LOCALIZATION ON 4 SITES FOR VERTEX-REINFORCED RANDOM WALKS ON $\mathbb{Z}$.

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Abstract. We characterize non-decreasing weight functions for which the associated one-dimensional vertex reinforced random walk (VRRW) localizes on 4 sites. A phase transition appears for weights of order $n \log \log n$: for weights growing faster than this rate, the VRRW localizes almost surely on at most 4 sites whereas for weights growing slower, the VRRW cannot localize on less than 5 sites. When w is of order $n \log \log n$, the VRRW localizes almost surely on either 4 or 5 sites, both events happening with positive probability.

1. Introduction

The model of the *vertex reinforced random walk* (VRRW) was first introduced by Pemantle [9] in 1992. It describes a discrete random walk $X = (X_n,\ n \geq 0)$ on a graph G, which jumps, at each unit of time n, from its actual position towards a neighboring site y with probability proportional to $w(Z_n(y))$, where $w : \mathbb{N} \to \mathbb{R}_+^*$ is some deterministic weight sequence and where $Z_n(y)$ is the local time of the walk at site y and time n. Thus, when w is non-decreasing, the walk tends to favor sites it has already visited many times in the past.

A striking feature of the model is that, depending on the reinforcement scheme w, it is possible for the walk to get "trapped" and visits only finitely many sites, even on an infinite graph. In this case, we say that the walk *localizes.* This unusual behaviour was first observed by Pemantle and Volkov [11] who proved that, with positive probability, the VRRW on the integer lattice $\mathbb{Z}$ with linear weight $w(n) = cn + 1$ visits only 5 sites infinitely often. This result was later completed by Tarrès [14] (see also [15] for a more recent and concise proof) who showed that localization of the walk on 5 sites occurs almost surely. More generally, Volkov [16] and more recently Benaïm and Tarrès [2] proved that the linearly reinforced VRRW localizes with positive probability on any graph with bounded degree. It is conjectured that this localization happens, in fact, with probability 1. However, this seems a very challenging question as it is usually difficult to prove almost sure asymptotics for VRRW (let us note that, even in the one-dimensional case, Tarrès's proof of almost sure localization is quite elaborate).

A seemingly closely related model is the so-called *edge reinforced random walk* (ERRW), introduced by Coppersmith and Diaconis [4] in 1987. The difference between VRRW and ERRW is only that the transition probabilities for ERRW depend on the *edge* local time of the walk instead of the *site* local time. In the

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one-dimensional case, Davis [5] proved that the ERRW with non-decreasing reinforcement weight function w is recurrent (i.e. the walk visits all sites infinitely often almost surely) i.f.f.

$$\sum_{i=0}^{\infty} \frac{1}{w(i)} = \infty. \tag{1}$$

Otherwise, the walk ultimately localizes on two consecutive sites almost surely.

It may seem natural to expect a similar simple criterion for VRRW. However, the picture turns out to be much more complicated than for ERRW because the walk may localize on subgraphs of cardinality larger than 2. Only partial results are currently available. For instance, when condition (1) fails and the sequence $(w(n), n \geq 0)$ is non-decreasing, the VRRW also gets stuck on two consecutive sites[1]. However, when (1) holds, the walk may or may not localize depending on the weight function w. In particular, it is conjectured that for reinforcements $w(n) \sim n^\alpha$, the walk is recurrent for $\alpha < 1$ and localizes on 5 sites for $\alpha = 1$. In this direction, it is proved in [17] that when $w(n)$ is of order n^α with $\alpha < 1$, the VRRW cannot localize. When $\alpha < 1/2$, this result was slightly refined by the second author in [12], who proved that the process is either a.s. recurrent or a.s. transient. Yet, a proof of the recurrence of the walk in this seemingly simple setting is still missing.

On the other hand (apart from the linear case) not much is known about the cardinality of the set of sites visited infinitely often when localization occurs and (1) holds. The aim of this paper is to partially answer this question by investigating under which conditions the VRRW ultimately localizes on less than 5 sites. In order to do so, we shall associate to each weight function w a number $\alpha_c(w) \in [0, \infty]$ (the precise definition of $\alpha_c(w)$ is given in the next section). The main result of the paper states that:

Theorem 1.1. *Assume that w is non-decreasing and that* (1) *holds. Denote by R' the set of sites which are visited infinitely often by the VRRW and by $|R'|$ its cardinality. Then, defining $\alpha_c(w)$ as in* (3)*, we have*

$$\begin{aligned} |R'| = 4 \textit{ with positive probability} &\iff \alpha_c(w) < \infty. \\ |R'| = 4 \textit{ almost surely} &\iff \alpha_c(w) = 0. \\ \begin{matrix} |R'| \textit{ equals } 4 \textit{ or } 5 \textit{ a.s., both events} \\ \textit{occurring with positive probability} \end{matrix} &\iff \alpha_c(w) \in (0, \infty). \end{aligned}$$

It is easy to see that a VRRW can never localize on 3 sites (see Proposition 4.2) therefore Theorem 1.1 combined with criterion (1) for localization on two sites covers all possible cases where a VRRW with non-decreasing weight localizes with positive probability on less than 5 sites.

The last part of the theorem shows that the size of R' can itself be random. Such a result was already observed for graphs like $\mathbb{Z}^d$, $d \geq 2$ (for linear reinforcement) where different non-trivial localization patterns may occur (see [2, 16]). Yet we find

[1] This result first appears at the end of [17]. However, there is a mistake in the original argument. For the sake of completness, we give an other proof of this result (for non-decreasing weight sequences) in Proposition 4.1.

this result more surprising in the one-dimensional setting since R' is necessarily an interval.

The parameter $\alpha_c(w)$ can be explicitly calculated for a large class of weights w. In particular, if $w(n) \sim n \log\log n$, then $\alpha_c(w) = 1$. Moreover,

Proposition 1.2. *For any non-decreasing weight function w such that* (1) *holds:*

$$\begin{aligned} w(n) \gg^{2} n \log\log n &\implies \alpha_c(w) = 0. \\ w(n) \asymp^{3} n \log\log n &\implies \alpha_c(w) \in (0,\infty). \\ w(n) \ll n \log\log n &\implies \alpha_c(w) = \infty. \end{aligned}$$

Let us mention that, when $\alpha_c(w) = \infty$, Theorem 1.1 simply states that, if the walk localizes, $|R'| \geq 5$ necessarily. In fact, it is proved in a forthcoming paper [1] that there exist non-decreasing weight functions w for which the VRRW localizes almost surely on finite sets of arbitrarily large cardinality (this result is, in a way, similar to those proved in [6, 7] for another related model of self-interacting random walks).

The proof of Theorem 1.1 is based on two main techniques. First we use martingales arguments which were introduced by Tarrès in [14, 15]. These martingales have the advantage of taking into account the facts that, on each site, the process is roughly governed by an urn process, but also the fact that all these urns are strongly correlated. The second tool is a continuous time construction of the VRRW, called Rubin's construction, which was already used by Davis [5] for urn processes, and by Sellke in [13] in the case of edge reinforcement. Tarrès introduced in [15] a variant of this construction, which allows for powerful couplings in the case of non-decreasing weights and which will be very useful in this study.

The remainder of the paper is organized as follows. In the next section we give some simple results concerning w-urns processes which will play an important role in the proof of the theorem. In Section 3, we recall some classical results concerning VRRW. The proof of Theorem 1.1 is provided in Section 4. Finally we prove Proposition 1.2 in the appendix along with other technical lemmas concerning properties of the critical parameter $\alpha_c(w)$.

2. w-URN PROCESSES

2.1. **Weight function w and the parameter α_c.** In the rest of the paper, we call *weight sequence* w a sequence $(w(n), n \geq 0)$ of positive real numbers. It will be convenient to extend w into a weight function $w : \mathbb{R}_+ \to \mathbb{R}_+^*$ by $w(t) = w(\lfloor t \rfloor)$ where $\lfloor t \rfloor$ stands for the integer part of t. Then, given w, we set

$$W(t) := \int_0^t \frac{1}{w(u)}\, du.$$

[2] we use the notation $f \gg g$, when $f(n)/g(n) \to \infty$.

[3] we say that $f \asymp g$ when there exists a constant $c > 0$, such that $c^{-1} f(n) \leq g(n) \leq c\, f(n)$, for all n large enough.

When condition (1) holds, we have $W(\infty)=\infty$ and W is an homeomorphism of $\mathbb{R}_+$ whose inverse we denote by W^{-1}. Then, for $\alpha>0$, we define the integral

$$I_\alpha(w) := \int_0^\infty \frac{dx}{w(W^{-1}(W(x)+\alpha))} = \int_0^\infty \frac{w(W^{-1}(y))}{w(W^{-1}(y+\alpha))}\,dy. \tag{2}$$

If furthermore we assume that w is non-decreasing, then $\alpha \to I_\alpha(w)$ is non-increasing and we can define the critical parameter $\alpha_c(w)$ by

$$\alpha_c(w) := \inf\{\alpha\geq 0 \ : \ I_\alpha(w)<\infty\} \in [0,\infty] \tag{3}$$

with the convention that $\inf\emptyset=\infty$.

2.2. **w-urn processes.** A w-urn is a process $(R_n,B_n)_{n\geq 0}$ defined on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that for all $n\geq 0$, $R_n+B_n=n$, $R_{n+1}\in\{R_n,R_n+1\}$, and

$$\mathbb{P}\{R_{n+1}=R_n+1\} = \frac{w(R_n)}{w(R_n)+w(B_n)}.$$

We call R_n (resp. B_n) the number of red (resp. blue) balls in the urn after the n-th draw. Set $R_\infty=\lim_{n\to\infty}R_n$ and $B_\infty=\lim_{n\to\infty}B_n$.

Our interest towards w-urn processes comes from fact that, if we consider a VRRW on the finite set $\{-1,0,1\}$ (*i.e.* the walk reflected at 1 and -1, see Section 3.4), then joint local times of the walk at sites 1 and -1 and at time $2n$ is exactly a w-urn process. The next proposition describes the asymptotic behavior of such an urn. Several arguments used during the proof of the result below will also play an important role when proving Theorem 1.1.

Proposition 2.1. *For any weight sequence w (not necessarily non-decreasing), we have*

$$\sum_{n\geq 0}\frac{1}{w(n)}<+\infty \quad\Longleftrightarrow\quad R_\infty<+\infty \text{ or } B_\infty<+\infty \ \text{ a.s.}$$

The process $\hat M=(\hat M_n,\, n\geq 0)$ defined by $\hat M_n := W(R_n)-W(B_n)$ is a martingale. Moreover,

(a) *If $\sum_n 1/w(n)=\infty$ and $\sum_n 1/w(n)^2<+\infty$, then $\hat M_n$ converges a.s. to some random variable $\hat M_\infty$, which admits a symmetric density with unbounded support.*
(b) *If $\sum_n 1/w(n)^2=\infty$ and $\inf_n w(n)>0$, then $\liminf \hat M_n=-\infty$ and $\limsup \hat M_n=+\infty$ a.s.*

Proof. The first equivalence is well known (see, for instance [10]) and follows immediately from Rubin's construction of the urn process which we will recall below. Let us note that we can rewrite $\hat M$ in the form

$$\hat M_n = \sum_{k=0}^{n-1}\left(\frac{1_{\{\text{the }(k+1)\text{-th draw is Red}\}}}{w(R_k)} - \frac{1_{\{\text{the }(k+1)\text{-th draw is Blue}\}}}{w(B_k)}\right).$$

Therefore, $\hat M$ is clearly a martingale. Let

$$\begin{aligned} V_n &:= \sum_{k\leq n}(\hat M_{k+1}-\hat M_k)^2 \\ &= \sum_{k\leq n}\left(\frac{1_{\{\text{the }(k+1)\text{-th draw is Red}\}}}{w(R_k)^2} + \frac{1_{\{\text{the }(k+1)\text{-th draw is Blue}\}}}{w(B_k)^2}\right). \end{aligned}$$

We have

$$\mathbb{E}[\hat{M}_n^2] = \mathbb{E}[V_{n-1}] \le 2\sum_{k=0}^{\infty} \frac{1}{w(k)^2}.$$

Thus, when assumption (a) holds, $\hat{M}$ converges almost surely and in L^2 towards some random variable $\hat{M}_\infty$. We now use Rubin's construction to identify this limit: let $(\xi_n, n \ge 0)$ and $(\xi'_n, n \ge 0)$ be two sequences of independent exponential random variables with mean 1. Define the random times $t_k = \xi_0/w(0) + \cdots + \xi_k/w(k)$ and $t'_k = \xi'_0/w(0) + \cdots + \xi'_k/w(k)$. We can construct the w-urn process (R_n, B_n) from these two sequences by adding a red ball in the urn at each instant $(t_k)_{k\ge 0}$ and a blue ball at each instant $(t'_k)_{k\ge 0}$ (see the appendix in [5] for details). Using this construction, we can rewrite $\hat{M}_n$ in the form

$$\hat{M}_n = \sum_{k=0}^{R_n-1} \frac{1-\xi_k}{w(k)} - \sum_{k=0}^{B_n-1} \frac{1-\xi'_k}{w(k)} + \sum_{k=0}^{R_n-1} \frac{\xi_k}{w(k)} - \sum_{k=0}^{B_n-1} \frac{\xi'_k}{w(k)}.$$

Observe that, by construction, for any n,

$$\left| \sum_{k=0}^{R_n-1} \frac{\xi_k}{w(k)} - \sum_{k=0}^{B_n-1} \frac{\xi'_k}{w(k)} \right| \le \max\left(\frac{\xi_{R_n}}{w(R_n)}, \frac{\xi'_{B_n}}{w(B_n)} \right).$$

Since a.s. $R_n \wedge B_n \to \infty$, we deduce that the r.h.s. above tends to 0 a.s. hence

$$\hat{M}_\infty = \sum_{k=0}^{\infty} \frac{1-\xi_k}{w(k)} - \sum_{k=0}^{\infty} \frac{1-\xi'_k}{w(k)}. \tag{4}$$

Both sums in the r.h.s. of the previous equation converge because they have a finite second moment. Thus, $\hat{M}_\infty$ admits a symmetric density with unbounded support since the ξ_n's and ξ'_n's are independent and ξ_0 has a non-vanishing density on $\mathbb{R}_+$.

