify the previous example by taking $p(x, x+1) < 1$ and adding $p(x, 0) := 1-p(x, x+1)$.

$\sum_{i\geq 0}\mu(i)(1-p_i)$, which yields

$$\begin{aligned}
\mu(0) &= \mu(0)(1-p_0)+\sum_{i=1}^{\infty}\mu(i)(1-p_i)\\
&= \mu(0)(1-p_0)+\sum_{i=1}^{\infty}(p_0\,\cdots\,p_{j-1})\,\mu(0)(1-p_i)\\
&= \mu(0)(1-p_0)+\lim_{n\to\infty}\sum_{i=1}^{n}(p_0\,\cdots\,p_{j-1})\,\mu(0)(1-p_i)\\
&= \lim_{n\to\infty}\mu(0)[1-p_0p_1\cdots p_n].
\end{aligned}$$

If $\prod_{i=0}^{\infty}p_i>0$, then $\mu(0)=0$ and thus $\mu(y)=0$, $\forall y\in\mathbb{N}$, which is impossible. So there is no invariant measure for the chain if $\prod_{i=0}^{\infty}p_i>0$.[3]

(iii) No uniqueness is guaranteed in general. For example, we have seen that for the biased random walk on $\mathbb{Z}$, both the counting measure on $\mathbb{Z}$, and $\mu(i):=(\frac{p}{q})^i$, $i\in\mathbb{Z}$, are invariant measures. $\square$

Exercise 26 (Records). Consider an i.i.d. sequence $(X_n,\,n\geq 0)$ of geometric random variables variables of parameter $0<p<1$. They are used to represent the lifetime of certain lamps. The lamps are numbered from 0; they are all lit on at the same time, and X_n denotes the lifetime of the lamp number n.

We define the times of successive records of lifetime of the lamps

$$\tau_0:=0,\qquad \tau_{n+1}:=\inf\{k>\tau_n : X_k>X_{\tau_n}\},\qquad n\geq 0,$$

as well as the successive records $Z_n:=X_{\tau_n}$, $n\geq 0$. So Z_n is the n-th record of lifetime that one sees at the sequence (X_n).

We assume that $(X_n,\,n\geq 0)$ is a Markov chain taking values in $S:=\mathbb{N}^*:=\{1,\,2,\,\cdots\}$ with transition probability

$$p(x,\,y)=q^{y-1}\,p,\qquad x,\,y\in\mathbb{N}^*.$$

We fix an $x\in\mathbb{N}^*$.

(i) Prove that under P_x, X_n, $n\geq 1$, are i.i.d. geometric random variables of parameter p. Are X_n, $n\geq 0$, independent? Are they i.i.d.?

[3]Since $p_0p_1\cdots p_n=P_0\{T_n<T_0\}\to P_0\{T_0=\infty\}$, the condition $\prod_{i=0}^{\infty}p_i>0$ is equivalent to saying that the chain is transient. On the other hand, if $\prod_{i=0}^{\infty}p_i=0$, then the chain is recurrent, so we know there are invariant measures. Conclusion: there is no invariant measure for the chain if and only if $\prod_{i=0}^{\infty}p_i>0$.

(ii) Compute $P_x(X_1 \le x, \cdots, X_{k-1} \le x, X_k > y)$ for k, $y \in \mathbb{N}^*$ such that $y \ge x$.

(iii) Let $\tau := \inf\{n \ge 1 : X_n > X_0\}$, $\inf \varnothing := \infty$. Compute $P_x(\tau = k,\ X_k > y)$ for k, $y \in \mathbb{N}^*$ such that $y \ge x$.

(iv) Prove under P_x, that τ is a geometric random variable of which we will determine the parameter, and that X_τ has the same law as $x + X_1$. Are τ and X_τ independent under P_x?

(v) Prove that $\pi(y) = q^{y-1}p$, $y \in \mathbb{N}^*$, is the unique invariant probability measure for p.

(vi) Prove that $P_\pi(\tau < \infty) = 1$ and that $E_\pi(\tau) = \infty$.

(vii) Prove that τ is a stopping time.

For the rest of the exercise, we can use the fact that for $(\tau_n,\, n \ge 1)$ defined as above, we have $\tau_{n+1} = \tau_n + \tau \circ \theta_{\tau_n}$ and thus each τ_n is a stopping time that is P_x-a.s. finite.

(viii) Prove that $(Z_n,\, n \ge 0)$ is a Markov chain under P_x, with respect to the filtration $(\mathscr{F}_{\tau_n})$. Compute its transition kernel and its initial law.

(ix) Let $f : \mathbb{N}^* \mapsto \mathbb{R}$ be bounded. Compute $E_x[f(Z_{n+1} - Z_n) \,|\, \mathscr{F}_{\tau_n})$. Prove that under P_x, the sequence $Z_n - Z_{n-1}$, $n \ge 1$ is i.i.d.

(x) Prove that $\frac{Z_n}{n}$ converges P_x-almost surely when $n \to \infty$, and determine the limit.

Solution. (i) Let us check that X_n, $n \ge 0$, are independent under P_x. By definition, it suffices to prove that for all $n \ge 1$, $X_0, \cdots, X_n$ are independent. For all $x_0, \cdots, x_n \in \mathbb{N}^*$,

$$\begin{aligned} P_x(X_0 = x_0, \cdots, X_n = x_n) &= \mathbf{1}_{\{x=x_0\}}\, p(x_0, x_1) \cdots p(x_{n-1}, x_n) \\ &= \mathbf{1}_{\{x=x_0\}}\, q^{x_1 - 1} p \cdots q^{x_n - 1} p\,, \end{aligned}$$

which means that $X_0, \cdots, X_n$ are independent, that $X_0 = x$ P_x-a.s., and that $X_1, \cdots, X_n$ are geometric random variables of parameter p.

Conclusion: Under P_x, X_n, $n \ge 1$ are i.i.d. geometric random variables of parameter p, and X_n, $n \ge 0$ are independent, but not i.i.d.

(ii) Let k, $y \in \mathbb{N}^*$ such that $y \ge x$. We have

$$P_x(X_1 \le x, \cdots, X_{k-1} \le x, X_k > y) = (1 - q^x)^{k-1}\, q^y = (1 - q^x)^{k-1} q^x\, q^{y-x}\,.$$

(iii) We have

$$\begin{aligned} P_x(\tau = k,\ X_k > y) &= P_x(X_1 \le X_0, \cdots, X_{k-1} \le X_0, X_k > y) \\ &= P_x(X_1 \le x, \cdots, X_{k-1} \le x, X_k > y) \\ &= (1 - q^x)^{k-1} q^x \cdot q^{y-x}\,. \end{aligned}$$

(iv) Taking $y = x$ in what we have obtained in (iii) yields

$$P_x(\tau = k) = (1 - q^x)^{k-1} q^x \,.$$

Summing over $k \geq 1$ gives that

$$P_x(X_\tau > y) = \sum_{k\geq 1} P_x(\tau = k,\ X_k > y) = \sum_{k\geq 1} (1 - q^x)^{k-1} q^x \cdot q^{y-x} = q^{y-x} \,.$$

Therefore, under P_x, τ is a geometric random variable of parameter q^x, and X_τ is distributed as $x + X_1$. For k, $y \in \mathbb{N}^*$ with $y \geq x$,

$$P_x(\tau = k,\ X_\tau > y) = P_x(\tau = k)\, P_x(X_\tau > y) \,,$$

which means that τ and X_τ are independent under P_x.

(v) A probability measure π on $\mathbb{N}^*$ is invariant means that for all $z \in \mathbb{N}^*$,

$$\pi(z) = \sum_{y\in\mathbb{N}^*} \pi(y)\, p(y, z) = \sum_{y\in\mathbb{N}^*} \pi(y)\, q^{z-1} p = q^{z-1} p \,.$$

This implies that $\pi(z) = q^{z-1}p$, $z \in \mathbb{N}^*$, is the unique invariant measure. [For uniqueness, we can also prove it by checking that the chain is irreducible and positive recurrent.]

(vi) We have

$$P_\pi(\tau < \infty) = \sum_{x\in\mathbb{N}^*} \pi(x)\, P_x(\tau < \infty) = \sum_{x\in\mathbb{N}^*} \pi(x) = 1 \,.$$

[We have used the fact that under P_x, τ being a geometric random variable is a.s. finite.] We also have

$$E_\pi(\tau) = \sum_{x\geq 1} E_x(\tau)\, q^{x-1}\, p = \sum_{x\geq 1} q^{-x}\, q^{x-1}\, p = \infty.$$

(vii) We have $\{\tau = 0\} = \varnothing \in \mathscr{F}_0$, and for $n \geq 1$,

$$\{\tau = n\} = \{X_1 \leq X_0, \cdots, X_{k-1} \leq X_0,\, X_k > X_0\} \in \mathscr{F}_n \,.$$

By definition, this means that τ is a stopping time.

(viii) By the Markov property,

$$\begin{aligned} E_x[f(X_{\tau_{n+1}}) \,|\, \mathscr{F}_{\tau_n}] &= E_x[f(X_{\tau_n + \tau\circ\theta_{\tau_n}})\, \mathbf{1}_{\{\tau_n<\infty\}} \,|\, \mathscr{F}_{\tau_n}] \\ &= E_{X_{\tau_n}}(f(X_\tau)) \\ &= \sum_{y\geq 1} f(X_{\tau_n} + y)\, q^{y-1}\, p \,, \end{aligned}$$

where we have used the fact that X_τ has the law of $x + X_1$ under P_x. As such,

$$E_x[f(Z_{n+1}) \,|\, \mathscr{F}_{\tau_n}] = \sum_{y\geq 1} f(Z_n + y)\, q^{y-1}\, p = \sum_{y \geq Z_n+1} f(y)\, q^{y-Z_n-1}\, p\,.$$

Therefore, $(Z_n,\, n \geq 0)$ is a Markov chain with transition kernel p^Z given by

$$p^Z(x,\, y) = \mathbf{1}_{\{y>x\}}\, q^{y-x-1}\, p\,.$$

(ix) Let $f : \mathbb{N}^* \mapsto \mathbb{R}$ be bounded. By the strong Markov property,

$$\begin{aligned} E_x[f(X_{\tau_{n+1}} - X_{\tau_n}) \,|\, \mathscr{F}_{\tau_n}] &= E_x[f(X_{\tau_n + \tau\circ\theta_{\tau_n}} - X_{\tau_n})\, \mathbf{1}_{\{\tau_n<\infty\}} \,|\, \mathscr{F}_{\tau_n}] \\ &= E_{X_{\tau_n}}[f(X_\tau - X_0)] \\ &= \sum_{y\geq 1} f(y)\, q^{y-1}\, p\,, \end{aligned}$$

where we have once again used the fact that under P_z, $X_\tau - z$ has the same law as X_1. This yields

$$E_x[f(Z_{n+1} - Z_n) \,|\, \mathscr{F}_{\tau_n}] = \sum_{y\geq 1} f(y)\, q^{y-1}\, p\,.$$

Taking expectation on both sides with respect to P_x, we obtain:

$$E_x(f(Z_{n+1} - Z_n)) = \sum_{y\geq 1} f(y)\, q^{y-1}\, p = E_x[f(Z_{n+1} - Z_n) \,|\, \mathscr{F}_{\tau_n}]\,.$$

Therefore, for all bounded functions $f : \mathbb{N}^* \mapsto \mathbb{R}$,

$$E_x[f(Z_{n+1} - Z_n) \,|\, \mathscr{F}_{\tau_n}] = E_x[f(Z_{n+1} - Z_n)]\,,$$

from which it follows that $Z_{n+1} - Z_n$ is independent of $\mathscr{F}_{\tau_n}$. Since $Z_i - Z_{i-1} = X_{\tau_i} - X_{\tau_{i-1}}$ is $\mathscr{F}_{\tau_i}$-measurable, this yields that $Z_i - Z_{i-1}$, $i \geq 1$, are independent. Furthermore, for all $i \geq 1$, $Z_i - Z_{i-1}$ is a geometric random variable of parameter P, which yields that $Z_i - Z_{i-1}$, $i \geq 1$, are i.i.d.

