On the speed of a cookie random walk

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Abstract

We consider the model of the one-dimensional cookie random walk when the initial cookie distribution is spatially uniform and the number of cookies per site is finite. We give a criterion to decide whether the limiting speed of the walk is non-zero. In particular, we show that a positive speed may be obtained for just 3 cookies per site. We also prove a result on the continuity of the speed with respect to the initial cookie distribution.

Keywords. Law of large numbers, cookie or multi-excited random walk, branching process with migration

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1 Introduction

We consider the model of the multi-excited random walk, also called cookie random walk, introduced by Zerner in [13] as a generalization of the model of the excited random walk described by Benjamini and Wilson in [3] (see also Davis [4] for a continuous time analogue). The aim of this paper is to study under which conditions the speed of a cookie random walk is strictly positive. In dimension $d \geq 2$, this problem was solved by Kozma [7, 8], who proved that the speed is always non-zero. In the one-dimensional case, the speed can either be zero or strictly positive. We give here a necessary and sufficient condition to determine if the walk's speed is strictly positive when the initial cookie environment is deterministic, spatially uniform and with a finite number of cookies per site. Let us start with an informal definition of such a process:

Let us put $M \geq 1$ "cookies" at each site of $\mathbb{Z}$ and let us pick $p_1, p_2, \dots, p_M \in [\frac{1}{2}, 1)$. We say that p_i represents the "strength" of the i^{th} cookie at any given site. Then, a cookie random walk $X = (X_n)_{n\geq 0}$ is simply a nearest neighbour random walk, eating the cookies it finds along its path by behaving in the following way:

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- If $X_n = x$ and there is no remaining cookie at site x, then X jumps at time $n+1$ to $x+1$ or $x-1$ with equal probability $\frac{1}{2}$.
- If $X_n = x$ and there remain the cookies with strengths $p_j, p_{j+1}, \dots, p_M$ at this site, then X eats the cookie with attached strength p_j (which therefore disappears from this site) and then jumps at time $n+1$ to $x+1$ with probability p_j and to $x-1$ with probability $1-p_j$.

This model is a particular case of self-interacting random walk: the position of X at time $n+1$ depends not only of its position at time n but also on the number of previous visits to its present site. Therefore, X is not a Markov process.

Let us now give a formal description of the general model. We define the set of cookie environments by $\Omega = [\frac{1}{2}, 1]^{\mathbb{N}^* \times \mathbb{Z}}$. Thus, a cookie environment is of the form $\omega = (\omega(i,x))_{i \geq 1, x \in \mathbb{Z}}$ where $\omega(i,x)$ represents the strength of the i^{th} cookie at site x. Given $x \in \mathbb{Z}$ and $\omega \in \Omega$, a cookie random walk starting from x in the cookie environment ω is a process $(X_n)_{n \geq 0}$ on some probability space $(\mathbf{\Omega}, \mathcal{F}, \mathbf{P}_{\omega,x})$ such that:

$$\begin{cases} \mathbf{P}_{\omega,x}\{X_0 = z\} = 1, \\ \mathbf{P}_{\omega,x}\{|X_{n+1} - X_n| = 1\} = 1, \\ \mathbf{P}_{\omega,x}\{X_{n+1} = X_n + 1 \mid X_1, \dots, X_n\} = \omega(j, X_n) \text{ where } j = \sharp\{0 \leq i \leq n \ , \ X_i = X_n\}. \end{cases}$$

In this paper, we restrict our attention to the set of environments $\Omega_M^u \subset \Omega$ which are spatially uniform with at most $M \geq 1$ cookies per site:

$$\omega \in \Omega_M^u \quad \Longleftrightarrow \quad \begin{cases} \text{for all } x \in \mathbb{Z} \text{ and all } i \geq 1 \ \omega(i,x) = \omega(i,0), \\ \text{for all } i > M \ \omega(i,0) = \frac{1}{2}, \\ \text{for all } i \geq 1 \ \omega(i,0) < 1. \end{cases}$$

The last condition $\omega(i,0) < 1$ is introduced only to exclude some possible degenerate cases but can be relaxed (see Remark 2.4). A cookie environment $\omega \in \Omega_M^u$ may be represented by $(M, \bar{p})$ where

$$\bar{p} = (p_1, \dots, p_M) = (\omega(1,0), \dots, \omega(M,0)).$$

In this case, we shall say that the associated cookie random walk is an $(M, \bar{p})$-cookie random walk and we will use the notation $\mathbf{P}_{(M,\bar{p})}$ instead of $\mathbf{P}_\omega$.

The question of the recurrence or transience of a cookie random walk was solved by Zerner in [13] for general cookie environments (even in the case where the initial cookie environment may itself be random). In particular, he proved that, if X is an $(M, \bar{p})$-cookie random walk, there is a phase transition according to the value of

$$\alpha = \alpha(M, \bar{p}) \stackrel{\text{def}}{=} \sum_{i=1}^{M} (2p_i - 1) - 1. \tag{1}$$

- If $\alpha \leq 0$ then the walk is recurrent *i.e.* $\limsup X_n = -\liminf X_n = +\infty$ a.s.
- If $\alpha > 0$ then X is transient toward $+\infty$ *i.e.* $\lim X_n = +\infty$ a.s.

In particular, for $M = 1$, the cookie random walk is always recurrent for any choice of $\bar{p}$. However, as soon as $M \geq 2$, the cookie random walk can either be transient or recurrent, depending on $\bar{p}$. Zerner [13] also proved that the speed of a $(M, \bar{p})$-cookie random walk X is always well defined (but may or may not be zero). Precisely,

- there exists a constant $v(M,\bar{p}) \geq 0$ such that

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v(M,\bar{p}) \quad \mathbf{P}_{(M,\bar{p})}\text{-almost surely.}$$

- The speed is monotonic in $\bar{p}$: if $\bar{p} = (p_1, \dots, p_M)$ and $\bar{q} = (q_1, \dots, q_M)$ are two cookie environments such that $p_i \leq q_i$ for all i, then $v(M,\bar{p}) \leq v(M,\bar{q})$.

- The speed of a $(2,\bar{p})$-cookie random walk is always 0.

The question of whether one can construct a $(M,\bar{p})$-cookie random walk with strictly positive speed was affirmatively answered by Mountford, Pimentel and Valle [9] who considered the case where all the cookies have the same strength $p \in [\frac{1}{2}, 1)$ *i.e.* the cookie vector $\bar{p}$ has the form $[p]_M \stackrel{\text{def}}{=} (p, \dots, p)$. They showed that:

- For any $p \in (\frac{1}{2}, 1)$, there exists an M_0 such that for all $M > M_0$ the speed of the $(M, [p]_M)$-cookie random walk is strictly positive.

- If $M(2p-1) < 2$, then the speed of the $(M, [p]_M)$-cookie random walk is zero.

They also conjectured that when $M(2p-1) > 2$, the speed should be non-zero. The aim of this paper is to prove that such is indeed the case.

Theorem 1.1. *Let X denote a $(M,\bar{p})$-cookie random walk, then*

$$\lim_{n\to\infty} \frac{X_n}{n} = v(M,\bar{p}) > 0 \quad \Longleftrightarrow \quad \alpha(M,\bar{p}) > 1$$

where $\alpha(M,\bar{p})$ is given by (1).