It remains to prove (b). Let us observe that, when $\inf_n w(n) > 0$, the martingale $\hat{M}_n$ has bounded increments. Using Theorem 2.14 in [8], it follows that, a.s., either $\hat{M}_n$ converges or $\limsup \hat{M}_n = +\infty$ and $\liminf \hat{M}_n = -\infty$. Moreover, when $\sum_n 1/w(n)^2 = \infty$, we have $\lim_{n\to\infty} V_n = \infty$ a.s. Therefore, we can define $k_n := \inf\{m : V_m \ge n\}$ and Theorem 3.2 of [8] states that $\hat{M}_{k_n}/\sqrt{n}$ converges in law towards a standard normal variable. In particular, $\hat{M}$ cannot converge. This completes the proof of the proposition. □

The next result illustrates how the parameter $\alpha_c(w)$ of Theorem 1.1 naturally appears in connection with w-urns.

Corollary 2.2. *Consider a w-urn (R_n, B_n). Assume that w is non-decreasing with $\sum_n 1/w(n) = \infty$ and $\sum_n 1/w(n)^2 < \infty$ and set*

$$Y^B := \sum_{k=0}^{\infty} \frac{1_{\{\text{the } (k+1)\text{-th draw is Blue}\}}}{w(k)}, \qquad Y^R := \sum_{k=0}^{\infty} \frac{1_{\{\text{the } (k+1)\text{-th draw is Red}\}}}{w(k)}.$$

Then, we have

(i) *If $\alpha_c(w) = 0$ then, a.s., $\min(Y^B, Y^R) < \infty$.*
(ii) *If $\alpha_c(w) \in (0, \infty)$ then $\mathbb{P}\{\min(Y^B, Y^R) < \infty\} \in (0,1)$.*
(iii) *If $\alpha_c(w) = \infty$ then, a.s., $\min(Y^B, Y^R) = \infty$.*

Proof. According to Proposition 2.1, $W(R_n) - W(B_n)$ converges to some random variable $\hat{M}_\infty$ with a symmetric density and unbounded support. Let $\delta = |\hat{M}_\infty|/2$. On the event $\{\hat{M}_\infty > 0\}$, we have, for n large enough,

$$W(n) \geq W(R_n) \geq W(B_n) + \delta$$

which yields, for some (random but finite) constant c

$$Y^B \leq c \sum_{k=0}^{\infty} \frac{1_{\{\text{the } (k+1)\text{-th draw is Blue}\}}}{w(W^{-1}(W(B_k) + \delta))} = \sum_{k=0}^{\infty} \frac{c}{w(W^{-1}(W(k) + \delta))}.$$

Thus, by symmetry and using $\mathbb{P}\{\hat{M}_\infty = 0\} = 0$, we get, a.s.,

$$\min(Y^B, Y^R) \leq \sum_{k=0}^{\infty} \frac{c}{w(W^{-1}(W(k) + \delta))}.$$

This proves (i) and also that $\mathbb{P}\{\min(Y^B, Y^R) < \infty\} > 0$ whenever $\alpha_c(w) \in (0, \infty)$. Conversely, set $\delta' = 2|\hat{M}_\infty|$. On the event $\{\hat{M}_\infty \geq 0\}$, we have, for n large enough

$$n = B_n + R_n \leq B_n + W^{-1}(W(B_n) + \delta') \leq R_n + W^{-1}(W(R_n) + \delta').$$

This gives

$$Y^B \geq c \sum_{k=0}^{\infty} \frac{1_{\{\text{the } (k+1)\text{-th draw is Blue}\}}}{w(B_k + W^{-1}(W(B_k) + \delta'))} = \sum_{k=0}^{\infty} \frac{c}{w(k + W^{-1}(W(k) + \delta'))}.$$

and the same bound also holds for Y^R. Therefore, by symmetry, we get, a.s.,

$$\min(Y^B, Y^R) \geq \sum_{k=0}^{\infty} \frac{c}{w(k + W^{-1}(W(k) + \delta'))}.$$

We conclude the proof using Lemma 5.6 of the appendix which insures that the sum above is infinite when $\delta' < \alpha_c(w)$. □

3. Vertex reinforced random walk

3.1. **The VRRW.** In the remainder of the paper, given the weight sequence w, $(X_n, n \geq 0)$ will denote a nearest neighbour random walk on the integer lattice $\mathbb{Z}$, starting from $X_0 = 0$ with transition probabilities given by

$$\mathbb{P}\{X_{n+1} = x \pm 1 \mid \mathcal{F}_n\} = \frac{w(Z_n(x \pm 1))}{w(Z_n(x+1)) + w(Z_n(x-1))}, \tag{5}$$

where $(\mathcal{F}_n, n \geq 0)$ is the natural filtration $\sigma(X_0, \ldots, X_n)$ and $Z_n(y)$ stands, up to a constant, for the local time of X at site y and at time n:

$$Z_n(y) := z_0(y) + \sum_{k=0}^{n} 1_{\{X_k = y\}}.$$

We call the sequence $\mathcal{C} := (z_0(y),\, y \in \mathbb{Z})$ the *initial local time configuration*. We say that X is a VRRW when it starts from the trivial configuration $\mathcal{C}_0 := (0, 0, \ldots)$. However, it will sometimes be convenient to consider the walk starting from some other configuration $\mathcal{C}$. In that case, we shall mention it explicitly and emphasize this fact by calling X a $\mathcal{C}$-VRRW.

In the rest of this section, we collect some important results concerning the VRRW which we will use during the proof of Theorem 1.1 in Section 4. For additional details, we refer the reader to [5, 10, 13, 14, 15] and the references therein.

3.2. **The martingales $M_n(x)$.** For $x \in \mathbb{Z}$, define $Z_\infty(x) := \lim_{n\to\infty} Z_n(x)$. Recall that R' stands for the set of sites visited infinitely often by the walk:

$$R' := \{x \in \mathbb{Z} \ : \ Z_\infty(x) = \infty\}.$$

The following quantities will be of interest:

$$Y_n^{\pm}(x) := \sum_{k=0}^{n-1} \frac{1_{\{X_k = x \text{ and } X_{k+1} = x\pm 1\}}}{w(Z_k(x\pm 1))}, \tag{6}$$

and

$$M_n(x) := Y_n^+(x) - Y_n^-(x).$$

It is a basic observation due to Tarrès [14, 15] that $(M_n(x), n \geq 1)$ is a martingale for each $x \in \mathbb{Z}$. Moreover, if

$$\sum_{n=0}^{\infty} \frac{1}{w(n)^2} < \infty, \tag{7}$$

then these martingales are bounded in L^2, and thus converge a.s. and in L^2 towards

$$M_\infty(x) := \lim_{n\to\infty} M_n(x).$$

We will also consider the (possibly infinite) limits:

$$Y_\infty^{\pm}(x) := \lim_{n\to\infty} Y_n^{\pm}(x).$$

From the definition of $Y^{\pm}$, we directly obtain the identity

$$Y_n^+(x-1) + Y_n^-(x+1) = W(Z_n(x)) - W(1)1_{\{x=0\}}, \tag{8}$$

which holds for all $x \in \mathbb{Z}$ and all $n \geq 0$. In particular, we get

$$\begin{aligned} W(Z_n(x+2)) - W(Z_n(x)) = Y_n^-(x+3) - Y_n^+(x-1) + M_n(x+1) \\ + W(1)(1_{\{x=-2\}} - 1_{\{x=0\}}). \end{aligned}$$

More generally, if we now consider a $\mathcal{C}$-VRRW starting from some arbitrary initial local time configuration $\mathcal{C}$, then $M_n(x)$ is still a martingale and the equation above takes the form

$$W(Z_n(x+2)) - W(Z_n(x)) = Y_n^-(x+3) - Y_n^+(x-1) + M_n(x+1) + c(x,\mathcal{C}). \tag{9}$$

where $c(x,\mathcal{C})$ is some constant depending only on x and on the configuration $\mathcal{C}$.

3.3. **Time-line construction of the VRRW.** We now describe a method to construct the VRRW for a collection of exponential random variables which is in a way similar to Rubin's algorithm for w-urns. This construction was introduced by Tarrès in [15] and may be seen as a variant for the VRRW of the continuous time construction previously described by Sellke in [13] for edge reinforced random walks. One of the main advantages of this construction is that it enables to create non-trivial coupling between VRRWs. Let us fix a sequence

$$\xi := (\xi_n^{\pm}(y), n \geq 0, y \in \mathbb{Z}) \in \mathbb{R}_+^{\mathbb{N}}$$

of positive real numbers. The value $\xi_n^-(y)$ (resp. $\xi_n^+(y)$) will be related to the duration of a clock attached to the oriented edge $(y, y-1)$ (resp. $(y, y+1)$). Given this sequence, we create a deterministic, integer valued, continuous-time process $(\widetilde{X}(t), t \geq 0)$ in the following way:

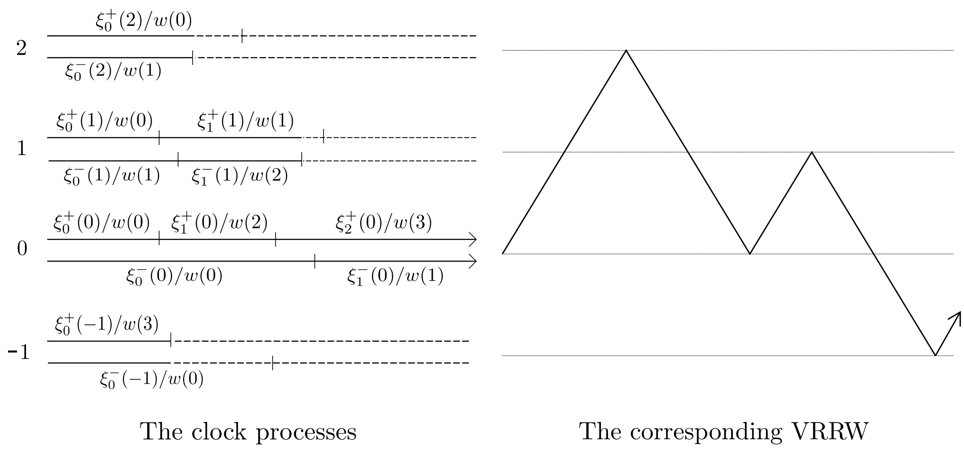


FIGURE 1. Illustration of the time-line construction.

- Set $\widetilde{X}(0) = 0$ and attach two clocks to the oriented edges $(0, -1)$ and $(0, 1)$ ringing respectively at times $\xi_0^-(0)/w(0)$ and $\xi_0^+(0)/w(0)$.
- When the first clock rings at time $\tau_1 := \xi_0^+(0)/w(0) \wedge \xi_0^-(0)/w(0)$, stop both clocks and set $\widetilde{X}(\tau_1) = \pm 1$ depending on which clock rung first. (if both clocks ring at the same time, we decide that $\widetilde{X}$ stays at 0 forever).

Assume that we have constructed $\widetilde{X}$ up to some time $t > 0$ at which time the process makes a right jump from some site $x - 1$ to x. Denote by k the number of jumps from x to $x - 1$ and by m the number of visits to $x - 1$ before time t. We follow the procedure below:

- Start a new clock attached to the oriented edge $(x, x - 1)$, which will ring after a time $\xi_k^-(x)/w(m)$.
- If the process already visited x some time in the past, restart the clock attached to the oriented edge $(x, x + 1)$ which had previously been stopped when the process last left site x. Otherwise, start the first clock for this edge which will ring at time $\xi_0^+(x)/w(0)$.
- As soon as one of these two clocks rings, stop both of them and let the process jump along the edge corresponding to the clock which rung first (if both clocks ring at the same time, we decide that $\widetilde{X}$ stays in x forever).