(x) Since $Z_0 = X_0 = x$ P_x-a.s., we have, by induction,

$$Z_n = x + \sum_{i=1}^{n} (Z_i - Z_{i-1})\,,$$

where $Z_i - Z_{i-1}$, $i \geq 1$, are i.i.d. geometric random variables of parameter p. By Kolmogorov's law of large numbers, $\frac{Z_n}{n} = \frac{x}{n} + \frac{1}{n}\sum_{i=1}^{n}(Z_i - Z_{i-1})$ converges P_x to $E_x(Z_1 - Z_0) = \frac{1}{p}$. $\square$

Exercise 27. Assume that $p(x, x) < 1$, $\forall x \in S$.

(i) Let $T := \inf\{n \geq 1 : X_n \notin X_0\}$, $\inf \varnothing := \infty$. Let $x \in S$. Determine the laws of T and of X_T under P_x.

(ii) Let $T_0 := 0$ and $T_{n+1} := T_n + T \circ \theta_{T_n}$. Prove that for any $x \in S$, $(X_n, n \in \mathbb{N})$ is an increasing sequence of P_x-almost surely finite stopping times with respect to the canonical filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

(iii) Let $Y_n := X_{T_n}$, $n \in \mathbb{N}$. Prove that $(Y_n, n \in \mathbb{N})$ is a Markov chain. Compute its transition probability.

(iv) Assume that p is irreducible and recurrent, with an invariant measure μ. Prove that $(Y_n, n \in \mathbb{N})$ is irreducible and recurrent, and that $\nu(x) := (1 - p(x, x))\, \mu(x)$, $x \in S$, is an invariant measure for $(Y_n, n \in \mathbb{N})$.

Solution. (i) By definition, for any $n \geq 1$,

$$P_x\{T \geq n\} = P_x\{x = X_0 = X_1 = \cdots = X_{n-1}\} = p(x, x)^{n-1},$$

which means T has a geometric distribution under P_x, with $P_x\{T < \infty\} = 1$ (recalling that $p(x, x) < 1$ by assumption).

Since $T < \infty$, P_x-a.s., X_T is P_x-a.s. well-defined, and its measurability is seen with the writing $X_T = \sum_{n=1}^{\infty} X_n \mathbf{1}_{\{T=n\}}$ (P_x-a.s.). To determine the law of X_T, we first note that $X_T \neq x$, P_x-a.s., and for any $y \neq x$,

$$\begin{aligned} P_x\{X_T = y\} &= \sum_{n=1}^{\infty} P_x\{T = n,\ X_n = y\} \\ &= \sum_{n=1}^{\infty} P_x\{x = X_1 = \cdots = X_{n-1},\ X_n = y\} \\ &= \sum_{n=1}^{\infty} p(x, x)^{n-1}\, p(x, y) \\ &= \frac{p(x, y)}{1 - p(x, x)}. \end{aligned}$$

(ii) We prove by induction (in n) that for any n and any x, T_n is P_x-a.s. finite and is a stopping time. For $n = 1$, this is already proved. Assume this holds for n. Then for any j,

$$\{T_{n+1} = j\} = \bigcup_{i=1}^{j-1} \{T_{n+1} = j,\ T_n = i\} = \bigcup_{i=1}^{j-1} \{X_i = X_{i+1} = \cdots = X_{j-1},\ X_j \neq X_i,\ T_n = i\}.$$

For any $i \leq j - 1$, $\{T_n = i\} \in \mathscr{F}_i$ (since T_n is a stopping time by induction assumption), which, a fortiori, is an element of $\mathscr{F}_j$, whereas $\{\{X_i = X_{i+1} = \cdots = X_{j-1},\ X_j \neq X_i\} \in \mathscr{F}_j$

by definition of $\mathscr{F}_j$. Therefore, $\{T_{n+1} = j\} \in \mathscr{F}_j$, proving that T_{n+1} is a stopping time. To show the finiteness, we see that by the strong Markov property,

$$\begin{aligned} P_x\{T_{n+1} < \infty\} &= P_x\{T_n < \infty,\, T_{n+1} < \infty\} \\ &= P_x\{T_n < \infty,\, T_n + T \circ \theta_{T_n} < \infty\} \\ &= E_x[\mathbf{1}_{\{T_n<\infty\}} P_{X_{T_n}}(T < \infty)]. \end{aligned}$$

We have proved in (i) that $P_{X_{T_n}}(T_n < \infty) = 1$ P_x-a.s., so $P_x\{T_{n+1} < \infty\} = P_x\{T_n < \infty\}$, which equals 1 by induction assumption.

As a consequence, for any n and any x, T_n is P_x-a.s. finite and is a stopping time. The monotonicity of $n \mapsto T_n$ is obvious by definition (and by the a.s. finiteness of T_n).

(iii) Let $n \in \mathbb{N}$ and let $x,\, y_0,\, y_1,\, \cdots,\, y_n \in S$. We compute

$$a_n := P_x\{Y_0 = y_0,\, \cdots,\, Y_n = y_n\}.$$

If $y_0 \neq x$, or if there exists $i \in \{1,\, 2,\, \cdots, n\}$ such that $y_i = y_{i-1}$, then the probability is obviously 0. So let us assume $y_0 = x$ and $y_i \neq y_{i-1}$, $\forall i \in \{1,\, 2,\, \cdots, n\}$. Then ($k_0 := 0$)

$$\begin{aligned} a_n &= \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_n=k_{n-1}+1}^{\infty} P_x\{Y_1 = y_1,\, \cdots,\, Y_n = y_n,\, T_1 = k_1,\, \cdots, T_n = k_n\} \\ &= \sum_{k_1=1}^{\infty} \sum_{k_2=k_1+1}^{\infty} \cdots \sum_{k_n=k_{n-1}+1}^{\infty} P_x\Big\{ \bigcap_{i=1}^{n} \{X_{k_{i-1}+1} = X_{k_{i-1}+2} = \cdots X_{k_i-1} = y_{i-1},\, X_{k_i} = y_i\} \Big\} \\ &= \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \prod_{i=1}^{n} p(y_{i-1},\, y_{i-1})^{k_i-k_{i-1}-1}\, p(y_{i-1},\, y_i) \\ &= \prod_{i=1}^{n} \frac{p(y_{i-1},\, y_i)}{1 - p(y_{i-1},\, y_{i-1})}. \end{aligned}$$

As a consequence, $(Y_n,\, n \in \mathbb{N})$ is a Markov chain, with transition kernel $q(x,\, y) := \frac{p(x,y)}{1-p(x,x)}$ for $y \neq x$, and $q(x,\, x) := 0$.

(iv) Under P_x, $X_i = Y_{T_{n-1}}$ for all $i \in [T_{n-1} + 1,\, T_n) \in \mathbb{Z}$, which implies that the sites visited by the new chain $(Y_n,\, n \in \mathbb{N})$ coincide with those visited by $(X_n,\, n \in \mathbb{N})$. So if $(X_n,\, n \in \mathbb{N})$ is irreducible and recurrent, then so is $(Y_n,\, n \in \mathbb{N})$.

Let $\nu(x) := (1 - p(x, x))\, \mu(x)$, $x \in S$. We have

$$\begin{aligned}
\sum_{y \in S} \nu(x)\, q(x, y) &= \sum_{y \in S \setminus \{x\}} (1 - p(x, x))\, \mu(x)\, \frac{p(x, y)}{1 - p(x, x)} \\
&= \sum_{y \in S \setminus \{x\}} \mu(x)\, p(x, y) \\
&= \sum_{y \in S} \mu(x)\, p(x, y) - \mu(x)\, p(x, x) \\
&= \mu(x) - \mu(x)\, p(x, x),
\end{aligned}$$

which is $\nu(x)$. So ν is an invariant measure for q. $\square$

Exercise 28 (Reflecting biased random walk on $\mathbb{N}$). In this exercise, $S = \mathbb{N}$ and

$$\begin{aligned}
&p(i, i+1) = p, \qquad p(i, i-1) = q = 1 - p, \qquad \text{if } i \geq 1, \\
&p(0, 1) = 1,
\end{aligned}$$

were $0 < p < 1$ is a fixed parameter.

(i) Prove that the chain is irreducible.

(ii) Prove that the measure μ, defined by $\mu(i) := (\frac{p}{q})^i$ (for $i \geq 1$) and $\mu(0) := 1$, is reversible.

(iii) Prove that the chain is positive recurrent if $p < \frac{1}{2}$.

(iv) Prove that the chain is null recurrent if $p = \frac{1}{2}$.

(v) Prove that the chain is transient if $p > \frac{1}{2}$.

Solution. (i) For any $x \geq 1$, $G(0, x) \geq p^x(0, x) \geq p(0, 1)\, p(1, 2) \cdots P(x-1, x) = p^x > 0$ and $G(x, 0) \geq p^x(x, 0) \geq p(1, 0)\, p(2, 1) \cdots P(x, x-1) = q^x > 0$. Therefore, the chain is irreducible.

(ii) We need to check, for $x < y$, that $\mu(x)\, p(x, y) = \mu(y)\, p(y, x)$.

If $y \geq x + 2$, the equality holds trivially ($0 = 0$). So we assume $y = x + 1$.

If $x = 0$ (thus $y = 1$), $\mu(0)\, p(0, 1) = p$ whereas $\mu(1)\, p(1, 0) = \frac{p}{q}\, q = p$: the equality is all right.

If $x \geq 1$, $\mu(x)\, p(x, x+1) = (\frac{p}{q})^x\, p$ whereas $\mu(x+1)$, $p(x+1, x) = (\frac{p}{q})^{x+1} q = (\frac{p}{q})^x\, p$: the equality is fine as well.

Conclusion: μ is reversible, and a fortiori, invariant.

(iii) If $p < \frac{1}{2}$, μ is a finite measure; so there is an invariant measure for the irreducible chain, which must be positive recurrent.

(iv) If $p = \frac{1}{2}$, μ is has infinite mass. The chain is irreducible. If we are able to prove that the chain is recurrent, then we will have shown that the chain is null recurrent. Let us prove that 0 is recurrent, which is equivalent to saying that $P_1\{T_0 < \infty\} = 1$ (since 0 is reflecting). However, the value of $P_1\{T_0 < \infty\}$ does not change if we replace the reflecting random walk by simple random walk, which is known to be recurrent (because the jump law has 0 expectation). So $P_1\{T_0 < \infty\} = 1$, and the reflecting random walk is null recurrent in case $p = \frac{1}{2}$.

(v) If $p > \frac{1}{2}$, again by checking $P_1\{T_0 < \infty\}$ for random walk without reflection at the origin (which is transient because the random walk has a non-centered jump law), we know that $P_1\{T_0 < \infty\} < 1$: the chain is transient. $\square$

Exercise 29. Consider an irreducible and aperiodic transition kernel on a finite state space. Prove that there exists $n_0 < \infty$ such that $p^n(x, y) > 0$, $\forall n \geq n_0$, $\forall x, y \in S$.

Solution. An irreducible chain on a finite state space is recurrent. Since it is aperiodic, we have seen in the class that for any $x, y \in S$, there exists $n_0 < \infty$ such that $p^n(x, y) > 0$, $\forall n \geq n_0$. The assumption that $\#S < \infty$ yields that we can choose a common $n_0 < \infty$ for all x and y such that $p^n(x, y) > 0$, $\forall n \geq n_0$. $\square$

Exercise 30 (Simple random walk on hypercube). In this exercise, $S := \{0, 1\}^d$. For $x := (x_1, \cdots, x_d) \in S$ and $i \in \{1, \cdots, d\}$, we denote by $x^{(i)} := (x_1^{(i)}, \cdots, x_d^{(i)})$ the unique element of S whose components coincide with those of x except for the i-th: $x_k^{(i)} = x_k$ for all $k \in \{1, \cdots, d\} \backslash \{i\}$ but $x_i^{(i)} \neq x_i$. Let

$$p(x, y) := \begin{cases} \frac{1}{d} & \text{if } y = x^{(i)} \text{ for some } i \in \{1, \cdots, d\} \\ 0 & \text{otherwise.} \end{cases}$$

(i) Prove that p is a transition kernel on S.