In particular, we see that a non-zero speed may be achieved for as few as 3 cookies per site. Comparing this result with the transience/recurrence criteria, we have a second order phase transition at the critical value $\alpha = 1$. In fact, it is proved in [2] that, in the zero speed case $0 < \alpha < 1$, the rate of transience of X_n is of order $n^{\frac{\alpha+1}{2}}$.

One would certainly like an explicit calculation of the limiting velocity in term of the cookie environment $(M,\bar{p})$ but this seems a challenging problem (one can still look at Corollary 3.7 where we give an implicit formula for the speed). However, one can prove that the speed is continuous in $\bar{p}$ and has a positive right derivative at all its critical points:

Theorem 1.2.

- *For each M, the speed $v(M,\bar{p})$ is a continuous function of $\bar{p}$ in Ω^u_M.*

- *For any environment $(M,\bar{p}_c)$ with $\alpha(M,\bar{p}_c) = 1$, there exists a constant $C > 0$ (depending on $(M,\bar{p}_c)$) such that*

$$\lim_{\substack{\bar{p}\to\bar{p}_c \\ \bar{p}\in\Omega^u_M \\ \alpha(\bar{p})>1}} \frac{v(M,\bar{p})}{\alpha(M,\bar{p}) - 1} = C.$$

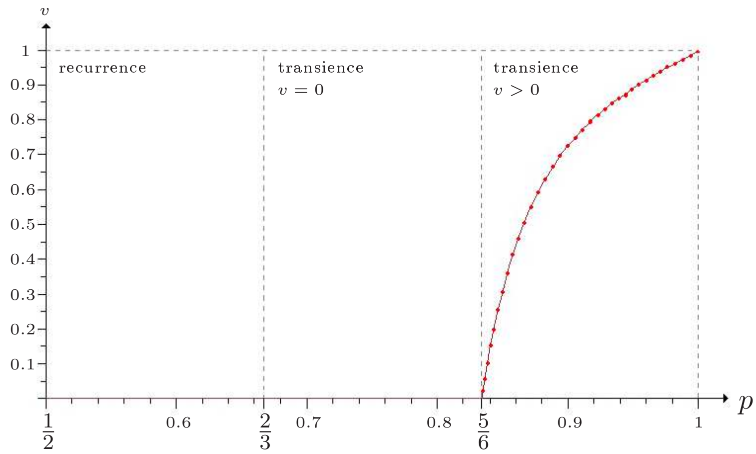

Figure 1: Simulation of the speed of a $(3, [p]_3)$-cookie random walk.

In particular, for $M \geq 3$, the (unique) critical value for an $(M, [p]_M)$-cookie random walk is $p_c = \frac{1}{M} + \frac{1}{2}$ and the function $v(p)$ is continuous, non-decreasing, zero for $p \leq p_c$, and admits a finite strictly positive right derivative at p_c (see figure 1).

The remainder of this paper is organized as follow. In the next section, we construct a Markov process associated with the hitting time of the cookie random walk. The method is similar to that used by Kesten, Kozlov and Spitzer [6] for the determination of the rates of transience of a random walk in a one-dimensional random environment. It turns out that, in our setting, the resulting process is a branching process with random migration. The study of this process and of its stationary distribution is undertaken in Section 3. This enables us to complete the proof of Theorem 1.1. Finally, the last section is dedicated to the proof of Theorem 1.2.

2 An associated branching process with migration

In the remainder of this paper, $X = (X_n)_{n \geq 0}$ will denote a $(M, \bar{p})$-cookie random walk. Since the speed of a recurrent cookie random walk is zero, we shall also assume that we are in the transient regime *i.e.*

$$\alpha(M, \bar{p}) = \sum_{i=1}^{M} (2p_i - 1) - 1 > 0. \tag{2}$$

For the sake of brevity, we simply write $\mathbf{P}_x$ for $\mathbf{P}_{(M,\bar{p}),x}$ and $\mathbf{P}$ instead of $\mathbf{P}_0$ (the process starting from 0). Let T_n stand for the hitting time of level $n \geq 0$ by X:

$$T_n = \inf(k \geq 0 \ , \ X_k = n). \tag{3}$$

For $0 \leq k \leq n$, let U_i^n denote the number of jumps of the cookie random walk from site i to site $i-1$ before reaching level n

$$U_i^n = \sharp\{0 \leq k < T_n, \ X_k = i \text{ and } X_{k+1} = i - 1\}.$$

Let K_n stand for the total time spent by X in the negative half-line up to time T_n

$$K_n = \sharp\{0 \leq k \leq T_n,\ X_k < 0\}.$$

A simple combinatorial argument readily yields

$$T_n = K_n - U_0^n + n + 2\sum_{k=0}^{n} U_k^n.$$

Notice that, as n tends to infinity, the random variable K_n increases toward K_∞, the total time spent by the cookie random walk in the negative half line. Similarly, U_0^n increases toward U_0^∞, the total number of jumps from 0 to -1. Since X is transient, $K_\infty + U_0^\infty$ is almost-surely finite and therefore

$$T_n \underset{n\to\infty}{\sim} n + 2\sum_{k=0}^{n} U_k^n. \tag{4}$$

Let us now prove that, for each n, the reverse process $(U_n^n, U_{n-1}^n, \ldots, U_1^n, U_0^n)$ has the same law as the n first steps of some branching process Z with random migration. We first need to introduce some notations. Let $(B_i)_{i\geq 1}$ denote a sequence of independent Bernoulli random variables under $\mathbf{P}$ with distribution:

$$\mathbf{P}\{B_i = 1\} = 1 - \mathbf{P}\{B_i = 0\} = \begin{cases} p_i & \text{if } i \leq M, \\ \frac{1}{2} & \text{if } i > M. \end{cases} \tag{5}$$

For $j \in \mathbb{N}$, define

$$k_j = \min(k \geq 1, \sharp\{1 \leq i \leq k, B_i = 1\} = j+1)$$

and

$$A_j = \sharp\{1 \leq i \leq k_j, B_i = 0\} = k_j - j - 1.$$

We have the following easy lemma:

Lemma 2.1. • *For any $i, j \geq 0$, we have $\mathbf{P}\{A_j = i\} > 0$.*

- *For all $j \geq M$, we have*

$$A_j \overset{law}{=} A_{M-1} + \xi_1 + \ldots + \xi_{j-M+1} \tag{6}$$

where $(\xi_i)_{i\geq 0}$ are i.i.d. random variables independent of A_{M-1} with geometrical distribution starting from 0 and with parameter $\frac{1}{2}$ i.e. $\mathbf{P}\{\xi_1 = i\} = (1/2)^{i+1}$.