We use a similar rule when the process makes a left jump from some site x to $x - 1$. We say that this construction *fails* if at some time, two clocks ring simultaneously. Let now τ_i stand for the time of the i-th jump of $\widetilde{X}$ (with the convention $\tau_0 = 0$ and $\tau_{n+1} = \tau_n$ if $\widetilde{X}$ does not move after time τ_n) and define the discrete time process $X = (X_n, n \geq 0)$ by

$$X_n := \widetilde{X}(\tau_n).$$

It is an elementary observation that if we now choose the $\xi_n^\pm(y)$ to be independent exponential random variables with mean 1, then the construction does not fail with probability 1 and the resulting process X is a VRRW with weight w.

Remark 3.1. *For the sake of clarity, we only describe the construction for the VRRW starting from the trivial configuration $\mathcal{C}_0$. However, it is clear that we can* *do a similar construction for any $\mathcal{C}$-VRRW by simply replacing the duration of the clocks $\xi_k^{\pm}(x)/w(m)$ with $\xi_k^{\pm}(x)/w(z_0(x\pm 1)+m)$.*

A remarkable feature of this construction comes from the fact that we can simultaneously create a family $(\widetilde{X}^{(u)},\ u\geq 0)$ of processes with nice monotonicity properties with respect to the u parameter. To this end, define, for $x\in\mathbb{Z}$,

$$\mathcal{H}_x := \big((\xi_n^{\pm}(y), n\geq 0, y\neq x), (\xi_n^{-}(x), n\geq 0), (\xi_n^{+}(x), n\geq 1)\big)\in\mathbb{R}_+^{\mathbb{N}}.$$

Then, given $\mathcal{H}_x$ together with a real number $u>0$, the pair $(\mathcal{H}_x, u)$ defines a deterministic process $X^{(u)} = (X_n^{(u)}, n\geq 0)$ using the construction above with $\xi_0^{+}(x)=u$. The following lemma is easily obtained by induction.

Lemma 3.2 (Tarrès [15]). *Suppose that w is non-decreasing. Fix $\mathcal{H}_x$ and $0<u\leq u'$ and assume that the construction for $X^{(u)}$ and $X^{(u')}$ both succeed. Given $y\in\mathbb{Z}$ and $k\geq 1$, denote by σ (resp. σ') the time when $X^{(u)}$ (resp. $X^{(u')}$) visits y for the k-th time. If σ and σ' are both finite, then*

$$Z_\sigma^{(u)}(y+1)\geq Z_{\sigma'}^{(u')}(y+1) \qquad \textit{and} \qquad Z_\sigma^{(u)}(y-1)\leq Z_{\sigma'}^{(u')}(y-1)$$

$$N_\sigma^{(u)}(y,y+1)\geq N_{\sigma'}^{(u')}(y,y+1) \qquad \textit{and} \qquad N_\sigma^{(u)}(y,y-1)\leq N_{\sigma'}^{(u')}(y,y-1),$$

where $Z_n^{(s)}$ stands the local time of $X^{(s)}$ and $N_n^{(s)}(y,y\pm 1)$ denotes the number of jumps from y to $y\pm 1$ up to time n. Moreover, denote by $\theta^{\pm}$ (resp. ${\theta'}^{\pm}$) the time when $X^{(u)}$ (resp. $X^{(u')}$) jumps for the k-th time from y to $y\pm 1$. If these quantities are finite, then

$$Y_{\theta^+}^{(u)+}(y)\leq Y_{\theta'^+}^{(u')+}(y) \qquad \textit{and} \qquad Y_{\theta^-}^{(u)-}(y)\geq Y_{\theta'^-}^{(u')-}(y),$$

where $Y^{(s)\pm}$ is defined as in (6) for the process $X^{(s)}$.

The combination of the time-line construction of the walk from i.i.d. exponential random variables together with Lemma 3.2 yields a simple proof of the following key result concerning the localization of the VRRW:

Lemma 3.3 (Tarrès [15]). *Assume that w is non-decreasing and that $\sum_n 1/w(n)^2$ is finite. Then, for any $x\in\mathbb{Z}$, a.s.,*

$$\{Y_\infty^+(x)<\infty\} = \{Y_\infty^-(x)<\infty\} \;=\; \{Z_\infty(x-1)<\infty\}\cup\{Z_\infty(x+1)<\infty\}.$$

Proof. This result is proved in [15] only for linear reinforcements w but the same arguments apply, in fact, for any non-decreasing weight function. However, since some details are omitted in [15], for the sake of completeness, we provide here a detailed proof (differing in some aspects from the original one). The first equality $\{Y_\infty^+(x)<\infty\} = \{Y_\infty^-(x)<\infty\}$ follows from the fact that the martingale $M_n(x)$ converges a.s. to some finite limit when $\sum_n 1/w(n)^2$ is finite. Concerning the second equality, the inclusion

$$\{Y_\infty^+(x)<\infty\} = \{Y_\infty^-(x)<\infty\} \;\supset\; \{Z_\infty(x-1)<\infty\}\cup\{Z_\infty(x+1)<\infty\}.$$

is straightforward (one of the sums $Y_\infty^{\pm}$ has only a finite number of terms). We use the time-line construction of the VRRW X from the sequence $(\xi_n^{\pm}(y), n\geq 0, y\in\mathbb{Z})$ to prove the converse inclusion. Denote by $N_k(x,x\pm 1)$ the number of jumps of X from x to $x\pm 1$ before time k, and set

$$T_x^{\pm} := \sum_{k\geq 0}\frac{1_{\{X_k=x,\ X_{k+1}=x\pm 1\}}\,\xi^{\pm}_{N_k(x,x\pm 1)}(x)}{w(Z_k(x\pm 1))}.$$

Thus, $T_x^{\pm}$ represents the total time consumed by the clocks attached to oriented edge $(x, x \pm 1)$. We claim that

$$\{Y_\infty^+(x) < \infty\} \cap \{Z_\infty(x-1) = \infty\} \cap \{Z_\infty(x+1) = \infty\} \subset \{T_x^+ = T_x^- < \infty\}. \tag{10}$$

We prove the result for $x < 0$ (the proof for $x > 0$ and $x = 0$ are similar). Let θ_k denote the time of the k-th jump from site $x+1$ to x and let i_k be the local time at site $x+1$ and at time θ_k. With this notation, on the event $\{Z_\infty(x+1) = \infty\}$, we can write

$$T_x^+ = \sum_{k \geq 1} \xi_{k-1}^+(x) \frac{1_{\{\theta_k < \infty\}}}{w(i_k)} \quad \text{and} \quad Y_\infty^+(x) = \sum_{k \geq 1} \frac{1_{\{\theta_k < \infty\}}}{w(i_k)}.$$

Define now

$$T_n^+(x) := \sum_{k=1}^{n-1} (\xi_{k-1}^+(x) - 1) \frac{1_{\{\theta_k < \infty\}}}{w(i_k)}.$$

Recall that $(\mathcal{F}_n, n \geq 0)$ stands for the natural filtration of X and notice that i_k is $\mathcal{F}_{\theta_k}$-measurable whereas $\xi_{k-1}^+(x)$ is independent of $\mathcal{F}_{\theta_k}$. Thus, $(T_n^+(x), n \geq 1)$ is a $\mathcal{F}_{\theta_n}$-martingale. Moreover, using that $w(i_k) \geq w(k)$ for all $k \geq 1$, it follows that the L^2-norm of this martingale is bounded by $\sum_{k \geq 0} w(k)^{-2} < \infty$. In particular, this implies that $Y_\infty^+(x)$ is finite if and only if T_x^+ is finite. It is also clear from the construction of the time-line process that, on the event $\{Z_\infty(x-1) = \infty\} \cap \{Z_\infty(x+1) = \infty\}$, we have $T_x^+ = T_x^-$. Thus we have established (10).

It remains to prove that the event

$$\mathcal{E}_x := \{T_x^+ = T_x^- < \infty\} \cap \{Z_\infty(x-1) = \infty\} \cap \{Z_\infty(x+1) = \infty\}$$

has probability 0. Recall the notation

$$\mathcal{H}_x := \big((\xi_n^{\pm}(y), n \geq 0)_{y \neq x}, (\xi_n^-(x), n \geq 0), (\xi_n^+(x), n \geq 1)\big) \in \mathbb{R}_+^{\mathbb{N}}.$$

and denote by μ the product measure on $\mathbb{R}_+^{\mathbb{N}}$ under which $\mathcal{H}_x$ is a collection of i.i.d. exponential random variables with mean 1. Given $\xi_0^+(x)$, the pair $(\mathcal{H}_x, \xi_0^+(x))$ defines the (deterministic) process $X = X(\mathcal{H}_x, \xi_0^+(x))$ via the time-line construction and X under the product law $\mathbb{P} := \mu \times \text{Exp}(1)$ is a VRRW.

Let us note that, for μ-a.e. realization of $\mathcal{H}_x$, the set of values of $\xi_0^+(x)$ such that the time-line construction fails is countable hence has zero Lebesgue measure. Moreover, Lemma 3.2 implies that, for any $(\mathcal{H}_x, u)$ and $(\mathcal{H}_x, u')$ in $\mathcal{E}_x$ with $u' > u$, we have

$$T_x^+(\mathcal{H}_x, u') > T_x^+(\mathcal{H}_x, u) \quad \text{and} \quad T_x^-(\mathcal{H}_x, u') \leq T_x^-(\mathcal{H}_x, u).$$

Thus, for any $\mathcal{H}_x$, there is at most one value of $\xi_0^+(x)$ such that $(\mathcal{H}_x, \xi_0^+(x)) \in \mathcal{E}_x$. This yields

$$\mathbb{P}\{\mathcal{E}_x\} = \mathbb{E}_\mu \left(\int_0^\infty e^{-u} 1_{\{(\mathcal{H}_x, u) \in \mathcal{E}_x\}} du \right) = \mathbb{E}_\mu(0) = 0.$$

□

A weaker statement can also be obtained when the assumptions of Lemma 3.3 do not hold.

Lemma 3.4. *For any weight sequence w and for any $x \in \mathbb{Z}$, we have, a.s.*

$$\{Z_\infty(x-1) < \infty\} \cup \{Z_\infty(x+1) < \infty\} \subset \{Y_\infty^+(x) < \infty\} \cap \{Y_\infty^-(x) < \infty\}.$$

Proof. By symmetry, we can assume without loss of generality that $Z_\infty(x-1) < \infty$. On the one hand, $Y^-_\infty(x)$ is finite since it is a sum with a finite number of terms. On the other hand, the conditional Borel-Cantelli lemma implies that $Y^+_\infty(x) < \infty$ (apply for instance the theorem of [3] with the sequence $1_{\{X_k=x-1\}}$). □

3.4. **VRRW restricted to a finite set.** In the sequel, it will be convenient to consider the vertex reinforced random walk restricted to some interval $[\![a,b]\!] := \{x \in \mathbb{Z} \ : \ a \le x \le b\}$ for some $a \le 0 \le b$, *i.e.* a walk with the same transition probabilities (5) as the VRRW X on $\mathbb{Z}$ except at the boundary sites a and b where it is reflected. We shall use the notation $\bar{X}$ to denote this reflected process. We also add a bar to denote all the quantities $\bar{Z}$,$\bar{Y}^{\pm}$,$\bar{M}$,... related with the reflected process $\bar{X}$.

Remark 3.5. *Let us emphasize the fact that, for $x \in]\!]a,b[\![$, the processes $\bar{M}_n(x) := \bar{Y}^+_n(x) - \bar{Y}^-_n(x)$ are still martingales, which are bounded in L^2 when (7) holds. In particular, Lemma 3.3 still holds for the reflected random walk for all site $x \in]\!]a,b[\![$. However, $\bar{M}_n(a)$ and $\bar{M}_n(b)$ are not martingales anymore. In particular, $\bar{Y}^+_\infty(a)$ or $\bar{Y}^-_\infty(b)$ can be infinite whereas $\bar{Y}^-_\infty(a)$ and $\bar{Y}^+_\infty(b)$ are, by construction, always equal to* 0.

We can construct the VRRW $\bar{X}$ restricted to the interval $[\![a,b]\!]$ using the same time-line construction used for X, choosing again the random variables $\xi^\pm_k(x)$ independent and exponentially distributed except for the two boundary r.v. $\xi^-_0(a)$ and $\xi^+_0(b)$ which are now chosen equal to ∞ (this prevent the walk from ever jumping from a to $a-1$ or from b to $b+1$). Let us note that, this construction depends only upon $(\xi^\pm_n(x), n \ge 0, x \in]\!]a,b[\![)$.