(ii) Is the kernel irreducible?

(iii) Is the kernel aperiodic?

(iv) Compute the invariant distribution.

Solution. (i) Since $p(x, y) \geq 0$ for all $x, y \in S$, and $\sum_{y \in S} p(x, y) = 1$ for all $x \in S$, p is a transition kernel on S.

(ii) Yes. For any $x \in S$, we see that $G(0, x) \geq p^n(0, x) > 0$, where $n := \sum_{i=1}^d |x_i|$: it suffices to the chain to increase at each step a component by one (this happens with probability $\frac{1}{d}$), and similarly, $G(x, 0) > 0$ by asking the chain to decrease at each step a component by one.

(iii) The chain is irreducible, and recurrent (because $\#S < \infty$).

Consider the set $I_x := \{n \in \mathbb{N} : p^n(x, x) > 0\}$ for $x = 0$. Clearly, $2 \in I_x$ (with $y := (1, 0, \cdots, 0)$, we see that $p^2(0, 0) \geq p(0, y)\, p(y, 0) = \frac{1}{d^2} > 0$) and for the same of parity, $n \notin I_x$ if n is odd, we conclude that $d_0 = 2$. So the period of the chain is $d = 2$: the kernel is not aperiodic.

(iv) The chain being irreducible on a finite state space, there is a unique invariant probability measure.

Since the chain is a special case of simple random walk on a graph with $A_x = d$, $\forall x \in S$, we have seen in the class that $\mu(x) := \frac{1}{\#A_x} = \frac{1}{d}$, $\forall x \in S$, is an invariant measure, so the uniform distribution $\mu(x) := \frac{1}{\#S} = \frac{1}{2^d}$, $\forall x \in S$, is the unique invariant probability measure. $\square$

Exercise 31 (Blocks of consecutive head runs in coin tossings). Let ξ_n, $n \geq 1$, be an i.i.d. sequence of Bernoulli random variables of parameter $\frac{1}{2}$. Let N_n denote the number of blocks of three successive zeros in ξ_i, $1 \leq i \leq n$.[4]

Consider the Markov chain $(X_n, n \geq 0)$ taking values in $S := \{0, 1, 2, 3\}$, with initial state 0, which records the number of consecutive zeros in the sequence $(\xi_n, n \geq 1)$, and which falls back to 0 after each counting of three successive zeros. For example, if $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \cdots) = (0, 1, 0, 0, 0, 0, 1, \cdots)$, then $Y_0 = 0$ (by definition) and $(Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, \cdots) = (1, 0, 1, 2, 3, 1, 0, \cdots)$, and $N_7 = 1 = \sum_{k=1}^{7} \mathbf{1}_{\{Y_k=3\}}$.

The transition matrix p for $(X_n, n \geq 0)$ is given by

$$p(0,0) = p(0,1) = \frac{1}{2}, \qquad p(1,0) = p(1,2) = \frac{1}{2}$$
$$p(2,0) = p(2,3) = \frac{1}{2}, \qquad p(3,0) = p(3,1) = \frac{1}{2}.$$

(i) Prove that $(X_n, n \geq 0)$ is irreducible and positive recurrent. Determine its invariant probability.

(ii) Prove that $\frac{N_n}{n}$ converges a.s. and compute the limit.

Solution. (i) Since every state is connected to state 0 in both ways, the chain is irreducible.

The space S being finite, $(X_n, n \geq 0)$ is irreducible and positive recurrent.

[4] We only count disjoint blocks, so the number of blocks of three successive zeros in 0, 0, 0, 0, 0, 1 is one, and that in 0, 0, 0, 0, 0, 0, 1 is two.

The equation $\pi(y) = \sum_{x\in S} \pi(x)\, p(x,\, y)$ leads to:

$$\begin{aligned}
\pi(0) &= \frac{1}{2}\pi(0) + \frac{1}{2}\pi(1) + \frac{1}{2}\pi(2) + \frac{1}{2}\pi(3),\\
\pi(1) &= \frac{1}{2}\pi(0) + \frac{1}{2}\pi(3),\\
\pi(2) &= \frac{1}{2}\pi(1),\\
\pi(3) &= \frac{1}{2}\pi(2).
\end{aligned}$$

Since $\pi(0) + \pi(1) + \pi(2) + \pi(3) = 1$, it follows that

$$\pi(0) = \frac{1}{2}, \quad \pi(1) = \frac{2}{7}, \quad \pi(2) = \frac{1}{7}, \quad \pi(3) = \frac{1}{14}.$$

(ii) Writing $N_n = \sum_{k=1}^{n} \mathbf{1}_{\{Y_k=3\}}$, it follows from the ergodic theorem that $\lim_{n\to\infty} \frac{N_n}{n} = \frac{1}{14}$ a.s. $\square$

Exercise 32. Consider a Markov chain $(X_n,\, n \geq 0)$ taking values in $S := \{1,\, 2,\, 3\}$ with transition matrix

$$p := \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

(i) Classify the states and determine all recurrence classes.

(ii) Compute an invariant probability and determine whether it is unique and if it is reversible.

(iii) Let $x \in S$, and let $T_x := \inf\{n \geq 1 :\, X_n = x\}$ (with $\inf \varnothing := \infty$). Compute $E_x(T_x)$.

(iv) Let $x \in S$. Compute the period of x. What can be said about $p^n(x,\, y)$ when $n \to \infty$?

Solution. (i) State 1 leads to state 2, which leads to state 3, which in turn leads to state 1, so any state x leads to any state y: the chain is irreducible. Since S is finite, all states are recurrent; they form a single recurrence class.

(ii) A probability measure π on S is invariant if and only if

$$(\pi(1), \pi(2), \pi(3)) \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix} = (\pi(1), \pi(2), \pi(3)), \qquad \pi(1) + \pi(2) + \pi(3) = 1\,,$$

with the condition that $\pi(x) \geq 0$ for all $x \in S$. We find the solution $\pi = (\frac{4}{9}, \frac{2}{9}, \frac{3}{9})$.

Since this is the unique solution for the system of equations, there is uniqueness of invariant probability (which is also a consequence of the fact that the chain is irreducible on a finite space).

The invariant probability π is not reversible: $\pi(2)\,p(2,\,3) = \frac{1}{9} \neq 0 = \pi(3)\,p(3,\,2)$.

(iii) We have $E_x(T_x) = \frac{1}{\pi(x)}$; so $E_1(T_1) = \frac{9}{4}$, $E_2(T_2) = \frac{9}{2}$ and $E_3(T_3) = \frac{9}{3} = 3$.

(iv) By irreducibility, the period is identical for all state $x \in S$. For $x = 1$, we have $p^2(1,\,1) > 0$ and $p^3(1,\,1) > 0$; the greatest commun divisor of $\{2,\,3\}$ being 1, we see that the chain is aperiodic. In particular, $p^n(x,\,y) \to \pi(y)$ for all $x,\,y \in S$. $\square$

Exercise 33 (Product of independent chains). Let $X = (X_n)$ and $Y = (Y_n)$ be two independent canonical Markov chains taking values in S^X and S^Y, with transition kernels p^X and p^Y, respectively. Let $Z_n = (X_n,\,Y_n)$, $n \geq 0$, which is a Markov chain with transition kernel

$$p^Z\big((x,\,y),\,(x',\,y')\big) := p^X(x,\,x')\,p^Y(y,\,y'), \qquad x,\,x' \in S^X, \quad y,\,y' \in S^Y .$$

The chain (Z_n) is called the product chain.

(i) Compute $(p^Z)^n$ in terms of $(p^X)^n$ and $(p^Y)^n$.

(ii) Prove that if both (X_n) and (Y_n) are irreducible, positive recurrent and aperiodic, then so is (Z_n).

(iii) Consider a checkerboard with 16 squares (4×4), numbered from 1 to 16 from left to right and from top to bottom. The squares are of alternating black and white colour, on which two mice move independently. Each mouse moves to one of the k neighbouring squares with equal probability $\frac{1}{k}$ (diagonal displacements being prohibited). Compute the expected waiting time between two successive meetings at square 7.

Solution. (i) By definition,

$$(p^Z)^n\big((x,\,y),\,(x',\,y')\big) := (p^X)^n(x,\,x')\,(p^Y)^n(y,\,y'), \qquad x,\,x' \in S^X,\,y,\,y' \in S^Y .$$

(ii) Since (X_n) is irreducible with period 1, we have $(p^X)^n(x,\,x') > 0$ for all $x,\,x' \in S^X$ and all sufficiently large n (Proposition 6.6). Something similar holds for $(p^Y)^n(y,\,y')$ as well. Therefore, $(p^Z)^n((x,\,y),\,(x',\,y')) > 0$ for all sufficiently large n. This yields that (Z_n) is irreducible and aperiodic (if recurrent).

To see positive recurrence of (Z_n), let π^X the (unique) invariant probability for (X_n), and π^Y be the invariant probability for (Y_n), then $\pi^Z(x,\,y) := \pi^X(x)\pi^Y(y)$, $(x,\,y) \in S^X \times S^Y$, is an invariant probability for (Z_n).

(iii) Let $(X_n,\, n \geq 0)$ be the sequence of numbers of the squares occupied by the first mouse. It is irreducible, and positive recurrent because of the finite state space. Let us compute its invariant probability measure. For obvious symmetry reason, write

$$\begin{aligned}
&a := \pi^X(1) = \pi^X(4) = \pi^X(13) = \pi^X(16)\,, \\
&b := \pi^X(2) = \pi^X(3) = \pi^X(5) = \pi^X(9) = \pi^X(14) = \pi^X(15) = \pi^X(8) = \pi^X(12)\,, \\
&c := \pi^X(6) = \pi^X(7) = \pi^X(10) = \pi^X(11)\,.
\end{aligned}$$

Equation $\sum_{x \in S^X} \pi^X(x)\, p^X(x,\, x') = \pi^X(x')$ for all $x' \in S^X$ gives

$$\begin{aligned}
a &= \frac{1}{3}\, b + \frac{1}{3}\, b\,, \\
b &= \frac{1}{2}\, a + \frac{1}{3}\, b + \frac{1}{4}\, c\,, \\
c &= \frac{1}{3}\, b + \frac{1}{3}\, b + \frac{1}{4}\, c + \frac{1}{4}\, c\,.
\end{aligned}$$

Moreover, $4a + 8b + 4c = 1$ because π^X is a probability measure. Il follows that

$$a = \frac{1}{24}, \qquad b = \frac{1}{16}, \qquad c = \frac{1}{12}.$$

Let $(Y_n,\, n \geq 0)$ be the sequence of numbers of the squares occupied by the second mouse. Then $\pi^Y := \pi^X$ is also the invariant probability for (Y_n), and $\pi^Z(x,\, y) := \pi^X(x)\pi^Y(y)$, $(x,\, y) \in S^X \times S^Y$, is the invariant probability for $(Z_n := (X_n,\, Y_n),\, n \geq 0)$. The product chain possesses two recurrent classes: class $\mathscr{C}_1$ of pairs of squares with identical colours, and class $\mathscr{C}_2$ of pairs of squares with different colours. The restriction of the chain to each of the two classes is irreducible and positive recurrent, and has the restriction of π^Z to the class as an invariant measure. The invariant probability on $\mathscr{C}_1$ is given by

$$\pi_1^Z(x,\, y) := \frac{\pi^Z(x,\, y)}{\pi^Z(\mathscr{C}_1)} = 2\pi^Z(x,\, y), \qquad x,\, y \in \mathscr{C}_1\,.$$

Indeed,

$$\pi^Z(\mathscr{C}_1) = \sum_{x,y \text{ of same colour}} \pi^X(x)\pi^X(y) = \sum_x \pi^X(x) \sum_{y \text{ of the same colour as } x} \pi^X(y) = \frac{1}{2}\,,$$

as the somme of $\pi^X(y)$ over all black squares y is identical to the somme of $\pi^X(y)$ over all white squares y.