Proof. The first part of the lemma is a direct consequence of the assumption that $\bar{p}$ is such that $p_k < 1$ for all k. To prove the second part, we simply notice that $k_{M-1} \geq M$ so that, for $j \geq M$, the random variable $A_j - A_{M-1}$ has the same law as the random variable

$$\min(k \geq 1, \sharp\{1 \leq i \leq k, \widetilde{B}_i = 1\} = j+1-M) - j - 1 + M \tag{7}$$

where $(\widetilde{B}_i)_{i\geq 0}$ is a sequence of i.i.d. random variables independent of A_{M-1}, with common Bernoulli distribution $\mathbf{P}\{\widetilde{B}_i = 0\} = \mathbf{P}\{\widetilde{B}_i = 1\} = \frac{1}{2}$. It is clear that (7) has the same law as $\xi_1 + \ldots + \xi_{j-M+1}$. $\square$

We now consider a process $Z = (Z_n, n \geq 0)$ and a family of probabilities $(\mathbb{P}_z)_{z\geq 0}$ such that, under $\mathbb{P}_z$, the process Z is a Markov chain starting from z, with transition probabilities:

$$\begin{cases} \mathbb{P}_z\{Z_0 = z\} = 1, \\ \mathbb{P}_z\{Z_{n+1} = k \mid Z_n = j\} = \mathbf{P}\{A_j = k\}. \end{cases}$$

Since the family of probabilities $(\mathbb{P}_z)$ depends on the law of the cookie environment $(M, \bar{p})$, we should rigourously write $\mathbb{P}_{(M,\bar{p}),z}$ instead of $\mathbb{P}_z$. However, when there is no possibility of confusion, we shall keep using the abbreviated notation. Furthermore, we simply write $\mathbb{P}$ for $\mathbb{P}_0$ and $\mathbb{E}$ stands for the expectation with respect to $\mathbb{P}$.

Let us now notice that, in view of the previous lemma, Z_n under $\mathbb{P}_z$ may be interpreted as the number of particles alive at time n of a branching process with random migration starting from z, that is a branching process which allows immigration and emigration (see Vatutin and Zubkov [11] for a survey of these processes). Indeed:

- If $Z_n = j \geq M-1$ then, according to Lemma 2.1, Z_{n+1} has the same law as $\sum_{k=1}^{j-M+1} \xi_k + A_{M-1}$, *i.e.* $M-1$ particles emigrate and the remaining particles reproduce according to a geometrical law with parameter $\frac{1}{2}$ and there is also an immigration of A_{M-1} new particles.

- If $Z_n = j \in \{0, \dots, M-2\}$ then Z_{n+1} has the same law as A_j *i.e.* all the j particles emigrate and A_j new particles immigrate.

We can now state the main result of this section:

Proposition 2.2. *For each $n \in \mathbb{N}$, $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ under $\mathbf{P}$ has the same law as $(Z_0, Z_1, \dots, Z_n)$ under $\mathbb{P}$.*

Proof. The argument is similar to the one given by Kesten *et al.* in [6]. Recall that U_i^n represents the numbers of jumps of the cookie random walk X from i to $i-1$ before reaching n. Then, conditionally on $(U_n^n, U_{n-1}^n, \dots, U_{i+1}^n)$, the number of jumps U_i^n from i to $i-1$ depends only on the number of jumps from $i+1$ to i, that is, depends only on U_{i+1}^n. This shows that $(U_n^n, U_{n-1}^n, \dots, U_0^n)$ is indeed a Markov process.

By definition, $Z_0 = 0$ $\mathbb{P}$-a.s. and $U_n^n = 0$ $\mathbf{P}$-a.s. It remains to compute $\mathbf{P}\{U_i^n = k \mid U_{i+1}^n = j\}$. Note that the number of jumps from i to $i-1$ before reaching level n is equal to the number of jumps from i to $i-1$ before reaching $i+1$ for the first time plus the sum of the number of jumps from i to $i-1$ between two consecutive jumps from $i+1$ to i which occur before reaching level n. Thus, conditionally on $\{U_{i+1}^n = j\}$, the random variable U_i^n has the same law as the number of failures (*i.e.* $B_k = 0$) in the Bernoulli sequence $(B_1, B_2, B_3, \dots)$ defined by (5) before obtaining exactly $j+1$ successes. This is precisely the definition of A_j and therefore $\mathbf{P}\{U_i^n = k \mid U_{i+1}^n = j\} = \mathbb{P}_j\{Z_1 = k\}$. $\square$

Since U_0^n is the number of jumps from 0 to -1 of the cookie random walk X before reaching level n and since we assumed that the cookie random walk X is transient, U_0^n increases almost surely toward the total number U_0^∞ of jumps of X from 0 to -1. In view of the previous proposition, this implies that under $\mathbb{P}$, Z_n converges in law toward a random variable which we denote by Z_∞.

Let us also note that Z is an irreducible Markov chain (this is a consequence of part 1 of Lemma 2.1). Since Z converges in law toward a limiting distribution, this shows that

Z is in fact a positive recurrent Markov chain. In particular, Z_n converges in law toward Z_∞ independently of its starting point (*i.e.* the law of Z_∞ is the same under any $\mathbb{P}_x$) and the law of Z_∞ is also the unique invariant probability for Z.

Corollary 2.3. *Recall that $v(M,\bar{p})$ denotes the limiting speed of the cookie random walk X. We have*

$$v(M,\bar{p}) = \frac{1}{1+2\mathbb{E}[Z_\infty]} \qquad \textit{(with the convention } 0 = \tfrac{1}{+\infty}\textit{).}$$

In particular, the speed of an $(M,\bar{p})$-cookie random walk is non zero i.i.f. the limiting random variable Z_∞ of its associated process Z has a finite expectation.

Proof. Since X is transient, we have the well known equivalence valid for $v \in [0,\infty]$:

$$\frac{X_n}{n} \underset{n\to\infty}{\longrightarrow} v \quad \mathbf{P}\text{-a.s.} \qquad \Longleftrightarrow \qquad \frac{T_n}{n} \underset{n\to\infty}{\longrightarrow} \frac{1}{v} \quad \mathbf{P}\text{-a.s.} \tag{8}$$

On the one hand, this equivalence and (4) yield

$$\frac{1}{n}\sum_{k=0}^{n} U_k^n \underset{n\to\infty}{\longrightarrow} \frac{1}{2v(M,\bar{p})} - \frac{1}{2} \quad \mathbf{P}\text{-a.s.} \tag{9}$$

On the other hand, making use of an ergodic theorem for the positive recurrent Markov chains Z with stationary limiting distribution Z_∞ (see for instance Theorem 1.10.2 on p53 of [10]), we find that

$$\frac{1}{n}\sum_{i=1}^{n} Z_k \underset{n\to\infty}{\to} \mathbb{E}[Z_\infty] \quad \mathbb{P}\text{-a.s.} \tag{10}$$

(note that this result is valid even if $\mathbb{E}[Z_\infty] = \infty$). Proposition 2.2 implies that the limits in (9) and (10) are the same. This completes the proof of the corollary. $\square$

Remark 2.4. *We assumed in the definition of an $(M,\bar{p})$ cookie environment that*

$$p_i \neq 1 \qquad \textit{for all } 1 \le i \le M.$$

This hypothesis is intended only to ensure that Z, starting from 0, is not almost surely bounded (for instance, if $p_1 = 1$ then 0 is a absorbing state for Z). More generally, one may check from the definition of the random variables A_j that Z starting from 0 is almost surely unbounded i.i.f.

$$\sharp\{1 \le j \le i\ ,\ p_j = 1\} \le \frac{i}{2} \quad \textit{for all } 1 \le i \le M. \tag{11}$$

When this condition fails, Z starting from 0 is almost surely bounded by $M-1$, thus $\mathbb{E}[Z_\infty] < \infty$ and the speed of the associated cookie random walk is strictly positive. Otherwise, when (11) is fulfilled, Z ultimately hits any level $x \in \mathbb{N}$ with probability 1 and the proof of Theorem 1.1 below remains valid.