Let $\bar{X}'$ denote another VRRW restricted to $[\![a,b']\!] \supset [\![a,b]\!]$ for some $b' \ge b$. Using the time-line construction for $\bar{X}$ and $\bar{X}'$ with the same random variables $\xi^\pm_k(x)$, except for $\xi^+_0(b)$, we directly deduce from Lemma 3.2 a monotonicity result between the local time processes of X and X' :

Lemma 3.6. *Assume that w is non-decreasing. Fix $z \in [\![a,b]\!]$ and $k \ge 1$, let σ, σ' be the times when $\bar{X}, \bar{X}'$ visit z for the k-th times. On the event $\{\sigma < \infty$ and $\sigma' < \infty\}$, we have, a.s.,*

$$\begin{array}{ccc} \bar{Z}_\sigma(z+1) \le \bar{Z}'_{\sigma'}(z+1) & \textit{and} & \bar{Z}_\sigma(z-1) \ge \bar{Z}'_{\sigma'}(z-1) \\ \bar{N}_\sigma(z,z+1) \le \bar{N}'_{\sigma'}(z,z+1) & \textit{and} & \bar{N}_\sigma(z,z-1) \ge \bar{N}'_{\sigma'}(z,z-1), \end{array}$$

where N and N' are defined as in Lemma 3.2 for $\bar{X}$ and $\bar{X}'$. Moreover, if we denote by $\theta^\pm$ (resp. $\theta'^\pm$) the time when $\bar{X}$ (resp. $\bar{X}'$) jump for the k-th time from z to $z \pm 1$, then, on the event of these quantities being finite, we have, a.s.,

$$\bar{Y}^+_{\theta^+}(z) \ge \bar{Y}'^+_{\theta'^+}(z) \qquad \textit{and} \qquad \bar{Y}^-_{\theta^-}(z) \le \bar{Y}'^-_{\theta'^-}(z).$$

We conclude this section with a simple lemma we will repeatedly invoke to reduce the study of the localization properties of the VRRW X on $\mathbb{Z}$ to those of the VRRW $\bar{X}$ restricted to a finite set.

Lemma 3.7. *For $N > 0$, let $\bar{X}$ be a VRRW on $[\![0,N]\!]$ and define the events*

$$\begin{array}{rcl} \mathcal{E} & = & \{\bar{Y}^+_\infty(0) < \infty\} \cap \{\bar{Y}^-_\infty(N) < \infty\}, \\ \mathcal{E}' & = & \mathcal{E} \cap \{\bar{X} \textit{ visits all sites of } [\![0,N]\!] \textit{ i.o.}\}. \end{array}$$

(i) *If $\mathcal{E}$ (resp. $\mathcal{E}'$) has positive probability, then the VRRW on $\mathbb{Z}$ has positive probability to localize on a subset of length at most (resp. equal to) $N+1$.*
(ii) *Reciprocally, if the VRRW on $\mathbb{Z}$ has positive probability to localize on a subset of length at most (resp. exactly) $N+1$, then there exists some initial local time configuration $\mathcal{C}$ such that, for the $\mathcal{C}$-VRRW on $[\![0,N]\!]$, the event $\mathcal{E}$ (resp. $\mathcal{E}'$) has positive probability.*

Proof. Define a sequence $(\chi_n)_{n\geq 0}$ of random variables which are, conditionally on $\bar{X}$, independent with law

$$\mathbb{P}\{\chi_n = 1 \mid \bar{X}\} = 1 - \mathbb{P}\{\chi_n = 0 \mid \bar{X}\} = \frac{w(0)1_{\{\bar{X}_n=0\}}}{w(0)+w(\bar{Z}_n(1))} + \frac{w(0)1_{\{\bar{X}_n=N\}}}{w(0)+w(\bar{Z}_n(N-1))}.$$

The Borel-Cantelli Lemma applied to the sequence (χ_n) yields

$$\mathcal{E} \subset \left\{\sum \chi_n < \infty\right\}. \tag{11}$$

Let us also note that if $\mathbb{P}\{\sum \chi_n < \infty\} > 0$, then necessarily $\mathbb{P}\{\sum \chi_n = 0\} > 0$ since we just need to change a finite number of χ_n. Moreover, it is clear that we can construct a VRRW X on $\mathbb{Z}$ and a reflected random walk $\bar{X}$ on $[\![0,N]\!]$ on the same probability space in such way that X and $\bar{X}$ coincide on the event $\{\sum \chi_n = 0\}$. Therefore, if $\mathbb{P}\{\mathcal{E}\} > 0$, it follows from (11) that the VRRW X on $\mathbb{Z}$ localizes on $[\![0,N]\!]$ with positive probability. Moreover, if $\mathbb{P}\{\mathcal{E}'\} > 0$, we find that

$$\mathbb{P}\left\{\{\sum \chi_n = 0\} \cap \{\bar{X} \text{ visits all sites of } [\![0,N]\!] \text{ i.o.}\}\right\} > 0$$

which implies that, with positive probability, X visits every site of $[\![0,N]\!]$ i.o. without ever exiting the interval.

Reciprocally, if the VRRW on $\mathbb{Z}$ has positive probability to localize on some interval $[\![x,x+N]\!]$, then, clearly, there exists some initial local time configuration $\mathcal{C}$ on $\mathbb{Z}$ such that the $\mathcal{C}$-VRRW on $\mathbb{Z}$ has positive probability never to exit the interval $[\![0,N]\!]$. On this event, the $\mathcal{C}$-VRRW on $\mathbb{Z}$ and the restricted $\mathcal{C}$-VRRW on $[\![0,N]\!]$ coincide. We conclude the proof using Lemma 3.4 which implies that, on this event, $Y^+_\infty(0)$ and $Y^-_\infty(N)$ are both finite. □

4. Proof of Theorem 1.1

We split the proof of the theorem into several propositions. We start with two elementary observations:

Proposition 4.1. *Let w be a weight sequence.*

- *If $\sum 1/w(k) = \infty$, then we have, a.s., $|R'| \neq 2$.*
- *Conversely, if the sum above is finite and the weight sequence w is non-decreasing, then, a.s., $|R'| = 2$.*

Proof. Assume that $\sum 1/w(k) = \infty$ and consider the reflected VRRW $\bar{X}$ on $[\![0,1]\!]$ starting from some initial configuration $\mathcal{C}$. We have

$$\bar{Y}^+_\infty(0) = \sum_{k\geq 0} \frac{1_{\{\bar{X}_{k+1}=1\}}}{w(\bar{Z}_k(1))} = \sum_{i=\bar{Z}_0(1)}^{\infty} \frac{1}{w(i)} = \infty.$$

Thus, Lemma 3.7 implies that $\mathbb{P}\{|R'| = 2\} = 0$.

Reciprocally, if w is non-decreasing and $\sum 1/w(k) < \infty$, then $\sum 1/w(k)^2 < \infty$ and we can invoke Lemma 3.3 to conclude. □

Proposition 4.2. *For any weight sequence w, we have, a.s., $|R'| \neq 3$.*

Proof. Consider the reflected VRRW $\bar{X}$ on $[\![0,2]\!]$ starting from some initial configuration $\mathcal{C}$. We distinguish two cases:

- if $\sum 1/w(n) < \infty$, then Rubin's construction at site 1 implies that either 0 or 2 is visited only finitely many times (notice that we do not require here w to be monotonic).
- if $\sum 1/w(n) = \infty$, then we have

$$\bar{Y}^{+}_{\infty}(0) + \bar{Y}^{-}_{\infty}(2) = \sum_{k\geq 0} \frac{1_{\{\bar{X}_{k+1}=1\}}}{w(\bar{Z}_k(1))} = \sum_{i=\bar{Z}_0(1)}^{\infty} \frac{1}{w(i)} = \infty.$$

Thus, in both cases, Lemma 3.7 implies that $\mathbb{P}\{|R'| = 3\} = 0$.

□

Remark 4.3. Let us note that the result above does not hold for edge-reinforced random walks: if, for instance, $w(n) = 1$ when n is even and $w(n) = n^2$ when n is odd, then $R' = \{-1, 0, 1\}$ a.s., see Sellke [13].

In the rest of the paper, given two sequences $(u_n)_{n\geq 1}$ and $(v_n)_{n\geq 1}$, we shall use Tarrès's notation [14, 15] and write $u_n \equiv v_n$, when $(u_n - v_n)_{n\geq 1}$ is a converging sequence.

Lemma 4.4. *Assume that w is non-decreasing. Then*

$$|R'| = 4 \text{ with positive probability} \quad \Longrightarrow \quad \sum_{n\geq 0} \frac{1}{w(n)^2} < \infty.$$

Proof. Assume that $\sum 1/w(n)^2 = \infty$. Let $\bar{X}$ be a VRRW on $[\![0,3]\!]$ starting from some initial configuration $\mathcal{C}$. Recall that (9) states that

$$\begin{aligned} W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) &= \bar{Y}^{-}_n(3) - \bar{Y}^{+}_n(-1) + \bar{M}_n(1) + c \\ &= \bar{Y}^{-}_n(3) + \bar{M}_n(1) + c, \end{aligned} \tag{12}$$

where c is some constant depending on the initial configuration $\mathcal{C}$. Assume now that $\bar{Y}^{-}_{\infty}(3)$ is finite and let us prove that necessarily $\bar{Y}^{+}_{\infty}(0) = \infty$. Equation (12) becomes

$$W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) \equiv \bar{M}_n(1). \tag{13}$$

According to Theorem 2.14 of [8], either $\bar{M}_n(1)$ converges or $\limsup \bar{M}_n(1) = -\liminf \bar{M}_n(1) = \infty$. On one hand, remark that

$$\sum_{n\geq 0} (\bar{M}_{n+1}(1) - \bar{M}_n(1))^2 \geq \sum_{n\geq 0} \frac{1_{\{\bar{X}_n=0\}}}{w(\bar{Z}_n(0))^2}.$$

Hence, on the event $\{\bar{Z}_{\infty}(0) = \infty\}$, we have $\limsup \bar{M}_n = -\liminf \bar{M}_n = \infty$. On the other hand, by periodicity, $\bar{Z}_{\infty}(2) \vee \bar{Z}_{\infty}(0) = \infty$. Recalling that $\lim_{x\to\infty} W(x) = \infty$, we deduce, using (13) that $\{\bar{M}_n(1) \text{ converges}\} \subset \{\bar{Z}_{\infty}(2) = \infty\} \cap \{\bar{Z}_{\infty}(0) = \infty\}$. Therefore, a.s.,

$$\liminf_n \bar{M}_n(1) = -\infty \quad \text{and} \quad \limsup_n \bar{M}_n(1) = +\infty.$$

In particular, there exists a.s. arbitrarily large integers n, such that $W(\bar{Z}_n(0)) \geq W(\bar{Z}_n(2)) + 1$. Pick such an n and let m be the largest integer smaller than n such that $W(\bar{Z}_m(0)) \leq W(\bar{Z}_m(2))$. For $k \in (m,n]$, we have $\bar{Z}_k(0) \geq \bar{Z}_k(2)$ hence

$$\bar{Z}_k(1) \leq \bar{Z}_k(0) + \bar{Z}_k(2) + \bar{Z}_0(1) \leq 3\bar{Z}_k(0),$$

assuming that n is large enough. Since w is non-decreasing, we get

$$\begin{aligned}\sum_{k\in(m,n]} \frac{1_{\{\bar{X}_k=0\}}}{w(\bar{Z}_k(1))} \geq \sum_{k\in(m,n]} \frac{1_{\{\bar{X}_k=0\}}}{w(3\bar{Z}_k(0))} &= \sum_{i=\bar{Z}_m(0)+1}^{\bar{Z}_n(0)} \frac{1}{w(3i)} \\ &\geq \frac{1}{3}\Big\{W(\bar{Z}_n(0)) - W(\bar{Z}_m(0)+1)\Big\} \geq \frac{1}{4}.\end{aligned}$$

As this holds for infinitely many n, we deduce that, a.s.

$$\bar{Y}^+_\infty(0) = \sum_k \frac{1_{\{\bar{X}_k=0\}}}{w(\bar{Z}_k(1))} = \infty.$$

We conclude by using Lemma 3.7. □

Let us note that Lemma 4.4 together with Lemma 5.3 of the appendix imply that, for any non-decreasing weight sequence w, we have

$$\mathbb{P}\{|R'| = 4\} > 0 \text{ or } \alpha_c(w) < \infty \implies \sum_n \frac{1}{w(n)^2} < \infty.$$

Hence, when proving Theorem 1.1, we can assume, without loss of generality, that $\sum_n 1/w(n)^2 < \infty$. In particular, the martingales introduced in Section 3.2 converge a.s. and in L^2.