Finally,

$$E_{(7,\,7)}(T_{(7,\,7)}) = \frac{1}{\pi^Z(7,\,7)} = \frac{1}{2c^2} = 72\,,$$

which is the expected waiting time between two successive meetings at square 7. $\square$

Exercise 34 (Metropolis algorithm). Assume $S < \infty$, and p irreducible and symmetric[5]. Let μ be a probability measure on S such that $\mu(x) > 0$, $\forall x \in S$. Let

$$q(x,\, y) := \begin{cases} p(x,\, y)\, \min\{\frac{\mu(y)}{\mu(x)},\, 1\} & \text{if } y \neq x, \\ 1 - \sum_{z\in S\backslash\{x\}} p(x,\, z)\, \min\{\frac{\mu(z)}{\mu(x)},\, 1\} & \text{if } y = x. \end{cases}$$

(i) Prove that μ is reversible for q.

(ii) Prove that q is irreducible.

(iii) From now on, we assume that there exist $x \neq y$ such that $\mu(x) \neq \mu(y)$. Let $\mathscr{M} := \{x \in S : \mu(x) = \max_{y\in S} \mu(y)\}$. Prove that there exists $x_0 \in \mathscr{M}$ such that $p(x_0,\, y) > 0$ for some $y \in S\backslash\mathscr{M}$. Prove that $q(x_0,\, z) < p(x_0,\, z)$ for some $z \in S$. Prove that $q(x_0,\, x_0) > 0$.

(iv) Is q aperiodic?

Solution. (i) Obviously, q is a transition kernel. Let us check $\mu(x)\, q(x,\, y) = \mu(y)\, q(y,\, x)$. There is nothing to prove if $x = y$, so let us assume $x \neq y$. Then

$$\mu(x)\, q(x,\, y) = p(x,\, y) \min\{\mu(y),\, \mu(x)\}, \qquad \mu(y)\, q(y,\, x) = p(y,\, x) \min\{\mu(x),\, \mu(y)\},$$

so they are identical by the assumption that $p(x,\, y) = p(y,\, x)$. As a consequence, μ is reversible for q, and a fortiori, invariant.

(ii) Let $x \neq y \in S$. Since p is irreducible, there exist $x_0 = x,\, x_1,\, \cdots,\, x_n = y$ elements of S such that $p(x_{i-1},\, x_i) > 0$, $\forall i \in \{1,\, \cdots,\, n\}$.

By definition, $p(x_{i-1},\, x_i) > 0$ implies $q(x_{i-1},\, x_i) > 0$. So if we write G_q for Green's function associated with q, then $G_q(x,\, y) > 0$: q is irreducible.

(iii) Suppose that $p(x,\, y) = 0$, $\forall x \in \mathscr{M}$, $\forall y \notin \mathscr{M}$. Then for all $x \in \mathscr{M}$, we would have $P_x\{X_n \in \mathscr{M},\, \forall n \in \mathbb{N}\} = 1$, which would contradict the irreducibility assumption (since $\mathscr{M}$ is a strict subset of S). Consequently, there exists $x_0 \in \mathscr{M}$ such that $p(x_0,\, y_0) > 0$ for some $y_0 \notin \mathscr{M}$.

Since $y_0 \notin \mathscr{M}$, we have $\mu(x_0) > \mu(y_0)$, so by definition,

$$q(x_0,\, y_0) = p(x_0,\, y_0)\, \frac{\mu(y_0)}{\mu(x_0)} < p(x_0,\, y_0).$$

Finally, since $y_0 \neq x_0$, we have

$$q(x_0,\, x_0) = 1 - \sum_{z\neq x_0} q(x_0,\, z) > 1 - \sum_{z\neq x_0} p(x_0,\, z) = p(x_0,\, x_0) \geq 0.$$

[5] Definition of symmetry of p: $p(x,\, y) = p(y,\, x)$, $\forall x,\, y \in S$.

(iv) Since $q(x_0, x_0) > 0$, q is aperiodic. $\square$

Exercise 35. Assume that the chain is irreducible and positive recurrent, with invariant probability measure π. Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be bounded functions. Let $h(x) := \sum_{y\in S} p(x, y)\, g(y)$, $\forall x \in S$. We assume that $f + h = g$. Let ν be an arbitrary probability measure on S.

(i) Let $S_n := \sum_{i=0}^n f(X_i)$, $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} \frac{1}{n} E_\nu(S_n)$ exists and determine its value.

(ii) Let $M_0 := 0$ and $M_{n+1} := \sum_{i=0}^n [g(X_{i+1}) - h(X_i)]$, $n \in \mathbb{N}$. Prove that $(M_n, n \in \mathbb{N})$ is a martingale under P_ν, with respect to the canonical filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$.

(iii) Prove that for all $n \in N$,

$$E_\nu(M_{n+1}^2) = E_\nu\Big\{ \sum_{i=0}^n [g(X_{i+1}) - h(X_i)]^2 \Big\}.$$

(iv) Prove that

$$\lim_{n\to\infty} \frac{1}{n} E_\nu\Big\{ \sum_{i=0}^{n-1} g(X_{i+1}) h(X_i) \Big\} = \int_S h^2 \, \mathrm{d}\pi.$$

(v) Prove that

$$\lim_{n\to\infty} \frac{E_\nu(M_n^2)}{n} = \int_S (g^2 - h^2) \, \mathrm{d}\pi.$$

(vi) Prove that

$$\lim_{n\to\infty} \frac{E_\nu(S_n^2)}{n} = \int_S (g^2 - h^2) \, \mathrm{d}\pi.$$

Solution. We assume in questions (i)–(v) without loss of generality that $\nu = \delta_x$ for some $x \in S$ (otherwise, if need be, we apply the dominated convergence theorem). This is obvious for questions (i)–(iv), but is also obvious for question (v) if we take the identity in (iii) into account and apply the dominated convergence theorem.

(i) The chain being irreducible and positive recurrent, we know that $\frac{1}{n} \sum_{i=0}^n f^+(X_i) \to \int f^+ \, \mathrm{d}\pi$. Since f is bounded, we can apply the dominated convergence theorem, to see that $\frac{1}{n} \sum_{i=0}^n E_x[f^+(X_i)] \to \int f^+ \, \mathrm{d}\pi$. Exactly for the same reason, $\frac{1}{n} \sum_{i=0}^n E_x[f^-(X_i)] \to \int f^- \, \mathrm{d}\pi$. As a consequence, $\frac{1}{n} E_x(S_n) \to \int f \, \mathrm{d}\pi$.

(ii) For any n, M_n is integrable and $\mathscr{F}_n$-measurable. Since $E_x[M_{n+1} - M_n \,|\, \mathscr{F}_n] = E_x[g(X_{n+1}) - h(X_n) \,|\, \mathscr{F}_n] = E_x[g(X_{n+1}) \,|\, \mathscr{F}_n] - h(X_n) = h(X_n) - h(X_n) = 0$, it follows that $(M_n, n \in \mathbb{N})$ is a martingale.

(iii) It suffices to check that for $j > i$, $E_x\{[g(X_{j+1}) - h(X_j)]\,[g(X_{i+1}) - h(X_i)]\} = 0$.

Since $g(X_{i+1}) - h(X_i)$ is $\mathscr{F}_j$-measurable, $E_x\{[g(X_{j+1}) - h(X_j)]\,[g(X_{i+1}) - h(X_i)] \mid \mathscr{F}_j\} = [g(X_{i+1}) - h(X_i)]\,E_x\{g(X_{j+1}) - h(X_j) \mid \mathscr{F}_j\}$ which vanishes, since we have noted that the last conditional expectation is 0, P_x-a.s.

(iv) Since $E[g(X_{n+1}) - h(X_n) \mid \mathscr{F}_n] = 0$, we have $E[(g(X_{n+1}) - h(X_n))h(X_n)] = 0$, so

$$\frac{1}{n}\,E_x\Big\{\sum_{i=0}^{n-1} g(X_{i+1})h(X_i)\Big\} = \frac{1}{n}\,E_x\Big\{\sum_{i=0}^{n-1} h(X_i)^2\Big\},$$

which converges P_x-a.s. to $\int h^2\,\mathrm{d}\pi$ (as the chain is irreducible and positive recurrent).

(v) We have

$$\frac{E_x(M_n^2)}{n} = \frac{1}{n}E_x\Big\{\sum_{i=0}^{n-1} g(X_{i+1})^2\Big\} + \frac{1}{n}E_x\Big\{\sum_{i=0}^{n-1} h(X_i)^2\Big\} - \frac{2}{n}E_x\Big\{\sum_{i=0}^{n-1} g(X_{i+1})h(X_i)\Big\}.$$

On the right-hand side, the first term converges to $\int g^2\,\mathrm{d}\pi$, the third to $2\int h^2\,\mathrm{d}\pi$, whereas the second (by question (iv)) to $\int h^2\,\mathrm{d}\pi$. So $\frac{E_x(M_n^2)}{n} \to \int g^2\,\mathrm{d}\pi + \int h^2\,\mathrm{d}\pi - 2\int h^2\,\mathrm{d}\pi = \int (g^2 - h^2)\,\mathrm{d}\pi$.

(vi) In this question, we work under P_ν instead of P_x.

By assumption, $f = g - h$, so $S_n = \sum_{i=0}^n [g(X_i) - h(X_i)] = M_{n+1} + g(x) - g(X_{n+1})$, and thus

$$\frac{E_\nu(S_n^2)}{n} = \frac{E_\nu(M_{n+1}^2)}{n} + \frac{1}{n}\,E_\nu\{[g(x) - g(X_{n+1})]^2\} + \frac{1}{n}\,E_\nu\{2M_{n+1}[g(x) - g(X_{n+1})]\}\,.$$

We let $n \to \infty$. On the right-hand side, the first term converges to $\int (g^2 - h^2)\,\mathrm{d}\pi$ (by (v)). For the second and third, we recall that g is bounded by assumption, so there exists a constant C such that $|g(x) - g(X_{n+1})| \le C$. The second term on the right-hand side thus tends to 0, being bounded by $\frac{C}{n}$. For the third, we note that by the Cauchy–Schwarz inequality (or Jensen's inequality), $|E_\nu\{M_{n+1}[g(x) - g(X_{n+1})]\}| \le C[E_\nu(M_{n+1}^2)]^{1/2}$. So by (v) again, $\frac{1}{n}E_\nu\{2M_{n+1}[g(x) - g(X_{n+1})]\} \to 0$.

Consequently, $\frac{E_\nu(S_n^2)}{n} \to \int (g^2 - h^2)\,\mathrm{d}\pi$. □

Exercise 36 (Renewal theorem). Ken enjoys wine. One day, he makes a dream: at the beginning, he has in the pocket an amount of money $x \in \mathbb{N}$ (in yuans); at each minute, he drinks a glass of wine, which costs him a yuan; each time there is no money left in the pocket, he finds a wallet containing a random integer number of pieces of one yuan each, and he instantly restarts buying and drinking wine without losing a second. The dream goes on, indefinitely.

We modelize the capital X_n in Ken's pocket at each time $n \in \mathbb{N}$ by means of a Markov chain $(X_n,\, n \geq 0)$ taking values in $\mathbb{N}$, with transition kernel

$$p(x,y) := \begin{cases} f(y+1) & x = 0,\ y \geq 0, \\ 1 & x > 0,\ y = x-1, \\ 0 & \text{otherwise}, \end{cases}$$

where f is a probability measure on $\mathbb{N}^* := \{1, 2, \cdots\}$. We assume that $f : \mathbb{N}^* \mapsto (0, 1)$ with $\sum_{n=1}^{\infty} f(n) = 1$, such that $f(y) > 0$ for all $y \in \mathbb{N}^*$.