3 Study of Z_∞

We proved in the previous section that the strict positivity of the speed of the cookie random walk X is equivalent to the existence of a finite first moment for the limiting distribution of its associated Markov chain Z. We shall now show that, for any cookie environment $(M,\bar{p})$ (with $\alpha(M,\bar{p})>0$), we have

$$\mathbb{E}[Z_\infty] \stackrel{\text{def}}{=} \mathbb{E}_{(M,\bar{p})}[Z_\infty] < \infty \qquad \Longleftrightarrow \qquad \alpha(M,\bar{p}) > 1.$$

This will complete the proof of Theorem 1.1. We start by proving that Z_∞ cannot have moments of any order.

Proposition 3.1. *We have*

$$\mathbb{E}\left[Z_\infty^{M-1}\right] = +\infty.$$

Proof. Let us introduce the first return time to 0 for Z:

$$\sigma = \inf(n \geq 1\ ,\ Z_n = 0).$$

Since Z is a positive recurrent Markov chain, we have $1 \leq \mathbb{E}_0[\sigma] < \infty$ and the invariant probability measure is given for any $y \in \mathbb{N}$ by

$$\mathbb{P}\{Z_\infty = y\} = \frac{\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} \mathbf{1}_{\{Z_k=y\}}\right]}{\mathbb{E}_0[\sigma]}.$$

A monotone convergence argument yields

$$\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] = \mathbb{E}_0[\sigma]\mathbb{E}[Z_\infty^{M-1}] \tag{12}$$

(where both side of this equality may be infinite). We can find $n_0 \in \mathbb{N}^*$ such that $\mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\} > 0$ (in fact, since we assume that $p_i < 1$ for all i, we can choose $n_0 = 1$). Therefore, making use of the Markov property of Z, we find that

$$\begin{aligned}
\mathbb{E}_0\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] &\geq \mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\}\mathbb{E}_M\left[\sum_{k=0}^{\sigma-1} Z_k^{M-1}\right] \\
&= \mathbb{P}_0\{Z_{n_0} = M,\ n_0 < \sigma\}\sum_{k=0}^{\infty} \mathbb{E}_M\left[Z_{k\wedge\sigma}^{M-1}\right]. \qquad (13)
\end{aligned}$$

In view of (12) and (13), we just need to prove that

$$\sum_{k=0}^{\infty} \mathbb{E}_M\left[Z_{k\wedge\sigma}^{M-1}\right] = \infty. \tag{14}$$

We now use a coupling argument. Let again $(\xi_i)_{i\geq 1}$ denote a sequence of i.i.d. geometrical random variables with parameter $1/2$. We define an inhomogeneous Markov chain $\widetilde{Z}$ such that, under $\mathbb{P}_z$:

- $\widetilde{Z}_0 = z$.
- $\widetilde{Z}_1$ has the same law as $\sum_{i=1}^{\widetilde{Z}_0} \xi_i$.
- For $n \geq 1$, $\widetilde{Z}_{n+1}$ has the same law as $\sum_{i=1}^{\min(0,\widetilde{Z}_n-(M-1))} \xi_i$ (with the convention $\sum_1^0 = 0$).

Thus, $\widetilde{Z}$ is a branching process with $\min(\widetilde{Z}_n, M-1)$ emigrants at each unit of time, except at time $n = 0$ where no emigration occurs.

Recall that Z is a branching process with migration, where at most $\min(Z_n, M-1)$ particles emigrate at each unit of time, and has the same offspring reproduction law as $\widetilde{Z}$. Therefore, for any $z \geq 0$, the process $\widetilde{Z}$ under $\mathbb{P}_z$ is stochastically dominated by Z under $\mathbb{P}_{z+M-1}$ (we need to shift the starting point by $M-1$ because $\widetilde{Z}$ has no emigration at time $n = 0$). Since 0 is an absorbing state for $\widetilde{Z}$, this implies that, for all $n \geq 0$,

$$\mathbb{E}_1[\widetilde{Z}_n^{M-1}] \leq \mathbb{E}_M[Z_{n\wedge\sigma}^{M-1}]. \tag{15}$$

The process $\widetilde{Z}$ belongs to the class of processes studied by Vinokurov in [12]. Moreover, all the assumptions of Theorem 2 and 3 of [12] are fulfilled (in the notation of [12], we have $\theta = M-1$). Therefore, there exist two constants $c_1, c_2 > 0$, such that, as n tends to infinity,

$$\mathbb{P}_1\{\widetilde{Z}_n > 0\} \sim \frac{c_1}{n^M} \quad \text{and} \quad \mathbb{P}_1\{\widetilde{Z}_n > n \mid \widetilde{Z}_n > 0\} \sim c_2.$$

Thus

$$\begin{aligned}\mathbb{E}_1[\widetilde{Z}_n^{M-1}] &= \mathbb{E}_1[\widetilde{Z}_n^{M-1} \,|\widetilde{Z}_n > 0]\mathbb{P}_1\{\widetilde{Z}_n > 0\} \\ &\geq n^{M-1}\mathbb{P}_1\{\widetilde{Z}_n > n \mid \widetilde{Z}_n > 0\}\mathbb{P}_1\{\widetilde{Z}_n > 0\} \sim \frac{c_1 c_2}{n}. \end{aligned} \tag{16}$$

The combination of (15) and (16) yields (14). □

Remark 3.2. *In view of the last proposition and Corollary 2.3, we recover the fact that for $M = 2$, the speed of the cookie random walk is always zero.*

In order to study more precisely the distribution of Z_∞, we need the following lemma:

Lemma 3.3. *We have*

$$\mathbf{E}\left[A_{M-1}\right] = 2\sum_{i=1}^{M}(1-p_i).$$

Proof. Recall that $(B_i)_{i\geq 1}$ denotes a sequence of independent Bernoulli random variables with distribution given by (5). Let $L = \sharp\{1 \leq i \leq M,\ B_i = 1\} = \sum_{i=1}^{M} B_i$, we have

$$\mathbf{E}[L] = \sum_{i=1}^{M} p_i.$$

Recall also that A_{M-1} denotes the number of failures in the sequence $(B_i)_{i\geq 1}$ before obtaining M successes. Furthermore, $M - L$ represents the number of failures in the subsequence $(B_i)_{1\leq i\leq M}$. So we may rewrite A_{M-1} in the form

$$\begin{aligned} A_{M-1} &= M-L+\Big(\inf\Big\{j\geq 0, \sum_{i=M+1}^{M+j} B_i = M-L\Big\} - (M-L)\Big) \\ &= \inf\Big\{j\geq 0, \sum_{i=M+1}^{M+j} B_i = M-L\Big\} \end{aligned}$$

(with the convention $\sum_{M+1}^{M} = 0$). Therefore, given L, the random variable A_{M-1} represents the number of trials needed to get $M-L$ successes along the unbiased coin tossing sequence $(B_i)_{i\geq M+1}$. Thus, given L, the random variable A_{M-1} has a negative binomial distribution with parameters $M-L$ and $p=1/2$. In particular, we have $\mathbf{E}[A_{M-1}\mid L] = 2(M-L)$ and we conclude that

$$\mathbf{E}[A_{M-1}] = \mathbf{E}[\mathbf{E}[A_{M-1}\mid L]] = \mathbf{E}[2(M-L)] = 2\sum_{i=1}^{M}(1-p_i).$$