Proposition 4.5. *Assume that w is non-decreasing and that* (1) *holds. We have*

$$|R'| = 4 \textit{ with positive probability} \iff \alpha_c(w) < \infty.$$

Proof. Let us first suppose that $|R'| = 4$ with positive probability. Thus, according to Lemma 3.7, there exists some initial local time configuration $\mathcal{C}$ such that, for the $\mathcal{C}$-VRRW $\bar{X}$ on $[\![0,3]\!]$, the event $\mathcal{E} := \{\bar{Y}^+_\infty(0) < \infty\} \cap \{\bar{Y}^-_\infty(3) < \infty\}$ has positive probability. Using (9), we find that

$$\begin{aligned}W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) &= \bar{Y}^-_n(3) - \bar{Y}^+_n(-1) + \bar{M}_n(1) + C\\ W(\bar{Z}_n(3)) - W(\bar{Z}_n(1)) &= \bar{Y}^-_n(4) - \bar{Y}^+_n(0) + \bar{M}_n(2) + C'.\end{aligned}$$

As we already noticed, we can assume without loss of generality that $\sum 1/w(n)^2 < \infty$ so the martingales $\bar{M}_n(1)$ and $\bar{M}_n(2)$ converge. Hence, there exist finite random variables α, β, such that, on the event $\mathcal{E}$,

$$\begin{aligned}W(\bar{Z}_n(1)) - W(\bar{Z}_n(3)) &= \alpha + o(1),\\ W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) &= \beta + o(1).\end{aligned} \tag{14}$$

For n large enough, this yields

$$\max(\bar{Z}_n(1), \bar{Z}_n(2)) \leq W^{-1}(W(\max(\bar{Z}_n(0), \bar{Z}_n(3))) + \gamma)$$

with $\gamma := |\alpha| + |\beta| + 1$. Hence, we have

$$\begin{aligned}
\bar{Y}^+_\infty(0) + \bar{Y}^-_\infty(3) &= \sum_{k\geq 0}\left(\frac{1_{\{\bar{X}_k=0\}}}{w(\bar{Z}_k(1))} + \frac{1_{\{\bar{X}_k=3\}}}{w(\bar{Z}_k(2))}\right)\\
&\geq \sum_{k\geq 0}\frac{1_{\{\bar{X}_k\in\{0,3\}\}}}{w(\max(\bar{Z}_k(1),\bar{Z}_k(2)))}\\
&\geq c\sum_{k\geq 0}\frac{1_{\{\bar{X}_k\in\{0,3\}\}}}{w(W^{-1}(W(\max(\bar{Z}_k(0),\bar{Z}_k(3)))+\gamma))}\\
&\geq c\sum_{k\geq 0}\frac{1}{w(W^{-1}(W(k)+\gamma))}.
\end{aligned}$$

Therefore, on the event $\mathcal{E}$, we have $I_\gamma(w) < \infty$. This shows that $\alpha_c(w) < \infty$.

We now prove the converse implication. Let us assume that $I_\delta(w) < \infty$ for some $\delta > 0$. In particular, $\sum_n 1/w(n)^2 < \infty$ (*c.f.* Lemma 5.3). In view of Lemma 3.7, we will show that, for the reflected VRRW $\bar{X}$ on $[\![0,3]\!]$, the event $\{\bar{Y}^+_\infty(0) < \infty\} \cap \{\bar{Y}^-_\infty(3) < \infty\}$ has positive probability. This will insure that the VRRW on $\mathbb{Z}$ localizes with positive probability on a subset of size less or equal to 4 which will complete the proof of the proposition since localization on 2 or 3 sites is not possible with our assumptions on w.

We use the time-line construction. As explained in the previous section we can construct $\bar{X}$ from a sequence $(\xi^\pm_n(y),\ n \geq 0,\ y \in \{1,2\})$ of independent exponential random variables with mean 1. Observe that the sequences $(\xi^\pm_n(1)/w(n), n \geq 0)$ define a w-urn process via Rubin's construction (choosing + for the red balls). Let $\hat{M}_\infty(1)$ denote the limit of this urn defined as in Proposition 2.1. Then, we have $\hat{M}_\infty(1) \geq \delta + 1$ with positive probability. Recall the definition of Y^R given in Corollary 2.2 and note that, on the event $\{\hat{M}_\infty(1) \geq \delta + 1\}$, the random variable Y^R is finite. Besides, using Lemma 3.6 to compare $\bar{X}$ with the walk restricted on $[\![0,2]\!]$ (which correspond to the urn process above), we get

$$\bar{Y}^+_\infty(0) \leq Y^R < \infty \quad \text{on the event } \{\hat{M}_\infty(1) \geq \delta + 1\}.$$

By symmetry, considering the limit $\hat{M}_\infty(2)$ of the urn process $(\xi^\pm_n(2)/w(n), n \geq 0)$, we also find that

$$\bar{Y}^-_\infty(3) < \infty \quad \text{on the event } \{\hat{M}_\infty(2) \leq -\delta - 1\}.$$

The random variables $\hat{M}_\infty(1)$ and $\hat{M}_\infty(2)$ being independent, we conclude that $\{\bar{Y}^+_\infty(0) < \infty\} \cap \{\bar{Y}^-_\infty(3) < \infty\}$ has positive probability. □

Proposition 4.6. *Assume that w is non-decreasing and that* (1) *holds. We have*

$$\alpha_c(w) \in (0,\infty) \implies |R'| = 5 \text{ with positive probability.}$$

Proof. Assume that $\alpha_c(w) \in (0,\infty)$. In particular, we have $\sum_n 1/w(n)^2 < \infty$. Since the walks associated with a weight w and any non-zero multiple of w have the same law, we will assume without loss of generality that $w(0) \geq 1$. Let $\bar{X}$ denote the VRRW reflected on $[\![0,4]\!]$. Let us prove that, with positive probability, $\bar{Y}^+_\infty(0)$ and $\bar{Y}^-_\infty(4)$ are both finite and $\bar{X}$ visits all sites of $[\![0,4]\!]$ infinitely often.

We use again the time-line representation explained in Section 3 except that we will change the construction slightly for the transition at site 2. Recall that according to the original construction, when the process jumps for the k-th time from 1 (resp. 3) to 2 and has made m visits to 1 (resp. 3) before time t, then we attached to the oriented edge $(2,1)$ (resp. $(2,3)$) a clock which rings after time $\xi_k^-(2)/w(m)$ (resp. $\xi_k^+(2)/w(m)$). In our new construction, we choose to attach instead a clock which rings after time $\xi_m^-(2)/w(m)$ (resp. $\xi_m^+(2)/w(m)$). The random variables $(\xi_k^{\pm}(2),\, k \geq 0)$ being i.i.d, this modification does not change the law of $\bar{X}$ (some random variables $\xi_k^{\pm}(2)$ are simply never used).

Fix some $0 < \varepsilon < 1$ and consider the two w-urn processes $u_1 := (\xi_n^{\pm}(1)/w(n), n \geq 0)$, and $u_3 := (\xi_n^{\pm}(3)/w(n), n \geq 0)$. Since $\bar{Y}_\infty^+(0)$ is stochastically smaller than it would be for the process reflected in $[\![0,2]\!]$, using similar arguments as in the proof of Proposition 4.5, we see that there exists a set $E_1 \subset (\mathbb{R}_+^2)^{\mathbb{N}}$, such that the event $\mathcal{E}_1 := \{u_1 \in E_1\}$ has positive probability and on which $\bar{Y}_\infty^+(0) \leq \varepsilon^3$. By symmetry, there exists a set E_2, such that $\mathcal{E}_2 := \{u_3 \in E_2\}$ has positive probability and on which $\bar{Y}_\infty^-(4) \leq \varepsilon^3$. By independence of the urns u_1 and u_3, the event $\mathcal{E}_1 \cap \mathcal{E}_2$ also has positive probability. In view of Lemma 3.7, it remains to prove that, on this event, $\bar{X}$ visits all the sites of $[\![0,4]\!]$ infinitely often with positive probability.

We now consider the urn process $u_2 := (\xi_n^{\pm}(2)/w(n), n \geq 0)$. Recall that, according to (4), we may express the limit $\hat{M}_\infty(2)$ of this urn in the form:

$$\hat{M}_\infty(2) = \sum_{n\geq 0}\left(\frac{1-\xi_n^+(2)}{w(n)}\right) - \sum_{n\geq 0}\left(\frac{1-\xi_n^-(2)}{w(n)}\right). \tag{15}$$

Similarly, it is not difficult to check that we can also express the limit of the martingale $\bar{M}_\infty(2) := \lim_{n\to\infty}(\bar{Y}_n^+(2) - \bar{Y}_n^-(2))$ in the form

$$\bar{M}_\infty(2) = \sum_{n\geq 0}\left(\frac{1-\xi_{c_n}^+(2)}{w(c_n)}\right) - \sum_{n\geq 0}\left(\frac{1-\xi_{d_n}^-(2)}{w(d_n)}\right). \tag{16}$$

where $(c_n, n \geq 0)$ and $(d_n, n \geq 0)$ are the increasing (random) sequences such that $\bar{Y}_n^+(2) = \sum_{c_k \leq n} 1/w(c_k)$ and $\bar{Y}_n^-(2) = \sum_{d_k \leq n} 1/w(d_k)$. The idea now is to compare $\bar{M}_\infty(2)$ and $\hat{M}_\infty(2)$ and prove that, on the event $\mathcal{E}_1 \cap \mathcal{E}_2$ their values are close. Then we will use the fact that $\hat{M}_\infty(2)$ has a density to deduce that $\bar{M}_\infty(2)$ can be smaller than $\alpha_c(w)$.

Subtracting (16) from (15), we find that

$$\hat{M}_\infty(2) - \bar{M}_\infty(2) = \sum_{n\geq 0}\left(\frac{1-\xi_{i_n}^+(2)}{w(i_n)}\right) + \sum_{n\geq 0}\left(\frac{1-\xi_{j_n}^-(2)}{w(j_n)}\right),$$

where $(i_n, n \geq 0)$ and $(j_n, n \geq 0)$ are the complementary sequences of $(c_n, n \geq 0)$ and $(d_n, n \geq 0)$. Moreover, using relation (8), we have

$$\bar{Y}_\infty^+(0) = \sum_n \frac{1}{w(j_n)} \quad \text{and} \quad \bar{Y}_\infty^-(4) = \sum_n \frac{1}{w(i_n)}.$$

But, using similar arguments as in the proof of (10), we obtain

$$\mathbb{E}\left[\left(\sum_{n\geq 0}\frac{1-\xi_{i_n}^+(2)}{w(i_n)}\right)^2 \;\Big|\; \mathcal{E}_1 \cap \mathcal{E}_2\right] = \mathbb{E}\left[\sum_{n\geq 0}\frac{1}{w(i_n)^2} \;\Big|\; \mathcal{E}_1 \cap \mathcal{E}_2\right]$$
$$\leq \frac{1}{w(0)}\mathbb{E}\left[\sum_{n\geq 0}\frac{1}{w(i_n)} \;\Big|\; \mathcal{E}_1 \cap \mathcal{E}_2\right] \leq \mathbb{E}[\bar{Y}_\infty^-(4) \mid \mathcal{E}_1 \cap \mathcal{E}_2] \leq \varepsilon^3.$$

Using Tchebychev's inequality, we deduce

$$\mathbb{P}\left\{|\hat{M}_\infty(2) - \bar{M}_\infty(2)| \geq 2\varepsilon \mid \mathcal{E}_1 \cap \mathcal{E}_2\right\} \leq \varepsilon.$$

Recalling that $\hat{M}_\infty(2)$ has a density with support on the whole of $\mathbb{R}$ (*c.f.* Proposition 2.1), we can pick $\eta > 0$ such that $\mathbb{P}\{|\hat{M}_\infty(2)| \leq \eta\} = 2\varepsilon$. This yields

$$\mathbb{P}\Big\{|\bar{M}_\infty(2)| \geq \eta + 2\varepsilon \mid \mathcal{E}_1 \cap \mathcal{E}_2\Big\} \leq 1 - 2\varepsilon + \varepsilon \leq 1 - \varepsilon,$$

so the set $\mathcal{E}_3 := \mathcal{E}_1 \cap \mathcal{E}_2 \cap \{|\bar{M}_\infty(2))| \leq \eta + 2\epsilon\}$ has positive probability. Moreover, we have,

$$W(\bar{Z}_n(1)) - W(\bar{Z}_n(3)) = \bar{Y}_n^+(0) - \bar{Y}_n^-(4) - \bar{M}_n(2),$$

and therefore, for all n large enough, on $\mathcal{E}_3$,

$$\bar{Z}_n(2) \leq \bar{Z}_n(1) + \bar{Z}_n(3) \leq \bar{Z}_n(1) + W^{-1}(W(\bar{Z}_n(1)) + \eta + 4\varepsilon).$$

Choosing ε small enough such that $\delta := \eta + 4\varepsilon < \alpha_c(w)$, we obtain, for N large

$$\begin{aligned}
\bar{Y}_\infty^+(1) &\geq \sum_{n\geq N} \frac{1_{\{\bar{X}_n=1,\bar{X}_{n+1}=2\}}}{w(\bar{Z}_n(1) + W^{-1}(W(\bar{Z}_n(1)) + \delta))} \\
&\geq \sum_{n\geq N} \frac{1}{w(n + W^{-1}(W(n) + \delta))} - \sum_{n\geq N} \frac{1_{\{\bar{X}_n=1,\bar{X}_{n+1}=0\}}}{w(\bar{Z}_n(1))} \\
&\geq \sum_{n\geq N} \frac{1}{w(n + W^{-1}(W(n) + \delta))} - \bar{Y}_\infty^+(0).
\end{aligned}$$

It follows from Lemma 5.6 of the Appendix that $\bar{Y}_\infty^+(1)$ is infinite on $\mathcal{E}_3$. By symmetry, we also have $\bar{Y}_\infty^-(3) = \infty$ on $\mathcal{E}_3$. Therefore, according to Lemma 3.3, on $\mathcal{E}_3$, the VRRW on $[\![0, 4]\!]$ visits every site infinitely often. This concludes the proof of the proposition. □

Proposition 4.7. *Assume that w is non-decreasing and that* (1) *holds. Then*

$$\alpha_c(w) < \infty \quad \Longrightarrow \quad |R'| \in \{4, 5\} \text{ almost surely.}$$

Proof. We first argue that, if $\alpha_c(w) < \infty$, then R' is a.s. finite and non-empty. Indeed, recalling Lemma 3.6, each time X visits a new site, say $x > 0$, as long as it does not visit $x + 2$, the restriction of X to the set $\{x - 1, x, x + 1\}$ can be coupled with a w-urn process in such a way that it always makes less jumps to $x + 1$ than the urn process. Then, Corollary 2.2 and Lemma 3.3 insure that X never visits $x + 2$ with a positive probability uniformly bounded from below by a constant depending only on this urn process (and therefore which does not depend on the past trajectory of X before its first visit to x). It follows that, a.s., $\limsup X_n < \infty$ and by symmetry $\liminf X_n > -\infty$. Hence the walk localizes on a finite set almost surely.