If $X_i = y > 0$ at time i, then $X_{i+1} = y - 1$ at time $i + 1$. If $X_i = 0$ at time i, Ken finds $y \geq 1$ yuans with probability $f(y)$, and he instantly spends a yuan to buy wine, so at time $i+1$, $X_{i+1} = y - 1$ with probability $f(y)$.

Let $T_0^{(0)} := 0$ and let $T_0^{(n)} := \inf\{i > T_0^{(n-1)} : X_i = 0\}$, $n \geq 1$. In words, $T_0^{(n)}$ is the n-th return to state 0.

(i) Prove that $P_0(T_0^{(1)} = n) = f(n)$, $n \geq 1$.

(ii) Classify the states, and determine all recurrence classes.

(iii) Let λ be the measure on $\mathbb{N}$ given by

$$\lambda(x) := \sum_{y=x+1}^{\infty} f(y), \qquad x \in \mathbb{N}\,.$$

Prove that λ is the unique (up to a constant multiple) invariant measure for p.

(iv) Give a necessary and sufficient condition for (X_n) to be positive recurrent. Prove that there is uniqueness of invariant probability if and only if

$$m := \sum_{n=1}^{\infty} n\, f(n) < \infty\,.$$

We assume $m < \infty$ in the rest of the exercise.

(v) Compute $\lim_{n\to\infty} P_x(X_n = y)$, for all $x,\, y \in \mathbb{N}$.

(vi) Let $u(n) := P_0(X_n = 0)$, $n \geq 1$. Prove that $\{X_0 = X_n = 0\} = \cup_{z=1}^{n}\{X_0 = X_n = 0,\, T_0^{(1)} = z\}$. Prove that

$$u(n) = \sum_{z=1}^{n} f(z)\, u(n-z) =: [f * u](n), \qquad n \geq 1.$$

(vii) Define $t_i := T_0^{(i)} - T_0^{(i-1)}$, $i \geq 1$. Prove that under P_0, $(t_i,\, i \geq 1)$ is a sequence of i.i.d. random variables. Compute $P_0(t_i = n)$ for all $n \geq 1$.

(viii) Prove that $P_0(T_0^{(i)} = n) = f^{i*}(n)$, $i \geq 1$, where $f^{i*} = f * \cdots * f$ is the i-th fold convolution of f.

(ix) Prove that $\{X_n = 0\} = \cup_{i=0}^{\infty}\{T_0^{(i)} = n\}$. Prove that

$$u(n) = \sum_{i=1}^{\infty} f^{i*}(n), \qquad n \geq 1.$$

(x) Prove that

$$\lim_{n\to\infty} u(n) = \frac{1}{m}\,.$$

Solution. (i) By definition,

$$P_0(T_0^{(1)} = n) = P_0(X_n = 0,\, X_{n-1} = 1,\, \cdots,\, X_1 = n-1) = P_0(X_1 = n-1) = f(n)\,.$$

(ii) State 0 leads to all state $x > 0$ because $p(0,\, x) = f(x) > 0$. All state $x > 0$ leads to state $x - 1$, and thus to state 0 after x steps. The chain is thus irreducible.

By (i), $P_0(T_0^{(1)} < \infty) = \sum_{n=1}^{\infty} f(n) = 1$; so that chain is (irreducible and) recurrent.

(iii) We have

$$\sum_{x\in\mathbb{N}} \lambda(x)\, p(x,\, y) = f(y+1) \sum_{z=1}^{\infty} f(z) + \sum_{z=y+2}^{\infty} f(z) = \sum_{z=y+1}^{\infty} f(z) = \lambda(y)\,,$$

which means that λ is invariant.

Let μ is an invariant measure, so

$$\mu(y) = \sum_{x\in\mathbb{N}} \mu(x)\, p(x,\, y) = f(y+1)\, \mu(0) + \mu(y+1)\,,$$

which yields

$$\mu(y) = \mu(0) - \mu(0)(f(y) + f(y-1) + \cdots + f(1)) = \mu(0) \sum_{z=y+1}^{\infty} f(z) = \mu(0)\, \lambda(y)\,,$$

as desired.

(iv) The chain (X_n) is positive recurrent if and only if $\infty > E_0(T_0^{(1)}) = \sum_{n=1}^{\infty} n\, f(n)$.

There exists an invariant probability if and only if $\sum_{x\in N} \lambda(x) < \infty$; on the other hand,

$$\sum_{x=0}^{\infty} \lambda(x) = \sum_{x=0}^{\infty} \sum_{z=x+1}^{\infty} f(z) = \sum_{z=1}^{\infty} f(z) \sum_{x=0}^{z-1} 1 = \sum_{z=1}^{\infty} z\, f(z) = m\,.$$

Hence, existence of an invariant probability is equivalent to $m < \infty$.

(v) Since $m < \infty$ by assumption, the unique invariant probability is

$$\pi(x) = \frac{1}{m} \sum_{z=x+1}^{\infty} f(z), \qquad x \in \mathbb{N}\,.$$

The chain is irreducible and positive recurrent, and is aperiodic because $p(0,\,0)=f(1)>0$. Thus for all $x,\,y\in\mathbb{N}$,

$$\lim_{n\to\infty}P_x(X_n=y)=\pi(y)=\frac{1}{m}\sum_{z=y+1}^{\infty}f(z)\,.$$

(vi) If $X_0=X_n=0$, then $T_0^{(1)}\in\{1,\cdots,n\}$, hence $\{X_0=X_n=0\}\subset\cup_{z=1}^n\{X_0=X_n=0,\,T_0^{(1)}=z\}$. The converse being trivial, we obtain $\{X_0=X_n=0\}=\cup_{z=1}^n\{X_0=X_n=0,\,T_0^{(1)}=z\}$.

Applying the strong Markov property at $T_0^{(1)}$ gives that

$$\begin{aligned}P_0(X_n=0) &= \sum_{z=1}^{n}P_0(T_0^{(1)}=z,\,X_n=0)\\ &= \sum_{z=1}^{n}E_0\Big(\mathbf{1}_{\{T_0^{(1)}=z\}}\,P_{X_z}(X_{n-z}=0)\Big)\\ &= \sum_{z=1}^{n}E_0(T_0^{(1)}=z)\,P_0(X_{n-z}=0)\,,\end{aligned}$$

which is $\sum_{z=1}^n f(z)\,u(n-z)$.

(vii) By definition, $t_{i+1}=T_0^{(1)}\circ\theta_{T_0^{(i)}}$. It follows from the strong Markov property that t_i, $i\ge 1$, are i.i.d. under P_0, and that

$$P_0(t_i=n)=P_0(t_1=n)=P_0(T_0^{(1)}=n)=f(n),\qquad n\ge 1.$$

(viii) Since $T_0^{(i)}=t_1+\cdots+t_i$, and $(t_i,\,i\ge 1)$ is a sequence of i.i.d. random variables under P_0 with common law f, we have $P_0(T_0^{(i)}=n)=f^{i*}(n)$.

(ix) The sequence $(T_0^{(i)},\,i\ge 0)$ representing successive hitting times at state 0, it follows that $\{X_n=0\}$ coincides with disjoint union $\cup_{i=1}^n\{T_0^{(i)}=n\}$; hence for all $n\ge 1$,

$$u(n)=P_0(X_n=0)=\sum_{i=1}^{\infty}P_0(T_0^{(i)}=n)=\sum_{i=1}^{\infty}f^{i*}(n)\,.$$

(x) By (v), $u(n)=P_0(X_n=0)\to\pi(0)=\frac{1}{m}\sum_{z=1}^{\infty}f(z)=\frac{1}{m}$. □

Exercise 37 (A criterion for recurrence and transience). In this exercise, $S=\mathbb{N}$. We assume that the chain is irreducible.

Question A. Prove that for any $x\in S$, $\limsup_{n\to\infty}X_n=\infty$, P_x-a.s.

Question B. Let $f : S \to \mathbb{R}_+$ be such that for some $k \in \mathbb{N}$,

$$\sum_{y \in S} p(x, y) f(y) \le f(x), \qquad \forall x > k.$$

Let $\tau := \inf\{n \in \mathbb{N} : X_n \le k\}$. Prove that for all $x > k$, $(f(X_{n \wedge \tau}), n \in \mathbb{N})$ under P_x is a supermartingale.

Question C. In this question (and only in this question), we assume that $\lim_{x \to \infty} f(x) = \infty$.

(C1) Prove that for all $x \in S$, $P_x\{\tau < \infty\} = 1$.

(C2) Prove that for all $x \in S$ and all $n \in \mathbb{N}$, $P_x\{\tau \circ \theta_n < \infty\} = 1$.

(C3) Let $x \in S$. Prove that $\sum_{n=0}^{\infty} \mathbf{1}_{\{X_n \le k\}} = \infty$, P_x-a.s.

(C4) Prove that the chain is recurrent.

Question D. In this question, we assume that $f(x) > 0$, $\forall x \in S$, and that $\lim_{x \to \infty} f(x) = 0$.

(D1) Prove that $\lim_{x \to \infty} P_x\{\tau < \infty\} = 0$.[6]

(D2) Prove that the chain is transient.

Question E. We apply our results to an example of birth-and-death chain. Assume $p(0, 1) = 1$ and for $x \ge 1$,

$$p(x, y) := \begin{cases} p_x & \text{if } y = x + 1, \\ q_x & \text{if } y = x - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $p_x > 0$ and $q_x := 1 - p_x > 0$, $\forall x$.

(E1) Prove that the chain is irreducible.

(E2) We assume from now on that for some $\lambda \in \mathbb{R}$, $p_x = \frac{1}{2} + (1 + \varepsilon(x))\frac{\lambda}{x}$, where $\varepsilon(x)$ is such that $\lim_{x \to \infty} \varepsilon(x) = 0$. Prove that for all $\alpha \in \mathbb{R}$,

$$\frac{E_x(X_1^\alpha)}{x^\alpha} = 1 + \frac{2\alpha}{x^2}(\lambda - \frac{1}{4} + \frac{\alpha}{4}) + \frac{o(1)}{x^2}, \qquad x \to \infty.$$

(E3) Prove that in case $\lambda < \frac{1}{4}$, the chain is recurrent.

(E4) Prove that in case $\lambda > \frac{1}{4}$, the chain is transient.

Solution. (A) If the chain is recurrent, then any state y is visited infinitely often P_x-a.s. by the chain, which implies $\limsup_{n \to \infty} X_n \ge y$, P_x-a.s. Since y is arbitrary, this means $\limsup_{n \to \infty} X_n = \infty$, P_x-a.s.

If the chain is transient, then any state $y \in \mathbb{N}$ is visited only finitely many times P_x-a.s., which implies $X_n \to \infty$, P_x-a.s.

[6] Hint: You could start by studying $E_x[f(X_\tau) \mathbf{1}_{\{\tau < \infty\}}]$.