$\square$

We now study the law of the limiting distribution Z_∞ of the Markov chain Z. This is done via the study of its probability generating function (p.g.f.)

$$G(s) = \mathbb{E}\left[s^{Z_\infty}\right] \qquad \text{for } s\in[0,1].$$

Lemma 3.4. *The p.g.f. G of Z_∞ is the unique p.g.f. solution of the following equation*

$$1-G\left(\frac{1}{2-s}\right) = a(s)(1-G(s))+b(s) \qquad \textit{for all } s\in[0,1], \tag{17}$$

with

$$a(s) = \frac{1}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]},$$

and

$$b(s) = 1-\frac{1}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]} + \sum_{k=0}^{M-2}\frac{G^{(k)}(0)}{k!}\left(\frac{\mathbf{E}\left[s^{A_k}\right]}{(2-s)^{M-1}\mathbf{E}\left[s^{A_{M-1}}\right]} - \frac{1}{(2-s)^k}\right).$$

Proof. The law of Z_∞ is a stationary distribution for the Markov chain Z, therefore

$$\begin{aligned} G(s) = \mathbb{E}\left[\mathbb{E}_{Z_\infty}\left[s^{Z_1}\right]\right] &= \sum_{k=0}^{\infty}\mathbb{P}\{Z_\infty = k\}\mathbb{E}_k\left[s^{Z_1}\right] \\ &= \sum_{k=0}^{M-2}\mathbb{P}\{Z_\infty = k\}\mathbb{E}_k\left[s^{Z_1}\right] + \sum_{k=M-1}^{\infty}\mathbb{P}\{Z_\infty = k\}\mathbb{E}_k\left[s^{Z_1}\right]. \end{aligned}$$

By the definition of Z, the random variable Z_1 under $\mathbb{P}_k$ has the same law as A_k under $\mathbf{P}$. Moreover, according to Lemma 2.1, for $k\geq M-1$, A_k has the same law as $A_{M-1}+\xi_1+$

$\ldots + \xi_{k-M+1}$ where $(\xi_i)_{i\geq 1}$ is a sequence of i.i.d. random variables independent of A_{M-1} and with geometric distribution with parameter $\frac{1}{2}$. Thus,

$$\begin{aligned}
G(s) &= \sum_{k=0}^{M-2} \mathbb{P}\{Z_\infty = k\}\mathbf{E}\left[s^{A_k}\right] + \sum_{k=M-1}^{\infty} \mathbb{P}\{Z_\infty = k\}\mathbf{E}\left[s^{A_{M-1}+\xi_1+\ldots+\xi_{k+1-M}}\right] \\
&= \sum_{k=0}^{M-2} \mathbb{P}\{Z_\infty = k\}\mathbf{E}\left[s^{A_k}\right] + \frac{\mathbf{E}\left[s^{A_{M-1}}\right]}{\mathbf{E}\left[s^{\xi}\right]^{M-1}} \sum_{k=M-1}^{\infty} \mathbb{P}\{Z_\infty = k\}\mathbf{E}\left[s^{\xi}\right]^k \\
&= \sum_{k=0}^{M-2} \mathbb{P}\{Z_\infty = k\}\left(\mathbf{E}\left[s^{A_k}\right] - \mathbf{E}\left[s^{A_{M-1}}\right]\mathbf{E}\left[s^{\xi}\right]^{k+1-M}\right) + \frac{\mathbf{E}\left[s^{A_{M-1}}\right]}{\mathbf{E}\left[s^{\xi}\right]^{M-1}} G\left(\mathbf{E}\left[s^{\xi}\right]\right).
\end{aligned}$$

Since $\mathbf{E}\left[s^{\xi}\right] = \frac{1}{2-s}$, and $k!\mathbb{P}\{Z_\infty = k\} = G^{(k)}(0)$, we get

$$G(s) = \sum_{k=0}^{M-2} \frac{G^{(k)}(0)}{k!}\left(\mathbf{E}\left[s^{A_k}\right] - \mathbf{E}\left[s^{A_{M-1}}\right](2-s)^{M-1-k}\right) + \mathbf{E}\left[s^{A_{M-1}}\right](2-s)^{M-1} G\left(\frac{1}{2-s}\right),$$

from which we deduce that G solves (17).

Furthermore, using the same arguments as above and going backward, we can check that if some p.g.f. satisfies (17), then the associated probability distribution is stationary for the irreductible Markov chain Z. In view of the uniqueness of the stationary distribution, we conclude that G is indeed the unique p.g.f. satisfying equation (17). $\square$

Given two functions f and g, we use the classical notation $f(x) = O(g(x))$ in the neighbourhood of zero if $|f(x)| \leq C|g(x)|$ for some constant C and all $|x|$ small enough.

Lemma 3.5. *The functions a and b of Lemma 3.4 are analytic on $(0,2)$. In particular, they admit a Taylor expansion of any order near point 1 and, as x goes to 0:*

$$\begin{aligned}
a(1-x) &= 1 - \alpha x + O(x^2), \\
b(1-x) &= O(x).
\end{aligned}$$

Proof. Recall the definitions of the random variables A_k given in Section 2. Since a geometric random variable with parameter $\frac{1}{2}$ admits exponential moments of order strictly smaller than 2, it follows that the p.g.f. $s \mapsto \mathbf{E}[s^{A_k}]$ are strictly positive and analytic on $(0,2)$. From the explicit form of the functions a and b given in the previous lemma, we conclude that these two functions are indeed analytic on $(0,2)$. A Taylor expansion of a near 1 gives

$$a(1-x) = 1 - \left(M - 1 - \mathbf{E}[A_{M-1}]\right)x + O(x^2) = 1 - \alpha x + O(x^2), \tag{18}$$

where we used Lemma 3.3 for the last equality. Since G is a p.g.f. we have $G(1) = 1$ which, in view of (17), yields $b(1) = 0$ and therefore $b(1-x) = O(x)$. $\square$

The following proposition relies on a careful study of equation (17) and is the key to the proof of Theorem 1.1.

Proposition 3.6. *Recall that*

$$\alpha = \sum_{i=1}^{M}(2p_i - 1) - 1 > 0.$$

The p.g.f. G of Z_∞ is such that, as $x > 0$ goes to 0:

- *if $0 < \alpha < 1$, then $1 - G(1-x) \sim cx^\alpha$, for some constant $c > 0$.*
  In particular $\mathbf{E}[Z_\infty] = +\infty$.
- *if $\alpha = 1$, then $1 - G(1-x) \sim cx|\ln x|$, for some constant $c > 0$.*
  In particular $\mathbf{E}[Z_\infty] = +\infty$.
- *if $\alpha > 1$, then $1 - G(1-x) = \frac{b''(1)}{2(\alpha-1)}x + O(x^{2\wedge\alpha})$.*
  In particular $\mathbf{E}[Z_\infty] = \frac{b''(1)}{2(\alpha-1)} < +\infty$.