Let us now assume, by contradiction, that $|R'| = N + 1 \geq 6$ with positive probability. Thus, according to Lemma 3.7, there exists some initial local time configuration $\mathcal{C}$ such that, for the $\mathcal{C}$-VRRW $\bar{X}$ on $[\![0, N]\!]$, the event

$$\mathcal{E} := \{\bar{Y}_\infty^+(0) + \bar{Y}_\infty^-(N) < \infty\} \cap \{\bar{X} \text{ visits } 0 \text{ and } N \text{ i.o.}\}$$

has positive probability. Moreover, using (9), we have

$$\begin{array}{rcl} W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) & \equiv & \bar{Y}_n^-(3) \\ W(\bar{Z}_n(3)) - W(\bar{Z}_n(1)) & \equiv & \bar{Y}_n^-(4) - \bar{Y}_n^+(0). \end{array}$$

On the event $\mathcal{E}$, the quantity $\bar{Y}_\infty^+(0)$ is finite whereas $\bar{Y}_\infty^-(3)$ and $\bar{Y}_\infty^-(4)$ are infinite according to Lemma 3.3 since $N \geq 5$. Thus, for all $A > 0$, the stopping time

$$T_A := \inf \left\{ n \geq 0 \ : \ \begin{array}{c} \bar{X}_n = 2 \\ W(\bar{Z}_n(0)) \leq W(\bar{Z}_n(2)) - A \\ W(\bar{Z}_n(1)) \leq W(\bar{Z}_n(3)) - A \end{array} \right\}$$

is finite on $\mathcal{E}$. We claim that

$$\mathbb{P}\{\bar{Y}_\infty^+(1) = \infty \mid T_A < \infty\} \to 0 \quad \text{as } A \to \infty. \tag{17}$$

For the time being, assume that (17) holds. As before, Lemma 3.3 states that, on $\mathcal{E}$, we have $\bar{Y}_\infty^+(1) = \infty$. Thus, for all $A > 0$, we get

$$\mathbb{P}\{\mathcal{E}\} \leq \mathbb{P}\left\{\{\bar{Y}_\infty^+(1) = \infty\} \cap \{T_A < \infty\}\right\} \leq \mathbb{P}\{\bar{Y}_\infty^+(1) = \infty \mid T_A < \infty\},$$

which yields $\mathbb{P}\{\mathcal{E}\} = 0$ and contradicts the initial assumption that the walk localizes with positive probability on more than 5 sites.

It remains to prove (17). For $A > 0$, consider a process $\bar{X}^A$ which is, up to time T_A, equal to the VRRW $\bar{X}$ on $[\![0, N]\!]$ and which, after time T_A, has the transition of the VRRW restricted on $[\![0, 2]\!]$. In view of Lemma 3.6, we can construct $\bar{X}^A$ together with $\bar{X}$ in such way that, with obvious notation,

$$\bar{Y}_\infty^+(0) \leq \bar{Y}_\infty^{A,+}(0) \quad \text{and} \quad \bar{Y}_\infty^+(1) \leq \bar{Y}_\infty^{A,+}(1).$$

After time T_A, the process $\bar{X}^A$ is simply an urn process. Hence, using the same arguments as in the proof of Proposition 2.1, for $n \geq T_A$, the process

$$\hat{M}_n^A := W(\bar{Z}_n^A(0)) - W(\bar{Z}_n^A(2))$$

is a martingale with quadratic variation bounded by $2\sum_n 1/w(n)^2$. Noticing that, by definition of T_A, we have $\hat{M}_{T_A}^A \leq -A$, the maximal inequality for martingales shows that, for any $\varepsilon > 0$, there exists a constant $C > 0$, such that for all $A > 0$,

$$\mathbb{P}\left\{ \sup_{n \geq T_A} W(\bar{Z}_n^A(0)) - W(\bar{Z}_n^A(2)) \geq -A + C \mid \mathcal{F}_{T_A} \right\} \leq \varepsilon. \tag{18}$$

Moreover, for every odd integer $n \geq T_A$, we have $\bar{X}_n^A = 1$ from which we deduce that, for all $n \geq T_A$,

$$\bar{Z}_n^A(1) \ \geq \ \bar{Z}_n^A(2) + \bar{Z}_n^A(0) + \bar{Z}_{T_A}^A(1) - \bar{Z}_{T_A}^A(0) - \bar{Z}_{T_A}^A(2) \ \geq \ \bar{Z}_n^A(2) - \bar{Z}_{T_A}^A(2).$$

On the event $\{\sup_{n \geq T_A} W(\bar{Z}_n^A(0)) - W(\bar{Z}_n^A(2)) < -A + C\}$, we get, for $n \geq T_A$,

$$\bar{Z}_n^A(1) \geq W^{-1}(W(\bar{Z}_n^A(0) + A - C) - \bar{Z}_{T_A}^A(2).$$

This yields

$$\begin{array}{rcl} \bar{Y}_\infty^{A,+}(0) & = & \bar{Y}_{T_A}^{A,+}(0) + \displaystyle\sum_{n > T_A} \frac{1_{\{\bar{X}_n^A = 0\}}}{w(\bar{Z}_n^A(1))} \\ & \leq & \bar{Y}_{T_A}^{A,+}(0) + \displaystyle\sum_{n \geq 0} \frac{1}{w(W^{-1}(W(n) + A - C) - \bar{Z}_{T_A}^A(2))} \end{array}$$

with the convention $w(x) = w(0)$ for $x \leq 0$. Thus, according to Lemma 5.5 of the appendix, for $A > \alpha_c(w) + C$, we have $\bar{Y}^{A,+}_\infty(0) < \infty$ on the event $\{T_A < \infty\} \cap \{\sup_{n \geq T_A} W(\bar{Z}^A_n(0)) - W(Z^A_n(2)) < -A + C\}$. Using (18), we obtain

$$\mathbb{P}\left\{\bar{Y}^{A,+}_\infty(0) = \infty \,|\, T_A < \infty\right\} \leq \varepsilon. \tag{19}$$

We can now choose $A_0 > \alpha_c(w) + C$ and $K > 0$ such that

$$\mathbb{P}\{\bar{Y}^{A_0,+}_\infty(0) \geq K \,|\, T_{A_0} < \infty\} \leq 2\varepsilon.$$

Notice that for $A > A_0$, the random variable $\bar{Y}^{A,+}_\infty(0)$ is stochastically dominated by $\bar{Y}^{A_0,+}_\infty(0)$ (this is again a consequence of Lemma 3.6 using the same time-line construction for $\bar{X}^A$ and $\bar{X}^{A_0}$). Moreover, by hypothesis,

$$\mathbb{P}\{T_A < \infty\} \geq \mathbb{P}\{\mathcal{E}\} := c > 0,$$

hence

$$\forall A > A_0, \quad \mathbb{P}\{\bar{Y}^{A,+}_\infty(0) \geq K \,|\, T_A < \infty\} \leq 2\varepsilon/c. \tag{20}$$

Finally, we consider a third process $\tilde{X}^A$ which coincides up to time T_A with $\bar{X}$ and $\bar{X}^A$, and which, after time T_A, has the transition of the VRRW restricted on $[\![0, 3]\!]$. Again, we can construct these processes in such way that $\bar{Y}^{A,+}_\infty(0)$ stochastically dominates $\tilde{Y}^{A,+}_\infty(0)$. This domination implies, that (20) also holds with $\tilde{Y}^{A,+}_\infty(0)$ in place of $\bar{Y}^{A,+}_\infty(0)$. Moreover,

$$\tilde{M}^A_n(2) := W(\tilde{Z}^A_n(3)) - W(\tilde{Z}^A_n(1)) + \tilde{Y}^{A,+}_n(0) \qquad n \geq T_A,$$

is a martingale with bounded quadratic variation. As before, we deduce from the maximal inequality for martingales, that, for some constant $C' > 0$ depending only on ε and the weight function w,

$$\mathbb{P}\left\{\inf_{n \geq T_A} \tilde{M}^A_n(2) - \tilde{M}^A_{T_A}(2) \leq -C' \;\middle|\; \mathcal{F}_{T_A}\right\} \leq \varepsilon.$$

Using the facts that $\tilde{M}^A_{T_A}(2) \geq A$ and that $\tilde{Y}^{A,+}_n(0) \geq K$ with probability smaller than $2\varepsilon/c$ on the event $\{T_A < \infty\}$, we obtain

$$\mathbb{P}\left\{\inf_{n \geq T_A} W(\tilde{Z}^A_n(3)) - W(\tilde{Z}^A_n(1)) \leq -C' - K + A \;\middle|\; T_A < \infty\right\} \leq \varepsilon',$$

with $\varepsilon' = \varepsilon(1 + 2/c)$. We now fix A large enough such that $A - C' - K > \alpha_c(w)$. Using the trivial relation $\tilde{Z}^A_n(2) \geq \tilde{Z}^A_n(3) - \tilde{Z}^A_{T_A}(3)$ for $n \geq T_A$, we deduce, in the same way as for the proof of (19), that

$$\mathbb{P}\left\{\tilde{Y}^{A,+}_\infty(1) = \infty \;\middle|\; T_A < \infty\right\} \leq \varepsilon'.$$

We conclude the proof of (17) by noticing that $\tilde{Y}^{A,+}_\infty(1)$ stochastically dominates $\bar{Y}^+_\infty(1)$. □

Lemma 4.8. *Assume that w is non-decreasing and that $\sum_n 1/w(n)^2 < \infty$. Fix $\infty \leq a < 0 < b \leq \infty$ and let $\bar{X}$ be a $\mathcal{C}$-VRRW on $[\![a, b]\!]$ for some initial local time configuration $\mathcal{C}$. Set*

$$\delta := \lim_{n \to \infty} W(\bar{Z}_n(1)) - W(\bar{Z}_n(-1)) \quad \textit{when the limit exists.}$$

Then, for any $\delta_0 \in \mathbb{R}$, we have

$$\mathbb{P}\big\{\{\delta \textit{ exists and equals } \delta_0\} \cap \{\bar{Z}_\infty(0) = \infty\}\big\} = 0.$$

Proof. Since the weights w and λw (for $\lambda > 0$) define the same VRRW, we assume, without loss of generality that $w(Z_0(1)) = 1$. Recalling the time-line construction described in Section 3, we create the $\mathcal{C}$-VRRW $\bar{X}$ on $[\![a,b]\!]$ from a collection $((\xi_n^{\pm}(y), n \geq 0), y \in]\!]a,b[\![)$. Set

$$\mathcal{H} := \big((\xi_n^{\pm}(y), n \geq 0, y \neq 0), (\xi_n^{-}(0), n \geq 0), (\xi_n^{+}(0), n \geq 1)\big) \in \mathbb{R}_+^{\mathbb{N}}$$

and let μ denote the product measure on $\mathbb{R}_+^{\mathbb{N}}$ under which $\mathcal{H}$ is a collection of i.i.d. exponential random variables with mean 1. Then, given $\mathcal{H}$ and some other variable $\xi_0^{+}(0)$, the pair $(\mathcal{H}, \xi_0^{+}(0))$ defines a process $\bar{X} = \bar{X}(\mathcal{H}, \xi_0^{+}(0))$ via the time-line construction which is a VRRW under the product probability $\mathbb{P} := \mu \times \mathrm{Exp}(1)$. For $u > 0$, define

$$\mathcal{B}_u = \{\mathcal{H} \in \mathbb{R}_+^{\mathbb{N}},\ \delta(\mathcal{H}, u) \text{ exists and equals } \delta_0 \text{ and } \bar{Z}_\infty(0) = \infty\}$$

and

$$\mathcal{B} = \{(\mathcal{H}, u) \in \mathbb{R}_+^{\mathbb{N}} \times \mathbb{R}_+,\ \mathcal{H} \in \mathcal{B}_u\}.$$