(B) This is essentially done in the class: for any n, since τ is a stopping time, we deduce that $f(X_{n\wedge\tau})$ is $\mathscr{F}_n$-measurable, and by writing

$$f(X_{(n+1)\wedge\tau}) = f(X_{n+1})\,\mathbf{1}_{\{\tau>n\}} + f(X_\tau)\,\mathbf{1}_{\{\tau\le n\}} = f(X_{n+1})\,\mathbf{1}_{\{\tau>n\}} + f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau\le n\}},$$

we see that $E_x[f(X_{n+1})\,\mathbf{1}_{\{\tau>n\}}\,|\,\mathscr{F}_n] = \mathbf{1}_{\{\tau>n\}}\,E_x[f(X_{n+1})\,|\,\mathscr{F}_n] = \mathbf{1}_{\{\tau>n\}}\sum_{y\in S} p(X_n,\,y)f(y)$, and $E_x[f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau\le n\}}\,|\,\mathscr{F}_n] = f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau\le n\}}$ (by measurability), so

$$\begin{aligned} E_x[f(X_{(n+1)\wedge\tau})\,|\,\mathscr{F}_n] &= \mathbf{1}_{\{\tau>n\}}\sum_{y\in S} p(X_n,\,y)f(y) + f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau\le n\}} \\ &\le \mathbf{1}_{\{\tau>n\}}\,f(X_n) + f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau\le n\}} \\ &= f(X_{n\wedge\tau}), \end{aligned}$$

where, to get the inequality in the middle, we used the fact that $X_n > k$ on the event $\{\tau > n\}$. By induction, we get $E_x[f(X_{n\wedge\tau})] \le E_x[f(X_{0\wedge\tau})] = f(x)$, which proves that $f(X_{n\wedge\tau})$ is P_x-integrable, and that $(f(X_{n\wedge\tau}),\,n\in\mathbb{N})$ is a supermartingale.

(C1) If $x\le k$, then $\tau = 0$, P_x-a.s., so $P_x\{\tau<\infty\} = 1$.

Assume now $x > k$. Since $(f(X_{n\wedge\tau}),\,n\in\mathbb{N})$ is a non-negative P_x-supermartingale, it converges P_x-a.s. to a finite limit, denoted by ξ. On the set $\{\tau=\infty\}$, $f(X_n)\to\xi$, P_x-a.s. However, by assumption, $f(y)\to\infty$ for $y\to\infty$, which yields that on the set $\{\tau=\infty\}$, (X_n) is P_x-a.s. bounded. In view of Question A, this happens with P_x-probability 0; i.e., $P_x\{\tau<\infty\} = 1$.

(C2) By the Markov property, $P_x\{\tau\circ\theta_n<\infty\} = P_{X_n}\{\tau<\infty\}$, which is P_x-a.s. 1 (see (C1)).

(C3) Let $A_n := \{\tau\circ\theta_n<\infty\}$, $n\in\mathbb{N}$, which is a non-increasing sequence of events, with $\cap_{n\in\mathbb{N}}A_n = \{\sum_{n=0}^\infty \mathbf{1}_{\{X_n\le k\}} = \infty\}$. By (C2), $P_x(A_n) = 1$, $\forall n\in\mathbb{N}$, so $P_x(\cap_{n\in\mathbb{N}}A_n) = 1$, which means $\sum_{n=0}^\infty \mathbf{1}_{\{X_n\le k\}} = \infty$, P_x-a.s.

(C4) We have proved $\sum_{n=0}^\infty \mathbf{1}_{\{X_n\le k\}} = \infty$, P_x-a.s. A fortiori, $E_x[\sum_{n=0}^\infty \mathbf{1}_{\{X_n\le k\}}] = \infty$, i.e., $\sum_{y\in k} G(x,\,y) = \infty$. So there exists $y\in\{0,\,1,\,\cdots,\,k\}$ such that $G(x,\,y)=\infty$. A fortiori, $G(y,\,y)=\infty$, i.e., y is a recurrent state. Since the chain is irreducible, it is recurrent.

(D1) Let $x\in S$. By Fatou's lemma,

$$E_x[f(X_\tau)\,\mathbf{1}_{\{\tau<\infty\}}] \le \liminf_{n\to\infty} E_x[f(X_{n\wedge\tau})\,\mathbf{1}_{\{\tau<\infty\}}],$$

which is bounded by $f(x)$ because $(f(X_{n\wedge\tau}),\,n\in\mathbb{N})$ is a P_x-supermartingale (see Question B).

On the other hand,

$$E_x[f(X_\tau)\,\mathbf{1}_{\{\tau<\infty\}}] \ge c\,P_x\{\tau<\infty\},$$

where $c := \min_{0 \le y \le k} f(y) > 0$. Thus $P_x\{\tau < \infty\} \le \frac{1}{4} f(x) c$. Letting $x \to \infty$ gives the desired conclusion.

(D2) If the chain were recurrent, we would have $P_x\{\tau < \infty\} = 1$, $\forall x \in S$, which would contract what we have proved in (D1). Conclusion: the chain is transient.

(E1) For any $x \ge 1$, $G(0, x) \ge p^x(0, x) \ge p(0, 1)\, p(1, 2) \cdots p(x-1, x) = p_0 p_1 \cdots p_{x-1} > 0$, and similarly, $G(x, 0) \ge p^x(x, 0) \ge p(x, x-1) \cdots p(2, 1)\, p(1, 0) = q_x \cdots q_2 q_1 > 0$. Therefore, the chain is irreducible.

(E2) For any $x \ge 1$,

$$E_x(X_1^\alpha) = p_x\,(x+1)^\alpha + q_x\,(x-1)^\alpha = x^\alpha [p_x (1 + x^{-1})^\alpha + q_x\,(1 - x^{-1})^\alpha].$$

Letting $x \to \infty$ and making an asymptotic development (under the second order) of the function $s \mapsto (1+s)^\alpha$ for s in the neighbourhood of 0, we get

$$\begin{aligned}
\frac{E_x(X_1^\alpha)}{x^\alpha} &= \Big(1 + \frac{\alpha}{x} + \frac{\alpha(\alpha-1) + o(1)}{2x^2}\Big)\Big(\frac{1}{2} + \frac{(1+\varepsilon(x))\lambda}{x}\Big) \\
&\quad + \Big(1 - \frac{\alpha}{x} + \frac{\alpha(\alpha-1) + o(1)}{2x^2}\Big)\Big(\frac{1}{2} - \frac{(1+\varepsilon(x))\lambda}{x}\Big) \\
&= 1 + \frac{4\alpha\lambda + \alpha(\alpha-1) + o(1)}{x^2}, \qquad x \to \infty.
\end{aligned}$$

(E3) If $\lambda < \frac{1}{4}$, then by choosing $\alpha > 0$ such that $\alpha - \frac{1}{4} + \frac{\alpha}{4} < 0$, we get $E_x(X_1^\alpha) \le x^\alpha$ for all sufficiently large x, say $\forall x > k$. So the function $f(x) := x^\alpha$ satisfies the conditions in Question C, which yields that the chain is recurrent.

(E4) If $\lambda > \frac{1}{4}$, then by choosing $\alpha < 0$ such that $\alpha - \frac{1}{4} + \frac{\alpha}{4} > 0$, we get $E_x(X_1^\alpha) \le x^\alpha$ for all sufficiently large x, say $\forall x > k$. So the function $f(x) := x^\alpha$ satisfies the conditions in Question D, which yields that the chain is transient. $\square$

Exercise 38 (Foster's criterion). Assume that the chain is irreducible. The aim of this exercise is to show that if there exist $\alpha > 0$, a finite set $A \subset S$ and $f : S \to \mathbb{R}_+$ such that

$$(*) \qquad \sum_{y \in S} p(x, y)\, f(y) < \infty, \quad \forall x \in S; \qquad \sum_{y \in S} p(x, y)\, f(y) \le f(x) - \alpha, \quad \forall x \in A^c,$$

then the chain is positive recurrent.

Notation: For any $A \subset S$, we write

$$\tau_A := \inf\{n \in \mathbb{N} : X_n \in A\}, \qquad T_A := \inf\{n \ge 1 : X_n \in A\}, \qquad \inf \varnothing := \infty.$$

Question A. In this question, we assume that there exists a finite set $A \subset S$ such that $E_x(T_A) < \infty$, $\forall x \in A$. Let $\varrho_0 := 0$ and

$$\varrho_{n+1} := \varrho_n + T_A \circ \theta_{\varrho_n}, \qquad n \in \mathbb{N}.$$

(A1) Prove that for all $x \in A$ and $n \in \mathbb{N}$, $P_x\{\varrho_n < \infty\} = 1$.

(A2) Let $Y_n := X_{\varrho_n}$, $n \in \mathbb{N}$. Prove that $(Y_n,\, n \in \mathbb{N})$ is a Markov chain, and determine its transition probability q.

(A3) Prove that $(Y_n,\, n \in \mathbb{N})$ is irreducible and positive recurrent.

(A4) Let $x \in A$. Let $T_x := \inf\{n \geq 1 : X_n = x\}$ and $T_x^Y := \inf\{n \geq 1 : Y_n = x\}$. Prove that

$$T_x = \sum_{n=0}^{\infty} \mathbf{1}_{\{T_x^Y > n\}}\, T_A \circ \theta_{\varrho_n},$$

and that $\{T_x^Y > n\} \in \mathscr{F}_{\varrho_n}$.

(A5) Prove that for any $x \in A$,

$$E_x(T_x) \leq \max_{y \in A} E_y(T_A)\, E_x(T_x^Y).$$

(A6) Prove that $(X_n,\, n \in \mathbb{N})$ is positive recurrent.

Question B. In this question, we assume $(*)$ for some $\alpha > 0$, finite set $A \subset S$ and $f : S \to \mathbb{R}_+$.

(B1) Let $Z_n := f(X_n) + n\alpha$, $n \in \mathbb{N}$. Prove that for all $x \in A^c$, $(Z_{n \wedge \tau_A},\, n \in \mathbb{N})$ is a P_x-supermartingale with respect to the canonical filtration $(\mathscr{F}_n)_{n \in \mathbb{N}}$.

(B2) Prove that for all $x \in S$, $E_x(\tau_A) \leq \frac{f(x)}{\alpha}$.

(B3) Prove that $(X_n,\, n \in \mathbb{N})$ is positive recurrent.

Solution. Several ingredients in the solution (to questions (A1), (A2) and (B1)) have been seen in previous exercises, so we only make a sketch.

(A1) By induction and the strong Markov property. [More details in Exercise 27 (ii).]

(A2) Check by definition, discussing on the value of the stopping times. The transition kernel is $q(x,\, y) = P_x\{X_{T_A} = y\}$. [More details in Exercise 27 (iii).]

(A3) Let $x \neq y \in S$. Since $(X_n,\, n \in \mathbb{N})$ is irreducible, there exist $x_0 := x, x_1, \cdots, x_n := y$ such that $p(x_{i-1}, x_i) > 0$, $\forall i \in \{1, 2, \cdots, n\}$. Let $x_{j_1}, \cdots, x_{j_m}$ be the sequence of $x_1, \cdots, x_{n-1}$ lying in A. Then

$$P_x\{Y_1 = x_{j_1}, \cdots, Y_m = x_{j_m}, Y_{m+1} = y\} \geq P_x\{X_1 = x_1, \cdots, X_{n-1} = x_{n-1}, X_n = y\} > 0,$$

which yields that $(Y_n,\, n \in \mathbb{N})$ is irreducible. Since the chain takes values in a finite space, it is positive recurrent.

(A4) Let $k \geq 1$. The event $T_x^Y = k$ means that on the occasion of its k-th return to A, the chain (X_n) returns to x for the first time. So $\{T_x^Y = k\} = \{T_x = \varrho_k\}$. Therefore, on the event $\{T_x^Y = k\}$, we have

$$T_x = \varrho_k = \sum_{n=0}^{k-1} (\varrho_{n+1} - \varrho_n) = \sum_{n=0}^{k-1} T_A \circ \theta_{\varrho_n} = \sum_{n=0}^{\infty} (T_A \circ \theta_{\varrho_n})\, \mathbf{1}_{\{k>n\}},$$

giving the first desired equality.

Furthermore, $\{T_x^Y \leq k\} = \cup_{n=0}^k \{T_x = \varrho_n\}$. Since $\{T_x = \varrho_n\} \in \mathscr{F}_{\varrho_n} \subset \mathscr{F}_{\varrho_k}$ for $n \in \{0,\, 1,\, \cdots,\, k\}$, we have $\{T_x^Y \leq k\} \in \mathscr{F}_{\varrho_k}$.