Proof. Since G is a p.g.f, it is completely monotonic and we just need to prove the proposition along the sequence $x = \frac{1}{n}$ with $n \in \mathbb{N}^*$. Making use of Lemma 3.4 with $s = 1 - \frac{1}{n}$, we get, for all $n \geq 1$

$$1 - G\left(1 - \frac{1}{n+1}\right) = a\left(1 - \frac{1}{n}\right)\left(1 - G\left(1 - \frac{1}{n}\right)\right) + b\left(1 - \frac{1}{n}\right).$$

Let us define the sequence $(u_n)_{n\geq 1}$ by

$$\begin{cases} u_1 \stackrel{\text{def}}{=} 1 - G(0) = 1 - \mathbf{P}\{Z_\infty = 0\} > 0, \\ u_n \stackrel{\text{def}}{=} \frac{1-G(1-1/n)}{\prod_{i=1}^{n-1} a(1-1/i)} \qquad \text{for } n \geq 2. \end{cases} \tag{19}$$

We also use the notation

$$r_n \stackrel{\text{def}}{=} \frac{b(1-1/n)}{\prod_{i=1}^{n} a(1-1/i)}.$$

Hence, (u_n) is a sequence of positive numbers which satisfies the relation

$$u_{n+1} = u_n + r_n,$$

thus

$$u_n = u_1 + \sum_{j=1}^{n-1} r_j.$$

This equality may be rewritten

$$1 - G\left(1 - \frac{1}{n}\right) = \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right)\left(1 - G(0) + \sum_{j=1}^{n-1} r_j\right). \tag{20}$$

In view of Lemma 3.5, we can write the Taylor expansion of a of order M near 1 in the form

$$a(1-x) = 1 - \alpha x + a_2 x^2 + \ldots + a_M x^M + O(x^{M+1}).$$

Using the classical result

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma_0 + \ldots + \frac{\gamma_M}{n^M} + O\left(\frac{1}{n^{M+1}}\right),$$

we deduce that

$$\prod_{i=1}^{n} a\left(1-\frac{1}{i}\right) = \frac{C}{n^\alpha}\left(1+\frac{a'_1}{n}+\frac{a'_2}{n^2}+\ldots+\frac{a'_{M-1}}{n^{M-1}}+O\left(\frac{1}{n^M}\right)\right) \quad \text{with } C>0. \tag{21}$$

Lemma 3.5 also states that, when b is not identically 0, there exists a unique $k \in \{1,2,\ldots\}$ such that

$$b(1-x) = D_k x^k + O(x^{k+1}), \qquad \text{with } D_k \neq 0. \tag{22}$$

If b is identically 0, we use the convention $k=+\infty$. In particular, when k is finite, combining (21) and (22) , we deduce that

$$r_n = D_k C^{-1} n^{\alpha-k} + O(n^{\alpha-k-1}). \tag{23}$$

This implies, whenever $\alpha - k > -1$ that

$$\sum_{j=1}^{n-1} r_j = \frac{D_k C^{-1}}{\alpha-k+1} n^{\alpha-k+1} + O(1 \vee n^{\alpha-k}). \tag{24}$$

Let us now assume that $k=1$. Combining (20), (21) and (24) we find that $1-G(1-\frac{1}{n})$ converges towards $\frac{D_1}{\alpha} \neq 0$ as n tends to infinity but this cannot happen because G is continuous at 1^- with $G(1)=1$. Thus, we have shown that in fact

$$k \geq 2.$$

We now consider the three cases $\alpha>1$, $\alpha=1$, $\alpha<1$ separately.

$\boxed{\alpha > 1}$

We have three sub-cases: either $\alpha > k-1$, or $\alpha < k-1$, or $\alpha = k-1$ with $k \geq 3$.

- $\underline{\alpha > k-1}$: Just as before, combining (20), (21) and (24), we now get

$$1 - G\left(1-\frac{1}{n}\right) = \frac{D_k}{(\alpha-k+1)n^{k-1}} + O\left(\frac{1}{n^{k\wedge\alpha}}\right).$$

  If k were strictly larger than 2, we would have

$$\lim_{n\to\infty} n(1-G(1-1/n)) = 0$$

  and therefore $G'(1) = \mathbb{E}[Z_\infty] = 0$ which cannot be true because Z is a positive random variable which is not equal to zero almost surely. Thus k must be equal to 2 and

$$1 - G\left(1-\frac{1}{n}\right) = \frac{D_2}{(\alpha-1)n} + O\left(\frac{1}{n^{2\wedge\alpha}}\right). \tag{25}$$

  Using the equality $D_2 = \frac{b''(1)}{2}$, we conclude that

$$\mathbf{E}[Z_\infty] = \frac{b''(1)}{2(\alpha-1)} < +\infty.$$

- $\underline{\alpha < k-1}$: We prove that this case never happens. Indeed, in view of (23) we find that
$$\sum_{j=1}^{\infty} r_j < \infty$$
(this result also trivially holds when $k = \infty$ since r_j is equally zero in this case). Combining this with (20) and (21) we see that
$$1 - G\left(1 - \frac{1}{n}\right) = O\left(\frac{1}{n^\alpha}\right).$$
Since $\alpha > 1$, this implies, just as in the previous case, that $\mathbb{E}[Z_\infty] = 0$, which is absurd.

- $\underline{\alpha = k-1 \text{ and } k \geq 3}$: Again, we prove that this case is empty. Using (23), we get
$$r_n \sim \frac{D_k C^{-1}}{n}.$$
With the help of (20) and (21), we conclude that
$$1 - G\left(1 - \frac{1}{n}\right) \sim D_k \frac{\ln n}{n^{k-1}}.$$
Since $k \geq 3$, we again obtain $\mathbb{E}[Z_\infty] = 0$, which is unacceptable.

Thus, we have completed the proof of the proposition when $\alpha > 1$ and we proved by the way that k must be equal to 2 and that $b''(1) > 0$.

$\boxed{\alpha = 1}$

We first prove, just as in the previous cases, that $k = 2$. Let us suppose that $k \geq 3$. In view of Lemma 3.5, we can write the Taylor expansion of b of order M near 1 in the form
$$b(1-x) = D_3 x^3 + \ldots + D_M x^M + O(x^{M+1}) \tag{26}$$
where $D_i \in \mathbb{R}$ for $i \in \{3, 4, \ldots, M\}$. Combining (21) and (26) we deduce that
$$\sum_{j=1}^{n-1} r_j = g_0 + \frac{g_1}{n} + \frac{g_2}{n^2} + \ldots + \frac{g_{M-2}}{n^{M-2}} + O\left(\frac{1}{n^{M-1}}\right). \tag{27}$$
Therefore, in view of (20), (21) and (27), we get
$$1 - G\left(1 - \frac{1}{n}\right) = \frac{\lambda_1}{n} + \frac{\lambda_2}{n^2} + \ldots + \frac{\lambda_{M-1}}{n^{M-1}} + O\left(\frac{1}{n^M}\right).$$
Comparing with the Taylor expansion of the p.g.f. G, we conclude that $\mathbf{E}(Z_\infty^{M-1}) < \infty$ which contradicts Proposition 3.1. Thus, $k = 2$ and (23) yields
$$r_n \sim \frac{D_2 C^{-1}}{n} \quad \text{with } D_2 \neq 0. \tag{28}$$