We will prove that, for almost every $u > 0$ and $h > 0$,

$$\mu\{\mathcal{B}_u \cap \mathcal{B}_{u+h}\} = 0. \tag{21}$$

This equation implies that $\{u \in \mathbb{R}^+, \mu\{\mathcal{B}_u\} > 0\}$ has zero Lebesgue measure. Hence

$$\mathbb{P}\{\{\delta = \delta_0\} \cap \{\bar{Z}_\infty(0) = \infty\}\} = \mathbb{P}\{\mathcal{B}\} = \int_0^\infty e^{-u} \mu\{\mathcal{B}_u\} du = 0.$$

It remains to prove (21). Given $(\mathcal{H}, u)$ such that $\bar{Z}_\infty(1) = Z_\infty(-1) = \infty$, we can define, as in the proof of Lemma 3.3, the increasing sequences $(i_k^{\pm}, k \geq 0)$ such that, for all $n \geq 0$,

$$Y_n^{\pm}(0) = \sum_{i_k^{\pm} \leq n} \frac{1}{w(i_k^{\pm})}.$$

For $t > 0$, define also

$$z_t^{\pm} := \inf\left\{ n \ :\ \sum_{i_k^{\pm} \leq n} \frac{\xi_k^{\pm}(0)}{w(i_k^{\pm})} > t \right\}. \tag{22}$$

Thus, $z_t^{\pm}$ represents the local time at site ± 1 when the clock process attached to site 0 has consumed a time t. Hence, another way to define $z_t^{\pm}$ is to consider the continuous-time process $(\tilde{X}(s), s \geq 0)$ associated with $(\mathcal{H}, u)$ via the time-line construction (recall that $\bar{X}$ is deduced from $\tilde{X}$ by a change of time). Defining

$$\tau_t := \inf\Big\{ s > 0, \int_0^s 1_{\{\tilde{X}_s = 0\}} ds > t \Big\},$$

we get that $z_t^{\pm} = \tilde{Z}_{\tau_t}(\pm 1)$.

Let us notice that, on $\mathcal{B}$, $\mathbb{P}$-a.s., we have $Z_\infty(-1) = Z_\infty(1) = \infty$ so the sequences $(i_k^{\pm})$ are well defined for all $k \geq 0$. Moreover, on the event $\{Z_\infty(-1) = Z_\infty(1) = \infty\}$, the total time consumed by the clock process at site 0 is infinite $\mathbb{P}$-a.s. (see the proof of Lemma 3.3). Hence, on $\mathcal{B}$, the random variables $z_t^{\pm}$ are finite for all $t > 0$, $\mathbb{P}$-a.s. Define

$$\delta_t := W(z_t^{+}) - W(z_t^{-}) = W(\tilde{Z}_{\tau_t}(1)) - W(\tilde{Z}_{\tau_t}(-1)).$$

By definition of δ, we get

$$\lim_{t \to \infty} \delta_t = \delta. \qquad \mathbb{P}\text{-a.s. on } \mathcal{B}.$$

Thus, for almost any $u > 0$ (with respect to the Lebesgue measure), we have

$$\lim_{t\to\infty} \delta_t(\mathcal{H}, u) = \delta(\mathcal{H}, u) \qquad \text{for } \mu\text{-a.e. } \mathcal{H} \in \mathcal{B}_u. \tag{23}$$

Now let $u, h > 0$ be fixed and such that (23) holds for u and $u + h$. Pick $\mathcal{H} \in \mathcal{B}_u \cap \mathcal{B}_{u+h}$. Lemma 3.2 implies that, for all $k \geq 0$, we have

$$i_k^+(\mathcal{H}, u+h) \leq i_k^+(\mathcal{H}, u) \quad \text{ and } \quad i_k^-(\mathcal{H}, u+h) \geq i_k^-(\mathcal{H}, u).$$

Recalling that $w(\bar{Z}_0(1)) = w(i_0^+) = 1$, we deduce from (22) that, for $t > 0$,

$$z_t^+(\mathcal{H}, u+h) \leq z_{t-h}^+(\mathcal{H}, u) \quad \text{ and } \quad z_t^-(\mathcal{H}, u+h) \geq z_t^-(\mathcal{H}, u).$$

This yields

$$\begin{aligned}
\delta_t(\mathcal{H}, u) - \delta_t(\mathcal{H}, u+h) &\geq W(z_t^+(\mathcal{H}, u)) - W(z_{t-h}^+(\mathcal{H}, u)) \\
&\geq \sum_{z_{t-h}^+ \leq k < z_t^+} \frac{1}{w(k)} \\
&\geq \sum_{z_{t-h}^+ \leq i_k^+ < z_t^+} \frac{1}{w(i_k^+)} := \Delta_{u,h}^+(t),
\end{aligned}$$

where z_{t-h}^+, z_t^+ and i_k^+ stand for $z_{t-h}^+(\mathcal{H}, u), z_t^+(\mathcal{H}, u)$ and $i_k^+(\mathcal{H}, u)$. In view of (23), we deduce that, for almost every $u, h > 0$, we have

$$\mu\{\mathcal{B}_u \cap \mathcal{B}_{u+h}\} \leq \mu\big\{\mathcal{B}_u \cap \{\limsup_{t\to\infty} \Delta_{u,h}^+(t) = 0\}\big\}.$$

It remains to prove that the r.h.s. in the previous inequality is equal to zero. For $\mathcal{H} \in \mathcal{B}_u$, the quantity

$$h_t^*(\mathcal{H}, u) := \sum_{z_{t-h}^+ \leq i_k^+ < z_t^+} \frac{\xi_k^+(0)}{w(i_k^+)}$$

is well defined. Moreover, it is clear that,

$$\lim_{t\to\infty} h_t^* = h \quad \mu\text{-a.s. on } \mathcal{B}_u. \tag{24}$$

On the other hand, we have,

$$|h_t^* - \Delta_{u,h}^+(t)|^2 1_{\mathcal{B}_u} \leq 2\left(\sum_{i_k^+ \geq z_{t-h}^+} \frac{1-\xi_k^+(0)}{w(i_k^+)} 1_{\{\theta_k<\infty\}}\right)^2 + 2\left(\sum_{i_k^+ \geq z_t^+} \frac{1-\xi_k^+(0)}{w(i_k^+)} 1_{\{\theta_k<\infty\}}\right)^2,$$

where θ_k denotes the time of the k-th jump of $\bar{X}$ from 1 to 0. Let $(\tilde{\mathcal{F}}_t, t > 0)$ denote the natural filtration of the continuous time process $\tilde{X}_t(\cdot, u)$. Using the same argument as in the proof of (10), we find that

$$\begin{aligned}
\mathbb{E}_\mu\left[(h_t^* - \Delta_{u,h}^+(t))^2 1_{\mathcal{B}_u} \;\middle|\; \tilde{\mathcal{F}}_{\tau_{t-h}}\right] &\leq 4\mathbb{E}_\mu\left[\sum_{i_k^+ \geq z_{t-h}^+} \frac{1_{\{\theta_k<\infty\}}}{w(i_k^+)^2} \;\middle|\; \tilde{\mathcal{F}}_{\tau_{t-h}}\right] \\
&\leq \sum_{k \geq z_{t-h}^+} \frac{4}{w(k)^2}.
\end{aligned}$$

This yields

$$\mu\left\{|h_t^* - \Delta_{u,h}^+(t)|1_{\mathcal{B}_u} \geq h/2 \;\Big|\; \tilde{\mathcal{F}}_{\tau_{t-h}}\right\} \leq \frac{16}{h^2}\sum_{k\geq z_{t-h}^+}\frac{1}{w(k)^2}.$$

Hence, by monotone convergence,

$$\lim_{t\to\infty}\mu\left\{|h_t^* - \Delta_{u,h}^+(t)|1_{\mathcal{B}_u} \geq h/2\right\} = 0 \tag{25}$$

Combining (24) and (25), we conclude that

$$\begin{aligned}&\lim_{t\to\infty}\mu\left\{\mathcal{B}_u\cap\{\Delta_{u,h}^+(t)\leq h/4\}\right\}\\ &\quad\leq \lim_{t\to\infty}\mu\left\{\mathcal{B}_u\cap\{|h_t^* - \Delta_{u,h}^+(t)|\geq h/2\}\right\} + \lim_{t\to\infty}\mu\left\{\mathcal{B}_u\cap\{h_t^*\leq 3h/4\}\right\} = 0,\end{aligned}$$

which implies $\mu\{\mathcal{B}_u\cap\{\limsup_{t\to\infty}\Delta_{u,h}^+(t)=0\}\} = 0$. □

We can now prove the last part of Theorem 1.1.

Proposition 4.9. *Assume that w is non-decreasing and that* (1) *holds. Then*

$$\alpha_c(w) = 0 \quad\Longrightarrow\quad |R'|\neq 5 \text{ almost surely.}$$

Proof. Assume by contradiction that $\alpha_c(w) = 0$ and that $|R'| = 5$ holds with positive probability. Thus there exists an initial local time configuration $\mathcal{C}$ such that, for the $\mathcal{C}$-VRRW $\bar{X}$ on $[\![0,4]\!]$, the event

$$\mathcal{E} := \{\bar{Y}_\infty^+(0) + \bar{Y}_\infty^-(4) < \infty\}\cap\{\bar{X}\text{ visits 0 and 4 i.o.}\}$$

has positive probability. Moreover, Equation (9) yields, for $n\geq 0$,

$$W(\bar{Z}_n(1)) - W(\bar{Z}_n(3)) = \bar{Y}_n^+(0) - \bar{M}_n(2) - \bar{Y}_n^-(4) + c,$$

for some constant c depending on the initial configuration. On the event $\mathcal{E}$, each term on the r.h.s. of this equation converges to a limit, thus

$$\lim_{n\to\infty} W(\bar{Z}_n(1)) - W(\bar{Z}_n(3)) = \bar{Y}_\infty^+(0) - \bar{M}_\infty(2) - \bar{Y}_\infty^-(4) + c =: \delta$$

exists and is finite. Moreover, Lemma 4.8 implies that $\mathbb{P}\{\{\delta = 0\}\cap\mathcal{E}\} = 0$. Let us now prove that the event $\mathcal{E}\cap\{\delta > 0\}$ has probability 0 (the same result holds for $\delta < 0$ by symmetry). On this event, for n large enough, we get

$$W(\bar{Z}_n(1)) \geq W(\bar{Z}_n(3)) + \delta', \tag{26}$$

with $\delta' = \delta/2$. Besides, we have

$$\begin{aligned}W(\bar{Z}_n(2)) - W(\bar{Z}_n(0)) &\equiv \bar{Y}_n^-(3)\\ W(\bar{Z}_n(2)) - W(\bar{Z}_n(4)) &\equiv \bar{Y}_n^+(1).\end{aligned}$$

Since, on $\mathcal{E}$, the quantities $\bar{Y}_\infty^-(3)$ and $\bar{Y}_\infty^+(1)$ are infinite (*c.f.* Lemma 3.3), we deduce that $\bar{Z}_n(2)$ is larger than $\bar{Z}_n(0)$ and $\bar{Z}_n(4)$ for n large enough. Moreover, by periodicity, the sum of these three quantities is, up to a constant, equal to $n/2$. Thus, for n large enough, we obtain

$$3\bar{Z}_n(2) \geq \frac{n}{2} \geq \bar{Z}_n(1).$$

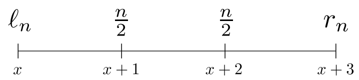
Using (26), we get, for n large enough, on the event $\mathcal{E}\cap\{\delta > 0\}$,

$$\frac{1_{\{\bar{X}_n=3,\,\bar{X}_{n+1}=2\}}}{w(\bar{Z}_n(2))} \quad\leq\quad \frac{1_{\{\bar{X}_n=3\}}}{w(\bar{Z}_n(1)/3)} \leq \frac{1_{\{\bar{X}_n=3\}}}{w(W^{-1}(W(\bar{Z}_n(3))+\delta')/3)}.$$

Since $\alpha_c(w) = 0$, it follows from Lemma 5.5 of the Appendix that $\bar{Y}^-_\infty(3)$ is finite which contradicts the fact that $\bar{X}$ visits all the sites of $[\![0,4]\!]$ infinitely often. □