(A5) By (A4) and then by the strong Markov property,

$$E_x(T_x) = \sum_{n=0}^{\infty} E_x[(T_A \circ \theta_{\varrho_n})\, \mathbf{1}_{\{T_x^Y > n\}}] = \sum_{n=0}^{\infty} E_x[\mathbf{1}_{\{T_x^Y > n\}}\, E_{X_{\varrho_n}}(T_A)].$$

Since $E_{X_{\varrho_n}}(T_A) \leq \max_{y \in A} E_y(T_A)$, this yields

$$E_x(T_x) \leq \max_{y \in A} E_y(T_A) \sum_{n=0}^{\infty} E_x[\mathbf{1}_{\{T_x^Y > n\}}].$$

It remains to note that $\sum_{n=0}^{\infty} E_x[\mathbf{1}_{\{T_x^Y > n\}}] = E_x(T_x^Y)$ by the Fubini–Tonelli theorem.

(A6) By (A3), $(Y_n,\, n \in \mathbb{N})$ is positive recurrent, so $E_x(T_x^Y) < \infty$. On the other hand, by assumption, $E_y(T_A) < \infty$, $\forall y \in A$. Since A is a finite set, this yields $\max_{y \in A} E_y(T_A) < \infty$. So by the inequality proved in (A5), we have $E_x(T_x) < \infty$, $\forall x \in A$. Since the chain $(X_n,\, n \in \mathbb{N})$ is irreducible, this means it is positive recurrent.

(B1) By writing $Z_{(n+1)\wedge\tau_A} = Z_{n+1}\, \mathbf{1}_{\{\tau_A > n\}} + Z_{\tau_A}\, \mathbf{1}_{\{\tau_A \leq n\}}$, one checks, using assumption $(*)$, that $(Z_{n\wedge\tau_A},\, n \in \mathbb{N})$ is a P_x-supermartingale. [More details in Exercise 37, Question B.]

(B2) If $x \in A$, the inequality holds trivially.

Assume now $x \in A^c$. Since $(Z_{n\wedge\tau_A},\, n \in \mathbb{N})$ is a P_x-supermartingale (see (B1)), we have $E_x[Z_{n\wedge\tau_A}] \leq E_x(Z_0) = f(x)$, $\forall n \in \mathbb{N}$. On the other hand, since $Z_n \geq n\alpha$, $\forall n \in \mathbb{N}$, we have $E_x[Z_{n\wedge\tau_A}] \geq \alpha\, E_x(n \wedge \tau_A)$. Therefore, $E_x(n \wedge \tau_A) \leq \frac{f(x)}{\alpha}$, $\forall n \in \mathbb{N}$. An application of the monotone convergence theorem yields the desired inequality.

(B3) Let $x \in A$. By definition, $T_A = 1 + \tau_A \circ \theta_1$. Hence

$$E_x(T_A) = 1 + E_x(\tau_A \circ \theta_1) = 1 + E_x[E_{X_1}(\tau_A)].$$

By (B2), $E_{X_1}(\tau_A) \leq \frac{f(X_1)}{\alpha}$, so that for all $x \in A$,

$$E_x(T_A) \leq 1 + \frac{E_x[f(X_1)]}{\alpha} = 1 + \frac{1}{\alpha} \sum_{y \in S} p(x,\, y)\, f(y),$$

which is finite by assumption. So the condition of Question A is satisfied: we are entitled to apply (A6) to conclude that $(X_n,\, n \in \mathbb{N})$ is positive recurrent. $\square$

Exercise 39 (Branching processes). Let $(Z_n,\, n \geq 0)$ be a branching process, defined by

$$Z_{n+1} := \sum_{j=1}^{Z_n} \xi_{n,\,j}\,, \qquad (\sum_{j=1}^{0} := 0)$$

where $\xi_{n,\,j}$, for $(n,\, j) \in \mathbb{N} \times \mathbb{N}^*$, are i.i.d. random variables taking values in $\mathbb{N}$, whose common distribution is denoted by ξ. We assume $\xi(0) + \xi(1) < 1$ and $\xi(0) > 0$.

Let $\mathscr{F}_n := \sigma(\xi_{i,\,j};\, 1 \leq i \leq n,\, j \geq 1)$ and $\mathscr{F}_0 = \sigma(X_0)$.

Let

$$m := \sum_{k=0}^{\infty} k\, \xi(k), \qquad \phi(r) := \sum_{k=0}^{\infty} \xi(k) r^k, \qquad r \in [0,\, 1]\,.$$

Let $\phi_0(r) = r$, and $\phi_{n+1}(r) := \phi_n(\phi(r))$, $n \in \mathbb{N}$.

Question A. We assume $Z_0 = 1$ in this question.

(A1) Compute $E(Z_{n+1} \,|\, \mathscr{F}_n)$ and $E(Z_n)$ for all $n \geq 0$. Prove that for $n \in \mathbb{N}$ and $r \in [0,\, 1]$,

$$E(r^{Z_{n+1}} \,|\, \mathscr{F}_n) = \phi(r)^{Z_n}\,.$$

Compute $E(r^{Z_n})$.

(A2) Determine the number of solutions of $\phi(r) = r$ on $[0,\, 1]$.

Let q be the smallest solution of $\phi(r) = r$ on $[0,\, 1]$. Prove that $\phi_n(0) \leq \phi_{n+1}(0)$ for all $n \in N$, and that $\phi_n(0) \to q$, $n \to \infty$.

(A3) Let

$$T := \inf\{n \in \mathbb{N} : Z_n = 0\}, \qquad \inf \varnothing = \infty\,.$$

Prove that $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$ for all $n \in \mathbb{N}$. Prove that

$$P(T < \infty) = q\,.$$

We call the event $\{T < \infty\}$ *extinction*, and the event $\{T = \infty\}$ *survival.* We say that the system is *critical* if $m = 1$, *supercritical* if $m > 1$, and *sub-critical* if $m < 1$.

Find a relation between extinction/survival and supercritical/critical/sub-critical cases.

Question B (Martingales in the supercritical branching process). We assume $Z_0 = 1$ and $m > 1$ in this question.

(B1) Define $M_n := q^{Z_n}$, $n \geq 0$. Prove that $(M_n,\, n \geq 0)$ is an $(\mathscr{F}_n)$-martingale, and study convergence (a.s., in L^1, in L^p for $1 < p < \infty$). Prove that

$$\lim_{n\to\infty} q^{Z_n} \mathbf{1}_{\{T=\infty\}} = 0 \qquad \text{a.s.}$$

(B2) Prove that

$$\mathbf{1}_{\{T=\infty\}} = \mathbf{1}_{\{\lim_{n\to\infty} Z_n = \infty\}} \qquad \text{a.s.}$$

Compute $P(Z_n \text{ converges})$.

(B3) Let $W_n := \frac{Z_n}{m^n}$, $n \geq 0$. Prove that $(W_n,\, n \geq 0)$ is an $(\mathscr{F}_n)$-martingale.

(B4) Assuming $\sigma^2 := \sum_{k=0}^{\infty} k^2 \xi(k) - m^2 < \infty$, prove that for all $n \in \mathbb{N}$,

$$E(Z_{n+1}^2 \,|\, \mathscr{F}_n) = m^2\, Z_n^2 + \sigma^2\, Z_n \qquad \text{a.s.}$$

Prove that $\sup_{n\in\mathbb{N}} E(W_n^2) < \infty$. Prove that W_n converges in L^2, to a limit denoted by W_∞.

(B5) Define $L(\lambda) := E(\mathrm{e}^{-\lambda W_\infty})$, $\lambda \geq 0$. Write a functional identity involving ϕ, L and m.

(B6) Prove that $P(W_\infty = 0) = \lim_{\lambda\to\infty} L(\lambda)$.

(B7) We assume $\sigma^2 < \infty$. Compute $P(W_\infty = 0)$. Prove that $\mathbf{1}_{\{T=\infty\}} = \mathbf{1}_{\{W_\infty > 0\}}$ a.s. Prove that on the set of the system's survival, Z_n grows at an exponential rate, and give an equivalent.

Question C (Martingales in the critical/subcritical branching process). We assume $Z_0 = 1$ and $m \leq 1$ in this question.

(C1) Prove that W_n converges almost surely, and study the limit.

(C2) Does W_n converge in L^1?

Question D (The branching process as a Markov process). Let $x \in \mathbb{N}^*$.

(D1) Prove that $(Z_n,\, n \geq 0)$ is a Markov chain taking values in $\mathbb{N}$. Compute its transition kernel.

(D2) Classify the states, and determine the recurrence classes.

(D3) Prove that

$$P_x(\exists n \in \mathbb{N} :\; Z_k = 0,\, \forall k \geq n) + P_x\Big(\lim_{n\to\infty} Z_n = \infty\Big) = 1.$$

Solution. (A1) We have

$$
\begin{aligned}
E(Z_{n+1} \,|\, \mathscr{F}_n) &= E\Big(\sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \sum_{j=1}^{k} \xi_{n,j} \,\Big|\, \mathscr{F}_n \Big) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \sum_{j=1}^{k} E(\xi_{n,j} \,|\, \mathscr{F}_n) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \sum_{j=1}^{k} E(\xi_{n,j}) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \, km \,,
\end{aligned}
$$

which is $m\,Z_n$. Taking expectation on both sides gives $E(Z_{n+1}) = m\,E(Z_n)$. So by induction and the fact that $Z_0 = 1$, we obtain $E(Z_n) = m^n$, $n \geq 0$.

Let $n \in \mathbb{N}$ and $r \in [0, 1]$. Since $r^{Z_n} \in [0, 1]$, it is integrable, and

$$
\begin{aligned}
E(r^{Z_{n+1}} \,|\, \mathscr{F}_n) &= E\Big(\sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} r^{\sum_{j=1}^{k} \xi_{n,j}} \,\Big|\, \mathscr{F}_n \Big) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \, E\Big(r^{\sum_{j=1}^{k} \xi_{n,j}} \,\Big|\, \mathscr{F}_n \Big) .
\end{aligned}
$$

Since $E(r^{\sum_{j=1}^{k} \xi_{n,j}} \,\big|\, \mathscr{F}_n) = \prod_{j=1}^{k} E(r^{\xi_{n,j}}) = \phi(r)^k$, it follows that

$$
E(r^{Z_{n+1}} \,|\, \mathscr{F}_n) = \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \, \phi(r)^k = \phi(r)^{Z_n} \qquad \text{a.s.}
$$

Taking expectation on both sides, and we obtain: $E(r^{Z_{n+1}}) = E(\phi(r)^{Z_n})$. Iterating the procedure, and since $Z_0 = 1$, we get $E(r^{Z_n}) = \phi_n(r)$, for $n \in \mathbb{N}$ and $r \in [0, 1]$.

(A2) The moment generating function $\phi(\,\cdot\,)$ is a power series with radius of convergence ≥ 1. As a consequence, for all $0 \leq r < 1$,

$$
\phi''(r) = \sum_{k\geq 2} k(k-1)\xi(k)\, r^{k-2} \,,
$$

and $\phi''(r) > 0$ for $0 < r < 1$ since there exists $k \geq 2$ such that $\xi(k) > 0$ (recalling that $\xi(0) + \xi(1) < 1$ by assumption). So the function $r \to \phi(r) - r$ is strictly convex. Since $m = \phi'(1-)$ (by the monotone convergence theorem when $r \uparrow 1$), we see that $\phi(r) - r$ is strictly above the slope m at $r = 1$ on $(0, 1)$, and

$$
\phi(r) - r > (m-1)(r-1) > 0 \,,
$$

if $r \in [0, 1)$ and $m \leq 1$. Since $\phi(1) = 1$, we conclude that in case $m \leq 1$, $q = 1$ is the unique fixed point of ϕ on $[0, 1]$.