In view of (20) and (21), this estimate implies

$$1 - G\left(1 - \frac{1}{n}\right) \sim D_2 \frac{\ln n}{n},$$

and therefore

$$\mathbb{E}[Z_\infty] = +\infty. \tag{29}$$

$\boxed{\alpha < 1}$

Since $k \geq 2$, equation (23) yields

$$\sum_{j=1}^{\infty} r_j < \infty$$

(of course, this is trivially true when $k = \infty$). Thus, the sequence (u_n) defined by (19) converges to a constant $c_1 \geq 0$. Suppose first that $c_1 = 0$. In this case, k cannot be infinite (because when $k = \infty$, the sequence (u_n) is constant and then $c_1 = u_1 > 0$). From (23) we deduce that

$$u_n = -\sum_{j=n}^{\infty} r_j \sim \frac{D_k C^{-1}}{(k - \alpha - 1)n^{k-\alpha-1}},$$

therefore, with the help of (21) we get

$$1 - G\left(1 - \frac{1}{n}\right) = u_n \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right) \sim \frac{D_k}{(k - \alpha - 1)n^{k-1}}.$$

Since $k \geq 2$, this implies that $n(1 - G(1 - 1/n))$ converges to a finite constant and so $\mathbf{E}[Z_\infty] < \infty$. We have already noticed that this implies a strictly positive speed for the cookie random walk in the associated cookie environment $(M, \bar{p})$. But (by possibly extending the value of M) we can always construct a cookie environment $(M, \bar{q})$ such that $\bar{p} \leq \bar{q}$ and $\alpha(\bar{q}) = 1$. In view of (29), the associated cookie random walk has zero speed and this contradicts a monotonicity result of Zerner (*c.f.* Theorem 17 of [13]). Therefore c_1 cannot be 0 and by (19) and (21), we get

$$1 - G\left(1 - \frac{1}{n}\right) = u_n \prod_{i=1}^{n-1} a\left(1 - \frac{1}{i}\right) \sim \frac{c_1 C}{n^\alpha}.$$

□

Theorem 1.1 is now a direct consequence of the last proposition and Corollary 2.3. Moreover, in view of the expression of $\mathbf{E}[Z_\infty]$ given in the previous proposition, we get the following expression for the limiting speed:

Corollary 3.7. *For any cookie environnement such that $\alpha \geq 1$, we have $b''(1) > 0$ and the speed of the walk is given by the formula*

$$v = \frac{\alpha - 1}{\alpha - 1 + b''(1)}.$$

In view of a classical Abelian/Tauberian Theorem (*c.f.* section XIII.5 of [5]), we also deduce from Proposition 3.6 the following estimate concerning the tail distribution of Z_∞ in the zero speed case:

Corollary 3.8. *When $\alpha \leq 1$, there exists a constant $c > 0$ such that*

$$\mathbb{P}\{Z_\infty > n\} \underset{n\to\infty}{\sim} \begin{cases} c/n^\alpha & \text{if } 0 < \alpha < 1, \\ (c\ln n)/n & \text{if } \alpha = 1. \end{cases} \tag{30}$$

Remark 3.9. *Recall that the random variable Z_∞ has the same distribution as the total number of jumps from 0 to -1 for the cookie random walk. We may also relate this quantity to the total number R of returns to the origin. Indeed, since U_0^n (resp. U_1^n) stands for the respective total number of jumps from 0 to -1 (resp. from 1 to 0) before reaching level n, the total number of returns to the origin before reaching level n is $U_0^n + U_1^n$ which, under $\mathbf{P}$, has the same distribution as $Z_n + Z_{n-1}$ under $\mathbb{P}$. Therefore, we may express the p.g.f. H of the random variable R in term of the p.g.f. G of Z_∞:*

$$\begin{aligned} H(s) &= \mathbb{E}\left[s^{Z_\infty}\mathbb{E}_{Z_\infty}\left[s^{Z_\infty}\right]\right] \\ &= \frac{1}{a(s)}G\left(\frac{s}{2-s}\right) + \sum_{k=0}^{M-2}\frac{G^{(k)}(0)}{k!}s^k\left(\mathbb{E}\left[s^{A_k}\right] - \frac{1}{a(s)(2-s)^k}\right). \end{aligned}$$

In particular, Proposition 3.6 holds for H and the tail distribution of the total number of returns to the origin when $\alpha \leq 1$ has the same form as in (30).

Remark 3.10. *In the particular case $M = 2$ (there are at most 2 cookies per site), the only unknown in the definition of the function b is $G(0)$. Since we know that $b'(1) = 0$ (*c.f.* the beginning of the proof of Proposition 3.6) we can explicitly calculate $G(0)$, that is the probability that the cookie random walk never jumps from 0 to 1, which is also the probability that the cookie random walk never hits -1. According to the previous remark, we can also calculate the probability that the cookie random walk never returns to 0. Hence, we recover Theorem 18 of [13] in the case of a deterministic cookie environment.*

4 Continuity of the speed and differentiability at the critical point

The aim of this section is to prove Theorem 1.2. Recall that Corollary 3.7 states that

$$v(M,\bar{p}) = \begin{cases} 0 & \text{if } \alpha(M,p) \leq 1, \\ \frac{\alpha-1}{\alpha-1+b''(1)} & \text{if } \alpha(M,p) > 1, \end{cases}$$

where $b''(1)$ stands for the second derivative at point 1 of the function b defined in Lemma 3.4. Furthermore, when $\alpha(M,\bar{p}) = 1$, then $b''(1)$ is strictly positive (*cf.* (28)). Hence, in order to prove Theorem 1.2, we just need to show that $b''(1) = b''_{(M,\bar{p})}(1)$ is a continuous function of $\bar{p}$ in Ω_M^u. It is also clear from the definition of the random variables A_k that the functions

$$\bar{p} \to \left(\mathbf{E}_{(M,\bar{p})}\left[s^{A_k}\right]\right)^{(i)}(1) \quad (i.e. \text{ the } i^{\text{th}} \text{ derivative at point } 1)$$

are continuous in $\bar{p}$ in Ω^u_M for all $k \geq 0$ and all $i \geq 0$ (they are polynomial functions in $p_1, \dots, p_M$). Therefore, it simply remains to prove that, for all $k \geq 0$, the functions

$$\bar{p} \to \mathbb{P}_{(M,\bar{p})}\left\{Z_\infty = k\right\}$$

are continuous in Ω^u_M. The following lemma is based on the monotonicity of the hitting times of a cookie random walk with respect to the environment.

Lemma 4.1. *Let $(M, \bar{p})$ be a cookie environment such that $\alpha(M, \bar{p}) > 0$. Then there exist $\varepsilon > 0$ and $f : \mathbb{N} \mapsto \mathbb{R}_+$ with $\lim_{n\to+\infty} f(n) = 0$ such that*

$$\forall \bar{q} \in B(\bar{p}, \varepsilon)\ \ \forall j \in \mathbb{N}\ \ \forall n \in \mathbb{N} \quad \left|\mathbb{P}_{(M,\bar{q})}\left\{Z_\infty = j\right\} - \mathbb{P}_{(M,\bar{q})}\left\{Z_n = j\right\}\right| \leq f(n),$$

where

$$B(\bar{p}, \varepsilon) = \left\{\bar{q} = (q_1, \dots, q_M),\ \frac{1}{2} \leq q_i < 1,\ \alpha(M, \bar{q}) > 0 \text{ and } \sum_{i=1}^{M} |p_i - q_i| \leq \varepsilon\right\}.$$