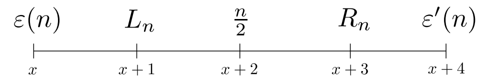
Theorem 1.1 is now a consequence of Propositions 4.5, 4.6, 4.7 and 4.9. Let us conclude this section by remarking that we can also describe the shape of the asymptotic local time configuration when $\alpha_c(w) < \infty$. Indeed, collecting the results obtained during the proof of the theorem, it is not difficult to check (the details being left out for the reader) that the asymptotic local time profile of the walk on R' at time n takes the form:

Localization on 4 sites



$W(n/2) - W(\ell_n) \to \delta$
$W(n/2) - W(r_n) \to \delta'$
$\delta, \delta' > \alpha_c(w)$

Localization on 5 sites



$L_n + R_n \sim n/2$
$W(R_n) - W(L_n) \to \delta$
$0 < |\delta| < \alpha_c(w)$

In particular, when the walk localizes on 4 sites, only the two central sites are visited a non-negligible proportion of time (this follows from (30) of Lemma 5.5 of the appendix). When the walk localizes on 5 sites, a more unusual behaviour may happen. If the weight function is regularly varying (for example $w(n) \sim n \log\log n$), then, again, the walk spends asymptotically all its time on two consecutive vertices:

$$\lim_{n\to\infty} \frac{L_n \wedge R_n}{n} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{L_n \vee R_n}{n} = \frac{1}{2}.$$

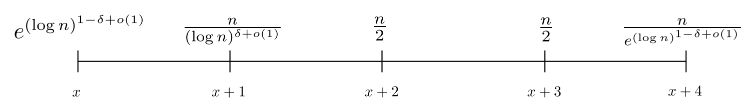
However, this result is not true for general weight functions. In fact, the ratio $Z_n(y)/n$ of time spent at site y may not converge. For instance, considering the weight sequence w_0 of Remark 5.4 of the appendix, we find that when the walk localizes on 5 sites:

$$\liminf_{n\to\infty} \frac{L_n \wedge R_n}{n} = 0 \qquad \text{but} \qquad \limsup_{n\to\infty} \frac{L_n \wedge R_n}{n} > 0.$$

Finally, let us mention that the functions $\varepsilon(n)$ and $\varepsilon'(n)$ can also be explicitly computed for particular weights sequences. For example, for $w(n) = n \log\log n$, using (9) and similar arguments as in the proof of (10) , we find that

$$W(\varepsilon(n)) \equiv Y_n^+(x+1) \equiv \sum_{k=0}^{n/2} \frac{p_k}{w(k)} \quad \text{and} \quad W(\varepsilon'(n)) \equiv W(n/2) - W(\varepsilon(n))$$

where p_k denotes the probability of the walk to jump to site $x+1$ at its k-th visit to $x+2$ (*i.e.* $p_k \sim L_{2k}/k$). Thus, after some (rather tedious) calculations, we deduce that, on the event $\delta := \lim_{n\to\infty} W(R_n) - W(L_n) \in (0,1)$, the asymptotic local time profile on R' takes the form:



5. Appendix

Proposition 5.1. *Let w and $\widetilde{w}$ denote two non-decreasing weight functions.*

(a) *For any $\lambda > 0$, we have $\alpha_c(w) = \lambda \alpha_c(\lambda w)$ **(scaling)**.*
(b) *If $w \le \widetilde{w}$, then $\alpha_c(w) \ge \alpha_c(\widetilde{w})$ **(monotonicity)**.*
(c) *If $w \sim \widetilde{w}$, then $\alpha_c(w) = \alpha_c(\widetilde{w})$ **(asymptotic equivalence)**.*

Proof. The scaling property (a) follows directly from the relation $\lambda I_\alpha(\lambda w) = I_{\lambda\alpha}(w)$. We now prove (b). For $x \ge 0$ and $\alpha > 0$, set $u(x,\alpha) := W^{-1}(W(x)+\alpha)$ and define $\tilde{u}(x,\alpha)$ similarly for $\tilde{w}$. We have,

$$\int_x^{u(x,\alpha)} \frac{1}{w(t)}\, dt = \int_x^{\widetilde{u}(x,\alpha)} \frac{1}{\widetilde{w}(t)}\, dt = \alpha. \tag{27}$$

When $w \le \widetilde{w}$, the equality above implies that $u(x,\alpha) \le \widetilde{u}(x,\alpha)$ for all x and all $\alpha > 0$. Since w is non-decreasing we get $w(u(x,\alpha)) \le \widetilde{w}(\widetilde{u}(x,\alpha))$, hence $I_\alpha(w) \ge I_\alpha(\widetilde{w})$. This establishes (b).

Suppose now that w and $\tilde{w}$ are two weight functions such that $w(x) = \tilde{w}(x)$ for all x larger than some x_0. Then, in view of (27), we see that $u(x,\alpha) = \tilde{u}(x,\alpha)$ for all $\alpha > 0$ and all $x \ge x_0$. Hence $\alpha_c(w) = \alpha_c(\tilde{w})$. This shows that $\alpha_c(w)$ does not depend upon the values taken by w on any compact interval $[0, x_0]$. Thus (c) follows directly from (a) and (b). □

Remark 5.2. *Theorem 1.1 states that when $\alpha_c(w)$ is finite and non-zero, the walk localizes on either 4 or 5 sites. It would certainly be interesting to estimate the probability of each of these events. This seems a difficult question. Let us remark that these probabilities are not (directly) related to $\alpha_c(w)$. Indeed, for any $\lambda > 1$, the weight functions w and λw define the same VRRW yet, $\alpha_c(w) \ne \alpha_c(\lambda w)$.*

Proof of Proposition 1.2. We just need to check that $\alpha_c(x \log\log x) = 1$. For $x \ge 1$, set

$$w(x) := x(1 + L(\log x)),$$

where L denotes the Lambert function defined as the solution of $L(x)e^{L(x)} = x$. Then it follows from elementary calculation that

$$\frac{w(W^{-1}(x))}{w(W^{-1}(x+\alpha))} = \frac{x^x(1+\log x)}{(x+\alpha)^{x+\alpha}(1+\log(x+\alpha))} \underset{x\to\infty}{\sim} \frac{e^{-\alpha}}{x^\alpha}.$$

Therefore $\alpha_c(w) = 1$. Using now the well known equivalence $L(x) \sim \log(x)$, we conclude using (c) of Proposition 5.1 that $\alpha_c(x \log\log x) = 1$. □

Lemma 5.3. *Assume that w is non-decreasing. We have*

$$\liminf_{x\to\infty} \frac{w(x)}{x} \ge \frac{1}{\alpha_c(w)}.$$

In particular, when $\alpha_c(w) < \infty$ then $\sum 1/w(n)^2 < \infty$ and if $\alpha_c(w) = 0$ then w has super-linear growth.

Proof. In view of the scaling property $\lambda I_\alpha(\lambda w) = I_{\lambda\alpha}(w)$, we just need to prove that $\liminf w(x)/x < 1$ implies $I_1(w) = \infty$. Thus, let us assume that for some $\varepsilon > 0$, there exist arbitrarily large x such that $w(x)/x \le 1 - \varepsilon$. Then, for such an x and for $y \le \varepsilon x$, we have, since w is non-decreasing,

$$W(y + (1-\varepsilon)x) - W(y) = \int_y^{y+(1-\varepsilon)x} \frac{dz}{w(z)} \ge \frac{(1-\varepsilon)x}{w(x)} \ge 1,$$

which we can rewrite as

$$\frac{1}{W^{-1}(W(y)+1)} \geq \frac{1}{y+(1-\varepsilon)x}.$$

Thus,

$$\int_{\varepsilon x/2}^{\varepsilon x} \frac{1}{w(W^{-1}(W(y)+1))} \geq \frac{\varepsilon}{2-\varepsilon}. \tag{28}$$

Since there exist arbitrarily large x such that (28) holds, we conclude that $I_1(w) = \infty$.

□

Remark 5.4. *The previous lemma cannot be improved without additional assumptions on w. Indeed, consider the weight function w_0 defined by $w_0(x) = (n!)^2$ for $x \in [((n-1)!)^2, (n!)^2)$, $n \in \mathbb{N}^*$. It is easily seen that*

$$\liminf_{x\to\infty} \frac{w_0(x)}{x} = 1 = \frac{1}{\alpha_c(w_0)}.$$

This provides an example of a weight function which does not uniformly grow faster than linearly and yet for which the VRRW localizes on 4 sites with positive probability. On the other hand, if w is assumed to be regularly varying, then using similar arguments to those in the proof above, one can check that the finiteness of $\alpha_c(w)$ implies $\lim_\infty w(x)/x = \infty$.

Lemma 5.5. *Assume that w is non-decreasing and $\alpha_c(w) < \infty$, for any $0 < \delta < \delta'$, we have*

$$\liminf_{x\to\infty} \frac{W^{-1}(W(x)+\delta')}{W^{-1}(W(x)+\delta)} \geq e^{\frac{\delta'-\delta}{\alpha_c(w)}}. \tag{29}$$

Furthermore, for $\delta > \alpha_c(w)$,

$$\lim_{x\to\infty} \frac{x}{W^{-1}(W(x)+\delta)} = 0. \tag{30}$$

As a consequence:

(a) *For any $\delta > \alpha_c(w)$ and any $c \in \mathbb{R}$,*

$$\sum_n^\infty \frac{1}{w(W^{-1}(W(n)+\delta)-c)} < \infty.$$

(b) *If $\alpha_c(w) = 0$, then, for any $\delta, \gamma > 0$,*

$$\sum_n^\infty \frac{1}{w(\gamma W^{-1}(W(n)+\delta))} < \infty.$$

Proof. Recall the notation $u(x,\delta) := W^{-1}(W(x)+\delta)$. We have

$$\int_{u(x,\delta)}^{u(x,\delta')} \frac{ds}{w(s)} = \delta' - \delta.$$

Using Lemma 5.3 and the fact that $u(x,\delta)$ tends to infinity as x goes to infinity. we get, for any $\alpha > \alpha_c(w)$,

$$\liminf_{x\to\infty} \log\left(\frac{u(x,\delta')}{u(x,\delta)}\right) = \liminf_{x\to\infty} \int_{u(x,\delta)}^{u(x,\delta')} \frac{ds}{s} \geq \lim_{x\to\infty} \int_{u(x,\delta)}^{u(x,\delta')} \frac{ds}{\alpha w(s)} = \frac{\delta'-\delta}{\alpha}$$

which yields (29). Assertion (a) now follows from (29) noticing that, for $\alpha_c(w) < \gamma < \delta$, we have $W^{-1}(W(n)+\delta) - c \geq W^{-1}(W(n)+\gamma)$ for all n large enough. The proof of Assertion (b) is similar.

It remains to prove (30). Let $\delta > \alpha_c(w)$ and pick $\varepsilon > 0$ small enough such that $\alpha_c(w) < \delta - \varepsilon$. Assume by contradiction that, for some $A > 0$, we can find x_0 arbitrarily large such that $u(x_0, \delta) \leq Ax_0$. Then, for all $x < x_0$,

$$\varepsilon = \int_{u(x,\delta-\varepsilon)}^{u(x,\delta)} \frac{dy}{w(y)} \leq \int_0^{Ax_0} \frac{dy}{w(u(x,\delta-\varepsilon))} = \frac{Ax_0}{w(u(x,\delta-\varepsilon))}$$

which, in turn, implies

$$\int_{x_0/2}^{x_0} \frac{dx}{w(u(x,\delta-\varepsilon))} \geq \frac{\varepsilon}{2A}$$

and contradicts the fact that $I_{\delta-\varepsilon}(w) < \infty$. □

Lemma 5.6. *Assume that w is non-decreasing. For any $\beta < \alpha_c(w)$, we have*

$$\sum_n^\infty \frac{1}{w(n + W^{-1}(W(n)+\beta))} = \infty.$$

Proof. Choose $\alpha \in (\beta, \alpha_c(w))$. Since w is non-decreasing, for any $t \geq 0$ and any $m \leq n$, we have

$$u(n,t) - u(m,t) \geq n - m. \tag{31}$$

Assume now that, for some large n, we have $n + u(n,\beta) \geq u(n,\alpha)$. Then, necessarily, there exists $k \in [u(n,\beta), u(n,\alpha)]$, such that $w(k) \leq n/(\alpha-\beta)$. In particular, since w is non-decreasing, $w(u(n,\beta)) \leq n/(\alpha-\beta)$. Moreover, using (31), we get $m + u(m,\beta) \leq u(n,\beta)$ for all $m \leq n/2$. Thus we also have $w(m+u(m,\beta)) \leq n/(\alpha-\beta)$, for all $m \leq n/2$. It follows that

$$\sum_{m=n/4}^{n/2} \frac{1}{w(m+u(m,\beta))} \geq \frac{\alpha-\beta}{4}.$$

Therefore if $n + u(n,\beta) \geq u(n,\alpha)$, for infinitely many n, the desired result follows. Conversely, if the inequality above holds only for finitely many n, then because w is non-decreasing, the result follows as well from the fact that $I_\alpha(w) = \infty$. □

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