By assumption $\phi(0) = \xi(0) > 0$. If $m > 1$, then the slope of $\phi(r) - r$ at 1 is negative, so $\phi(1 - \varepsilon) - (1 - \varepsilon) < 0$ for some sufficiently small $\varepsilon > 0$. The continuity of ϕ on $[0, 1]$ yields the existence of $0 < c < 1$ such that $\phi(c) - c = 0$. By convexity, c is unique. As a consequence, in case $m > 1$, there are two fixed points of ϕ on $[0, 1]$ given by $q = c \in (0, 1)$ and 1.

Summarizing, we have $\phi(r) > r$ if $r < q$, and $\phi(r) < r$ if $r > q$, so the sequence $(\phi_n(0),\, n \geq 0)$ is non-decreasing, and its limit is q, the smallest fixed point.

(A3) The inclusion $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$ is clear by definition. Since $\{T < \infty\} = \cup_{n=0}^{\infty} \{Z_n = 0\}$, this yields

$$P(T < \infty) = \lim_{n \to \infty} \uparrow P(Z_n = 0)\,.$$

On the other hand, $\phi_n(r) = E(r^{Z_n}) = E(\mathbf{1}_{\{Z_n = 0\}}) + E(r^{Z_n} \mathbf{1}_{\{Z_n \geq 1\}})$; in particular, $\phi_n(0) = P(Z_n = 0)$. By (A2), $P(T < \infty) = \lim_{n \to \infty} \uparrow \phi_n(0) = q$.

We have seen in (A2) that $q < 1$ if and only if $m > 1$; hence $P(\text{extinction}) < 1$ if and only if $m > 1$ (critical).

(B1) For any $n \geq 0$, M_n is $\mathscr{F}_n$-measurable, and integrable (because $0 \leq M_n \leq 1$ a.s.); moreover, $E(M_{n+1} \,|\, \mathscr{F}_n) = \phi(q)^{Z_n} = q^{Z_n} = M_n$. Consequently, $(M_n,\, n \geq 0)$ is an $(\mathscr{F}_n)$-martingale.

Since for any $1 \leq p < \infty$, $\sup_{n \in \mathbb{N}} E(|M_n|^p) \leq 1 < \infty$, it follows that the $(M_n,\, n \geq 0)$ converges p.s., in L^1, and in L^p for all $1 \leq p < \infty$, to a random variable M_∞ taking values in $[0, 1]$.

Observe that

$$M_{n \wedge T} = M_n \, \mathbf{1}_{\{T = \infty\}} + M_{n \wedge T} \, \mathbf{1}_{\{T < \infty\}}\,;$$

taking expectation on both sides yields

$$E(M_0) = q = E(M_n \, \mathbf{1}_{\{T = \infty\}}) + E(M_{n \wedge T} \, \mathbf{1}_{\{T < \infty\}}).$$

Since $0 \leq M_{n \wedge T} \leq 1$ a.s., we are entitled to apply the dominated convergence theorem to see that when $n \to \infty$, $E(M_{n \wedge T} \, \mathbf{1}_{\{T < \infty\}}) \to E(M_T \, \mathbf{1}_{\{T < \infty\}}) = P(T < \infty) = q$, and $E(M_n \, \mathbf{1}_{\{T = \infty\}}) \to E(M_\infty \, \mathbf{1}_{\{T = \infty\}})$. Accordingly,

$$E(M_\infty \, \mathbf{1}_{\{T = \infty\}}) = 0\,.$$

The a.s. non-negativity of the integrand yields that it vanishes a.s. Hence $\lim_{n \to \infty} q^{Z_n} \, \mathbf{1}_{\{T = \infty\}} = 0$ a.s., as desired.

(B2) For almost all $\omega \in \{T = \infty\}$, we have $\lim_{n\to\infty} q^{Z_n(\omega)} = 0$ (by (B1)), hence $\lim_{n\to\infty} Z_n(\omega) = \infty$. This yields $\mathbf{1}_{\{T=\infty\}} \le \mathbf{1}_{\{\lim_{n\to\infty} Z_n=\infty\}}$ a.s. The converse being trivial, we obtain $\mathbf{1}_{\{T=\infty\}} = \mathbf{1}_{\{\lim_{n\to\infty} Z_n=\infty\}}$ a.s., as desired.

In particular, almost surely on the survival of the system, the size of the population goes to infinity, whereas on the extinction of the system, the size eventually vanishes by definition. So $P(Z_n \text{ converges}) = 0$.

(B3) Clearly, $(W_n,\, n \in \mathbb{N})$ is $(\mathscr{F}_n)$-adapted, with $E(W_n) = 1$, $\forall n \in \mathbb{N}$ (which is easily seen by induction). We have seen in (A1) that $E(Z_{n+1} \,|\, \mathscr{F}_n) = m\, Z_n$, which is equivalent to saying that $E(W_{n+1} \,|\, \mathscr{F}_n) = W_n$: $(W_n,\, n \in \mathbb{N})$ is indeed an $(\mathscr{F}_n)$-martingale.

(B4) By definition,

$$\begin{aligned}
E(Z_{n+1}^2 \,|\, \mathscr{F}_n) &= E\Big[\sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \Big(\sum_{j=1}^{k} X_{n,\,j}\Big)^2 \,\Big|\, \mathscr{F}_n\Big] \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \sum_{j_1=1}^{k}\sum_{j_2=1}^{k} E(X_{n,\,j_1} X_{n,\,j_2} \,|\, \mathscr{F}_n) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \sum_{j_1=1}^{k}\sum_{j_2=1}^{k} E(X_{n,\,j_1} X_{n,\,j_2}) \\
&= \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \Big(k(\sigma^2+m^2) + k(k-1)m^2\Big),
\end{aligned}$$

which is $\sigma^2\, Z_n + m^2\, Z_n^2$, as desired.

Taking expectation on both sides and dividing by $m^{2(n+1)}$, we see that

$$E(W_{n+1}^2) = E(W_n^2) + \frac{\sigma^2}{m^{2+n}}\,.$$

The sequence (u_n) defined by $u_0 := 1$ and $u_{n+1} = u_n + \frac{\sigma^2}{m^{2+n}}$ (for all $n \in \mathbb{N}$) converges (to $1 + \frac{\sigma^2}{m(m-1)}$; recalling that $m > 1$ by assumption), and is thus bounded. In other words,

$$\sup_n E(W_n^2) < \infty\,.$$

The martingale (W_n) being bounded in L^2, it converges (a.s. and) in L^2.

(B5) By definition,

$$E(\mathrm{e}^{-\lambda\, m\, W_{n+1}}) = E[(\mathrm{e}^{-\lambda/m^n})^{Z_{n+1}}] = \phi_{n+1}(\mathrm{e}^{-\lambda/m^n}) = \phi\Big(\phi_n(\mathrm{e}^{-\lambda/m^n})\Big)\,,$$

which is $= \phi(E[\mathrm{e}^{-\lambda Z_n/m^n}]) = \phi(E[\mathrm{e}^{-\lambda W_n}])$. We let $n \to \infty$. By continuity of ϕ and the dominated convergence theorem, we obtain: for $\lambda \ge 0$,

$$L(\lambda\, m) = \phi(L(\lambda)).$$

(B6) We write

$$L(\lambda) = E(\mathbf{1}_{\{W_\infty=0\}}) + E(\mathrm{e}^{-\lambda W_\infty}\,\mathbf{1}_{\{W_\infty>0\}}),$$

and let $\lambda \to \infty$. By dominated convergence, $E(\mathrm{e}^{-\lambda W_\infty}\,\mathbf{1}_{\{W_\infty>0\}}) \to 0$, $\lambda \to \infty$, so that $P(W_\infty = 0) = \lim_{\lambda\to\infty} L(\lambda)$.

(B7) Let $\ell := \lim_{\lambda\to\infty} L(\lambda)$. By (B5), $\phi(\ell) = \ell$ (recalling the continuity of ϕ); hence $\ell \in \{q,\, 1\}$ (recalling that $m > 1$ by assumption). However, ℓ cannot be 1 because (W_n) is uniformly integrable (see (B4)) under the assumption $\sigma^2 < \infty$. Therefore, $P(W_\infty = 0) = q$.

Let us prove $\mathbf{1}_{\{W_\infty>0\}} = \mathbf{1}_{\{T=\infty\}}$ a.s. Since $\{W_\infty > 0\} \subset \{Z_n \to \infty\}$, we have $\mathbf{1}_{\{T=\infty\}} - \mathbf{1}_{\{W_\infty>0\}} \geq 0$ a.s. This non-negative random variable having expectation $E(\mathbf{1}_{\{T=\infty\}} - \mathbf{1}_{\{W_\infty>0\}}) = (1-q) - (1-q) = 0$, it vanishes a.s., proving the desired almost sure identity.

On $\{T = \infty\}$ (which is a.s. $\{W_\infty > 0\}$), we have $Z_n \sim m^n W_\infty$ a.s.

(C1) By assumption, $m < 1$, so $T < \infty$ a.s. (see (A3)). This means that $W_n = 0$ a.s. for all sufficiently large n. In particular, $W_n \to W_\infty := 0$ a.s.

(C2) Since $E(W_\infty) = 0 < E(W_n)$, $\forall n$, it follows that the martingale is not uniformly integrable: there is no convergence in L^1 for (W_n).

(D1) Recall from (A1) that $E(r^{Z_{n+1}} \,|\, \mathscr{F}_n) = \phi(r)^{Z_n}$ a.s., i.e.,

$$\sum_{j\in\mathbb{N}} r^j E(\mathbf{1}_{\{Z_{n+1}=j\}} \,|\, \mathscr{F}_n) = \sum_{j\in\mathbb{N}} r^j \sum_{k\in\mathbb{N}} \mathbf{1}_{\{Z_n=k\}} \alpha_j(k)\,,$$

where $\alpha_j(k)$ is the coefficient of r^j in the power series ϕ^k. Consequently,

$$E(\mathbf{1}_{\{Z_{n+1}=j\}} \,|\, \mathscr{F}_n) = \alpha_j(Z_n)\,, \qquad \text{a.s.}$$

By definition, $\phi(r)^k = \sum_{j\in\mathbb{N}} \xi^{*k}(j) r^j$, where ξ^{*k} is the k-th fold convolution of ξ (with $\xi^{*k} := \delta_0$ if $k = 0$). Hence $\alpha_j(k) = \xi^{*k}(j)$, and

$$E(\mathbf{1}_{\{Z_{n+1}=j\}} \,|\, \mathscr{F}_n) = \xi^{*Z_n}(j) = p(Z_n,\, j)\,.$$

In words, $(Z_n,\, n \in \mathbb{N})$ is a Markov chain with transition kernel $p(x,\, y) := \xi^{*x}(y)$, for x, $y \in \mathbb{N}$, in agreement with what was seen in the class.

(D2) Let $x \in \mathbb{N}^*$. We have $P_x(T_0 < \infty) \geq p(x,\, 0) = \xi(0)^x > 0$, so state x leads to state 0; on the other hand, state x is not accessible from state 0 which is absorbant. We conclude that 0 is the only recurrent state (so $\{0\}$ is the only recurrence class); all other states are transient.

(D3) Starting from transient state x, consider T_0, the first hitting time of any recurrent state. There are two possibilities.

First possibility: $T_0 < \infty$, in which case the chain stays absorbed at state 0 after T_0: $Z_k = 0$ for all $k \geq T_0$.

Second (and last) possibility: $T_0 = \infty$, in which case the classification theorem tells us that the chain, though never hitting 0, visits each of the transient states only a finite number of times P_x-a.s. So for P_x-almost all ω and for all $N \in \mathbb{N}^*$, there exists $n_0(\omega) < \infty$ such that $Z_n \notin \{1, 2, \cdots, N\}$, $\forall n \geq n_0(\omega)$. This yields that P_x-almost surely on $\{T_0 = \infty\}$, $\lim_{n\to\infty} Z_n = \infty$.

As a consequence,

$$P(\exists n \in \mathbb{N}, \forall k \geq n, Z_k = 0) + P\Big(\lim_{n\to\infty} Z_n = \infty\Big) = 1,$$

as desired. $\square$