Proof. Let us fix $(M, \bar{p})$ with $\alpha(M, \bar{p}) > 0$. For $\varepsilon > 0$, define the vector $\bar{p}^\varepsilon = (p^\varepsilon_1, \dots, p^\varepsilon_M)$ by $p^\varepsilon_i = \max(\frac{1}{2}, p_i - \varepsilon)$. We can choose $\varepsilon > 0$ such that $\alpha(M, \bar{p}^\varepsilon) > 0$. Then, for all $\bar{q} \in B(\bar{p}, \varepsilon)$, we have

$$\bar{p}^\varepsilon \leq \bar{q} \tag{31}$$

(where $\leq$ denotes the canonical partial order on $\mathbb{R}^M$). Let us now pick $\bar{q} \in B(\bar{p}, \varepsilon)$, $j \in \mathbb{N}$ and $n \in \mathbb{N}$. Recall that U^∞_0 denotes the total number of jumps of the cookie random walk from 0 to -1 and

$$\mathbb{P}_{(M,\bar{q})}\{Z_\infty = j\} = \mathbf{P}_{(M,\bar{q})}\{U^\infty_0 = j\} = \mathbf{P}_{(M,\bar{q})}\{X \text{ jumps } j \text{ times from 0 to -1}\},$$

and

$$\begin{aligned}\mathbb{P}_{(M,\bar{q})}\{Z_n = j\} = \mathbf{P}_{(M,\bar{q})}\{U^n_0 = j\}& \\ = \mathbf{P}_{(M,\bar{q})}\{X \text{ jumps } j \text{ times from 0 to -1 before reaching } n\}&.\end{aligned}$$

Hence

$$\begin{aligned}\left|\mathbb{P}_{(M,\bar{q})}\{Z_\infty = j\} - \mathbb{P}_{(M,\bar{q})}\{Z_n = j\}\right| &= \left|\mathbf{P}_{(M,\bar{q})}\{U^\infty_0 = j\} - \mathbf{P}_{(M,\bar{q})}\{U^n_0 = j\}\right| \\ &\leq \mathbf{P}_{(M,\bar{q})}\{U^n_0 \neq U^\infty_0\} \\ &= \mathbf{P}_{(M,\bar{q})}\{A\},\end{aligned} \tag{32}$$

where A is the event "X visits -1 at least once after reaching level n". Recall the notation $\omega = \omega(i, x)_{i\geq 1, x\in\mathbb{Z}}$ for a general cookie environment given in the introduction. Let now $\omega_{X,n}$ denote the (random) cookie-environment obtained when the cookie random walk X hits level n for the first time and shifted by n, *i.e.* for all $x \in \mathbb{Z}$ and $i \geq 1$, if the initial cookie environment is ω, then

$$\omega_{X,n}(i, x) = \omega(j, x + n) \quad \text{ where } j = i + \sharp\{0 \leq k < T_n, X_k = x + n\}.$$

With this notation we have

$$\mathbf{P}_{(M,\bar{q})}\left\{A\right\} = \mathbf{E}_{(M,\bar{q})}\left[\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -(n+1) \text{ at least once}\}\right].$$

Besides, X has not eaten any cookie at the sites $x \geq n$ before time T_n. Thus, the environment $\omega_{X,n}$ satisfies

$$\omega_{X,n}(i,x) = q_i, \quad \text{for all } x \geq 0 \text{ and } i \geq 1 \text{ (with the convention } q_i = \tfrac{1}{2} \text{ for } i > M).$$

Hence, in view of (31), the random cookie environment $\omega_{X,n}$ is larger (for the canonical partial order) than the deterministic environment $\omega_{\bar{p}^\varepsilon}$ defined by

$$\begin{cases} \omega_{\bar{p}^\varepsilon}(i,x) = \frac{1}{2}, & \text{for all } x < 0 \text{ and } i \geq 1, \\ \omega_{\bar{p}^\varepsilon}(i,x) = p_i^\varepsilon, & \text{for all } x \geq 0 \text{ and } i \geq 1 \text{ (with the convention } p_i^\varepsilon = \frac{1}{2} \text{ for } i \geq M). \end{cases}$$

Thus, Lemma 15 of [13] yields

$$\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -(n+1) \text{ at least once}\} \leq \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X \text{ visits } -(n+1) \text{ at least once}\}$$

In view of (32) we deduce that

$$|\mathbb{P}_{(M,\bar{q})}\{Z_\infty = j\} - \mathbb{P}_{(M,\bar{q})}\{Z_n = j\}| \leq f(n),$$

where $f(n) = \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X \text{ visits } -(n+1) \text{ at least once}\}$ does not depend of $\bar{q}$. It remains to prove that $f(n)$ tends to 0 as n goes to infinity. Let us first notice that

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{\forall n \geq 0 \quad X_n \geq 0\} = \mathbf{P}_{(M,\bar{p}^\varepsilon)}\{\forall n \geq 0 \quad X_n \geq 0\},$$

since these probabilities depend only on the environments on the half line $[0, +\infty)$. Recall also that the cookie random walk in the environment $(M, \bar{p}^\varepsilon)$ is transient (we have chosen ε such that $\alpha(M, \bar{p}^\varepsilon) > 0$), thus

$$\mathbf{P}_{(M,\bar{p}^\varepsilon)}\{\forall n \geq 0 \quad X_n \geq 0\} = \mathbf{P}_{(M,\bar{p}^\varepsilon)}\{U_0^\infty = 0\} = \mathbb{P}_{(M,\bar{p}^\varepsilon)}\{Z_\infty = 0\} > 0.$$

Hence

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{\forall n \geq 0 \quad X_n \geq 0\} > 0,$$

which implies

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n = 0 \text{ infinitely often}\} < 1,$$

and a 0-1 law (*c.f.* Proposition 5 of [13]) yields

$$\mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n = 0 \text{ infinitely often}\} = \mathbf{P}_{\omega_{\bar{p}^\varepsilon}}\{X_n \leq 0 \text{ infinitely often}\} = 0.$$

Therefore, $\lim_{n\to\infty} f(n) = 0$. □

Recall that the transition probabilities of the Markov chain Z are given by the law of the random variables A_k:

$$\mathbb{P}_{(M,\bar{p})}\left\{Z_{n+1} = j \mid Z_n = i\right\} = \mathbf{P}_{(M,\bar{p})}\left\{A_i = j\right\}.$$

It is therefore clear that for each fixed n and each k, the function $\bar{p} \to \mathbb{P}_{(M,\bar{p})}\left\{Z_n = k\right\}$ is continuous in $\bar{p}$ in Ω_M^u. Writing

$$\begin{aligned} |\mathbb{P}_{(M,\bar{q})}\left\{Z_\infty = k\right\} - \mathbb{P}_{(M,\bar{p})}\left\{Z_\infty = k\right\}| &\leq |\mathbb{P}_{(M,\bar{q})}\left\{Z_\infty = k\right\} - \mathbb{P}_{(M,\bar{q})}\left\{Z_n = k\right\}| \\ &+ |\mathbb{P}_{(M,\bar{q})}\left\{Z_n = k\right\} - \mathbb{P}_{(M,\bar{p})}\left\{Z_n = k\right\}| + |\mathbb{P}_{(M,\bar{p})}\left\{Z_\infty = k\right\} - \mathbb{P}_{(M,\bar{p})}\left\{Z_n = k\right\}| \end{aligned}$$

and in view of the previous lemma, we conclude that for each k the function $\bar{p} \to \mathbb{P}_{(M,\bar{p})}\left\{Z_\infty = k\right\}$ is also continuous in $\bar{p}$ in Ω_M^u, which completes the proof of Theorem 1.2.

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